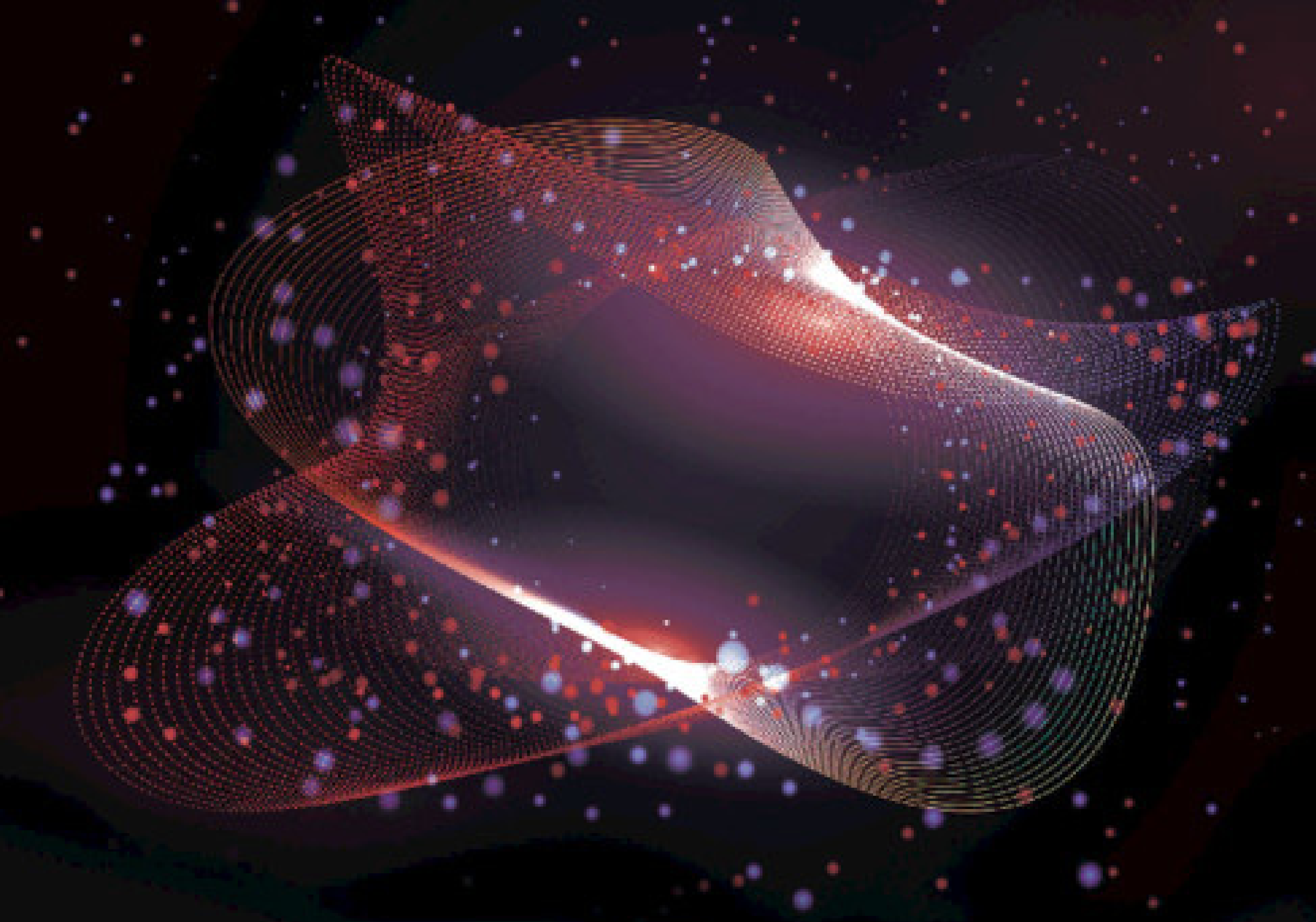

THE GENERAL THEORY OF RELATIVITY

A Mathematical Approach

FAROOK RAHAMAN

The General Theory of Relativity

The general theory of relativity, Einstein's theory of gravitation, has been included as a compulsory subject in undergraduate and graduate courses in Physics and Applied Mathematics all over the world. However, the physics-first approach that is taken by many textbooks is not universally used, as the approach often depends on the instructors' or students' background. Conceived from the lecture notes made by the author over a teaching career spanning 18 years, this book introduces the general theory of relativity for advanced students with a strong mathematical background.

The proposed book takes a 'math-first approach', for which the mathematical formalism comes first and is then applied to physics. It presents a concise yet comprehensive and structured understanding of the general theory of relativity. The book discusses the mathematical foundation of the general theory of relativity and focuses heavily on topics such as tensor calculus, geodesics, Einstein field equations, linearized gravity, Lie derivatives and their applications, the causal structure of spacetime, rotating black holes, and basic knowledge of cosmology and astrophysics. All of these are explained through a large number of worked examples and exercises.

Farook Rahaman is a Professor of Mathematics at Jadavpur University, Kolkata. Besides writing a book, *The Special Theory of Relativity*, he has published numerous research papers on galactic dark matter, wormhole geometry, charged fluid model, topological defects in the early universe, gravastars, black hole physics, star modeling, and the cosmological model of the universe.

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To
my parents
Majeda Rahaman and Late Obaidur Rahaman
and
my son and wife
Md Rahil Miraj and Pakizah Yasmin

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Preface

At the beginning of the twentieth century, Einstein spent many years developing a new theory in physics. The newly developed theory is known as the theory of relativity. This is basically a combination of two theories: the first one is known as the special theory of relativity and latter one is dubbed as the general theory of relativity. The special theory of relativity is based on two postulates, namely the principle of relativity or equivalence, that is, the laws of physics are the same in all inertial systems, which means no preferred inertial system exists, while the second postulate is the principle of the constancy of the speed of light. The general theory of relativity asserts that there is no difference between the local effects of a gravitational field and that of acceleration of an inertial system. In other words, spacetime is warped or distorted by the matter and energy in it as an effect of gravity. According to the general theory of relativity, massive objects cause the outer space to twist due to gravity like a heavy ball bending a thin rubber sheet that is holding the ball. Heavier balls bend spacetime far more than lighter ones. Like the special theory of relativity, the general theory of relativity attracted scientists a lot, immediately after its discovery by Einstein. As a result, it has been included as a compulsory subject in graduate and postgraduate courses of physics and applied mathematics all over the globe. Einstein proposed the field equations for the general theory of relativity by applying his own intuition. Later, many other methods were developed to construct Einstein's field equations.

This book on the general theory of relativity is an outcome of a series of lectures delivered by me, over several years, to postgraduate students of mathematics at Jadavpur University. I should mention that it is not a fundamental book. This book has been written, from a mathematical point of view, after consulting several books existing in the literature. I have provided the list of the reference books. During my lectures, many students asked questions that helped me know their needs as well as the shortcomings in their understanding. Therefore, it is a well-planned textbook that has been organized in a logical order and every topic has been dealt with in a simple and lucid manner. A number of problems with hints, taken from the question papers of different universities, are included in each chapter.

The book is organized as follows:

In Chapter One a brief overview of tensor calculus, including the different types of tensors as well as operations on tensors, is given. Generalized Kronecker delta, Christoffel symbols, affine connection, covariant derivatives, geodesic coordinate, and various forms of tensors are described, with examples, as a foreground to understand the basics of general relativity. Chapter Two starts with a discussion of the geodesic equation in curved spacetime. In addition, several problems for different spacetimes are provided on geodesics. Chapter Three begins with the statement of three basic principles, namely Mach's principle, equivalence principle, and the principle of covariance. Next, the Einstein gravitational field equations are derived from the variational principle.

Also, in this chapter, the outline of some modified theories of gravity, such as f(R) theory of gravity, Gauss–Bonnet gravity, f(G) theory of gravity or modified Gauss–Bonnet gravity, f(T) theory of gravity, f(R,T) theory of gravity, Brans–Dicke theory of gravity, and Weyl gravity, are provided. A discussion on linearized gravity is given in Chapter Four. Newtonian limit of Einstein field equations or weak field approximation of Einstein field equations is derived. It is shown that Poisson's equation can be viewed as an approximation of Einstein field equations. A short mathematical description of gravitational wave is also provided. Chapter Five is dedicated to a short discussion on Lie derivatives and their applications. Killing equations and Killing vectors are also discussed with several examples. A short note on conformal Killing vector is also provided. Chapter Six is devoted to discussions on spacetimes of spherically symmetric distributions of matter. The exact exterior and interior solutions of Einstein field equations in spherically symmetric spacetimes are discussed. The proof of Birkoff's theory is provided. It states that a spherically symmetric gravitational field in vacuum is necessarily static and must have Schwarzschild form. The Tolman–Oppenheimer–Volkov (TOV) equation is discussed. Isotropic coordinate system is a new coordinate system whose spatial distance is proportional to the Euclidean square of the distances. Some static spherically symmetric spacetimes are rewritten in an isotropic coordinate system. A short discussion on interaction between the gravitational and electromagnetic fields are provided. Reissner–Nordström solution is a static solution of the gravitational field outside of a spherically symmetric charged body. Particle and photon orbits in the Schwarzschild spacetime are discussed in Chapter Seven. Also, in this chapter, using the trajectory in the gravitational field of sun (i.e., in the Schwarzschild spacetime), several tests of the theory of general relativity, namely the precession of the perihelion motion of mercury, bending of light, radar echo delay, and gravitational redshift, are explained. A discussion on the stable circular orbits in the Schwarzschild spacetime is given. A general treatment is provided for the experimental test of general theory of relativity for a general static and spherically symmetric configuration. Causal structure in the special theory of relativity, i.e., in Minkowski spacetime or flat spacetime, is characterized so that no massive particle can travel faster than light. In general relativity, locally there is no difference of the causality relation with Minkowski spacetime. However, globally, the causality relation is significantly different due to various spacetime topologies. A short discussion on causal structure of spacetimes is given in Chapter Eight. Several basic definitions and some standard theorems related to causality are explained. Chapter Nine deals with discussions on causal structures of specific spacetimes, which are the standard exact solutions of Einstein field equations such as Minkowski spacetime, de Sitter and anti-de Sitter spacetimes, Robertson–Walker spacetime, Bianchi-I spacetime, Schwarzschild spacetime, and Reissner–Nordström black hole. A short elementary discussion on rotating black holes is given in Chapter Ten. After introducing the tetrad, an outline of the derivation of the Kerr and Kerr–Newman solutions is illustrated through the complex transformation algorithm for both in four and higher dimensions. Some of the different forms of the Kerr solution are mentioned. Some elementary properties of the Kerr solution including the maximal extension of Kerr spacetime are discussed. Finally, brief discussions on Hawking radiation, Penrose process of extraction of energy from a Kerr black hole, and laws of black hole thermodynamics are given. Chapters Eleven and Twelve provide some simple applications of general theory of relativity in astrophysics and cosmology, respectively. Some preliminary concepts of extrinsic curvature, Lagrangian formalism of the general theory of relativity, and $3 + 1$ decomposition of spacetime are given as appendices.

Acknowledgments

This book has been made possible through the support, contributions, and assistance of many people and various organizations. I take this opportunity to express my sincere gratitude to all of them. I would like to deeply and sincerely thank my mother (Majeda), wife (Pakizah), and son (Rahil), without whose loving support and encouragement this book could not have been completed. I also express my sincere gratitude to my father-in-law (Abdul Hannan), mother-in-law (Begum Nurjahan), brother-in-law (Dr. Ruhul Amin), younger brother (Mafrook Rahaman), and niece (Ayat Nazifa) for their patience and support during the entire period of the preparation of the manuscript. It is a pleasure to thank Dr. Nupur Paul, Dr. Sayeedul Islam, Dr. Banashree Sen, Dr. Mosiur Rahaman, Dr. Indrani Karar, Monsur Rahaman, Dr. Shyam Das, Lipi Baskey, Nayan Sarkar, Md Rahil Miraj, Dr. Arkopriya Mallick, Sabiruddin Molla, Dr. Ayan Banerjee, Dr. Tuhina Manna, Dr. Amna Ali, Dr. Nasarul Islam, Ksh. Newton Singh, Somi Aktar, Bidisha Samanta, Dr. Sourav Roychowdhury, Dr. Debabrata Deb, Dr. Amit Das, Dr. Abdul Aziz, Dr. Anil Kumar Yadav, Monimala Mandal, Antara Mapdar, Dr. Saibal Ray, Dr. Mehedi Kalam, Susmita Sarkar, Dr. Piyali Bhar, Dr. Gopal Chandra Shit, Dr. Ranjan Sharma, Dr. Shounak Ghosh, and Dr. Iftikar Hossain Sardar for their technical assistance in the preparation of the book. I remain thankful to all the professors and non-teaching staff members of the Department of Mathematics, Jadavpur University for providing me with all the available facilities and services whenever needed. Particularly I would like to mention the library staff for their excellent support. Finally, I am also thankful to the authority of the Inter-University Centre for Astronomy and Astrophysics (IUCAA), Pune, India for providing all kinds of working facility and hospitality under the Associateship Scheme.

CHAPTER 1

Tensor Calculus — A Brief Overview

1.1 Introduction

The principal target of tensor calculus is to investigate the relations that remain the same when we change from one coordinate system to any other. The laws of physics are independent of the frame of references in which physicists describe physical phenomena by means of laws. Therefore, it is useful to exploit tensor calculus as the mathematical tool in which such laws can be formulated.

1.2 Transformation of Coordinates

Let there be two reference systems, S with coordinates $(x^1, x^2, \ldots, x^n)$ and $\overline{S}$ with coordinates $(\overline{x}^1, \overline{x}^2, \ldots, \overline{x}^n)$ (Fig. 1). The new system $\overline{S}$ depends on the old system S as

$$\overline{x}^i = \phi^i(x^1, x^2, \ldots, x^n); \quad i = 1, 2, \ldots, n. \tag{1.1}$$

Here ϕ^i are single-valued continuous differentiable functions of $x^1, x^2, \ldots, x^n$ and further the Jacobian

$$\left|\frac{\partial \phi^i}{\partial x^j}\right| = \begin{vmatrix} \frac{\partial \phi^1}{\partial x^1} & \frac{\partial \phi^1}{\partial x^2} & \frac{\partial \phi^1}{\partial x^3} & \cdots & \frac{\partial \phi^1}{\partial x^n} \\ \frac{\partial \phi^2}{\partial x^1} & \frac{\partial \phi^2}{\partial x^2} & \frac{\partial \phi^2}{\partial x^3} & \cdots & \frac{\partial \phi^2}{\partial x^n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\partial \phi^n}{\partial x^1} & \frac{\partial \phi^n}{\partial x^2} & \frac{\partial \phi^n}{\partial x^3} & \cdots & \frac{\partial \phi^n}{\partial x^n} \end{vmatrix} \neq 0.$$

Differentiation of Eq. (1.1) yields

$$d\overline{x}^i = \sum_{r=1}^{n} \frac{\partial \phi^i}{\partial x^r} dx^r = \sum_{r=1}^{n} \frac{\partial \overline{x}^i}{\partial x^r} dx^r = \sum_{r=1}^{n} \overline{a}^i_r dx^r.$$

Now and onward, we use the Einstein summation convention, i.e., omit the summation symbol $\sum$ and write the above equations as

$$d\overline{x}^i = \frac{\partial \overline{x}^i}{\partial x^r} dx^r = \overline{a}^i_r dx^r, \tag{1.2}$$

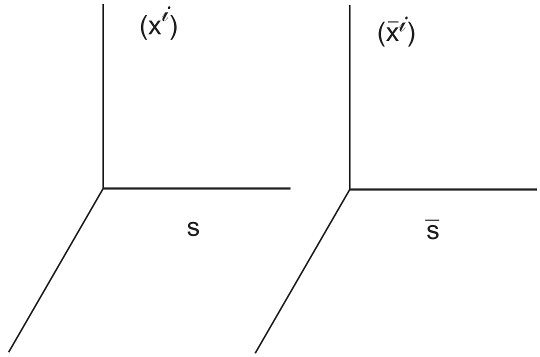

Figure 1 S and $\bar{S}$ frames.

or

$$dx^i = \frac{\partial x^i}{\partial \bar{x}^m} d\bar{x}^m = a^i_m d\bar{x}^m. \tag{1.3}$$

The repeated index r or m is known as **dummy index**. The index i is not dummy and is known as **free index**.

The transformation matrices are inverse to each other

$$\bar{a}^i_r \, a^m_i = \delta^m_r. \tag{1.4}$$

The symbol δ^m_r is Kronecker delta, is defined as

$$\begin{aligned} \delta^m_r &= 1 \;\; if \;\; m = r \\ &= 0 \;\; if \;\; m \neq r \end{aligned}$$

Obviously vectors in $(\bar{S})$ system are linked with (S) system.

1.3 Covariant and Contravariant Vector and Tensor

Usually one can describe the tensors by means of their properties of transformation under coordinate transformation. There are two possible ways of transformations from one coordinate system (x^i) to the other coordinate system $(\bar{x}^i)$.

Let us consider a set of n functions A_i of the coordinates x^i. The functions A_i are said to be the components of **covariant vector** if these components transform according to the following rule

$$\bar{A}_i = \frac{\partial x^j}{\partial \bar{x}^i} A_j. \tag{1.5}$$

Also, one can find by multiplying $\frac{\partial \bar{x}^i}{\partial x^k}$ and using $\frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^j}{\partial \bar{x}^i} = \delta^j_k$ and $\delta^j_k A_j = A_k$

$$A_k = \frac{\partial \bar{x}^i}{\partial x^k} \bar{A}_i.$$

Exercise 1.1

Gradient of a scalar B, i.e., $B_i = \frac{\partial B}{\partial x_i}$ is a covariant vector.

Here, A_i is known as the **covariant tensor of first order or of the type** (0, 1)**.**

The functions A^i are said to be the components of the **contravariant vector** if these components transform according to the following rule

$$\bar{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j \tag{1.6}$$

Also, one can find by multiplying both sides with $\frac{\partial x^k}{\partial \bar{x}^i}$ and using $\delta^k_j A^j = A^k$

$$A^k = \frac{\partial x^k}{\partial \bar{x}^i} \bar{A}^i.$$

Here, A^i is known as the **contravariant tensor of first order or of the type** (1, 0)**.**

Exercise 1.2

Tangent vector $\frac{dx^i}{du}$ of the curve $x^i = x^i(u)$ is a contravariant vector.

Exercise 1.3

Let components of velocity vector in Cartesian coordinates are $\dot{x}$ and $\dot{y}$. Find corresponding components in polar coordinates.

Hint: Here, $x^1 = x$, $x^2 = y$, and $\overline{x}^1 = r$, $\overline{x}^2 = \theta$ with $x = r\cos\theta$, $y = r\sin\theta$, i.e., $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}(\frac{y}{x})$.

Let $A^1 = \dot{x}$, $A^2 = \dot{y}$. We will have to find $\overline{A}^1$, $\overline{A}^2$.

("dot" denotes differentiation with respect to t.)

Using the definition $\overline{A}^i = \frac{\partial \bar{x}^i}{\partial x^j} A^j$, we have

$$\overline{A}^1 = \frac{\partial \bar{x}^1}{\partial x^1} A^1 + \frac{\partial \bar{x}^1}{\partial x^2} A^2 \; or, \; \overline{A}^1 = \frac{\partial r}{\partial x}\dot{x} + \frac{\partial r}{\partial y}\dot{y} = \dot{r}.$$

Similarly,

$$\overline{A}^2 = \frac{\partial \theta}{\partial x}\dot{x} + \frac{\partial \theta}{\partial y}\dot{y} = \dot{\theta}.$$

Exercise 1.4

Let components of acceleration vector in Cartesian coordinates be $\ddot{x}$ and $\ddot{y}$. Find corresponding components in polar coordinates.

Hint: Let $A^1 = \ddot{x}$, $A^2 = \ddot{y}$. We will have to find $\overline{A}^1, \overline{A}^2$.
Here,

$$\overline{A}^1 = \frac{\partial r}{\partial x}\ddot{x} + \frac{\partial r}{\partial y}\ddot{y} = \ddot{r} - r\dot{\theta},\ \overline{A}^2 = \frac{\partial \theta}{\partial x}\ddot{x} + \frac{\partial \theta}{\partial y}\ddot{y} = \ddot{\theta} + \frac{2}{r}\dot{\theta}\dot{r}.$$

1.3.1 Invariant

Let ϕ be a function of coordinate system (x^i) and $\overline{\phi}$ be its transform in another coordinate system $(\overline{x}^i)$. Then, ϕ is said to be **invariant** if $\overline{\phi} = \phi$.

Exercise 1.5

The expression A^iB_i is an invariant or scalar, i.e.,

$$\bar{A}^i\bar{B}_i = A^iB_i. \tag{1.7}$$

Hint: Use definitions given in Eqs. (1.5) and (1.6).

An **invariant or scalar** is known as the **tensor of the type** $(0, 0)$**.**

1.3.2 Contravariant and covariant tensors of rank two

Let C^i and B^j be two contravariant vectors with n components, then $C^iB^j = A^{ij}$ has n^2 quantities, i.e., A^{ij} are the set of n^2 functions of the coordinates x^i. If the transformation of A^{ij} is like

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial \bar{x}^j}{\partial x^l}A^{kl}, \tag{1.8}$$

then A^{ij} is known as **contravariant tensor of rank two**. Here, A^{ij} is also known as the **contravariant tensor of order two or of the type** $(2, 0)$**.**

If we multiply both sides of (1.8) by $\frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}$, then

$$A^{rs} = \frac{\partial x^r}{\partial \bar{x}^i}\frac{\partial x^s}{\partial \bar{x}^j}\bar{A}^{ij}.$$

Again, if C_i and B_j are two covariant vectors with n components, then $C_iB_j = A_{ij}$ form n^2 quantities, i.e., A_{ij} are the set of n^2 functions of the coordinates x^i.

If the transformation of A_{ij} is like

$$\bar{A}_{ij} = \frac{\partial x^k}{\partial \bar{x}^i}\frac{\partial x^l}{\partial \bar{x}^j}A_{kl}, \tag{1.9}$$

then A_{ij} is known as **covariant tensor of rank two**.

Here, A_{ij} is also known as the **covariant tensor of order two or of the type** $(0, 2)$**.**

If we multiply both sides of (1.9) by $\frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}$, then

$$A_{rs} = \frac{\partial \bar{x}^i}{\partial x^r}\frac{\partial \bar{x}^j}{\partial x^s}\bar{A}_{ij}.$$

1.3.3 Mixed tensor of order two A^i_j

Suppose A^i_j is a set of n^2 functions of n coordinates. If the transformation obeys the following rule

$$\bar{A}^i_j = \frac{\partial \bar{x}^i}{\partial x^k}\frac{\partial x^l}{\partial \bar{x}^j}A^k_l,$$

then A^k_l is known as the **mixed tensor of order two or of the type** $(1, 1)$**.**

Thus, mixed tensor of order two can be obtained by taking a covariant vector A_i and a contravariant vector B^j, i.e., $C^j_i = A_i B^j$.

Exercise 1.6

Kronecker delta δ^j_i is a mixed tensor of order two.
Hint: If δ^j_i can be combined with components of two vectors to form a scalar, then δ^j_i will be a tensor. Now

$$A^i B_j \delta^j_i = A^i B_i = scalar.$$

If the transformation obeys the following rule

$$\bar{A}^{i_1 i_2 \ldots i_p}_{j_1 j_2 \ldots j_q} = \frac{\partial \bar{x}^{i_1}}{\partial x^{k_1}}\frac{\partial \bar{x}^{i_2}}{\partial x^{k_2}}\cdots\frac{\partial \bar{x}^{i_p}}{\partial x^{k_p}}\frac{\partial x^{l_1}}{\partial \bar{x}^{j_1}}\frac{\partial x^{l_2}}{\partial \bar{x}^{j_2}}\cdots\frac{\partial x^{l_q}}{\partial \bar{x}^{j_q}}A^{k_1 k_2 \ldots k_p}_{l_1 l_2 \ldots l_q},$$

then $A^{k_1 k_2 \ldots k_p}_{l_1 l_2 \ldots l_q}$ is known as **mixed tensor of the type** (p, q)**.**

1.3.4 Symmetric and skew-symmetric tensors

If a tensor is unaltered after changing every pair of contravariant or covariant indices, then it is said to be a symmetric tensor. Let $T_{\alpha\beta}$ be a covariant tensor of rank two.

If $T_{\alpha\beta} = T_{\beta\alpha}$, *then it is known as* **symmetric tensor**.

If a tensor is altered in its sign but not in magnitude after changing every pair of contravariant or covariant indices, then it is said to be a skew-symmetric tensor.

If $T_{\alpha\beta} = -T_{\beta\alpha}$, *then it is known as* **antisymmetric or skew-symmetric tensor**.

Exercise 1.7

Kronecker delta δ_{ij} is a symmetric tensor.

Exercise 1.8

If A_i is covariant vector, then $curlA_i = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i}$ is a skew-symmetric tensor.

Hint: Use $curlA_i = \frac{\partial A_i}{\partial x_j} - \frac{\partial A_j}{\partial x_i} = B_{ij}$ and show that $B_{ij} = -B_{ji}$.

Note 1.1

Symmetry property of a tensor is independent of the coordinate system.

Note 1.2

A symmetric tensor of order two in n-dimensional space has at most $\frac{n(n+1)}{2}$ independent components whereas an antisymmetric tensor of order two has at most $\frac{n(n-1)}{2}$ independent components.

1.4 Operations on Tensors

i. The addition and subtraction of two tensors of the same type is a tensor of same type.

Exercise 1.9

$$A_{ij} \pm B_{ij} = C_{ij},\ A^{ij} \pm B^{ij} = C^{ij},\ A_i^j \pm B_i^j = C_i^j$$

Exercise 1.10

Any covariant or contravariant tensor of second order can be expressed as a sum of a symmetric and a skew-symmetric tensor of order two.

Hint:

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}), \quad \text{etc.}$$

ii. The type of the tensor remains invariant by multiplication of a scalar α.

Exercise 1.11

$$\alpha A_{ij} = C_{ij}, \quad \alpha A^{ij} = C^{ij}, \quad \alpha A_i^j = C_i^j$$

iii. **Outer product:** The outer product of two tensors is a new tensor whose order is the sum of the orders of the given tensors.

Exercise 1.12

Let two tensors of types (2,3) and (1,2) be respectively, A^{ij}_{klm} and B^{a}_{bc}, then the outer product of these tensors has type (3,5), i.e.,

$$A^{ij}_{klm}B^{a}_{bc} = T^{ija}_{klmbc}$$

iv. **Contraction:** The particular type of operation by which the order (r) of a mixed tensor is lowered by order ($r-2$) is known as contraction.

Exercise 1.13

Let A^{ij}_{klm} be a mixed tensor of order five. The new tensor A^{ij}_{kim} can be obtained by replacing lower index l by the upper index i and taking summation over i, one gets the tensor of order three.

$$A^{ij}_{kim} = B^{j}_{km}$$

v. **Inner product:** The outer product of two tensors followed by contraction with respect to an upper index and a lower index of the other results in a new tensor which is called an inner product.

Exercise 1.14

$$A^{ij}_{k}\,B^{k}_{mn} \equiv C^{ijk}_{kmn} = D^{ij}_{mn}, \quad A^{ij}_{k}\,B^{m}_{ij} = D^{m}_{k}$$

1.4.1 Test for tensor character: Quotient Law

An entity whose inner product by an arbitrary tensor (covariant or contravariant) always gives a tensor is itself a tensor.

Exercise 1.15

If $C(i,j)A^iB^j$ is an invariant, then $C(i,j) = C_{ij}$ is a tensor of the type (0,2).

Exercise 1.16

If $C(p,q,r)B^{qs}_{r} = A^{s}_{p}$, then $C(p,q,r) = C^{r}_{pq}$ is a tensor of the type (1,2).

Exercise 1.17

Let λ^i, μ^i be the components of two arbitrary vectors with $a_{hijk}\lambda^h\mu^i\lambda^j\mu^k = 0$, then prove that

$$a_{hijk} + a_{hkji} + a_{jihk} + a_{jkhi} = 0.$$

Hint: Given that

$$A = a_{hijk}\lambda^h \mu^i \lambda^j \mu^k = 0.$$

Differentiating with respect to λ^h, we get

$$\frac{\partial A}{\partial \lambda^h} = a_{hijk}\mu^i \lambda^j \mu^k + a_{pihk}\lambda^p \mu^i \mu^k = 0.$$

Again, differentiating with respect to λ^j, we get

$$\frac{\partial^2 A}{\partial \lambda^h \partial \lambda^j} = a_{hijk}\mu^i \mu^k + a_{jihk}\mu^i \mu^k = 0.$$

Now, differentiating with respect to μ^i and μ^k, one will find, respectively,

$$\frac{\partial^3 A}{\partial \lambda^h \partial \lambda^j \partial \mu^i} = a_{hijk}\mu^k + a_{hkji}\mu^k + a_{jihk}\mu^k + a_{jkhi}\mu^k = 0,$$
$$\frac{\partial^4 A}{\partial \lambda^h \partial \lambda^j \partial \mu^i \partial \mu^k} = a_{hijk} + a_{hkji} + a_{jihk} + a_{jkhi} = 0.$$

Exercise 1.18

If A^i is an arbitrary contravariant vector and $C_{ij}A^iA^j$ is an invariant, then show that $C_{ij} + C_{ji}$ is a covariant tensors of the second order.
Hint: Given $C_{ij}A^iA^j$ is an invariant for arbitrary contravariant vector A^i, therefore,

$$C_{ij}A^iA^j = C'_{ij}A'^iA'^j.$$

Tensor law of transformation yields

$$C_{ij}A^iA^j = C'_{ij}\frac{\partial x'^i}{\partial x^\alpha}A^\alpha \frac{\partial x'^j}{\partial x^\beta}A^\beta.$$

Now interchanging the suffix i and j

$$C_{ji}A^jA^i = C'_{ji}\frac{\partial x'^j}{\partial x^\alpha}\frac{\partial x'^i}{\partial x^\beta}A^\alpha A^\beta = C'_{ji}\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}A^\alpha A^\beta.$$

(interchanging the dummy suffixes α and β)
Thus,

$$(C_{ji} + C_{ij})A^iA^j = (C'_{ji} + C'_{ij})\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}A^\alpha A^\beta,$$
$$\Rightarrow (C_{\alpha\beta} + C_{\beta\alpha})A^\alpha A^\beta = (C'_{ji} + C'_{ij})\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}A^\alpha A^\beta,$$
$$\Rightarrow \left[(C_{\alpha\beta} + C_{\beta\alpha}) - (C'_{ij} + C'_{ji})\frac{\partial x'^i}{\partial x^\alpha}\frac{\partial x'^j}{\partial x^\beta}\right] A^\alpha A^\beta = 0.$$

Since A^α is arbitrary, therefore, the expression within the square bracket vanishes. Hence, $C_{\alpha\beta} + C_{\beta\alpha}$ is a $(0, 2)$-tensor.

1.4.2 Conjugate or reciprocal tensor of a tensor

Consider a symmetric covariant tensor of second order a_{ij}, i.e., of the type (0,2) whose determinant, $|a_{ij}|$ is nonzero; then

$$b^{ij} = \frac{\text{cofactor of } a_{ij} \text{ in } |a_{ij}|}{|a_{ij}|}$$

is known as reciprocal tensor of a_{ij}. It is of the type (2,0).

Note 1.3

Reciprocal tensor exists for any tensor. Only condition being its determinant is nonzero. Here, $a_{ij}b^{ik} = \delta_j^k$ and $|a_{ij}||b^{ik}| = |\delta_j^k| = 1$. Usually, conjugate of a_{ij} is written as a^{ij} and $a_{ij}a^{ij} = \delta_j^j = n$.

Note 1.4

Tensor equations in one system (x^i) remain valid in all other coordinate systems $(\overline{x}^i)$, e.g., if $T^i_{jkl} = 2T^i_{ljk}$, then $\overline{T}^i_{jkl} = 2\overline{T}^i_{ljk}$.

1.5 Generalized Kronecker Delta

The generalized Kronecker Delta $\delta^{\alpha\beta}_{\mu\nu}$ is defined as follows:

$$\begin{aligned}\delta^{\alpha\beta}_{\mu\nu} &= \begin{vmatrix} \delta^\alpha_\mu & \delta^\beta_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu \end{vmatrix} \\ &= +1,\ \alpha \neq \beta,\ \alpha = \mu,\ \beta = \nu \\ &= -1,\ \alpha \neq \beta,\ \alpha = \nu,\ \beta = \mu \\ &= 0,\ \textit{otherwise}.\end{aligned}$$

We can define $\delta^{\alpha\beta\gamma}_{\mu\nu\xi}$ and $\delta^{\alpha\beta\gamma\rho}_{\mu\nu\xi\omega}$ as follows:

$$\delta^{\alpha\beta\gamma}_{\mu\nu\xi} = \begin{vmatrix} \delta^\alpha_\mu & \delta^\beta_\mu & \delta^\gamma_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu & \delta^\gamma_\nu \\ \delta^\alpha_\xi & \delta^\beta_\xi & \delta^\gamma_\xi \end{vmatrix},$$

$$\delta^{\alpha\beta\gamma\rho}_{\mu\nu\xi\omega} = \begin{vmatrix} \delta^\alpha_\mu & \delta^\beta_\mu & \delta^\gamma_\mu & \delta^\rho_\mu \\ \delta^\alpha_\nu & \delta^\beta_\nu & \delta^\gamma_\nu & \delta^\rho_\nu \\ \delta^\alpha_\xi & \delta^\beta_\xi & \delta^\gamma_\xi & \delta^\rho_\xi \\ \delta^\alpha_\omega & \delta^\beta_\omega & \delta^\gamma_\omega & \delta^\rho_\omega \end{vmatrix}.$$

Exercise 1.19

$$\delta^{123}_{123} = \delta^{123}_{231} = 1,$$
$$\delta^{123}_{213} = \delta^{123}_{132} = -1.$$

Exercise 1.20

Show that

$$\delta^{\alpha\beta}_{\mu\beta} = 3\delta^{\alpha}_{\mu}.$$

Exercise 1.21

Show that

$$\delta^{\alpha}_{\alpha} = 4.$$

Exercise 1.22

Show that

$$\delta^{\alpha\beta\tau}_{\mu\gamma\tau} = 2\delta^{\alpha\beta}_{\mu\gamma}.$$

Hint:

$$\delta^{\alpha\beta\tau}_{\mu\gamma\tau} = \begin{vmatrix} \delta^{\alpha}_{\mu} & \delta^{\beta}_{\mu} & \delta^{\tau}_{\mu} \\ \delta^{\alpha}_{\gamma} & \delta^{\beta}_{\gamma} & \delta^{\tau}_{\gamma} \\ \delta^{\alpha}_{\tau} & \delta^{\beta}_{\tau} & \delta^{\tau}_{\tau} \end{vmatrix}.$$

Now, expand along third row and use $\delta^{\tau}_{\tau} = 4$

Exercise 1.23

Show that

$$\delta^{\alpha\beta\tau\rho}_{\mu\nu\gamma\rho} = -\begin{vmatrix} \delta^{\alpha}_{\mu} & \delta^{\beta}_{\mu} & \delta^{\tau}_{\mu} \\ \delta^{\alpha}_{\nu} & \delta^{\beta}_{\nu} & \delta^{\tau}_{\nu} \\ \delta^{\alpha}_{\gamma} & \delta^{\beta}_{\gamma} & \delta^{\tau}_{\gamma} \end{vmatrix}.$$

Exercise 1.24

Show that

$$\delta^{\alpha\beta\tau\rho}_{\mu\nu\tau\rho} = -2(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}).$$

Exercise 1.25

Show that

$$\delta^{\alpha\beta\tau\rho}_{\mu\beta\tau\rho} = -6\delta^{\alpha}_{\mu}.$$

Exercise 1.26

Show that

$$\delta^{\alpha\beta\tau\rho}_{\alpha\beta\tau\rho} = -24.$$

Symbols: Symmetric and skew-symmetric tensors of second order:

$$T_{(ab)} = \frac{1}{2}(T_{ab} + T_{ba}), \quad T_{[ab]} = \frac{1}{2}(T_{ab} - T_{ba}).$$

For the tensors of third order, we can construct symmetric and skew-symmetric tensors as

$$T_{(abc)} = \frac{1}{3!}(T_{abc} + T_{bca} + T_{cab} + T_{bac} + T_{acb} + T_{cba}),$$
$$T_{[abc]} = \frac{1}{3!}(T_{abc} + T_{bca} + T_{cab} - T_{bac} - T_{acb} - T_{cba}).$$

We can express skew-symmetry symbols by means of generalized Kronecker delta as

$$T_{[ab]} = \frac{1}{2!}T_{cd}\delta^{cd}_{ab},$$
$$T_{[abc]} = \frac{1}{2!}T_{cde}\delta^{cde}_{abc}.$$

1.6 The Line Element

The distance between two neighboring points $P(\vec{r}(x^i))$ and $F(\vec{r}(x^i) + d\vec{r}(x^i))$ (x^i are the coordinates of the space) in an n-dimensional space is given by (see Fig. 2)

$$ds^2 = d\vec{r} \cdot d\vec{r} = g_{ab}dx^a dx^b \tag{1.10}$$

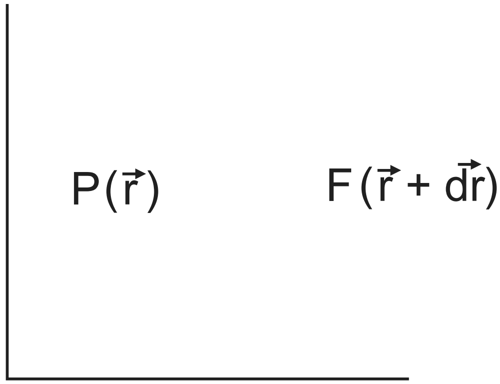

Figure 2 Two neighboring points in a space.

Here,

$$d\vec{r}(x^i) = \frac{\partial \vec{r}}{\partial x^1}dx^1 + \frac{\partial \vec{r}}{\partial x^2}dx^2 + \dots\dots\dots + \frac{\partial \vec{r}}{\partial x^n}dx^n = \alpha_1 dx^1 + \alpha_2 dx^2 + \dots + \alpha_n dx^n$$

with

$$\alpha_i = \frac{\partial \vec{r}}{\partial x^i} \text{ and } g_{ab} = \alpha_a \cdot \alpha_b.$$

The distance between two neighboring points is referred as **line element** and is given by Eq. (1.10).

Here, g_{ab} are known as metric tensor, which are functions of x^a. If $g = |g_{ab}| \neq 0$ and ds is adopted to be invariant, then the space is called **Riemannian space**.

In mathematics, Riemannian space is used for a positive-definite metric tensor, whereas in theoretical physics, spacetime is modeled by a pseudo-Riemannian space in which the metric tensor is indefinite.

The metric tensor g_{ab} is also called **fundamental tensor** (covariant tensor of order two).

In Euclidean space:

$$ds^2 = dx^2 + dy^2 + dz^2.$$

In Minkowski flat spacetime, the line element

$$ds^2 = dx^{0^2} - dx^{1^2} - dx^{2^2} - dx^{3^2}.$$

Since the distance ds between two neighboring points is real, the Eq. (1.10) will be amended to

$$ds^2 = eg_{ij}dx^i dx^j,$$

where e is known as the indicator and assumes the value $+1$ or -1 in order that ds^2 be always positive.

The contravariant tensor g^{ij} is defined by

$$g^{ij} = \frac{\Delta^{ij}}{g},$$

here Δ^{ij} is the cofactor of g_{ij} and g is the determinant of g_{ij}.

Obviously

$$g_{ab}\, g^{bc} = g_a^c = \delta_a^c.$$

With the help of g^{ab} and g_{ab}, one can raise or lower the indices of any tensor as

$$\begin{aligned} g_{ac}T^{ab} &= T_c^b \\ T_{ab}\, g^{ac} &= T_b^c \\ g_{ab}A^b &= A_a \\ g^{ab}\, A_a &= A^b \end{aligned}$$

Here, A_a and A^a are known as **associated vectors.**

$$g^{ab}\, g^{cd}\, g_{bd} = g^{ac}.$$

$$A_a B^b = g^{ab} A_a B_b = g_{ab} A^a B^b.$$

Exercise 1.27

Show that the determinant of the metric tensor is not a scalar. Also, prove that the expression $\sqrt{-g}\, d^4x$ where $d^4x = dx^1 dx^2 dx^3 dx^4$ is an invariant volume element.

Hints: We know

$$g'_{ab} = \frac{\partial x^c}{\partial x'^a}\frac{\partial x^d}{\partial x'^b}\, g_{cd}$$

$$\Rightarrow \quad det(g'_{ab}) = \left|\frac{\partial x}{\partial x'}\right|^2 det\,(g_{cd})$$

$$\Rightarrow \quad g' = \left|\frac{\partial x}{\partial x'}\right|^2 g \tag{1.11}$$

This indicates that the determinant of the metric tensor is not a scalar.

Also the volume element d^4x transform into d^4x' as

$$d^4x' = \left|\frac{\partial x'}{\partial x}\right| d^4x \tag{1.12}$$

From (1.11) and (1.12), we get

$$\sqrt{-g'}\, d^4x' = \sqrt{-g} d^4x.$$

Exercise 1.28

Find out the metric tensor of a three-dimensional Euclidean space in cylindrical and polar coordinates.
Hint: Here, for cylindrical coordinates,

$$y^1 = x^1 \cos x^2,\ y^2 = x^1 \sin x^2,\ y^3 = x^3$$

The metric tensor in three-dimensional Euclidean space is

$$ds^2 = {dy^1}^2 + {dy^2}^2 + {dy^3}^2$$

Now,

$$dy^1 = dx^1 \cos x^2 - x^1 \sin x^2 dx^2,\ dy^2 = dx^1 \sin x^2 + x^1 \cos x^2 dx^2,\ dy^3 = dx^3$$

Substituting these we get

$$ds^2 = {dx^1}^2 + {x^1}^2 {dx^2}^2 + {dx^3}^2$$

For polar coordinates,

$$y^1 = x^1 \sin x^2 \cos x^3,\ y^2 = x^1 \sin x^2 \sin x^3,\ y^3 = x^1 \cos x^2$$

Using the same procedure, one can find

$$ds^2 = {dx^1}^2 + {x^1}^2 {dx^2}^2 + {x^1}^2 \sin^2 x^2 {dx^3}^2$$

Exercise 1.29

A curve in spherical coordinates x^i is given by

$$x^1 = t, \quad x^2 = sin^{-1}\left(\frac{1}{t}\right), \quad x^3 = 2\sqrt{t^2 - 1}.$$

Find the length of arc for $1 \leq t \leq 2$.

Hint: In a spherical coordinate, the metric is given by

$$\begin{aligned} ds^2 &= (dx^1)^2 + (x^1)^2(dx^2)^2 + (x^1 sinx^2)^2(dx^3)^2 \\ &= (dt)^2 + t^2\left(-\frac{dt}{t\sqrt{t^2-1}}\right)^2 + \left(t.\frac{1}{t}\right)^2\left(\frac{2t}{\sqrt{t^2-1}}dt\right)^2 \\ &= \frac{5t^2}{t^2-1}(dt)^2 \end{aligned}$$

Therefore, the required length of the arc $1 \leq t \leq 2$ is given by

$$\int_{t_1}^{t_2} ds = \sqrt{5}\int_1^2 \frac{t}{\sqrt{t^2-1}}dt = \sqrt{15} \text{ units.}$$

1.6.1 Norm

Let A^μ (A_μ) be any contravariant (covariant) vector. Then **norm or magnitude or length** l **of the vector** A^μ **(**A_μ**)** is defined as

$$l^2 = A^\mu A_\mu = g_{\mu\nu}A^\mu A^\nu = g^{\mu\nu}A_\mu A_\nu.$$

Exercise 1.30

Magnitude l of a vector is an invariant.
Hint: Try to show

$$A^\mu A_\mu = \overline{A}^\mu \overline{A}_\mu$$

1.6.2 Unit vector

A vector is said to be unit vector (unit covariant or unit contravariant) if

$$g^{ij}A_iA_j = 1 = g_{ij}A^iA^j.$$

1.6.3 Null vector

A vector is said to be null vector (covariant or contravariant) if

$$g^{ij}A_iA_j = 0 = g_{ij}A^iA^j.$$

1.6.4 Time-like vector

A vector is said to be time-like vector (covariant or contravariant) if

$$g^{ij}A_iA_j = g_{ij}A^iA^j > 0 \ \ \textit{with signature}\ (+, -, -, -).$$

Alternatively,

A vector is said to be time-like vector (covariant or contravariant) if

$$g^{ij}A_iA_j = g_{ij}A^iA^j < 0 \; with \; signature \; (-, +, +, +).$$

1.6.5 Space-like vector

A vector is said to be space-like vector (covariant or contravariant) if

$$g^{ij}A_iA_j = g_{ij}A^iA^j < 0 \;\; with \; signature \; (+, -, -, -).$$

Alternatively, a vector is said to be space-like vector (covariant or contravariant) if

$$g^{ij}A_iA_j = g_{ij}A^iA^j > 0 \;\; with \; signature \; (-, +, +, +).$$

The time-like, space-like, and null vectors have important physical relevance as follows: Two events are causally connected by a time-like vector when they lie within a light cone, whereas a space-like vector connects two events that lie outside the light cone, i.e., the events are causally disconnected. Two events that lie on the light cone are connected by a null vector. Actually, collection of all null vectors in a Lorentzian space forms a light cone.

Note 1.5

The signature (p, q) of a metric tensor g is defined as the number of positive and negative eigenvalues of the real symmetric matrix g_{ab} of the metric tensor, with respect to a certain basis. However, in practice, the signature of a nondegenerate metric tensor is denoted by a single number $s = p - q$, e.g., $s = 1 - 3 = -2$ for $(+, -, -, -)$ and $s = 3 - 1 = +2$ for $(-, +, +, +)$. A metric with a positive definite signature (p, 0) is known as a Riemannian metric, whereas a metric with signature $(p, 1)$ or $(1, q)$ is called a Lorentzian metric.

A light cone in special and general relativity is the surface describing the temporal evolution of a blaze of light originating from a sole event and roving in all directions in spacetime.

Two events are causally connected if one event in spacetime can influence the other event; in other words, one can join one event to the other event with a time-like or null vector.

Exercise 1.31

$(1, 0, 0, -1)$ is a null vector, whereas $(1, 0, 0, \sqrt{2})$ is a unit vector in Minkowski space

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

Hint: Here in the first case,

$$g_{00} = 1, \;\; g_{11} = -1, \;\; g_{22} = -1, \;\; g_{33} = -1,$$

and

$$A^0 = 1, \;\; A^1 = 0, \;\; A^2 = 0, \;\; A^3 = -1.$$

Now,

$$l^2 = g_{ij}A^iA^j = g_{00}A^0A^0 + g_{11}A^1A^1 + g_{22}A^2A^2 + g_{33}A^3A^3 = 0, \text{ etc.}$$

1.6.6 Angle between two vectors A^μ and B^μ

In ordinary vector algebra, we know angle between two vectors $\vec{A}$ and $\vec{B}$ is defined as

$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}.$$

Similarly, one can define the angle between two vectors A^μ and B^μ as

$$\begin{aligned}\cos\theta &= \frac{\textit{scalar product of } A^\mu \textit{ and } B^\mu}{\textit{length of } A^\mu \times \textit{length of } B^\mu} \\ &= \frac{A^\mu B_\mu}{\sqrt{(A^\mu A_\mu)(B^\mu B_\mu)}} \\ &= \frac{g^{\mu\nu}A_\mu B_\nu}{\sqrt{(g^{\alpha\beta}A_\alpha A_\beta)(g^{\rho\sigma}B_\rho B_\sigma)}}.\end{aligned}$$

1.6.7 Orthogonal vectors

Two covariant vectors A_i, B_j or contravariant vectors A^i, B^j are said to be orthogonal if

$$g^{ij}A_iB_j = 0 = g_{ij}A^iB^j.$$

Exercise 1.32

If θ be the angle between two non-null vectors A^i and B^i at a point, show that

$$sin^2\theta = \frac{\left(g_{ij}g_{pq} - g_{ip}g_{jq}\right)A^iB^pA^jB^q}{\left(g_{ij}A^iA^j\right)\left(g_{pq}B^pB^q\right)}.$$

Hint: Let θ be the angle between two non-null vectors A^i and B^i at a point; then from the above definition

$$\cos\theta = \frac{g_{ij}A^iB^j}{\sqrt{g_{ij}A^iA^j}\sqrt{g_{pq}B^pB^q}}.$$

Now,

$$\sin^2\theta = 1 - \cos^2\theta = 1 - \frac{g_{ij}A^iB^j \;\; g_{pq}A^pB^q}{(g_{ij}A^iA^j)(g_{pq}B^pB^q)}$$

$$= \frac{g_{ij}g_{pq}A^iA^jB^pB^q - g_{ij}g_{pq}A^iA^pB^jB^q}{(g_{ij}A^iA^j)(g_{pq}B^pB^q)}$$

$$= \frac{g_{ij}g_{pq}A^iB^pA^jB^q - g_{ip}g_{jq}A^iB^pA^jB^q}{(g_{ij}A^iA^j)(g_{pq}B^pB^q)}$$

(Replacing the dummy indices j and p by p and j)

$$= \frac{\left(g_{ij}g_{pq} - g_{ip}g_{jq}\right)A^iB^pA^jB^q}{\left(g_{ij}A^iA^j\right)\left(g_{pq}B^pB^q\right)}.$$

Exercise 1.33

$(1, 1, 0, -1)$ and $(1, 0, 1, -1)$ are orthogonal vectors in Minkowski space

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

Hint: Here

$$g_{ij}A^iB^j = g_{00}A^0B^0 + g_{11}A^1B^1 + g_{22}A^2B^2 + g_{33}A^3B^3 = 0, \text{ etc.}$$

1.7 Levi-Civita Tensor or Alternating Tensor

Levi-Civita tensor is a tensor of order three in three dimensions and is denoted by ϵ_{abc} and defined as

$$\epsilon_{abc} = +1,$$

if a,b,c is an even permutation of $1, 2, 3$, i.e., in cyclic order.

$$= -1,$$

if a,b,c is odd permutation of $1, 2, 3$, i.e., not in cyclic order.

$$= 0$$

if any two indices are equal.

Levi-Civita tensor is a tensor of order four in four dimensions and denoted by ϵ^{abcd}.

$$\epsilon^{abcd} = +1,$$

if a,b,c,d is an even permutation of 0, 1, 2, 3, i.e., in cyclic order.

$$= -1,$$

if a,b,c,d is odd permutation of 0, 1, 2, 3, i.e., not in cyclic order.

$$= 0$$

if any two indices are equal.

The components of ϵ_{abcd} can be found from ϵ^{abcd} by lowering the indices in a typical way, just multiplying it by $(-g)^{-1}$:

$$\epsilon_{abcd} = g_{a\mu}\, g_{b\nu}\, g_{c\gamma}\, g_{d\sigma}(-g)^{-1}\epsilon^{\mu\nu\gamma\sigma}.$$

For example,

$$\begin{aligned} \epsilon_{0123} &= g_{0\mu}\, g_{1\nu}\, g_{2\gamma}\, g_{3\sigma}\, (-g)^{-1}\epsilon^{\mu\nu\gamma\sigma} \\ &= (-g)^{-1} det\, g_{\mu\nu} = -1. \end{aligned}$$

In general,

$$\epsilon_{abcd} = 1,$$

if a,b,c,d is an even permutation of 0, 1, 2, 3.

$$= -1,$$

if a,b,c,d is odd permutation of 0, 1, 2, 3.

$$= 0 \;\; otherwise.$$

Here,

$$\epsilon_{abcd}\epsilon^{abcd} = -24.$$

Hints: The explicit form of $\epsilon_{abcd}\epsilon^{pqnm}$ is

$$\begin{aligned} \epsilon_{abcd}\epsilon^{pqnm} = &-g^p_a g^q_b g^n_c g^m_d + g^q_a g^n_b g^m_c g^p_d - g^n_a g^m_b g^p_c g^q_d + g^m_a g^p_b g^q_c g^n_d + g^q_a g^p_b g^n_c g^m_d - g^p_a g^n_b g^m_c g^q_d \\ &+ g^n_a g^m_b g^q_c g^p_d - g^m_a g^q_b g^p_c g^n_d + g^n_a g^q_b g^p_c g^m_d - g^q_a g^p_b g^m_c g^n_d + g^p_a g^m_b g^n_c g^q_d - g^m_a g^n_b g^q_c g^p_d \\ &+ g^m_a g^q_b g^n_c g^p_d - g^q_a g^n_b g^p_c g^m_d + g^n_a g^p_b g^m_c g^q_d - g^p_a g^m_b g^q_c g^n_d + g^p_a g^n_b g^q_c g^m_d - g^n_a g^q_b g^m_c g^p_d \\ &+ g^q_a g^m_b g^p_c g^n_d - g^m_a g^p_b g^n_c g^q_d + g^p_a g^q_b g^m_c g^n_d - g^q_a g^m_b g^n_c g^p_d + g^m_a g^n_b g^p_c g^q_d - g^n_a g^p_b g^q_c g^m_d. \end{aligned}$$