Introduction to Turbulent Transport of Particles, Temperature and Magnetic Fields

Analytical Methods for Physicists and Engineers

Igor Rogachevskii

INTRODUCTION TO TURBULENT TRANSPORT OF PARTICLES, TEMPERATURE AND MAGNETIC FIELDS

Turbulence and the associated turbulent transport of scalar and vector fields present a classical physics problem that has dazzled scientists for over a century, yet many fundamental questions remain. Igor Rogachevskii, in this concise book, systematically applies various analytical methods to the turbulent transport of temperature, particles and magnetic fields. Introducing key concepts in turbulent transport including essential physics principles and statistical tools, this interdisciplinary book is suited for a range of readers, such as theoretical physicists, astrophysicists, geophysicists, plasma physicists and researchers in fluid mechanics and related topics in engineering. With an overview to various analytical methods, such as mean-field approach, dimensional analysis, multi-scale approach, quasi-linear approach, spectral tau approach, path-integral approach and analysis based on budget equations, it is also an accessible reference tool for advanced graduates, PhD students and researchers.

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Preface

Turbulence, and the associated turbulent transport of scalar and vector fields, is one of the classical problems of physics. It has been studied systematically for more than a hundred years. However, many fundamental questions related to the nature of turbulence remain. Turbulent transport is a part of the field related to turbulence. Many excellent books on turbulence that describe velocity fluctuations in detail have been published over the last hundred years. Some of these books include problems related to the turbulent transport of passive fields. However, there are no books that systematically apply different analytical methods to turbulent transport of temperature, particles and magnetic fields.

The current book is an introduction to the various analytical methods of theoretical physics and applied mathematics used to develop the mean-field theories for studying the turbulent transport of particles, temperature and magnetic fields. In particular, the following analytical methods are systematically applied in this book: the dimensional analysis, the multi-scale approach, the quasi-linear approach, the tau approach (the relaxation approach), the path-integral approach and analyses based on the budget equations. One-way and two-way couplings between turbulence and particles, or temperature, or magnetic fields are described.

This book is written for theoretical physicists, astrophysicists, geophysicists, plasma physicists, space science physicists and also for researchers working in fluid mechanics in engineering sciences. The current book can be useful for postgraduate students, specialist researchers of turbulence and turbulent transport and nonspecialist researchers from related fields. It can be presented as a source of advanced teaching material for specialized seminars, courses and schools. This book has appeared as a development and extension of the material of graduate and postgraduate lecture courses given by the author at Ben-Gurion University of the Negev. Some of these lectures have been given at the Nordic Institute of Theoretical Physics (Nordita) of KTH Royal Institute of Technology and Stockholm University.

The current book assumes prior knowledge of the basic equations and principles of fluid mechanics and magnetohydrodynamics, as well as initial knowledge about turbulence. For instance, the first part of the textbook *Turbulence* by Peter Davidson (Oxford University Press, 2015) provides a solid introduction to the field. For educational purposes, the current book is written with detailed analytical calculations and numerous practice problems and exercises. Every chapter ends with a "Further Reading" section containing a short review in the field. The book contains 60 exercises of varying difficulty with solutions.

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1

Turbulent Transport of Temperature Fields

In this chapter, we consider a turbulent transport of temperature field in an isotropic homogeneous and incompressible turbulence. We discuss the Kolmogorov theory of hydrodynamic turbulence and obtain spectrum of velocity fluctuations for fully developed turbulence using the dimensional analysis. We study isotropic and anisotropic spectra of temperature fluctuations in different subranges of turbulent scales and different Prandtl numbers applying the dimensional analysis. We derive mean-field equations for the temperature field and obtain expressions for turbulent heat flux, turbulent diffusion and level of temperature fluctuations for small and large Péclet numbers by means of various analytical methods, namely the dimensional analysis, the quasi-linear approach and the spectral tau approach (the relaxation approach).

1.1 Hydrodynamic Turbulence: Dimensional Analysis

In this section, we consider a theory of hydrodynamic isotropic homogeneous and incompressible turbulence using the dimensional analysis.

1.1.1 Governing Equations and Basic Parameters

The fluid velocity field in an incompressible flow is determined by the Navier[1]-Stokes[2] equation (Landau and Lifshits, 1987; Batchelor, 1967; Lighthill, 1986; Tritton, 1988; Faber, 1995; Falkovich, 2011):

$$\boxed{\frac{\partial \boldsymbol{U}}{\partial t} + (\boldsymbol{U} \cdot \nabla)\boldsymbol{U} = -\frac{\nabla P}{\rho} + \nu \, \Delta \boldsymbol{U} + \boldsymbol{f}.} \tag{1.1}$$

[1] Claude-Louis Navier (1785–1836) was a French engineer and physicist well-known for his works in mechanics, fluid dynamics, theory of elasticity and structural analysis.

[2] Sir George Gabriel Stokes (1819–1903) was a mathematician and physicist (who was born in Ireland and worked at the University of Cambridge) well-known for his works in fluid dynamics, optics and mathematical physics.

Equation (1.1) is the second law of Newton[3] for a unit mass of a fluid:

$$\rho \, \frac{\mathrm{d}\boldsymbol{U}}{\mathrm{d}t} = -\nabla P + \nabla \cdot \left(2\nu \, \rho \, \mathbf{S}^{(\mathrm{U})}\right) + \rho \boldsymbol{f}, \tag{1.2}$$

where according to the chain rule of differentiation of the function $\boldsymbol{U}[t, \boldsymbol{r}(t)]$, the substantial time derivative $\mathrm{d}\boldsymbol{U}/\mathrm{d}t$ for the moving fluid element is the sum of a local time derivative $\partial \boldsymbol{U}/\partial t$ and convective derivative $(\boldsymbol{U}\cdot\nabla)\boldsymbol{U}$. We take into account here that most fluids obey Newton's law of viscosity [see the second term on the right-hand side of Eq. (1.2)], where $\mathsf{S}_{ij}^{(\mathrm{U})} = \frac{1}{2}(\nabla_j U_i + \nabla_i U_j)$ are the components of the rate-of-strain-tensor $\mathbf{S}^{(\mathrm{U})}$ for incompressible fluid, ν is the kinematic viscosity, $\rho \boldsymbol{f}$ is the external force, that, e.g., creates a turbulent random velocity field, and P and ρ are the fluid pressure and density, respectively. The operators ∇ and $\Delta = \nabla^2$ in the Cartesian coordinates are defined as

$$\nabla = \boldsymbol{e}_x \, \frac{\partial}{\partial x} + \boldsymbol{e}_y \, \frac{\partial}{\partial y} + \boldsymbol{e}_z \, \frac{\partial}{\partial z}, \qquad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \tag{1.3}$$

and $\boldsymbol{e}_x, \boldsymbol{e}_y$ and $\boldsymbol{e}_z$ are unit vectors along the x-, y- and z-axes. When the viscosity ν tends to zero, Eq. (1.1) is reduced to the Euler[4] equation. The fluid pressure and density are the macroscopic variables that determine the internal state of the fluid, and they are related by the equation of state for the perfect gas, $P = (k_B/m_\mu) \, \rho \, T \equiv (R/\mu) \, \rho \, T$, where $k_B = R/N_A$ is the Boltzmann constant, R is the gas constant, N_A is the Avogadro number, $\mu = m_\mu N_A$ is the molar mass and m_μ is the mass of the molecules of the surrounding fluid. Generally for arbitrary fluid flows, the continuity equation which is the conservation law for the fluid mass reads

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \, \boldsymbol{U}) = 0 \, .} \tag{1.4}$$

This equation implies that for any volume, the change of the fluid mass inside the volume is compensated by the fluid flux through this volume.

For an incompressible fluid flow, the continuity equation (1.4) is reduced to

$$\boxed{\operatorname{div} \boldsymbol{U} \equiv \nabla \cdot \boldsymbol{U} = 0 \, ,} \tag{1.5}$$

where the fluid density ρ is constant in time and space. The second term $(\boldsymbol{U} \cdot \nabla)\boldsymbol{U}$ on the left-hand side of Eq. (1.1) is a nonlinear term that describes inertia. The

[3] Sir Isaac Newton (1642–1726) was an English mathematician, physicist and astronomer who made key contributions to the foundations of classical mechanics, optics and the infinitesimal calculus and built the first practical reflecting telescope.

[4] Leonhard Euler (1707–1783) is a Swiss mathematician, physicist, astronomer, geographer and engineer who made influential discoveries in mathematics (infinitesimal calculus and graph theory, topology, analytic number theory and mathematical analysis), mechanics, fluid dynamics, optics, astronomy and music theory.

dimensionless ratio of the nonlinear term to the viscous term in Eq. (1.1) is the Reynolds[5] number, which is a key parameter in the system:

$$\mathrm{Re} = \frac{|(\boldsymbol{U} \cdot \nabla)\boldsymbol{U}|}{|\nu\, \Delta \boldsymbol{U}|}. \tag{1.6}$$

For very large Reynolds numbers, the fluid flow is turbulent. There are many examples of turbulent flows in nature, laboratory experiments and industrial applications (Landau and Lifshits, 1987; Batchelor, 1953; Monin and Yaglom, 1971, 2013; Tennekes and Lumley, 1972; Frisch, 1995; Pope, 2000; Bernard and Wallace, 2002; Lesieur, 2008; Davidson, 2015). For instance, turbulence in laboratory experiments is produced, e.g., by oscillating grids, propellers, shear flows, etc. The atmospheric turbulence is produced by convective motions and large-scale shear flow (a non-uniform wind). Turbulence inside the Sun is produced by convection in the solar convective zone located under the solar surface. Turbulence in galaxies is produced by random supernova explosions. In astrophysics, turbulence can be also produced by shear motions and various plasma instabilities. Various pictures of turbulent flows can be found in the book by Van Dyke (1982).

In turbulent flows, the fluid velocity is a random field. Large-scale effects caused by small-scale turbulence can be studied using a mean-field approach. In the framework of this approach velocity field can be decomposed into the mean velocity and fluctuations, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, where according to the Reynolds rule velocity fluctuations $\boldsymbol{u}$ have zero mean value, $\langle \boldsymbol{u} \rangle = 0$ and $\overline{\boldsymbol{U}} = \langle \boldsymbol{U} \rangle$ is the mean fluid velocity. The angular brackets $\langle \ldots \rangle$ denote an averaging. Different kinds of averaging procedures will be discussed in the next section.

The Reynolds number defined by Eq. (1.6) can be estimated using the dimensional analysis. In particular, in Eq. (1.6) we replace operators $|\nabla|$ by ℓ_0^{-1} and Δ by ℓ_0^{-2}. This yields

$$\boxed{\mathrm{Re} = \frac{\ell_0\, u_0}{\nu},} \tag{1.7}$$

where ℓ_0 is the integral (energy-containing or maximum) scale of turbulence and $u_0 = \left[\langle \boldsymbol{u}^2 \rangle_{\ell=\ell_0}\right]^{1/2}$ is the characteristic turbulent velocity in the integral scale of turbulence ℓ_0. For example, in Table 1.1 we give turbulence parameters for various flows, e.g., for laboratory experiments in air flows, industrial flows in a wind tunnel and a diesel engine, atmospheric turbulence in the low troposphere (about 1 or 2 kilometers height from the Earth surface), and astrophysical turbulence, e.g., in the solar convective zone located under the solar surface with the depth about 1/3 of the solar radius and inside a galactic disk with a high concentration of stars. Here

[5] Osborne Reynolds (1842–1912) was an engineer (who was born in Ireland and worked at Owens College in Manchester, now the University of Manchester), well-known for his works in fluid dynamics and heat transfer.

Table 1.1 *Parameters for engineering, geophysical and astrophysical turbulence*

	ℓ_0 (cm)	u_0 (cm/s)	τ_0 (s)	ν (cm^2/s)	Re
Laboratory experiments	1–10	10–10^2	10^{-2}–1	10^{-1} (air)	10^2–10^4
Diesel engine	0.3	3×10^2	10^{-3}	10^{-2}	10^4
Wind tunnel	$(1–3) \times 10^2$	$(1–3) \times 10^3$	0.03–0.3	10^{-1}	10^6–10^7
Atmospheric turbulence	10^4	10^2	10^2	10^{-1}	10^7
Sun ($r \approx R_\odot$)	3×10^7	10^5	3×10^2	3×10^{-2}	10^{14}
Sun $\left(r \approx \frac{2}{3} R_\odot\right)$	5×10^9	2×10^3	3×10^6	10^{-1}	10^{14}
Galactic disk	10^{20}	10^6	10^{14}	10^{18}	10^8

$\tau_0 = \ell_0/u_0$ is the characteristic turbulent time in the scale ℓ_0, the radius $r = \frac{2}{3} R_\odot$ corresponds to the bottom of the solar convective zone and $R_\odot = 6.96 \times 10^{10}$ cm is the solar radius.

A fully developed turbulence for very large Reynolds numbers can be qualitatively regarded as a sea of eddies, i.e., an ensemble of turbulent eddies of different scales varying from the integral energy-containing scale ℓ_0 to very small viscous scale ℓ_ν. Turbulent eddy can be considered as a blob of vorticity $\boldsymbol{\nabla} \times \boldsymbol{u}$. In the scale ℓ_ν, the viscous dissipation of the turbulent kinetic energy becomes important. The dynamics of the turbulent eddies is as follows. The large eddies are unstable, and they break down into the small eddies. The new small eddies are also unstable and continue to breakdown into the very small eddies. This process is called the Richardson[6] energy cascade and implies the transfer of the turbulent kinetic energy from the integral scale to smaller ones (Richardson, 1920). The energy cascade stops when the size of the small eddies is of the order of the viscous scale of turbulence. At this scale, turbulent kinetic energy is dissipated into thermal energy. The rate of the dissipation of the turbulent kinetic energy ε can be estimated as

$$\varepsilon = \frac{u_0^2}{\tau_0} = \frac{u_0^3}{\ell_0}. \tag{1.8}$$

1.1.2 Kolmogorov Theory of Hydrodynamic Turbulence

In this section, we consider Kolmogorov[7] theory of hydrodynamic turbulence. Let us assume that

[6] Lewis F. Richardson (1881–1953) was an English mathematician, physicist and meteorologist, well-known for his works in turbulence, mathematical physics and mathematical techniques of weather forecasting.

[7] Andrey N. Kolmogorov (1903–1987) was a Russian mathematician, well-known for his works in theory of random processes and probability theory, theory of turbulence, topology, theory of differential equations, functional analysis and information theory.

- turbulence is homogeneous, i.e., $\nabla \langle \boldsymbol{u}^2 \rangle = 0$;
- turbulence is isotropic, i.e., there is no preferential direction;
- turbulent flow is incompressible, i.e., $\nabla \cdot \boldsymbol{u} = 0$ and the fluid density ρ is constant in time and in space;
- interactions in the turbulence are local, i.e., there are only interactions between turbulent eddies of the same size, and there are no interactions between the eddies of different sizes;
- in a subrange of turbulent scales $\ell_\nu \le \ell \le \ell_0$, the dissipation rate of the turbulent kinetic energy density is constant

$$\boxed{\varepsilon = \frac{u_0^3}{\ell_0} = \frac{u_\ell^3}{\ell} = \ldots = \frac{u_\nu^3}{\ell_\nu} = \text{const},} \tag{1.9}$$

where $u_\ell = \left[\langle \boldsymbol{u}^2 \rangle_\ell\right]^{1/2}$ is the characteristic turbulent velocity at the scale ℓ inside the inertial subrange of turbulence scales $\ell_\nu \le \ell \le \ell_0$ and $u_\nu = \left[\langle \boldsymbol{u}^2 \rangle_{\ell=\ell_\nu}\right]^{1/2}$ is the characteristic velocity at the viscous scale ℓ_ν. For the simplicity we assume here that the constant fluid density is unity. Equation (1.9) allows us to determine turbulent velocities in different scales,

$$u_0 = (\varepsilon\, \ell_0)^{1/3}, \quad u_\ell = (\varepsilon\, \ell)^{1/3}, \quad u_\nu = (\varepsilon\, \ell_\nu)^{1/3}. \tag{1.10}$$

Equation $u_\ell = (\varepsilon\, \ell)^{1/3}$ implies that the scaling for u_ℓ^2 in the inertial subrange of turbulent scales $\ell_\nu \ll \ell \ll \ell_0$ is given by

$$\boxed{u_\ell^2 = \varepsilon^{2/3}\, \ell^{2/3}} \tag{1.11}$$

[see Kolmogorov (1941), and its English translation in Kolmogorov (1991)], and the characteristic time $\tau_\ell = \ell / u_\ell$ in the inertial subrange of scales is

$$\boxed{\tau_\ell = \varepsilon^{-1/3}\, \ell^{2/3}.} \tag{1.12}$$

Using Eq. (1.10), we rewrite the Reynolds number as

$$\text{Re} = \frac{\ell_0\, u_0}{\nu} = \frac{\varepsilon^{1/3}\, \ell_0^{4/3}}{\nu}. \tag{1.13}$$

We introduce the local Reynolds number:

$$\text{Re}_\ell = \frac{\ell\, u_\ell}{\nu} = \frac{\varepsilon^{1/3}\, \ell^{4/3}}{\nu}. \tag{1.14}$$

Equations (1.13)–(1.14) allow us to determine the ratio $\text{Re}_\ell / \text{Re}$ as

$$\frac{\text{Re}_\ell}{\text{Re}} = \left(\frac{\ell}{\ell_0}\right)^{4/3}. \tag{1.15}$$

The viscous scale ℓ_ν (the Kolmogorov scale) is defined as the scale in which the local Reynolds number is 1. This implies that in the Kolmogorov scale, the nonlinear term in the Navier-Stokes equation is of the order of the viscous term. Therefore, Eq. (1.15) with the condition $\mathrm{Re}_{\ell=\ell_\nu} = 1$ allow us to relate the Kolmogorov scale ℓ_ν with the integral scale ℓ_0 of turbulence as

$$\boxed{\ell_\nu = \frac{\ell_0}{\mathrm{Re}^{3/4}}\,.} \tag{1.16}$$

Substituting the Kolmogorov scale ℓ_ν given by Eq. (1.16) into Eq. (1.10) for $u_\nu = (\varepsilon\,\ell_\nu)^{1/3}$, we obtain the characteristic velocity in the Kolmogorov scale as $u_\nu = (\varepsilon\,\ell_0)^{1/3}\,\mathrm{Re}^{-1/4}$, so that

$$\boxed{u_\nu = \frac{u_0}{\mathrm{Re}^{1/4}}\,,} \tag{1.17}$$

where $u_0 = (\varepsilon\,\ell_0)^{1/3}$. Therefore, the characteristic viscous time $\tau_\nu = \ell_\nu/u_\nu$ (the Kolmogorov time) is given by

$$\boxed{\tau_\nu = \frac{\tau_0}{\mathrm{Re}^{1/2}}\,.} \tag{1.18}$$

Next, we determine the spectrum of velocity fluctuations in the inertial subrange of scales (the Kolmogorov-Obukhov[8] spectrum). We define the energy spectrum function of the velocity field as

$$u_\ell^2 = \int_{k_0}^{k} E_{\mathrm{u}}(k')\,dk', \tag{1.19}$$

where wave numbers $k_0 = \ell_0^{-1}$ and $k = \ell^{-1}$. Using the dimensional analysis, we rewrite Eq. (1.19) as $u_\ell^2 = E_{\mathrm{u}}(k)\,k$. Therefore, the Kolmogorov-Obukhov spectrum $E_{\mathrm{u}}(k)$ in the inertial subrange of turbulent scales $k_0 \ll k \ll k_\nu$ is given by (Kolmogorov, 1941; Obukhov, 1941)

$$\boxed{E_{\mathrm{u}}(k) = \varepsilon^{2/3}k^{-5/3},} \tag{1.20}$$

where $k_\nu = \ell_\nu^{-1}$ and we take into account that $u_\ell = (\varepsilon/k)^{1/3}$. Equation (1.20) also directly follows from Eq. (1.11) using the relations $\ell = k^{-1}$ and $E_{\mathrm{u}}(k) = u_\ell^2/k$. Since $\varepsilon = u_\ell^2/\tau_\ell = E_{\mathrm{u}}(k)\,k/\tau(k)$, we obtain the scaling for the characteristic time $\tau(k)$ in the inertial subrange of turbulent scales as

$$\boxed{\tau(k) = \varepsilon^{-1/3}k^{-2/3}.} \tag{1.21}$$

[8] Alexander M. Obukhov (1918–1998) was a Russian geophysicist well-known for his works in atmospheric physics, meteorology, turbulence and mathematical statistics.

Equation (1.21) also directly follows from Eq. (1.12) using the relation $\ell = k^{-1}$. The Kolmogorov-Obukhov spectrum has been detected in many laboratory experiments where turbulence is produced by various sources. This spectrum also has been observed in atmospheric turbulence, space experiments with solar wind, and solar and galactic turbulence. The Kolmogorov-Obukhov spectrum can be considered as a universal spectrum since it is observed in various turbulent systems of different origins.

1.2 Spectra of Temperature Fluctuations: Dimensional Analysis

In this section, we obtain various spectra of temperature fluctuations in a hydrodynamic isotropic homogeneous and incompressible turbulence using the dimensional analysis.

1.2.1 Governing Equations, Averaging and Basic Parameters

The equation for the evolution of fluid temperature field $T(t, \boldsymbol{x})$ in an incompressible fluid velocity field $\boldsymbol{U}(t, \boldsymbol{x})$ reads (Landau and Lifshits, 1987; Batchelor, 1967)

$$\boxed{\frac{\partial T}{\partial t} + (\boldsymbol{U} \cdot \nabla) T = D^{(\theta)} \, \Delta T + I_T,} \tag{1.22}$$

where $D^{(\theta)}$ is the coefficient of the molecular diffusion of the temperature field and I_T is the heat source/sink that for simplicity is neglected below. Equation (1.22) is the convective diffusion equation. The continuity equation for the incompressible fluid velocity field is $\nabla \cdot \boldsymbol{U} = 0$. We apply a mean-field approach, i.e., all quantities are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values. For example, the temperature field $T = \overline{T} + \theta$, where $\overline{T} = \langle T \rangle$ is the mean fluid temperature, θ are temperature fluctuations, and $\langle \theta \rangle = 0$. The angular brackets $\langle \ldots \rangle$ denote an averaging. Similarly, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, where $\overline{\boldsymbol{U}} = \langle \boldsymbol{U} \rangle$ is the mean fluid velocity, $\boldsymbol{u}$ are velocity fluctuations and $\langle \boldsymbol{u} \rangle = 0$. There are three main ways of averaging:

- The time averaging (i.e., the averaging over the time):

$$\overline{T} = \frac{1}{t_M} \int_0^{t_M} T(t, \boldsymbol{x}) \, dt, \tag{1.23}$$

  where t_M is the total time of measurements (e.g., in the case of laboratory or field experiments) or the total time of calculations (e.g., in the case of numerical simulations).

- The spatial (volume) averaging:

$$\overline{T} = \frac{1}{L_x L_y L_z} \int_0^{L_x} dx \int_0^{L_y} dy \int_0^{L_z} T(t, \boldsymbol{x})\, dz, \tag{1.24}$$

  where L_x, L_y, L_z are the sizes of the box along x, y, z directions. The plane averaging,

$$\overline{T} = \frac{1}{L_x L_y} \int_0^{L_x} dx \int_0^{L_y} T(t, \boldsymbol{x})\, dy, \tag{1.25}$$

  is used or even the averaging along one direction,

$$\overline{T} = \frac{1}{L_y} \int_0^{L_y} T(t, \boldsymbol{x})\, dy, \tag{1.26}$$

  is also used.
- The ensemble averaging (e.g., averaging over independent spatial distributions $T_n = T(t_n, \boldsymbol{x})$ of temperature fields taken in different times: $t_1, t_2, \ldots, t_N$):

$$\overline{T} = \frac{1}{N} \sum_{n=1}^{N} T_n(\boldsymbol{x}), \tag{1.27}$$

  where t_n are the instants of measurements and N is the total number of data points.

Averaging Eq. (1.22) over an ensemble of turbulent velocity field, we arrive at the mean-field equation for the mean temperature field:

$$\boxed{\frac{\partial \overline{T}}{\partial t} + \nabla \cdot \left(\overline{T}\, \overline{\boldsymbol{U}} + \langle \theta\, \boldsymbol{u} \rangle\right) = D^{(\theta)}\, \Delta \overline{T},} \tag{1.28}$$

where $\langle \theta\, \boldsymbol{u} \rangle$ is the turbulent heat flux. In our derivation of Eq. (1.28), we take into account that

- various operators, like the averaging $\langle \ldots \rangle$, the partial derivative over time, the spatial partial derivatives, the operators ∇ and Δ, are linear commutative operators;
- $\langle \boldsymbol{u} \overline{T} \rangle = \overline{T} \langle \boldsymbol{u} \rangle = 0$ and $\langle \overline{\boldsymbol{U}} \theta \rangle = \overline{\boldsymbol{U}} \langle \theta \rangle = 0$.

Let us consider for simplicity the case $\overline{\boldsymbol{U}} = 0$. The obtained results will be the same for the constant mean fluid velocity due to the Galilean[9] invariance.

[9] Galileo Galilei (1564–1642) was an Italian astronomer, physicist, engineer, philosopher and mathematician, well-known for his works in physics, astronomy and applied science.

The equation for temperature fluctuations $\theta = T - \overline{T}$ is obtained by subtracting Eq. (1.28) from Eq. (1.22):

$$\boxed{\frac{\partial \theta}{\partial t} + \nabla \cdot [\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle] - D^{(\theta)} \Delta \theta = -(\boldsymbol{u} \cdot \nabla) \overline{T}.} \tag{1.29}$$

The second term, $\nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle)$, on the left-hand side of Eq. (1.29) is the nonlinear term, while the first term, $-(\boldsymbol{u} \cdot \nabla)\overline{T}$, on the right-hand side of Eq. (1.29) is the source of temperature fluctuations produced by the tangling of the gradient of the mean temperature $\nabla \overline{T}$ by random velocity fluctuations $\boldsymbol{u}$. The dimensionless ratio of the nonlinear term to the diffusion term in Eq. (1.29) is the Péclet[10] number which is a key parameter in the system:

$$\mathrm{Pe} = \frac{|\nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle)\,|}{|D^{(\theta)} \Delta \theta|}. \tag{1.30}$$

The Péclet number, defined by Eq. (1.30), can be estimated using dimensional analysis as

$$\boxed{\mathrm{Pe} = \frac{\ell_0\, u_0}{D^{(\theta)}}.} \tag{1.31}$$

Using Eq. (1.10) for the turbulent velocity $u_0 = (\varepsilon\, \ell_0)^{1/3}$ at the integral scale, we rewrite the Péclet number as

$$\mathrm{Pe} = \frac{\varepsilon^{1/3} \ell_0^{4/3}}{D^{(\theta)}}. \tag{1.32}$$

Next, we introduce the local Péclet number $\mathrm{Pe}_\ell = \ell\, u_\ell / D^{(\theta)}$ at the scale ℓ and use Eq. (1.10) for the turbulent velocity $u_\ell = (\varepsilon\, \ell)^{1/3}$, so that the local Péclet number is

$$\mathrm{Pe}_\ell = \frac{\ell\, u_\ell}{D^{(\theta)}} = \frac{\varepsilon^{1/3} \ell^{4/3}}{D^{(\theta)}}. \tag{1.33}$$

We determine the ratio $\mathrm{Pe}_\ell / \mathrm{Pe}$ as

$$\frac{\mathrm{Pe}_\ell}{\mathrm{Pe}} = \left(\frac{\ell}{\ell_0}\right)^{4/3}. \tag{1.34}$$

We introduce a diffusion scale ℓ_D defined as the scale in which the local Péclet number is 1. This implies that at the scale ℓ_D, the nonlinear terms in the equation

[10] Jean Claude Eugène Péclet (1793–1857) was a French physicist well-known for his works in fluid dynamics, heat transfer and theory of combustion.

for temperature fluctuations equal the diffusion term. Therefore, Eq. (1.34) yields the diffusion scale ℓ_D as

$$\boxed{\ell_D = \frac{\ell_0}{\mathrm{Pe}^{3/4}}.} \tag{1.35}$$

Let us consider the case when the diffusion scale is inside the inertial subrange of turbulent scales, $\ell_0 \geq \ell_D \geq \ell_\nu$. This implies that $u_D = (\varepsilon\, \ell_D)^{1/3}$ [see Eqs. (1.9)–(1.10)]. Substituting the diffusion scale (1.35) into the equation for u_D, we obtain the characteristic velocity at the diffusion scale as $u_D = (\varepsilon\, \ell_0)^{1/3}\ \mathrm{Pe}^{-1/4}$, so that

$$\boxed{u_D = \frac{u_0}{\mathrm{Pe}^{1/4}},} \tag{1.36}$$

where $u_0 = (\varepsilon\, \ell_0)^{1/3}$. Using Eqs. (1.35) and (1.36), we determine the characteristic diffusion time $\tau_D = \ell_D/u_D$ as

$$\boxed{\tau_D = \frac{\tau_0}{\mathrm{Pe}^{1/2}} \equiv \frac{\ell_D^2}{D^{(\theta)}},} \tag{1.37}$$

where we take into account that $\mathrm{Pe}_{\ell=\ell_D} = 1$, i.e., $u_D \ell_D = D^{(\theta)}$. Let us determine the ratio of the diffusion scale to the viscous scale ℓ_D/ℓ_ν:

$$\frac{\ell_D}{\ell_\nu} = \left(\frac{\mathrm{Re}}{\mathrm{Pe}}\right)^{3/4} = \left(\frac{D^{(\theta)}}{\nu}\right)^{3/4} = \mathrm{Pr}^{-3/4}, \tag{1.38}$$

where

$$\boxed{\mathrm{Pr} = \frac{\nu}{D^{(\theta)}}} \tag{1.39}$$

is the Prandtl[11] number. Small Prandtl numbers $\mathrm{Pr} \ll 1$ implies that $\ell_\nu \ll \ell_D$, i.e., the viscous scale ℓ_ν is the smallest scale. In the opposite case of large Prandtl numbers $\mathrm{Pr} \gg 1$, the diffusion scale $\ell_D \ll \ell_\nu$ is the smallest scale.

1.2.2 Isotropic Temperature Fluctuations

In this section, we consider the case of small Prandtl numbers ($\mathrm{Pr} \ll 1$) and study temperature fluctuations in the inertial subrange of turbulence, $\ell_\nu \ll \ell_D < \ell < \ell_0$. The energy spectrum function of the velocity field is defined as $u_\ell^2 = \int_{k_0}^{k} E_\mathrm{u}(k')\, dk'$, where $k_0 = \ell_0^{-1}$ and $k = \ell^{-1}$. The Kolmogorov-Obukhov spectrum of velocity fluctuations in the inertial subrange of turbulent scales is given by

$$E_\mathrm{u}(k) = \frac{u_\ell^2}{k} = \varepsilon^{2/3}\, k^{-5/3}, \tag{1.40}$$

[11] Ludwig Prandtl (1875–1953) was a German engineer and physicist well-known for his works in fluid dynamics, aerodynamics, shock waves, plasticity, structural mechanics and meteorology.

and the scaling for the turbulent time $\tau(k)$ is

$$\tau(k) = \frac{\ell}{u_\ell} = \frac{\ell}{(\varepsilon \ell)^{1/3}} = \varepsilon^{-1/3}\, k^{-2/3}. \tag{1.41}$$

Equations (1.40)–(1.41) are only valid when the rate of dissipation of the turbulent kinetic energy density is constant inside the inertial subrange of turbulent scales, i.e.,

$$\varepsilon \equiv \frac{u_\ell^2}{\tau_\ell} = \frac{E_{\rm u}(k)\, k}{\tau(k)} = \text{const}, \tag{1.42}$$

and $u_\ell = (\varepsilon/k)^{1/3}$.

Spectrum function of temperature fluctuations is defined as

$$\langle \theta^2 \rangle_\ell = \int_{k_0}^{k} \tilde{E}_\theta(k')\, dk'. \tag{1.43}$$

Using the dimensional analysis, we rewrite this expression as $\langle \theta^2 \rangle_\ell = \tilde{E}_\theta(k)\, k$ and assume that the rate of dissipation of temperature fluctuations is constant inside the subrange of scales $\ell_D < \ell < \ell_0$, i.e.,

$$\varepsilon_\theta \equiv \frac{\langle \theta^2 \rangle_\ell}{\tau_\ell} = \frac{\tilde{E}_\theta(k)\, k}{\tau(k)} = \text{const}. \tag{1.44}$$

The condition (1.44) for temperature fluctuations is analogous to condition (1.42) for velocity fluctuations in the inertial range of turbulence. Equations (1.42) and (1.44) yield the spectrum of isotropic temperature fluctuations inside the scale-dependent turbulent diffusion range of scales, $\ell_D < \ell < \ell_0$:

$$\boxed{\tilde{E}_\theta(k) \sim E_{\rm u}(k) \sim \varepsilon^{2/3}\, k^{-5/3}.} \tag{1.45}$$

This spectrum was obtained by Obukhov (1949) and Corrsin (1951).[12]

Let us consider the case, $\mathrm{Pr} > 1$, and study temperature fluctuations in the viscous subrange of scales, $\ell_D < \ell < \ell_\nu$. In this range of scales, Eq. (1.44) is valid, but the time $\tau(k)$ does not have a universal scaling. If $\tau(k) = \text{const}$, the spectrum of temperature fluctuations is

$$\boxed{\tilde{E}_\theta(k) \sim k^{-1}.} \tag{1.46}$$

This spectrum was obtained by Batchelor (1959)[13] and Kraichnan (1968).[14]

[12] Stanley Corrsin (1920–1986) was an American physicist, well-known for his works in experimental and theoretical fluid dynamics, turbulence and turbulent mixing.

[13] George Keith Batchelor (1920–2000) was an applied mathematician (who was born in Australia and worked at the University of Cambridge) well-known for his works in fluid dynamics, theory of turbulence and turbulent transport.

[14] Robert Harry Kraichnan (1928–2008) was an American theoretical physicist well-known for his works in the theory of turbulence, turbulent transport and magnetohydrodynamics.

1.2.3 Anisotropic Temperature Fluctuations in the Inertial-Diffusion Range

We consider anisotropic temperature fluctuations caused by the tangling of the mean temperature gradient by velocity fluctuations in the inertial-diffusion subrange of scales, $\ell_\nu < \ell < \ell_D$. This subrange of scales corresponds to the Prandtl numbers, Pr < 1. We use dimensional analysis, taking into account that molecular diffusion is a key effect in this subrange of scales. This implies that the molecular diffusion term, $D^{(\theta)}\Delta\theta$, in Eq. (1.29) should be balanced by the source term, $(\boldsymbol{u}\cdot\nabla)\overline{T}$, for temperature fluctuations, i.e.,

$$|D^{(\theta)}\Delta\theta| \sim |(\boldsymbol{u}\cdot\nabla)\overline{T}|, \tag{1.47}$$

which yields

$$\left\langle\theta^2\right\rangle_\ell \sim u_\ell^2 \left(\frac{\ell^2\nabla\overline{T}}{D^{(\theta)}}\right)^2 . \tag{1.48}$$

In the $\boldsymbol{k}$ space, Eq. (1.48) implies that

$$\tilde{E}_\theta(k) \sim E_\mathrm{u}(k)\, k^{-4} \left(\frac{\nabla\overline{T}}{D^{(\theta)}}\right)^2 , \tag{1.49}$$

where $E_\mathrm{u}(k)$ is the spectrum function of velocity fluctuations. Since the subrange of scales $\ell_\nu < \ell < \ell_D$ corresponds to the inertial range of scales, velocity fluctuations have the Kolmogorov-Obukhov spectrum (1.40). Therefore, the spectrum of the anisotropic temperature fluctuations in the inertial-diffusion range of scales is

$$\boxed{\tilde{E}_\theta(k) \sim \varepsilon^{2/3}\, k^{-17/3} \left(\frac{\nabla\overline{T}}{D^{(\theta)}}\right)^2 .} \tag{1.50}$$

This spectrum was obtained by G. Batchelor, I. Howells and A. Townsend[15] (Batchelor et al., 1959).

1.2.4 Anisotropic Temperature Fluctuations in the Inertial-Turbulent Diffusion Range

We consider anisotropic temperature fluctuations caused by the tangling of the mean temperature gradient by velocity fluctuations in the inertial-turbulent diffusion range of scales, $\ell_D < \ell < \ell_0$. This subrange of scales corresponds to the small Prandtl numbers, Pr < 1. We take into account that the main effect of turbulence

[15] Albert Alan Townsend (1917–2010) was a physicist (who was born in Australia and worked at the University of Cambridge) well-known for his works in fluid dynamics, experimental study of turbulence and turbulent transport, meteorology and nuclear physics.

on temperature fluctuations in incompressible flow is the scale-dependent turbulent diffusion that is much larger than the molecular diffusion for large Péclet numbers. Let us average Eq. (1.29) over an ensemble up to the scale ℓ_* that is inside the interval: $\ell_D \ll \ell_* \ll \ell_0$. This yields the renormalized equation for temperature fluctuations:

$$\frac{\partial \theta}{\partial t} - D_{_T}(\ell)\, \Delta\theta = -(\boldsymbol{u} \cdot \nabla)\overline{T}, \tag{1.51}$$

where $D_{_T}(\ell)$ is the scale-dependent turbulent diffusion coefficient that can be estimated as

$$\boxed{D_{_T}(\ell) = \ell\, u_\ell.} \tag{1.52}$$

In the subrange of scales $\ell_D \ll \ell_*$ the turbulent diffusion term $D_{_T}(\ell)\Delta\theta$ in Eq. (1.51) should be balanced by the source term, $(\boldsymbol{u}\cdot\nabla)\overline{T}$, for temperature fluctuations, i.e.,

$$|D_{_T}(\ell)\, \Delta\theta| \sim |(\boldsymbol{u} \cdot \nabla)\overline{T}|. \tag{1.53}$$

This implies that

$$\left\langle \theta^2 \right\rangle_\ell \sim u_\ell^2 \left(\frac{\ell^2\, \nabla\overline{T}}{D_{_T}(\ell)} \right)^2 \sim u_\ell^2 \left(\frac{\ell^2\, \nabla\overline{T}}{\ell\, u_\ell} \right)^2 \sim \left(\ell\, \nabla\overline{T}\right)^2, \tag{1.54}$$

where we used Eq. (1.52). Equation (1.54) written in the $\boldsymbol{k}$ space yields the spectrum of anisotropic temperature fluctuations in the inertial-turbulent diffusion range of scales $\ell_D \ll \ell \ll \ell_0$:

$$\boxed{\tilde{E}_\theta(k) \sim k^{-3}\, \left(\nabla\overline{T}\right)^2,} \tag{1.55}$$

where we take into account in Eq. (1.54) that according to dimensional analysis, $\langle\theta^2\rangle_\ell = \tilde{E}_\theta(k)\, k$ and $k = \ell^{-1}$. This spectrum is independent of the spectrum of the turbulent velocity field because u_ℓ^2 is canceled in Eq. (1.54). The spectrum (1.55) was obtained by A. Wheelon using the dimensional analysis (Wheelon, 1957) and by T. Elperin, N. Kleeorin and I. Rogachevskii, applying the renormalization approach (Elperin et al., 1996a).

1.3 Turbulent Transport of Temperature Fields: Dimensional Analysis

1.3.1 Governing Equations

We consider incompressible fluid velocity field $\boldsymbol{U}(t, \boldsymbol{x})$ satisfying the continuity equation: $\nabla \cdot \boldsymbol{U} = 0$. Since the velocity field is incompressible, Eq. (1.22) for the fluid temperature field $T(t, \boldsymbol{x})$ can be rewritten in the following form:

$$\boxed{\frac{\partial T}{\partial t} + \nabla \cdot (T\, \boldsymbol{U}) = D^{(\theta)}\, \Delta T.} \tag{1.56}$$

The velocity field is a random turbulent field created, e.g., by external forcing.

1.3.2 Mean-Field Approach

Our goal is to study the long-term evolution of the temperature field in the large scales, i.e., in spatial scales $L_T \gg \ell_0$, and the time scales $t_T \gg \tau_0$, where τ_0 is the characteristic turbulent time in the integral turbulent scale ℓ_0, L_T is the characteristic spatial scale of variations of the mean temperature field and t_T is the characteristic time-scale of variations of the mean temperature field. We use a mean-field approach in which all quantities are decomposed into the mean and fluctuating parts, where the fluctuating parts have zero mean values. In particular, the temperature field $T = \overline{T} + \theta$, where $\overline{T} = \langle T \rangle$ is the mean fluid temperature, θ are temperature fluctuations and $\langle \theta \rangle = 0$. The angular brackets $\langle \ldots \rangle$ denote ensemble averaging. In similar fashion, we decompose a velocity field, $\boldsymbol{U} = \overline{\boldsymbol{U}} + \boldsymbol{u}$, where $\overline{\boldsymbol{U}} = \langle \boldsymbol{U} \rangle$ is the mean fluid velocity, $\boldsymbol{u}$ are velocity fluctuations and $\langle \boldsymbol{u} \rangle = 0$. This decomposition corresponds to the Reynolds rules. Averaging Eq. (1.56) over an ensemble of turbulent velocity field, we arrive at the mean-field equation (1.28) for the mean temperature field.

1.3.3 Equation for Temperature Fluctuations

Equation (1.28) is not closed because we do not know the turbulent heat flux $\boldsymbol{F}^{(\theta)} = \langle \boldsymbol{u}\theta \rangle$. To determine the turbulent heat flux, we use the equation for temperature fluctuations that is obtained by subtracting Eq. (1.28) from Eq. (1.56):

$$\boxed{\frac{\partial \theta}{\partial t} + \nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle) - D^{(\theta)} \Delta\theta = -(\boldsymbol{u} \cdot \nabla)\overline{T}.} \tag{1.57}$$

The terms, $\nabla \cdot (\theta\, \boldsymbol{u} - \langle \theta\, \boldsymbol{u} \rangle)$, on the left-hand side of Eq. (1.57) are the nonlinear terms, while the first term, $-(\boldsymbol{u} \cdot \nabla)\overline{T}$, on the right-hand side of Eq. (1.57) determines the source of temperature fluctuations produced by the tangling of the gradient of the mean temperature, $\nabla\overline{T}$, by random velocity fluctuations, $\boldsymbol{u}$. Since Eq. (1.57) is nonlinear equation for temperature fluctuations, it cannot be solved exactly for the arbitrary range of parameters and arbitrary velocity field. Therefore, we have to use different approximate methods for the solution of Eq. (1.57). First, we consider a one-way coupling, i.e., we take into account the effect of the turbulent velocity on the temperature field, but neglect the feedback effect of the temperature field on the turbulent fluid flow. This implies that the temperature field is a passive scalar.

1.3.4 Dimensional Analysis

The first method that we apply here is the dimensional analysis. The dimension of the left-hand side of Eq. (1.57) is the rate of change of temperature fluctuations, i.e., θ/τ_θ, where τ_θ is the characteristic time of temperature fluctuations. We replace the left-hand side of Eq. (1.57) by θ/τ_θ, that yields:

$$\theta = -\tau_\theta\, (\boldsymbol{u} \cdot \nabla)\overline{T}. \tag{1.58}$$

We consider two cases of large and small Péclet numbers:

- *Large Péclet numbers*, $\mathrm{Pe} = u_0\, \ell_0/D^{(\theta)} \gg 1$. We also consider the case of large Reynolds numbers, $\mathrm{Re} = u_0\, \ell_0/\nu \gg 1$. This implies that we consider fully developed turbulence. In this case, the characteristic time of temperature fluctuations, τ_θ, can be identified with the correlation time τ_0 of the turbulent velocity field. Therefore, in the framework of the dimensional analysis, we replace the left-hand side of Eq. (1.57) with θ/τ_0. This yields the expression for temperature fluctuations:

$$\theta = -\tau_0\, (\boldsymbol{u} \cdot \nabla)\overline{T}. \tag{1.59}$$

- *Small Péclet numbers*, $\mathrm{Pe} \ll 1$. In this case, the nonlinear terms are much smaller than the diffusion terms. This implies that the molecular diffusion for $\mathrm{Pe} \ll 1$ is the main process, which determines the dynamics of temperature fluctuations. Therefore, we assume that the time τ_θ can be identified with the molecular diffusion time $\tau_D = \ell_0^2/D^{(\theta)}$, and the solution of Eq. (1.57) for $\mathrm{Pe} \ll 1$ reads

$$\theta = -\frac{\ell_0^2}{D^{(\theta)}}\, (\boldsymbol{u} \cdot \nabla)\overline{T}. \tag{1.60}$$

1.3.5 Turbulent Heat Flux and Level of Temperature Fluctuations

Large Péclet Numbers

Let us consider the case of large Péclet numbers $\mathrm{Pe} \gg 1$ and determine the turbulent heat flux and level of temperature fluctuations. Multiplying Eq. (1.59) by velocity fluctuations, u_i, and averaging over an ensemble of turbulent velocity field, we obtain the turbulent heat flux:

$$\langle \theta\, u_i \rangle = -\tau_0\, \langle u_i (\boldsymbol{u} \cdot \nabla)\overline{T} \rangle = -\tau_0\, \langle u_i u_j \rangle\, \nabla_j \overline{T}, \tag{1.61}$$

where we took into account that $\boldsymbol{u} \cdot \nabla \equiv u_j\, \nabla_j = u_1\, \nabla_1 + u_2\, \nabla_2 + u_3\, \nabla_3 \equiv u_x\, \nabla_x + u_y\, \nabla_y + u_z\, \nabla_z$ (i.e., there is summation in the repeating indexes). In isotropic turbulence, $\langle u_i u_j \rangle = \delta_{ij} \, \langle \boldsymbol{u}^2 \rangle / 3$, where δ_{ij} is the Kronecker tensor (or the unit matrix), that is defined as $\delta_{ij} = 1$ for $i = j$, and $\delta_{ij} = 0$ for $i \neq j$. Therefore, for an isotropic turbulence, the turbulent heat flux reads

$$\boxed{\boldsymbol{F}^{(\theta)} \equiv \langle \theta \, \boldsymbol{u} \rangle = -D_{_T} \, \boldsymbol{\nabla} \overline{T},} \tag{1.62}$$

with the coefficient of turbulent diffusion of the temperature field for large Péclet numbers:

$$\boxed{D_{_T} = \frac{1}{3} \, \tau_0 \, \left\langle \boldsymbol{u}^2 \right\rangle .} \tag{1.63}$$

Using Eq. (1.59) for θ^2 and averaging over an ensemble of turbulent velocity field, we determine the level of temperature fluctuations $\langle \theta^2 \rangle$:

$$\left\langle \theta^2 \right\rangle = \tau_0^2 \left\langle \left[(\boldsymbol{u} \cdot \nabla) \overline{T} \, \right]^2 \right\rangle = \tau_0^2 \, \langle u_i u_j \rangle \, \left(\nabla_i \overline{T} \right) \left(\nabla_j \overline{T} \right) . \tag{1.64}$$

Therefore, for an isotropic turbulence, $\langle u_i u_j \rangle = \delta_{ij} \, \langle \boldsymbol{u}^2 \rangle / 3$, the level of temperature fluctuations for large Péclet numbers is given by

$$\boxed{\left\langle \theta^2 \right\rangle = \frac{1}{3} \ell_0^2 \, \left(\boldsymbol{\nabla} \overline{T} \right)^2 ,} \tag{1.65}$$

where $\ell_0 = \tau_0 \, u_0$ and $u_0 \equiv u_{\rm rms} = \sqrt{\langle \boldsymbol{u}^2 \rangle}$ is the r.m.s. velocity fluctuations (characteristic turbulent velocity).

Small Péclet Numbers

Now we consider the case of small Péclet numbers Pe $\ll 1$ and determine the turbulent heat flux and level of temperature fluctuations. Multiplying Eq. (1.60) by u_i and averaging this equation over a statistics of a random velocity field, we obtain

$$\langle \theta \, u_i \rangle = - \frac{\ell_0^2}{D^{(\theta)}} \, \langle u_i u_j \rangle \, (\nabla_j \overline{T}). \tag{1.66}$$

Therefore, for isotropic turbulence, $\langle u_i u_j \rangle = \delta_{ij} \, \langle \boldsymbol{u}^2 \rangle / 3$, the turbulent heat flux $F_i^{(\theta)}$ for small Péclet numbers is given by $\boldsymbol{F}^{(\theta)} \equiv \langle \theta \, \boldsymbol{u} \rangle = -D_{_T} \, \boldsymbol{\nabla} \overline{T}$, which coincides with Eq. (1.62) derived for large Péclet numbers, but with a different coefficient of turbulent diffusion:

$$\boxed{D_T = \frac{u_0 \, \ell_0}{3} \, \text{Pe}.} \tag{1.67}$$

Since Pe $\ll 1$, the coefficient of turbulent diffusion D_T is much smaller than the molecular diffusion coefficient $D^{(\theta)}$. Using Eq. (1.60) for θ^2 and averaging over an ensemble of turbulent velocity field, we determine the level of temperature fluctuations $\langle \theta^2 \rangle$:

$$\langle\theta^2\rangle = \left(\frac{\ell_0^2}{D^{(\theta)}}\right)^2 \left\langle\left[(\boldsymbol{u}\cdot\nabla)\overline{T}\right]^2\right\rangle = \left(\frac{\ell_0^2}{D^{(\theta)}}\right)^2 \langle u_i u_j\rangle\,(\nabla_i\overline{T})(\nabla_j\overline{T}). \quad (1.68)$$

Therefore, for isotropic turbulence the level of temperature fluctuations for small Péclet numbers is given by

$$\boxed{\langle\theta^2\rangle = \frac{1}{3}\mathrm{Pe}^2\ell_0^2\left(\nabla\overline{T}\right)^2.} \quad (1.69)$$

1.3.6 Mean-Field Equation

Substituting the turbulent heat flux (1.62) into Eq. (1.28), and taking into account that for homogeneous turbulence the coefficient of turbulent diffusion is independent of coordinate, so that $\nabla\cdot\left(D_{_T}\,\nabla\overline{T}\right) = D_{_T}\,\Delta\overline{T}$, we obtain the mean-field equation for temperature field for homogeneous, isotropic and incompressible turbulence:

$$\frac{\partial\overline{T}}{\partial t} = \left(D^{(\theta)} + D_{_T}\right)\,\Delta\overline{T}. \quad (1.70)$$

Since the coefficient of turbulent diffusion $D_{_T}$ is positive, Eq. (1.70) implies that *the main effect of turbulence is enhancement of the diffusion of the mean temperature field, i.e., turbulence enhances the mixing*.

1.3.7 Solving the Diffusion Equation

Let us solve the diffusion equation (1.70) with the initial condition $T_0(\boldsymbol{r}) = \overline{T}(t=0,\boldsymbol{r})$. We use the Fourier[16] transform in the $\boldsymbol{k}$ space:

$$\overline{T}(t,\boldsymbol{r}) = \frac{1}{(2\pi)^3}\int \overline{T}(t,\boldsymbol{k})\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k}, \quad (1.71)$$

$$\overline{T}(t,\boldsymbol{k}) = \int \overline{T}(t,\boldsymbol{r})\exp(-\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{r}, \quad (1.72)$$

where $\boldsymbol{k}$ is the wave vector. We seek a solution for Eq. (1.70) in the form given by Eq. (1.71). Substituting solution (1.71) into Eq. (1.70), we obtain

$$\int\left(\frac{d\overline{T}(t,\boldsymbol{k})}{dt} + D_*\boldsymbol{k}^2\overline{T}(t,\boldsymbol{k})\right)\exp(\mathrm{i}\boldsymbol{k}\cdot\boldsymbol{r})\,d\boldsymbol{k} = 0, \quad (1.73)$$

[16] Jean Baptiste Joseph Fourier (1768–1830) was a French mathematician well-known for his works in mathematical physics, algebra, etc.

where $D_* = D^{(\theta)} + D_T$ is the total diffusion coefficient that is independent of the coordinate. Equation (1.73) yields the following ordinary differential equation where $\boldsymbol{k}$ is considered as a parameter, and the time t is a variable:

$$\frac{d\overline{T}(t,\boldsymbol{k})}{dt} = -D_* \boldsymbol{k}^2 \overline{T}(t,\boldsymbol{k}). \tag{1.74}$$

Equation (1.74) with the initial condition $\overline{T}(t=0,\boldsymbol{k}) = \overline{T}_0(\boldsymbol{k})$ has the following solution:

$$\overline{T}(t,\boldsymbol{k}) = \overline{T}_0(\boldsymbol{k}) \, \exp(-D_* \boldsymbol{k}^2 t). \tag{1.75}$$

The Fourier transform (1.72) for the initial temperature distribution $\overline{T}_0(\boldsymbol{k})$ reads:

$$\overline{T}_0(\boldsymbol{k}) = \int \overline{T}_0(\boldsymbol{r}') \exp(-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}')\, d\boldsymbol{r}'. \tag{1.76}$$

Substituting Eq. (1.76) into Eq. (1.75), we obtain

$$\overline{T}(t,\boldsymbol{k}) = \int \overline{T}_0(\boldsymbol{r}') \exp(-\mathrm{i}\,\boldsymbol{k}\cdot\boldsymbol{r}') \, \exp(-D_* \boldsymbol{k}^2 t)\, d\boldsymbol{r}'. \tag{1.77}$$

Now we substitute Eq. (1.77) for $\overline{T}(t,\boldsymbol{k})$ into Eq. (1.71), which yields

$$\overline{T}(t,\boldsymbol{r}) = \frac{1}{(2\pi)^3} \int d\boldsymbol{k} \int \overline{T}_0(\boldsymbol{r}') \exp\left[\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}') - D_* \boldsymbol{k}^2 t\right] d\boldsymbol{r}'. \tag{1.78}$$

We use the following identity:

$$\begin{aligned}
\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}') - D_* \boldsymbol{k}^2 t &= -D_* t \left[\boldsymbol{k}^2 - \frac{\mathrm{i}\,\boldsymbol{k}\cdot(\boldsymbol{r}-\boldsymbol{r}')}{D_* t} + \left(\frac{\mathrm{i}\,(\boldsymbol{r}-\boldsymbol{r}')}{2D_* t}\right)^2\right] - \frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4D_* t} \\
&= -D_* t \left(\boldsymbol{k} - \frac{\mathrm{i}\,(\boldsymbol{r}-\boldsymbol{r}')}{2D_* t}\right)^2 - \frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4D_* t}.
\end{aligned} \tag{1.79}$$

This identity allows us to rewrite solution (1.78) of the diffusion equation (1.70) in the final form:

$$\boxed{\overline{T}(t,\boldsymbol{r}) = \frac{1}{(4\pi\, D_* t)^{3/2}} \int \overline{T}_0(\boldsymbol{r}') \exp\left[-\frac{(\boldsymbol{r}-\boldsymbol{r}')^2}{4D_* t}\right] d\boldsymbol{r}',} \tag{1.80}$$

where we calculate the following integral,

$$\begin{aligned}
I &= \frac{1}{(2\pi)^3} \int \exp\left[-D_* t \left(\boldsymbol{k} - \frac{\mathrm{i}\,(\boldsymbol{r}-\boldsymbol{r}')}{2D_* t}\right)^2\right] d\boldsymbol{k} = \frac{1}{(2\pi)^3} \int \exp(-a^2 \tilde{\boldsymbol{k}}^2)\, d\tilde{\boldsymbol{k}} \\
&= \frac{1}{(2\pi)^3} \int_0^\infty \exp(-a^2 \tilde{k}^2)\, \tilde{k}^2 \, d\tilde{k} \int_0^\pi \sin\theta \, d\theta \int_0^{2\pi} d\varphi = \frac{1}{(4\pi\, D_* t)^{3/2}},
\end{aligned} \tag{1.81}$$