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TRANSCENDENTAL NUMBER THEORY

INTRODUCTION BY DAVID MASSER

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TRANSCENDENTAL NUMBER THEORY

First published in 1975, this classic book gives a systematic account of transcendental number theory, that is, the theory of those numbers that cannot be expressed as the roots of algebraic equations having rational coefficients. Their study has developed into a fertile and extensive theory, which continues to see rapid progress today. Expositions are presented of theories relating to linear forms in the logarithms of algebraic numbers, of Schmidt's generalization of the Thue–Siegel–Roth theorem, of Shidlovsky's work on Siegel's E-functions and of Sprindžuk's solution to the Mahler conjecture.

This edition includes an introduction written by David Masser describing Baker's achievement, surveying the content of each chapter and explaining the main argument of Baker's method in broad strokes. A new afterword lists recent developments related to Baker's work.

Alan Baker was one of the leading British mathematicians of the past century. He took great strides in number theory by, among other achievements, obtaining a vast generalization of the Gelfond–Schneider Theorem and using it to give effective solutions to a large class of Diophantine problems. This work kicked off a new era in transcendental number theory and won Baker the Fields Medal in 1970.

David Masser is Professor Emeritus in the Department of Mathematics and Computer Science at the University of Basel. He is a leading researcher in transcendence methods and applications and helped correct the proofs of the original edition of *Transcendental Number Theory* as Baker's student.

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ALAN BAKER F.R.S.

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UNIVERSITY PRESS

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UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of
education, learning, and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781009229944
DOI: 10.1017/9781009229937

First published 1975
Reprinted with additional material 1979
Reissued as a paperback with updated material in the Cambridge Mathematical Library series 1990
Reprinted with introduction 2022

A catalogue record for this publication is available from the British Library.

ISBN 978-1-009-22994-4 Paperback

Contents

Introduction

David Masser

On the first page of the Bibliography are listed earlier works about some of the topics treated in this monograph. In particular the books of Gelfond, Schneider and Siegel are universally regarded as milestones in the development of the theory of transcendental numbers. Each book was based largely on the author's own breakthroughs.

The present monograph represented a similar milestone. Chapters 2, 3, 4, 5, 9, 10, and to a lesser extent Chapters 6, 8, cover material due to the author Alan Baker. This material and Baker's own further developments of it earned him a Fields Medal in 1970.

Of course it is the material in Chapter 2 that constitutes the heart of his achievement. This is explained in the first two pages with a characteristic brevity and modesty. Here we wish to complement this with the following less brief and modest account.

The essential ideas can be conveyed through the special case of his Theorem 2.1 for $n = 2, 3$, even ignoring the extra 1 that appears there.

We start with $n = 2$. It amounts to the impossibility of

$$\beta \log \alpha = \log \alpha' \tag{1}$$

for α, α' non-zero algebraic numbers and β irrational algebraic. Of course this is the Gelfond–Schneider Theorem of 1934. It also follows from Theorem 6.1 of Chapter 6, and we proceed to sketch the argument.

We assume (1) and we will obtain a contradiction. Following Gelfond we construct a non-zero polynomial F, say in $\mathbf{Z}[x, y]$, such that

$$f(z) = F(e^z, e^{\beta z}) \tag{2}$$

has many zeroes. More precisely we need the derivatives

$$f^{(t)}(s \log \alpha) = 0 \tag{3}$$

for a certain range of integers

$$t < T, \ \ 1 \leq s \leq S \tag{4}$$

with T, S integers to be suitably chosen later. Thus we have zeroes at $\log\alpha, \dots, S\log\alpha$ and moreover of multiplicity at least T. The point is that the functions $e^z, e^{\beta z}$ in (2) take the values

$$e^{\log\alpha} = \alpha, \ \ e^{\beta\log\alpha} = e^{\log\alpha'} = \alpha'$$

at say $z = \log\alpha$; and these are algebraic numbers. Similarly at $s\log\alpha$ and with multiplicities. Thus the conditions (3) are homogeneous linear equations in the coefficients of F. Under appropriate assumptions relating T, S to the degree of F, these can be solved non-trivially; and using things like Lemma 1 of Chapter 2 or Lemma 1 of Chapter 6 one can make sure that the resulting coefficients are not too large.

Next, Gelfond used analytic techniques to show that the values $f^{(t)}(s\log\alpha)$ are very small on a range larger than (3). Compare (8) of Chapter 2 and the use of Cauchy's Theorem in section 5 of Chapter 6. These values are still algebraic numbers, and then arithmetic techniques show that they are in fact zero. Compare Lemma 3 of Chapter 2 or Lemma 3 of Chapter 6 (nowadays one tends to use heights, with a definition slightly different from that of Chapter 1).

Together these lead to (3) for the new range

$$t < 2T, \ \ 1 \leq s \leq 2S \tag{5}$$

slightly larger than (4) (this is not quite consistent with Baker's remark near the end of section 1 of Chapter 2 or the method of section 5 of Chapter 6, but it simplifies the proof a little).

And now the step from (4) to (5) can be iterated, and even indefinitely. This provides infinite multiplicities, and so this f must be identically zero. Taking into account the irrationality of β, we see that this implies that F is also identically zero; our required contradiction.

Next for $n = 3$ we have to reach a similar contradiction from

$$\beta_1\log\alpha_1 + \beta_2\log\alpha_2 = \log\alpha' \tag{6}$$

instead of (1), where $\alpha_1, \alpha_2, \alpha'$ are non-zero algebraic and β_1, β_2 are algebraic, this time with $1, \beta_1, \beta_2$ linearly independent over $\mathbf{Q}$. Even this was a new result.

Baker's first step looks natural: to construct now

$$f(z_1, z_2) = F(e^{z_1}, e^{z_2}, e^{\beta_1 z_1 + \beta_2 z_2}) \tag{7}$$

instead of (2) with many zeroes; but no-one had written this down before. Still less had anyone considered multiplicities, now defined by the partial derivatives

$$\left(\frac{\partial}{\partial z_1}\right)^{t_1}\left(\frac{\partial}{\partial z_2}\right)^{t_2} f(s\log\alpha_1, s\log\alpha_2) = 0 \tag{8}$$

instead of (3). Note that there is no $(s_1 \log\alpha_1, s_2 \log\alpha_2)$ here, because we do not have the Cartesian product situation for $\mathbf{C}^2$ mentioned in section 1 of Chapter 2. In fact our $(s\log\alpha_1, s\log\alpha_2)$ lie on a complex line in $\mathbf{C}^2$.

Baker took a range

$$t_1 + t_2 < T, \ \ 1 \le s \le S \tag{9}$$

instead of (4), and the problem is then to increase this as in (5).

Now the experts know that the world of two complex variables is very different from that of a single variable. Possibly Baker did not know this. Anyway, to this day no-one knows how to reach $t_1 + t_2 < 2T$ as in (5).

He probably started by reducing to $\mathbf{C}$ via

$$g(z) = f(z\log\alpha_1, z\log\alpha_2). \tag{10}$$

Then we deduce

$$g^{(t)}(s) = 0 \quad (t < T, \ \ 1 \le s \le S).$$

The twin analytic-arithmetic argument then shows that $g(s) = 0$ for a larger range of s, that is, $f(s\log\alpha_1, s\log\alpha_2) = 0$. However, as it stands we cannot deduce even $g'(s) = 0$ because differentiation in (10) introduces transcendental numbers, so we cannot get at, say,

$$\left(\frac{\partial}{\partial z_1}\right) f(s\log\alpha_1, s\log\alpha_2) \tag{11}$$

in this way.

Now that we have set up the scene, Baker's solution to this problem may seem in retrospect obvious: we use (10) with f replaced by $(\partial/\partial z_1)f$. For the new g we get almost (9), but now only for $t < T - 1$. However this tiny loss does not affect the argument, and we find indeed that (11) vanishes on the larger range of s.

And what about higher derivatives? To get at some

$$\left(\frac{\partial}{\partial z_1}\right)^{\tau_1}\left(\frac{\partial}{\partial z_2}\right)^{\tau_2} f(s\log\alpha_1, s\log\alpha_2)$$

we use (10) with f replaced by $(\partial/\partial z_1)^{\tau_1}(\partial/\partial z_2)^{\tau_2} f$. We then get (9) for $t < T - \tau_1 - \tau_2$. If we aim for all τ_1, τ_2 just with $\tau_1 + \tau_2 < T$, then hardly anything remains of the multiplicity; so it is wiser to restrict to say $\tau_1 + \tau_2 < T/2$, thus securing $t < T/2$. Now the loss is less tiny, but still acceptable.

We end up with (8) on the range, say,

$$t_1 + t_2 < T/2, \;\; 1 \leq s \leq 8S. \tag{12}$$

As (9) is about $T^2S/2$ conditions and (12) is about T^2S conditions, we do actually have more zeroes.

But now another problem arises: we cannot iterate indefinitely the step from (9) to (12).

In fact a related problem had turned up before Baker; for example when trying to show that the two sides of (1) cannot even be too near to each other. And indeed Baker was able to extend the classical methods; in this case the argument of section 5 of Chapter 2 amounts to the use of a non-vanishing Vandermonde determinant.

All this extends to n logarithms, and then to include 1 as in Theorem 2.1. In our notation assuming

$$\beta_0 + \beta_1 \log \alpha_1 + \cdots + \beta_{n-1} \log \alpha_{n-1} = \log \alpha', \tag{13}$$

we have to use

$$f(z_0, z_1, \ldots, z_{n-1}) = F(z_0, e^{z_1}, \ldots, e^{z_{n-1}}, e^{\beta_0 z_0 + \beta_1 z_1 + \cdots + \beta_{n-1} z_{n-1}})$$

in place of (7) at $(s, s\log \alpha_1, \ldots, s \log \alpha_{n-1})$ – compare Lemma 2 of Chapter 2.

This completes our account of Chapter 2. On the way, we have mentioned the problem of approximate versions of (1), and the corresponding generalizations to (6) and (13) are treated in Chapter 3. It is these that are needed for the applications in Chapters 4 and 5.

It is these applications, to Diophantine equations and class numbers, that were the most spectacular of his achievements. It is enough here to cite the first ever upper bounds for the solutions of Mordell's equation $y^2 = x^3 + k$ with a history going back to 1621, and the verification of Gauss's conjectures from 1801 about imaginary quadratic fields with class numbers $h = 1$ and $h = 2$.

But one should not overlook the less spectacular material in Chapter 6, whose subsequent developments (by others) will be described in the afterword.

Preface

Fermat, Euler, Lagrange, Legendre ... introitum ad penetralia huius divinae scientiae aperuerunt, quantisque divitiis abundent patefecerunt

Gauss, Disquisitiones Arithmeticae

The study of transcendental numbers, springing from such diverse sources as the ancient Greek question concerning the squaring of the circle, the rudimentary researches of Liouville and Cantor, Hermite's investigations on the exponential function and the seventh of Hilbert's famous list of 23 problems, has now developed into a fertile and extensive theory, enriching widespread branches of mathematics; and the time has seemed opportune to prepare a systematic treatise. My aim has been to provide a comprehensive account of the recent major discoveries in the field; the text includes, more especially, expositions of the latest theories relating to linear forms in the logarithms of algebraic numbers, of Schmidt's generalization of the Thue–Siegel–Roth theorem, of Shidlovsky's work on Siegel's E-functions and of Sprindžuk's solution to the Mahler conjecture. Classical aspects of the subject are discussed in the course of the narrative; in particular, to facilitate the acquisition of a true historical perspective, a survey of the theory as it existed at about the turn of the century is given at the beginning. Proofs in the subject tend, as will be appreciated, to be long and intricate, and thus it has been necessary to select for detailed treatment only the most fundamental results; moreover, generally speaking, emphasis has been placed on arguments which have led to the strongest propositions known to date or have yielded the widest application. Nevertheless, it is hoped that adequate references have been included to associated works.

Notwithstanding its long history, it will be apparent that the theory of transcendental numbers bears a youthful countenance. Many topics would certainly benefit by deeper studies and several famous longstanding problems remain open. As examples, one need mention only the celebrated conjectures concerning the algebraic independence of e and π and the transcendence of Euler's constant γ, the solution to either of which would represent a major advance. If this book should play some small rôle in promoting future progress, the author will be well satisfied.

The text has arisen from numerous lectures delivered in Cambridge, America and elsewhere, and it has also formed the substance of an Adams Prize essay.

I am grateful to Dr D. W. Masser for his kind assistance in checking the proofs, and also to the Cambridge University Press for the care they have taken with the printing.

Cambridge, 1974 A.B.

1

THE ORIGINS

1. Liouville's theorem

The theory of transcendental numbers was originated by Liouville in his famous memoir† of 1844 in which he obtained, for the first time, a class, *très-étendue*, as it was described in the title of the paper, of numbers that satisfy no algebraic equation with integer coefficients. Some isolated problems pertaining to the subject, however, had been formulated long before this date, and the closely related study of irrational numbers had constituted a major focus of attention for at least a century preceding. Indeed, by 1744, Euler had already established the irrationality of e, and, by 1761, Lambert had confirmed the irrationality of π. Moreover, the early studies of continued fractions had revealed several basic features concerning the approximation of irrational numbers by rationals. It was known, for instance, that for any irrational α there exists an infinite sequence of rationals p/q $(q > 0)$ such that‡ $|\alpha - p/q| < 1/q^2$, and it was known also that the continued fraction of a quadratic irrational is ultimately periodic, whence there exists $c = c(\alpha) > 0$ such that $|\alpha - p/q| > c/q^2$ for all rationals p/q $(q > 0)$. Liouville observed that a result of the latter kind holds more generally, and that there exists in fact a limit to the accuracy with which any algebraic number, not itself rational, can be approximated by rationals. It was this observation that provided the first practical criterion whereby transcendental numbers could be constructed.

Theorem 1.1. *For any algebraic number α with degree $n > 1$, there exists $c = c(\alpha) > 0$ such that $|\alpha - p/q| > c/q^n$ for all rationals p/q $(q > 0)$.*

The theorem follows almost at once from the definition of an algebraic number. A real or complex number is said to be algebraic if it is a zero of a polynomial with integer coefficients; every algebraic

† *C.R.* **18** (1844), 883–5, 910–11; *J. Math. pures appl.* **16** (1851), 133–42. For abbreviations see page 130.

‡ This is in fact easily verified; for any integer $Q > 1$, two of the $Q+1$ numbers 1, $\{q\alpha\}$ $(0 \leqslant q < Q)$, where $\{q\alpha\}$ denotes the fractional part of $q\alpha$, lie in one of the Q subintervals of length $1/Q$ into which $[0, 1]$ can be divided, and their difference has the form $q\alpha - p$.

number α is the zero of some such irreducible† polynomial, say P, unique up to a constant multiple, and the degree of α is defined as the degree of P. It suffices to prove the theorem when α is real; in this case, for any rational p/q ($q > 0$), we have by the mean value theorem:

$$-P(p/q) = P(\alpha)-P(p/q) = (\alpha-p/q)\,P'(\xi)$$

for some ξ between p/q and α. Clearly one can assume that $|\alpha-p/q| < 1$, for the result would otherwise be valid trivially; then $|\xi| < 1+|\alpha|$ and thus $|P'(\xi)| < 1/c$ for some $c = c(\alpha) > 0$; hence

$$|\alpha-p/q| > c\,|P(p/q)|.$$

But, since P is irreducible, we have $P(p/q) \neq 0$, and the integer $|q^n P(p/q)|$ is therefore at least 1; the theorem follows. Note that one can easily give an explicit value for c; in fact one can take

$$c^{-1} = n^2(1+|\alpha|)^{n-1} H,$$

where H denotes the height of α, that is, the maximum of the absolute values of the coefficients of P.

A real or complex number that is not algebraic is said to be transcendental. In view of Theorem 1.1, an obvious instance of such a number is given by $\xi = \sum_{n=1}^{\infty} 10^{-n!}$. For if we write

$$p_j = 10^{j!}\sum_{n=1}^{j} 10^{-n!}, \qquad q_j = 10^{j!} \quad (j = 1, 2, \ldots),$$

then p_j, q_j are relatively prime rational integers and we have

$$\begin{aligned}|\xi-p_j/q_j| &= \sum_{n=j+1}^{\infty} 10^{-n!}\\ &< 10^{-(j+1)!}(1+10^{-1}+10^{-2}+\ldots) = \tfrac{10}{9}\,q_j^{-j-1} < q_j^{-j}.\end{aligned}$$

Many other transcendental numbers can be specified on the basis of Liouville's theorem; indeed any non-terminating decimal in which there occur sufficiently long blocks of zeros, or any continued fraction in which the partial quotients increase sufficiently rapidly, provides an example. Numbers of this kind, that is real ξ which possess a sequence of distinct rational approximations p_n/q_n ($n = 1, 2, \ldots$) such that $|\xi-p_n/q_n| < 1/q_n^{\omega_n}$, where $\limsup \omega_n = \infty$, have been termed Liouville numbers; and, of course, these are transcendental. But other,

† That is, does not factorize over the integers or, equivalently, by Gauss' lemma, over the rationals.

less obvious, applications of Liouville's idea to the construction of transcendental numbers have been described; in particular, Maillet[†] used an extension of Theorem 1.1 concerning approximations by quadratic irrationals to establish the transcendence of a remarkable class of quasi-periodic continued fractions.[‡]

In 1874, Cantor introduced the concept of countability and this leads at once to the observation that 'almost all' numbers are transcendental. Cantor's work may be regarded as the forerunner of some important metrical theory about which we shall speak in Chapter 9.

2. Transcendence of *e*

In 1873, there appeared Hermite's epoch-making memoir entitled *Sur la fonction exponentielle*[§] in which he established the transcendence of e, the natural base for logarithms. The irrationality of e had been demonstrated, as remarked earlier, by Euler in 1744, and Liouville had shown in 1840, directly from the defining series, that in fact neither e nor e^2 could be rational or a quadratic irrational; but Hermite's work began a new era. In particular, within a decade, Lindemann succeeded in generalizing Hermite's methods and, in a classic paper,[‖] he proved that π is transcendental and solved thereby the ancient Greek problem concerning the quadrature of the circle. The Greeks had sought to construct, with ruler and compasses only, a square with area equal to that of a given circle. This plainly amounts to constructing two points in the plane at a distance $\sqrt{\pi}$ apart, assuming that a unit length is prescribed. But, since all points capable of construction are defined by the intersection of lines and circles, it follows easily that their co-ordinates in a suitable frame of reference are given by algebraic numbers. Thus the transcendence of π implies that the quadrature of the circle is impossible.

The work of Hermite and Lindemann was simplified by Weierstrass[¶] in 1885, and further simplified by Hilbert,[††] Hurwitz[‡‡] and Gordan[§§] in 1893. We proceed now to demonstrate the transcendence of e and π in a style suggested by these later writers.

† See Bibliography.

‡ Cf. *Mathematika*, **9** (1962), 1–8.

§ *C.R.* **77**; = *Oeuvres* III, 150–81.

‖ *M.A.* **20** (1882), 213–25.

¶ *Werke* II, 341–62.

†† *Ges. Abh.* I, 1–4.

‡‡ *Göttingen Nachrichten* (1893), 153–5.

§§ *M.A.* **43** (1893), 222–5.