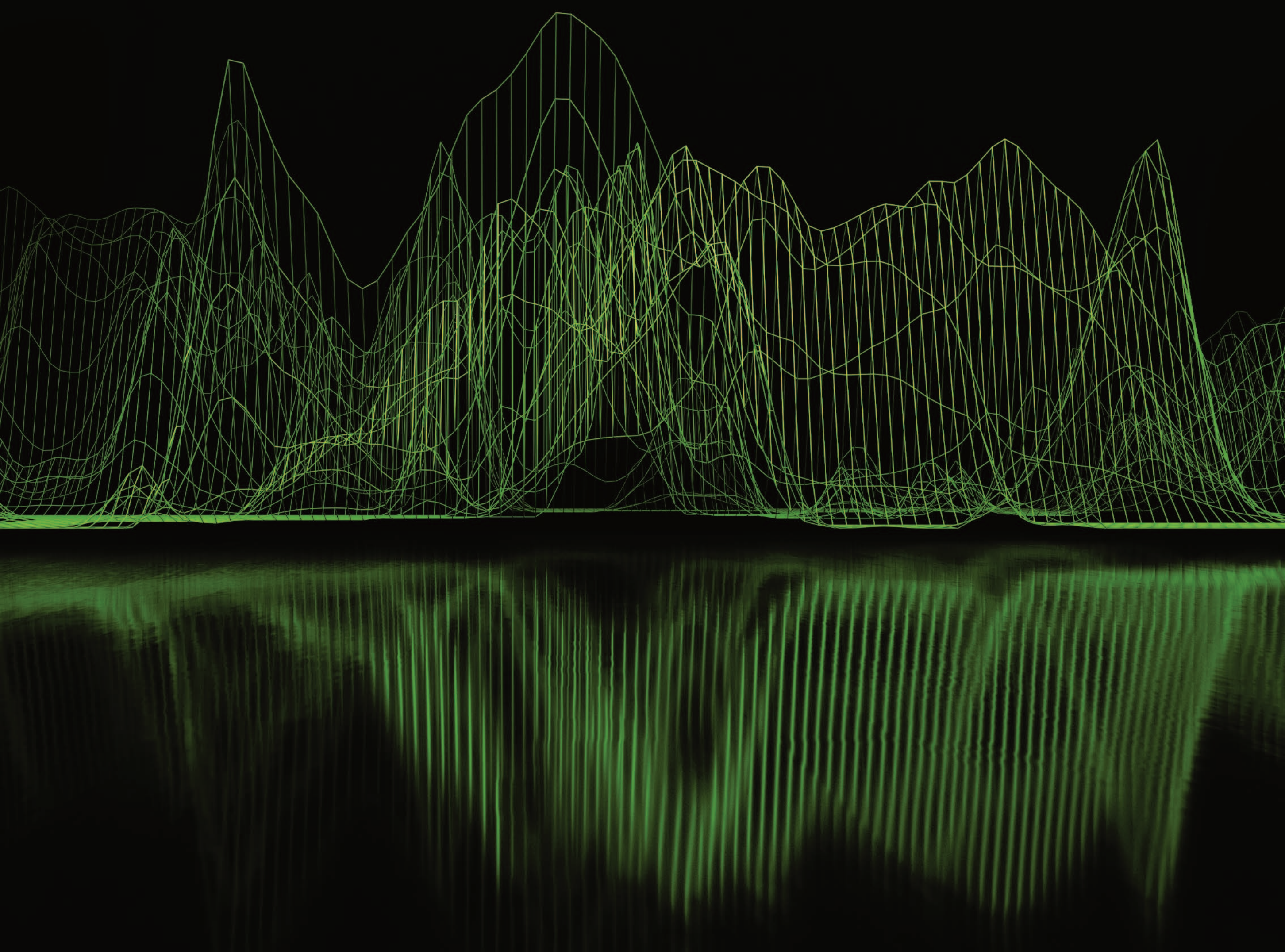
Amanda Turner | Dirk Zeindler

STOCHASTIC FINANCE

An introduction with examples



Stochastic Finance

Stochastic Finance provides an introduction to mathematical finance that is unparalleled in its accessibility. Through classroom testing, the authors have identified common pain points for students, and their approach takes great care to help the reader to overcome these difficulties and to foster understanding where comparable texts often do not.

Written for advanced undergraduate students, and making use of numerous detailed examples to illustrate key concepts, this text provides all the mathematical foundations necessary to model transactions in the world of finance. A first course in probability is the only necessary background.

The book begins with the discrete binomial model and the finite market model, followed by the continuous Black–Scholes model.

It studies the pricing of European options by combining financial concepts such as arbitrage and self-financing trading strategies with probabilistic tools such as sigma algebras, martingales and stochastic integration.

All these concepts are introduced in a relaxed and user-friendly fashion.

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Preface

Traders in the financial industry make decisions every day about whether to buy or sell securities such as stocks, gold and oil. While they know the prices of these securities today, their values tomorrow could be higher or lower. Another difficulty that arises is that some financial products depend on the future prices of other financial products. Examples are put and call options. The question here is how do you determine the current price of such a financial product. Financial mathematics, which is a branch of probability theory, allows us to answer questions like this and the answer is usually not the price one would expect at first glance. In this book, we introduce the subject of financial mathematics at a level that will be accessible to undergraduate students. One of our main aims is to highlight the underlying ideas and concepts and to motivate each step. We will also present these ideas and concepts as clearly and simply as possible, illustrating them with examples and pointing out typical misunderstandings. A solid acquaintance with probability is a prerequisite, so this textbook will most likely be of interest to third- or fourth-year students. Basic knowledge of stochastic processes and measure theory would be a helpful foundation, but is not essential as we will introduce the necessary definition and results.

In this book we examine several stochastic models for a stock market. We start with the simple binomial model and develop more complexity as the chapters progress, finally concluding with the Black–Scholes model, which is a reasonably realistic model for a real stock market. Each chapter is clearly explained so as to be accessible to a reader with no previous experience of the topic, and illustrated with numerous examples that explain both the motivation, and how it is used to answer concrete problems.

This textbook consists of two parts and, aside from Chapter 1, which is a prerequisite for both parts, each one can be read largely independently of the other. In Part I, we discuss discrete time models: the binomial model, the finite market model and the discrete Black–Scholes model. We introduce the risk neutral measure and prove the two fundamental theorems of asset pricing. The probability theory required for Part I is covered in Chapter 2. This chapter may be skipped by those who already have a good grounding in probability. Part II describes continuous-time models. We introduce Brownian motion, explain stochastic integration and discuss the Black–Scholes model. In particular, we derive the Nobel Prize-winning Black–Scholes formula for the pricing of call and put options. The probability theory required for Part II is covered in Chapter 6. Again, this chapter is optional for those familiar with the prerequisites.

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Part I

Discrete-Time Models for Finance

1 Introduction to Finance

In this chapter we introduce some basic notions from finance. We explain the assumptions on financial markets which will be used in the rest of the book, and define key concepts such as arbitrage.

1.1 Motivation

One of the main aims of this book is to answer questions such as:

What is the correct price for a financial product like a call option*?*

Surprisingly, the answer turns out not to be the average payoff for the option. We illustrate this with a simple example. Suppose you would like to buy a *stock*. Let S_0 denote the price of the stock today and S_1 the price tomorrow. Suppose that S_0 is £100 and that tomorrow S_1 will be either £99 or £101 with

$$\mathbb{P}[S_1 = £99] = 0.3 \text{ and } \mathbb{P}[S_1 = £101] = 0.7. \tag{1.1.1}$$

If you would like to buy the stock today, what price should you pay? Suppose a trader offered you the stock for the price $P = \mathbb{E}[S_1] = 100.4$. Would you accept this? As the market price for the stock is £100, you should not pay £100.4 as you can get the stock cheaper directly from the market. The average price is clearly the wrong answer and the correct price is, of course, £100 as this is the market price today. In this situation, we know the price today so can easily answer the question. However, suppose you would like to buy a financial product that tomorrow gives the payoff

$$\Phi_1 = \begin{cases} £2, & \text{if } S_1 = £101, \\ £0, & \text{if } S_1 = £99. \end{cases} \tag{1.1.2}$$

What price should you pay for this product? In this case the product cannot be bought directly from the market, so we do not immediately know the price. We can compute $\mathbb{E}[\Phi_1] = 1.4$. Is this the correct price? Consider a different situation. Suppose you buy one stock today by borrowing £99, which you need to return tomorrow, and investing £1 of your own money. Assume that there is no interest rate for borrowing money. Tomorrow your portfolio will have the value

$$S_1 - 99 = \begin{cases} £2, & \text{if } S_1 = £101, \\ £0, & \text{if } S_1 = £99. \end{cases} \tag{1.1.3}$$

This trading strategy gives the same payoff as the product (1.1.2), for an initial investment of £1. We should therefore definitely not pay $\mathbb{E}[\Phi_1] = 1.4$ for the financial product in (1.1.2) as we can achieve the same payoff for less money.

At this point we see that the approach with average prices does not work. Instead, one prices products in financial markets so that neither the investor nor the trader can make a risk-free profit. An example of a risk-free profit is buying a security in one market and simultaneously selling it in another market at a higher price. A chance to make a risk-free profit is called an arbitrage opportunity. Intuition tells us that the price for the financial product in (1.1.2) has to be £1, otherwise this would create an arbitrage opportunity in the market. This is indeed correct and will be justified in Example 3.3.9.

1.2 Financial Markets

The term 'financial market' is a general term for all marketplaces where trading in financial products takes place. This includes, in particular, all markets in which stocks, bonds, foreign exchange and derivatives are traded. One of the reasons for the existence of financial markets is to facilitate the flow of capital. If, for instance, a company needs money to build a new factory then it can sell shares of stocks to investors and these investors will receive as a future reward dividends or a rise in the stock price. The financial market can be divided into the primary and secondary markets.

1.2.1 Primary Markets

On a primary market one trades securities such as

- **Stocks**: risky assets whose future value is unpredictable. We will generally denote the value of a single stock at time t by S_t. This value is commonly called the **spot** price. If we have more than one stock, we will use the vector notation $\left(S_t^1, \ldots, S_t^d\right)$.
- **Bonds**: risk-free products with a given or predictable value in the future. In simple terms, a bond is a loan from an investor to a borrower. The borrower pays the investor a fixed rate of return over a specific timeframe. Bond prices will take one of the following two forms in this book.
  - The **risk-free bond** with price $B_t = B_0 (1 + r)^t$ at time t.
  - The bond with **continuously compounded interest rate** with price $B_t = B_0 e^{rt}$ at time t.
- **Foreign eXchange (FX)**: pounds, euros, dollars, ...
- **Commodities**: oil, metal, grain, ...
- **Fixed income**: credit products, such as mortgages, and interest rates.

Transactions occur directly between the investors and the companies that issue these securities.
We will mainly focus on the assets and financial contracts related to stocks. Thus let us have a brief look at stocks. Stocks are also called shares and are units of ownership in a corporation.

The owner of a stock is called a shareholder. A shareholder participates in profits generated by the company. These profits are distributed in the form of dividends. Stocks can be in the form of a physical paper certificate or these days usually in digital form. The two main types of stocks are common stocks and preferred stocks. We will not discuss the different types of stocks in this book. Also, we will not discuss how to buy or sell stocks. It is sufficient to know that

- We can buy, sell and borrow stocks.
- Stocks have a non-negative value which changes over time.

Also, it is useful to keep in mind that stocks are units of ownership in a corporation. For simplicity, we will assume that

- Stocks do not generate any dividends.
- We can buy, sell and borrow as many stocks (and bonds and any other securities) as we would like.
- We do not have to pay transaction fees and the transactions happen instantly.
- Our interaction with the market has no influence on the pricing in the market.

These assumptions are not entirely realistic, but they make it easier to understand the basic concepts. In more complicated models, these assumptions are replaced by more realistic ones but many of the general principles still hold.

One further term which we would like to introduce at this point is the concept of a **portfolio**. A portfolio is a collection of financial assets like stocks, bonds, commodities and cash. A portfolio can of course also contain contingent claims such as the call and put options, which we define in Section 1.2.2 below. When we speak of our portfolio, we are referring to the collection of all assets we own.

1.2.2 Secondary Markets

On a secondary market, investors trade securities which they already hold with each other, as opposed to a primary market on which securities are sold directly by the entities who created them. When a company issues new stocks, it does this on a primary market, and the proceeds of such a sale go to the company. Investors then sell on these stocks to other investors. These sales take place on a secondary market and any proceeds go to the investor, not to the original company. One can think of a secondary market as one on which the securities being traded are ‘second-hand’. The **stock market** is one such example.

On a secondary market, securities which originated on a primary market can be repackaged in elaborate ways, for example in the form of contracts, to create new securities. The values of such securities depend on the values of the underlying securities from the primary market and are called **derivatives** or **contingent claims**. Some examples are

- The **forward contract** (or short *forward*) is an agreement to buy or sell an asset at a certain price at a certain future time.

- A **future contract** is similar to the *forward* and both contracts allow people to buy or sell a specific type of asset at a specific time at a given price. However, *future contracts* (or simply *futures*) differ from *forwards* in a number of aspects. The main difference is that *futures* are traded on the stock exchange while *forwards* are not. Thus *futures* are more regulated than *forwards*.
- An **option** is a financial product that gives you the right to buy or sell an asset at a certain price called the **strike price**. The difference from a *forward* is that one has the right, but not the obligation, to buy or sell. A **call option** gives the option holder the right to buy at a given **strike price** K and a **put option** gives the option holder the right to sell at a given **strike price** K. The cash flow at the time of exercise is called the **payoff** and determines the option. There are two main types of options.
  - **European options:** Exercising happens at a fixed time T (called **maturity**).
  - **American options:** Exercising can happen at any time until T.

We will mainly focus on the assets and financial contracts related to stocks. Such products are called **equities**. This is the standard entry point to mathematical finance.

Sometimes people have trouble seeing the difference between the price of an option and the **strike price** of an option. To better understand the difference, imagine a European call option as a *piece of paper*, on which person A writes the inscription

> The holder of this piece of paper can purchase ⟨**number**⟩ stocks from the company ⟨**company name**⟩ from person A on date ⟨**maturity**⟩ at the price of ⟨**strike price**⟩.

Suppose that person A then sells the call option (piece of paper) to person B. The price of the option is the amount of money person B pays to person A in order to obtain the option in the first place, and the strike price is the amount which, at maturity, person B will pay to person A should they wish to buy the underlying security. The strike price is therefore a fixed value written into the contract. The option price, on the other hand, must be a value that B is willing to pay and A is willing to accept, and may depend on the current price of the stock, the time remaining until maturity, the riskiness of the stock, etc.

The question at this point is therefore: At what price should person A sell this call option? One of the main aims of this book is to answer this question. As a precursor, we should at least convince ourselves that this piece of paper does have some value. Would person A give this *piece of paper* to person B for free? Person B will only use this *piece of paper* if the strike price is below the market price at maturity, otherwise it would be cheaper to just buy the stocks directly on the market on that date. However, should person B buy from person A in this case, person A will lose money as they have to sell their stock at below market value. Therefore, person A can lose money from this contract, but not make money. The answer is therefore no, person A would not give the *piece of paper* to person B for free. Person B has to pay person A some money to get this contract. The European call option therefore has some (positive) value. The following example shows that in general the strike price is also not the correct price.

Example 1.2.1. *Assume that person A would like to sell a call option with maturity tomorrow and strike price £5 to person B.*

Is it a good idea to sell this call option to person B for £5?

Suppose that the price of the underlying asset is £100 today. It is very likely that tomorrow the price of this asset will still be around £100. Thus, person B is very likely to exercise this option tomorrow, as he can receive an asset worth £100 for only £5. Person A would lose around £95 in this case. Therefore, person A should not sell this call option for £5.

It is usual in finance to have in an option the right to buy/sell several units of the underlying security. However, we assume always that *call* and *put options* are to buy/sell only one unit of the security. This assumption only simplifies some formulas and has no influence on the pricing. Also note that in an option only the writer is specified. In the above example this is person A. On the other hand, the holder of the option is not specified. Thus person B can sell this option to somebody else.

1.2.3 Payoffs

In this book we mainly study *European contingent claims*, that is **contingent claims** with a fixed **maturity**. When we have a *European contingent claim* like a *call option* then it is natural to ask:

What is the payoff Φ_T of the *contingent claim* if the *maturity* is T and
what is the profit $\widetilde{\Phi}_T$ of the *contingent claim* if the *maturity* is T?

For clarity, the payoff Φ_T denotes the money we earn from the *contingent claim* at maturity, not taking into account the price we had to pay for the contingent claim. The profit $\widetilde{\Phi}_T$, on the other hand, denotes the amount we earn from the *contingent claim* at maturity minus the price we had to pay for the *contingent claim*.

Example 1.2.2. *Suppose you play the lottery and buy five tickets at £10 each today. A week later, one of the tickets wins £200 while the other four win nothing. Then the payoff of the five tickets is £200 while the profit is £150 = £200 – £50.*

Most people are primarily interested in the profit of a *contingent claim*. However, one of the main aims in this book is to determine the initial price of a *contingent claim* in a given model. In this situation, it is much more natural to work with the payoff than with the profit. We thus will work only with the payoff Φ_T of a *contingent claim*.

Let us consider some examples of payoffs. We use the notation for bonds and stocks we introduced in Section 1.2.1.

Example 1.2.3.

- Stocks*: The payoff of one stock at time T is* $\Phi_T = S_T$.
- Bonds*: The payoff of one bond at time T is* $\Phi_T = B_T$.
- Forward contract*: Suppose we have agreed on a forward contract to sell one stock at time T for the price* P_0*. Then the payoff is* $\Phi_T = P_0 - S_T$*. If we have, on the other hand, agreed on a forward contract to buy one stock at time T for the price* P_0 *then the payoff is* $\Phi_T = S_T - P_0$.
- European call option*: Suppose that the strike price of this call option is* $K \in (0, \infty)$ *and the maturity is* T*. Then the payoff is*

$$\Phi_T = \max\{S_T - K, 0\} = \begin{cases} S_T - K, & \text{if } S_T \geq K, \\ 0, & \text{if } S_T < K, \end{cases} \tag{1.2.1}$$

where S_T *is the price of the underlying asset at time* T*. Let us briefly explain this. If* $S_T > K$ *then the holder of the option will exercise it since he can buy the stock for the price K and can then sell it for the price* S_T *on the market. The payoff per option is then* $S_T - K$*. However, if* $S_T < K$ *then the holder of the option will not exercise it as he can get the stock cheaper on the market. Thus the payoff is* 0 *in this case.*
- *European put option: When the strike price is* $K \in (0, \infty)$ *then the payoff is*

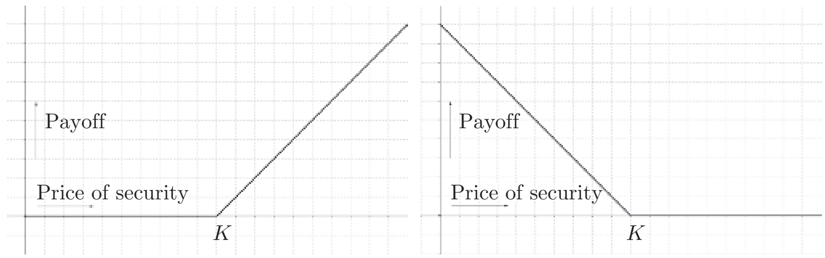
$$\Phi_T = \max\{K - S_T, 0\} = \begin{cases} 0, & \text{if } S_T \geq K, \\ K - S_T, & \text{if } S_T < K. \end{cases} \tag{1.2.2}$$

Note that one could be tempted to define the payoff Φ_T of a *contingent claim* as the money one earns when it is exercised. However, if we consider a call or put option then we see it is not always reasonable to exercise it. Thus the payoff has to be viewed as the money we earn from the *contingent claim* by making the 'best' decision, that is the one for which we get the most money out of the *contingent claim*. For instance, in Example 1.2.2 we have the choice between cashing and not cashing the winning ticket. The best choice in this case is of course to cash the winning ticket.

We have to point out here another typical misconception. Suppose you own a European call option with maturity T and would like to sell it at a time t with $t < T$. At what price should you sell it? In view of (1.2.1), one could be tempted to think that one should sell this option for the price $P = \max\{S_t - K, 0\}$. Let us consider an example.

Example 1.2.4. *Suppose you own a European call option with maturity in one year and strike price £60, and you would like to sell this call option today. Suppose that the price of the underlying asset is £50 today. In this case*

$$\max\{50 - 60, 0\} = 0.$$



(a) Payoff of a *call option* (b) Payoff of a *put option*

Figure 1.1 Payoff diagrams for *call* and *put options*.

This price would make sense if the payoff of the call option would always be 0*. However, suppose that we have* $90 \leq S_T \leq 110$*, no matter how the market evolves. In this case the call option gives a payoff of at least £30 at maturity. Obviously, you should not sell this call option for £0 (i.e. give it away for free).*

Here is another point to consider. If we own a call option with strike price K and would like to exercise it at maturity then we require money. In view of the assumptions in Section 1.2.1, we can do the following: Borrow £K at maturity, exercise the call option, sell the obtained security and return the borrowed money. Since we only exercise the call option if the price of the underlying security is larger than the strike price, we are always able to return the borrowed money. Thus, we are always able to exercise a call option under the assumptions in Section 1.2.1. For the sake of simplicity, from now on when we say that we are exercising a call option, we will implicitly assume that the holder would follow the procedure above.

1.2.4 Payoff Diagrams

Many *contingent claims* depend only on one security at a given time. Examples are *call* and *put options*. The payoff Φ_T of such *contingent claims* can be illustrated with a graph in $\mathbb{R}^2$. One uses the x-axis for the price of the underlying security and the y-axis for the payoff of the *contingent claim*. Such a graph is called a *payoff diagram*. The payoff diagrams for *call options* and *put options* are given in Figure 1.1.

There are of course also *contingent claims* depending on more than one security or on the value of a security at different times. An example is the so-called *lookback options*. For instance, the payoff of a *lookback call option* with fixed strike K is

$$\Phi_T = \max\{S_{\max} - K, 0\} \text{ with } S_{\max} = \max_{t \leq T} S_t. \tag{1.2.3}$$

1.3 Arbitrage Opportunities and Liquid Markets

An important term in finance is **arbitrage opportunity** (AO). An **arbitrage** is an opportunity to make money out of nothing or to make profit without risk. This is of course not a mathematically rigorous definition. We will give precise definitions later. As an illustration, let us consider an example of an arbitrage opportunity.

Example 1.3.1. *Suppose the price of an iPhone in the EU is* €600 *and in the US it is* \$600. *The exchange rate between euros and dollars is* €1= \$1.15. *An* arbitrage opportunity *can then be used as follows:*

- *Borrow* \$600 *and buy the iPhone on the US market for* \$600.
- *Sell the iPhone on the European market for* €600.
- *Exchange* €600→ \$690.
- *Pay back your debt and keep the profit of* \$90.

Such an arbitrage opportunity *is called a* **cash-and-carry arbitrage**.

A market is called **liquid** if one can buy and sell large quantities of an asset at any time, the transaction costs are low and transactions happen quickly. In this book, we are only interested in **liquid** markets. To simplify the arguments, we will always assume that we can sell and borrow as much of each security as we want, that we do not have to pay transaction fees and that transactions are immediate.

If a market is **liquid**, prices normally move very fast and eliminate *arbitrage opportunities*. The basic line of reasoning in mathematical finance is that the absence of *arbitrage opportunities* forces relations between prices of *forwards*, *futures*, *calls* and *puts* on a stock. One of the goals of mathematical finance is to establish these relations. However, unlike in physics, very few laws are available. One rule in mathematical finance is that financial products with larger payoffs must have larger prices.

Proposition 1.3.2. Suppose we have an arbitrage-free and liquid market, and consider two assets in this market. Suppose that at a future time T, the price of the second asset is greater than or equal to the price of the first asset, regardless of how the market evolves. Then this is also true at any prior time. Expressed in formulas,

$$P_T^{(1)} \leq P_T^{(2)} \implies P_t^{(1)} \leq P_t^{(2)} \ \forall 0 \leq t \leq T, \tag{1.3.1}$$

where $P_t^{(1)}$ denotes the price of the first asset and $P_t^{(2)}$ the price of the second asset at time t.

Note that in this proposition we only consider liquid markets. From now on we will no longer explicitly state this assumption, but assume that the reader is aware that we consider only liquid markets.

Proof. We argue by contradiction. Suppose that the market is arbitrage-free and liquid and (1.3.1) does not hold. Thus there exists a $t \leq T$ with $P_t^{(1)} > P_t^{(2)}$. We must have that $t < T$ since by assumption $P_T^{(1)} \leq P_T^{(2)}$. We now use the following strategy: *Buy cheap and sell expensive*. Explicitly:

- Do nothing until time t.
- At time t, borrow one unit of the first asset and sell it to buy one unit of the second asset. Since $P_t^{(1)} > P_t^{(2)}$, we have earned $P_t^{(1)} - P_t^{(2)}$. We then do nothing until time T.
- At time T, sell the unit of the second asset, buy a unit of the first asset and return it to the lender, and keep the remaining money. Since $P_T^{(1)} \leq P_T^{(2)}$, we have earned $P_T^{(2)} - P_T^{(1)}$ and have no debt left.

Our total profit is therefore $(P_t^{(1)} - P_t^{(2)}) + (P_T^{(2)} - P_T^{(1)}) > 0$. Thus this strategy is an arbitrage opportunity, which is a contradiction.

□

We can use Proposition 1.3.2 to give upper and lower bounds for the prices of options.

Example 1.3.3. *Consider an arbitrage-free and liquid market. Suppose in this market there is a bond B_t with interest rate $r = 0$ and a stock S_t. Now consider a European call option with maturity T and strike price K, written on S_t. Let C_t denote the price of this call option at time t for $t \leq T$. We then have*

$$S_t - K \leq C_t \leq S_t \ \text{for all } t \leq T. \tag{1.3.2}$$

Let us justify this. The payoff of the call option at time T is $\max\{S_T - K, 0\}$. Thus $C_T = \max\{S_T - K, 0\}$. Furthermore, at time T

$$S_T - K \leq \max\{S_T - K, 0\} \leq S_T. \tag{1.3.3}$$

To apply Proposition 1.3.2 to this inequality, we have to determine the arbitrage-free price at time t of a contingent claim with payoffs $S_T - K$ and S_T.

- *The arbitrage-free price of the stock at time t is S_t.*
- *Since $r = 0$, money does not lose its value over time. Thus the value of £K at time t is the same as its value at time T.*

Combining these two observations, we get that the arbitrage-free price of a contingent claim with a payoff $S_T - K$ is $S_t - K$ at time t. Thus Proposition 1.3.2 implies (1.3.2).

Similarly, if we consider a European put option with maturity T and strike price K and denote by P_t the price of this put option at time t for $t \leq T$, then

$$K - S_t \leq P_t \leq K \ \text{for all } t \leq T. \tag{1.3.4}$$

The assertion that the arbitrage-free price of £K is always £K only holds because the interest rate $r = 0$. To illustrate this point, let us consider an example.

Example 1.3.4. *Consider an arbitrage-free market with a bond B_t of the form*

$$B_t = B_0(1 + r)^t \ \text{with} \ B_0 > 0,\ r \geq 0. \tag{1.3.5}$$

The time $t = 0$ corresponds to today. Consider £K at time T in the future. This can be viewed as the contingent claim giving a payoff of £K at time T, no matter how the market evolves. What is the arbitrage-free price of these £K at time t? The answer is

$$\frac{K}{(1+r)^{T-t}}. \tag{1.3.6}$$

At first glance, this looks a little bit strange. One might believe that the price of £K is always £K. Thus let us illustrate why (1.3.6) is the correct answer.

- *Let us first show that £K is the wrong price. Suppose that we can borrow £K from somebody at time t and have to return these £K at time T. Then we invest these £K at time t into bonds and sell these bonds at time T. Thus we obtain £$K(1+r)^{T-t}$ at time T, which is strictly larger than K. We then return the borrowed £K and keep the rest. This is clearly an arbitrage opportunity, which is a contradiction.*
- *Similarly, suppose that we can borrow £K_t with $K_t > K(1+r)^{T-t}$ from somebody at time t and have to return £K at time T. In this case we can use the same argument as above.*
- *Finally, suppose that we can lend somebody £K_t with $K_t < K(1+r)^{T-t}$ at time t and will get back £K at time T. In this case, at time t we borrow bonds worth $K(1+r)^{t-T}$, sell them and lend £K_t. Thus we have earned £$K(1+r)^{T-t} - K_t$. At time T, we get £K back. We then use this money to buy bonds and return the borrowed bonds. Thus we have earned in total £$K(1+r)^{T-t} - K_t$ and have no debt left. In other words, we have found an arbitrage opportunity, which is a contradiction.*

This means that (1.3.6) gives the only possible value for which there is no arbitrage opportunity, and hence it must be the time t price in an arbitrage-free market.

In view of this example, we see that we can interpret the interest of the bond in a financial model as the inflation rate. Proposition 1.3.2 has some further consequences. In particular, it implies that the prices of call and put options are always non-negative since the payoff of call and put options is always non-negative. However, this does not hold for all contingent claims. Suppose that a contingent claim gives a payoff

$$\Phi_T = \min\{K - S_T, 0\}. \tag{1.3.7}$$

This payoff is always non-positive. Thus nobody will pay money to get this contingent claim. Instead, the writer of this contingent claim has to pay somebody money for taking this contingent claim. A further consequence of Proposition 1.3.2 is the *law of one price*.

Proposition 1.3.5 (Law of one price). Suppose that a market is liquid and arbitrage-free. If two securities have the same value at a (future) time T, then they must have the same value at any prior time. Expressed in formulas,

$$P_T^{(1)} = P_T^{(2)} \implies P_t^{(1)} = P_t^{(2)} \;\; \forall t \le T, \tag{1.3.8}$$

with $P_t^{(1)}$ and $P_t^{(2)}$ as in Proposition 1.3.2.

This proposition follows immediately from Proposition 1.3.2. A natural question at this point is if the law of one price can also be applied to the past. In other words, if two securities agree at some point in the past, do they also agree prior to that time? The answer is no, they can agree, but they do not have to agree. The reason is that we can apply the law of one price if and only if $P_T^{(1)} = P_T^{(2)}$ holds no matter how the market evolves. This condition is not necessarily fulfilled if two securities have the same price at some point in the past.

The law of one price can also be applied to combinations of securities. An example is the *put–call parity*.

Example 1.3.6. *Consider an arbitrage-free market containing a bond B_t with interest rate $r = 0$ and a stock S_t. Consider a European call option and a European put option, both with maturity T and strike price K, written on S_t. Let C_t denote the price of the call option and P_t the price of the put option at time t for $t \leq T$. Then*

$$C_t - P_t = S_t - K \text{ for all } t \leq T. \tag{1.3.9}$$

Let us justify (1.3.9). *It is straightforward to see that at time T*

$$S_T - K = \max\{S_T - K, 0\} - \max\{K - S_T, 0\} = C_T - P_T. \tag{1.3.10}$$

The law of one price and the observation in Example 1.3.3 then imply (1.3.9).

A typical mistake people make when they are asked to show equation (1.3.9) is that they believe $C_t = \max\{S_t - K, 0\}$ and $P_t = \max\{K - S_t, 0\}$. However, in almost all situations

$$C_t \neq \max\{S_t - K, 0\} \text{ and } P_t \neq \max\{K - S_t, 0\} \text{ for } t < T, \tag{1.3.11}$$

see for instance Example 1.2.4. Explicitly, the prices of a call and a put option are determined by the expected behaviour for the price of the underlying security. In the special case of the Black–Scholes model, the prices of call and put options are given by the Black–Scholes formula, see Theorem 5.5.2.

The law of one price can be used in some cases to determine arbitrage-free prices of a security in a given market. Let us consider an example.

Example 1.3.7. *Consider an arbitrage-free market containing a stock S_t and a contingent claim Φ_T. This contingent claim Φ_T has maturity T and is written on the stock S_T. Further, the payoff of Φ_T is*

- *£30 if S_T is greater than or equal to £20.*
- *£15 if S_T is less than or equal to £5.*
- *Interpolated linearly in the interval* $[5, 20]$.

As illustration, the payoff diagram of Φ_T can be found in Figure 1.2.

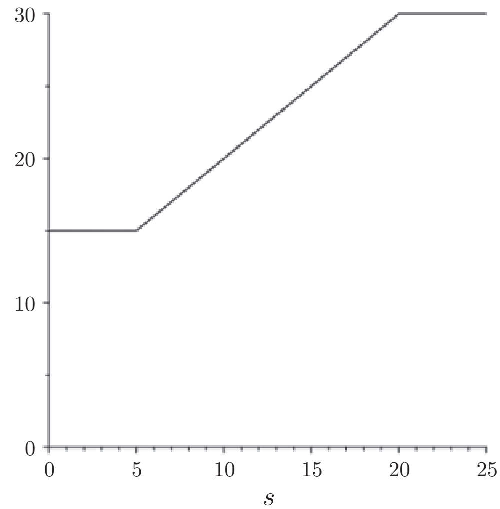


Figure 1.2 Payoff diagram of the contingent claim in Example 1.3.7.

Suppose that we can buy and sell put and call options according to Table 1.1.

Table 1.1 Option prices in Example 1.3.7

Strike price	*£5*	*£10*	*£15*	*£20*
Call option	*£12*	*£8*	*£5*	*£4*
Put option	*£1*	*£2*	*£4*	*£8*

All put and call options in Table 1.1 are written on S_t and have maturity T. We now determine the arbitrage-free price of the contingent claim Φ_T. For this, we first show that we can reproduce the payoff of Φ_T by buying and selling put and call options. Then we apply the law of one price.

The payoffs of call and put options are $\max\{S_T - K, 0\}$ *and* $\max\{K - S_T, 0\}$. *Thus their payoffs are continuous functions in S_T and are linear in the intervals* $[0, K]$ *and* $[K, \infty[$. *Since $\Phi_T = f(S_T)$ with f continuous and linear in the intervals* $[0, 5]$, $[5, 20]$ *and* $[20, \infty[$, *we use the approach*

$$\Phi_T = a \max\{S_T - 5, 0\} + b \max\{5 - S_T, 0\} + c \max\{S_T - 20, 0\} + d \max\{20 - S_T, 0\},$$

where a, b, c, $d \in \mathbb{R}$. This expression is continuous and linear in the intervals $[0, 5]$, $[5, 20]$ *and* $[20, \infty[$. *To determine the values of a, b, c, $d \in \mathbb{R}$, we enter the values $S_T = 0$, $S_T = 5$, $S_T = 20$ and $S_T = 25$. This then leads to the system of equations*

$$5b + 20d = 15, \quad 15d = 15, \quad 15a = 30, \quad 20a + 5c = 30. \tag{1.3.12}$$

A direct computation shows that this system has the solution

$$a = 2, \quad b = -1, \quad c = -2, \quad d = 1. \tag{1.3.13}$$

In words, we can achieve the payoff of Φ_T by buying two call options with strike price 5, selling one put option with strike price 5, selling two call options with strike price 20 and buying one put option with strike price 20. If we would like to produce the payoff of Φ_T with call and put options then we use Table 1.1 and see that we have to invest

$$2 \cdot 12 - 1 \cdot 1 - 2 \cdot 4 + 1 \cdot 8 = 23. \tag{1.3.14}$$

By assumption, the market is liquid and arbitrage-free. Thus the law of one price implies that the arbitrage-free price of Φ_T is 23.

The assumption that a market is arbitrage-free and liquid can only be used to derive laws similar to the *law of one price*. These are not, in general, enough to determine the actual prices of financial products. We need to have some additional information about prices, such as in Example 1.3.7, in order to use these laws to determine the prices. Financial products such as *forwards* or *options* are based on the future prices of underlying securities. These prices have a strong influence on whether a financial product makes a profit or a loss. In particular, prices such as in Table 1.1 are determined by the expected behaviour of the underlying security. Thus it is important to have a good stochastic model for the behaviour of the prices in a market and to know how arbitrage-free prices of a financial product are determined in a given model. In this book, we will focus on various stochastic models for a market. The main aim of this book is to show how the prices of financial products are determined in these models, and to develop the necessary tools to do that.

1.4 Exercises

Exercise 1.1. *Two financial products based on options are the strangle and the butterfly spread.*

- *A* strangle *is a combined option consisting of a long position (we are the* buyer *of the option) in a European call option with strike K_1 and a long position in a European put option with strike $K_2 < K_1$.*
- *A butterfly spread is a combined option consisting of a long position in a European call option with strike K_1, a long position in a European call option with strike $K_2 \neq K_1$ and short position (we are the* seller *of the option) in two European call options with strike $\frac{K_1+K_2}{2}$.*

It is assumed that all options in these combined options are written on the same stock and have the same maturity.

(a) Determine the payoff of the strangle and the butterfly spread.
(b) Draw payoff diagrams for the strangle and the butterfly spread with $K_1 = 15, K_2 = 5$.

Exercise 1.2. *The price of a security is £10 today. Suppose that you are the only person in the world who knows that the price of the security will be £25 in 3 months, while everybody else expects that the price will stay around £10.*