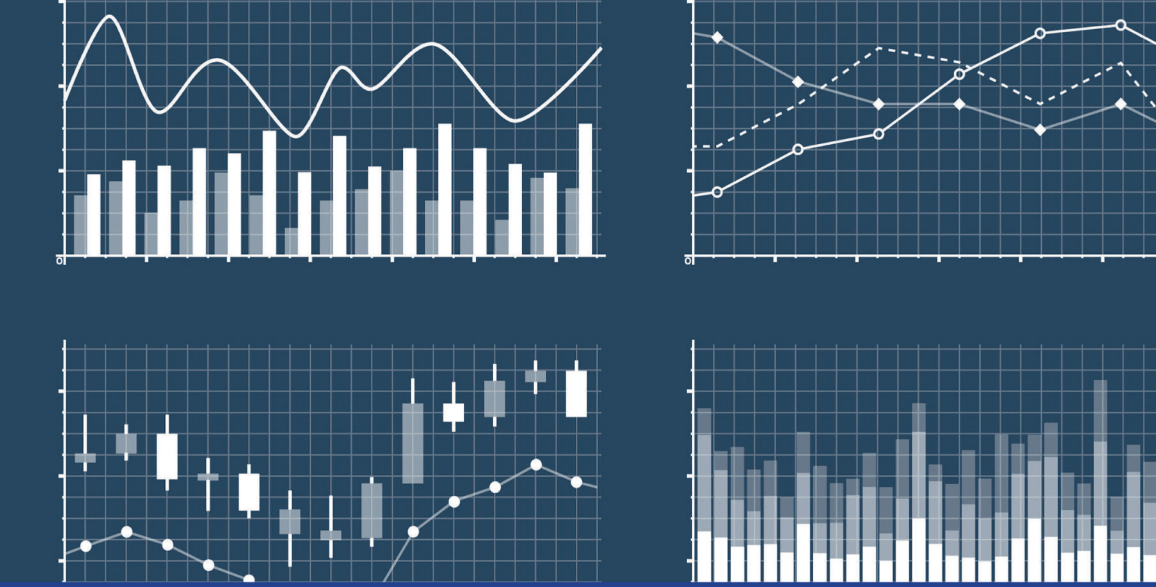

MULTI-SCALE SPECTRAL ANALYSIS IN HYDROLOGY

FROM THEORY TO PRACTICE

Adarsh S and M Janga Reddy

CRC Press
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Dedication

To Our Parents

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Preface

Let the time series speak for itself ...Time-series characterization is quite essential for capturing the behavior of the series and for the efficient forecasts. Proper understanding of the nonlinear and nonstationary features is an essential prerequisite for the effective simulation and prediction. The spectral analysis methods are suitable tools for time-series characterization and the classical methods like Fourier spectral analysis displays shortcomings in performance, if the time-series signal is complex nonlinear (NL) and nonstationary (NS) in characteristics. The Fourier methods of spectral analysis converts the signal from time to frequency domain and may suffer from compromise in quality, on dealing with the simultaneous transformation of complex NL-NS signals like hydrological signals. In this context, the advanced spectral analysis methods like Wavelet transforms (WT) and Hilbert Huang Transform (HHT) are found to be appropriate for the time-frequency (TF) characterization of complex signals. Both of these transforms facilitate multiscale feature extraction through effective decomposition of the candidate time series to signals of specific periodicity. The discrete version of wavelet transform found specific applications like trend analysis, hybrid predictive modeling while continuous variants are more popular in periodicity estimation and teleconnection studies in hydrology. HHT was proposed as a complementary tool to WT, which was reported to be successful in dealing with the inherent complexities of selection of appropriate mother wavelet and decomposition level, during the utilization of WT for T-F characterization of complex signals. The so called criticism on empirical mode decomposition (EMD) phas of HHT, such as lack of strong mathematical background can be considered as an advantage, as it allows the *time series to speak for itself* by resulting in decomposition of specific number of modes including the trend, even without fixing the decomposition levels *a priori*. Within two decades of its introduction, many theoretical advancements have reported, including robust noise-assisted variants of EMD, advanced Hilbert transform algorithms, HHT based running correlation analysis method like Time Dependent Intrinsic Correlation (TDIC) and Multivariate Empirical Mode Decomposition (MEMD) facilitating the simultaneous decomposition of multiple time-series signals. However, the enormous potential of these HHT-based methods in hydrology are not well debated in literature. The theoretical development of advanced spectral analysis methods like WT and HHT facilitate the use of such techniques for multiscale spectral characterization in hydrology and hence helps in overcoming the general critique on lack of real field practical applications of spectral analysis methods in hydrology. This book bridges this lacunae between theory and practical applications of spectral analysis methods in hydrology, by demonstrating number of case studies on trend analysis, hydroclimatic teleconnections, developing hydrologic frequency tool, fractal characterization, simulation of hydrological variables of different spatio-temporal scales, etc.

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Above all, we would like to express a deep sense of gratitude to GOD Almighty for making our dream a reality.

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1 Introduction

1.1 BACKGROUND

Modeling of different components of hydrological cycle such as precipitation, evaporation, streamflow, etc. are important in water resources planning and management. The real field data of most of the hydrological variables are represented as a time series, which are often nonlinear and nonstationary in characteristics. More specifically, the changes in statistical moments or covariance over the time domain refers to the nonstationarity of time series, while nonlinearity refers to data series possessing features such as asymmetric cycles, bimodality, nonlinear relationship between lagged variables, time irreversibility, and sensitivity to initial conditions (Fan and Yao 2003). Stationary assumption is the fundamental rationale in hydrological modeling, but such an assumption may lead to wrong estimates when applied to prediction problems in hydrology under the climate change scenario. This forces researchers to revisit the traditional practices of modeling of hydrological processes accounting for nonstationarity. Spectral analysis techniques can be used to get better insight into the characteristics of time series, by the determination of the frequency content of a time series, inference of the physical mechanisms responsible for this frequency content, evaluation of the performance of simulation models (Fleming et al. 2002). Such information may eventually lead to improvements in the prediction of hydrological variables in a nonstationary environment.

1.2 SCOPE OF MULTISCALE SPECTRAL ANALYSIS IN HYDROLOGY

The Fourier spectral analysis (FSA) has been traditionally used for the spectral characterization of a hydrological time series. But the FSA is suitable only for analyzing stationary time series, as there may be significant loss of information during the transformation to the frequency domain. However, most of the signals in practical hydrology contain numerous nonstationary characteristics such as drift, trends, abrupt changes, etc., and the FSA is not suitable to detect such features. One solution to this issue is to decompose the time series into time-frequency space. The Short Time Fourier Transform (STFT) introduced by Dennis Gabor (Gabor 1946) is helpful to map a signal into time-frequency space. In this method, the Fourier transform is used to analyze only a small section of the signal at a time (popularly known as windowing of the signal); but it was noted that the precision with which the information obtained is highly dependent on the size of the window chosen (Misiti et al. 2008). Overcoming the shortcomings of the Wavelet Transforms (WT) was evolved in 1980s, which include discrete and continuous variants. Both classes of wavelets finds numerous applications in the domain of time series analysis and their applications drawn considerable attention by the researchers in hydrology for the past two decades (Sang 2013; Nourani et al. 2014). Wavelet transforms can be used to extract short-term and long-term fluctuations by decomposing the time series into different subcomponents. From past literature, it is noticed that the multiscale decomposition of time series using wavelets are useful in understanding underlying character of the hydrological processes, which may thus help to forecast the hydrological variables accurately (Nourani et al. 2014).However, the choice between a large- or small-scale wavelet function and the selection of appropriate wavelet functions and decomposition levels are reported to be two major challenges in the use of wavelet transforms for the spectral analysis of a time series (Sang et al. 2016). In addition, the results obtained by wavelet analysis will be accurate only if the nonstationary series is linear in nature. By addressing these shortcomings, Huang et al. (1998) proposed a novel data adaptive decomposition method namely Empirical Mode Decomposition (EMD) and combined it with the traditional Hilbert Transform (HT) to develop a new spectral analysis method, namely, Hilbert-Huang Transform (HHT). By using HHT, the complexity in selection of appropriate mathematical functional form and decomposition level can be addressed. Over the two decades after its introduction, HHT is established as a potential tool for spectral analysis of a complex time series including a hydrologic series. The estimation of dominant periodicities of hydrological time series and climatic oscillations, extraction of nonlinear trend of hydrological series, hydroclimatic teleconnections in multiple time scales, use of single and multivariate EMD for decomposition of hydroclimatic signals into different time scales and its application in simulation or prediction of hydrological variables by hybrid methods involving data-driven methods, etc. are some possible domains where HHT has been applied. Both the wavelets and HHT are capable of providing the information in multiple time scales, which is an added advantage in prediction and feature extraction problems in hydrology. The information on multiscaling behaviors of different complex hydrological processes could help in improved hydrological predictions. The prediction of hydrological variables could be improved by finding the possible association of hydrologic processes with climate indices having specific periodicity. Such challenging issues can be handled in a better way by following an efficient multiscale decomposition process coupled with an appropriate spectral analysis technique. But most of the past studies investigated such associations based on computation of periodicities alone. However, in order to investigate the association between two time series having multiscale characteristics in a better way, a technique which enables a running correlation analysis in multiple time scales is more appropriate. One such technique that works based on the HHT is the Time Dependent Intrinsic Correlation (TDIC) method, which can be explored for hydroclimatic teleconnection studies. It is also well known that multiple variables may influence the hydrological processes, but multiscale decomposition methods such as EMD or its variants are univariate in nature and decomposition of multiple variables of concern using such methods may not give the same number of modes. As a result, at a specific time scale, the frequency content pertaining to the modes of different variables may be different, which may lead to erroneous interpretations in teleconnection studies. To rectify such problems the teleconnection analysis can be performed effectively using Multivariate Empirical Mode Decomposition (MEMD) method, here the common scales present in multiple variables of concern can be identified in a single step operation.

The hydroclimatic variables often possess multiscaling character. Therefore, to model the hydrological processes, a decomposition-based technique may be a better alternative. Eventhough many hybrid decomposition models were proposed for simulation and prediction of hydrologic variables, only few of them considered multiple variables in the modeling process. Also, many such studies considered appropriate lags at different time scales in the modeling exercise. Moreover, many of the decomposition-based hybrid models have used the decomposed components directly as inputs, by which the significant information from specific process scales are not accounted in modeling. In this context, the potential of MEMD can be utilized in modeling, as it facilitates to account both multiple inputs and associated features in multiple time scales. In addition, MEMD-based decomposition can be used as a useful mean to determine the representative scaling exponent of rainfall intensity series of different durations, which in turn may help in develop the hourly rainfall intensity duration frequency (IDF) relationships from longer duration rainfalls (such as monthly/daily) by using the scaling theory. In short, the usefulness of advanced spectral analysis methods such as WT or HHT and its algorithmic variations needs to be investigated in the context of characterization, teleconnection, and prediction of hydrological variables. This clearly bridges the gap between theoretical principles and practice. In this perspective, this book gives a comprehensive presentation of such practical frameworks along with the demonstration through a number of case study applications in the Indian context.

1.3 ORGANIZATION OF THE BOOK CONTENT

Chapter 2 first provides the brief theoretical description of conventional spectral analysis methods followed by detailed description of advanced spectral analysis methods such as wavelet transform and HHT. The descriptions of single and multivariate EMD, recent algorithmic developments like Arbitrary Order Hilbert Spectral Analysis (AOHSA) and Time Dependent Intrinsic Correlation (TDIC) are also presented in the chapter. Three novel frameworks for hydrological applications, the MEMD-TDIC approach for multiscale teleconnection, the MEMD-scaling theory approach for developing rainfall intensity-duration-frequency (IDF) curves, and the MEMD-based hybrid modeling for simulation and prediction of hydrological variables, are presented in the chapter. Chapter 3 is devoted for wavelet transform applications for hydrologic characterization. The extraction of trend using Discrete Wavelet Transform (DWT) and application of Continuous Wavelet Transform (CWT) for teleconnection are two major applications considered in this chapter. Chapter 4 considers HHT applications on rainfall time series. This chapter covers the time-frequency characterization, teleconnections, trend analysis and development of IDF curves of rainfall. Analysis of multiscale teleconnections of streamflow with sediment load and climate variables, fractal characterization using HHT, etc. are described in Chapter 5. Chapter 6 is exclusively devoted to the simulation of different hydrological time series such as rainfall, streamflow, and suspended sediment using MEMD-based hybrid models.

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2 The Theory of Advanced Spectral Analysis Methods

2.1 BACKGROUND

The spectral analysis tools can be used to characterize the time series signals and understand the processes involved. The classical Fourier spectral analysis is perhaps the most popular among these tools. But its efficacy in performance is limited to linear and stationary time series while the practical hydrologic time series rarely possess the properties of linearity or stationarity. The introduction of Short Time Fourier Transform (STFT) put forwarded the concept of time and frequency localization, but the constant and *a priori* fixation of window size was a problem for the modeler to work with the technique. Also it was rather found to be a difficult task to maintain the quality of localization in one of the domain without compromising the quality of localization in the other domain during the implementation of STFT. Overcoming such limitations Wavelet transforms (WT) evolved as a potential alternative and the Continuous Wavelet Transform (CWT) and Discrete Wavelet Transform (DWT) have received attention in various applications in processing hydroclimatic time series data. But the difficulties in choosing the appropriate wavelet type and level along with its inferior capabilities in handling non-stationarity lead researchers for a data adaptive decomposition method. Hilbert Huang Transform (HHT) is one such multiscale spectral analysis method suitable for time-frequency characterization of nonlinear and nonstationary time series. HHT first finds appropriate inputs for Hilbert transform by performing a multistage decomposition process called Empirical Mode Decomposition (EMD) to evolve orthogonal subseries called Intrinsic Mode Functions (IMFs) from a given time series data. Then the IMFs are subjected to Hilbert transform to give instantaneous amplitudes and instantaneous frequencies. This chapter presents a brief information on Fourier transform and wavelet transform followed by the detailed theoretical description of EMD, its noise-assisted variants, its multivariate extension, Hilbert Spectral Analysis (HSA), the procedure of HHT based Time Dependent Intrinsic Correlation (TDIC) analysis, etc. The chapter also gives the details of proposed methods such as TDIC based approach for hydroclimatic teleconnection studies, MEMD-Stepwise Linear Regression (SLR) hybrid model for time series prediction and MEMD-based procedure for developing rainfall Intensity-Duration-Frequency relationships.

2.2 CONVENTIONAL SPECTRAL ANALYSIS METHODS

2.2.1 Fourier Transform

Fourier transform can be viewed as a transformation in function space from the time domain to the frequency domain which contains trigonometric functions like *sines* and *cosine*s as basis functions that are localized in frequency only. The time-series signal can then be analyzed for its frequency content as the Fourier coefficients of the transformed function which represents the contribution of each *sine* and *cosine* function at different frequency. The power density spectrum of the signal shows how the power of the periodic signal is distributed among various frequency components.

In a Fourier transform, the mapping from the time domain to the frequency domain, is done by means of the complex periodic plain wave functions of the form $y(f,t) = e^{\omega t}$ where $\omega = 2\pi f$.

The basic definition of a Fourier transform of a continuous time signal $x(t)$ is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-i\omega t} dt \tag{2.1}$$

As most of the continuous time-series data contains discrete-time data, the infinite integral can be replaced by finite summation as follows:

$$X(\omega_k) = \sum_{n=0}^{N-1} x(t_n) e^{-i\omega_k t_n}, n = 0,\ 1,\ 2, \ldots,\ N-1 \tag{2.2}$$

where N is the number of time samples and ω_k is the k^{th} frequency. It is a Fourier representation of a finite length of sequence, which corresponds to samples equally spaced in frequency of the Fourier transform of the signal. For real valued signal $x(t)$, the Fourier transform is complex and in polar notation it can be represented as

$$X(\omega) = A(\omega) e^{i\theta(\omega)} \tag{2.3}$$

where $A(\omega)$ is the spectral amplitude and $\theta(\omega)$ is the phase angle. These properties can be represented as

$$A(\omega) = \sqrt{R(\omega)^2 + I(\omega)^2} \tag{2.4}$$

and

$$\theta(\omega) = \tan^{-1}\left(\frac{I(\omega)}{R(\omega)}\right) \tag{2.5}$$