PSYCHOLOGY REVIVALS

Addition and Subtraction

A Cognitive Perspective

Edited by

Thomas P. Carpenter, James M. Moser and Thomas A. Romberg

REVIVALS

Psychology Revivals

Addition and Subtraction

A hallmark of much of the research on children's thinking in the 1970s had been the focus on explicit content domains. Much of this research had been represented by an eclectic collection of studies sampled from a variety of disciplines and content areas. However, in the few years before this publication, research in several content domains has begun to coalesce into a coherent body of knowledge. Originally published in 1982, the chapters in this work represent one of the first attempts to bring together the perspectives of a variety of different researchers investigating a specific, well defined content domain.

This book presents theoretical views and research findings of a group of international scholars who are investigating the early acquisition of addition and subtraction skills by young children. Together, the contributors bring a blend of psychology, educational psychology, and mathematics education to this topic. Fields of interest such as information processing, artificial intelligence, early childhood, and classroom teaching and learning are included in this blend.

Taylor & Francis
Taylor & Francis Group
http://taylorandfrancis.com

Addition and Subtraction

A Cognitive Perspective

Edited by

Thomas P. Carpenter, James M. Moser and Thomas A. Romberg

Routledge
Taylor & Francis Group
LONDON AND NEW YORK

First published in 1982
by Lawrence Erlbaum Associates, Inc.

This edition first published in 2020 by Routledge
2 Park Square, Milton Park, Abingdon, Oxon OX14 4RN

and by Routledge
52 Vanderbilt Avenue, New York, NY 10017

Routledge is an imprint of the Taylor & Francis Group, an informa business

Publisher's Note
The publisher has gone to great lengths to ensure the quality of this reprint but points out that some imperfections in the original copies may be apparent.

Disclaimer
The publisher has made every effort to trace copyright holders and welcomes correspondence from those they have been unable to contact.

A Library of Congress record exists under ISBN: 0898591716

ISBN: 978-0-367-49544-2 (hbk)
ISBN: 978-1-003-04658-5 (ebk)

ADDITION and SUBTRACTION: A Cognitive Perspective

Edited by
THOMAS P. CARPENTER
JAMES M. MOSER
THOMAS A. ROMBERG
Wisconsin Research and Development Center for Individualized Schooling

LAWRENCE ERLBAUM ASSOCIATES, PUBLISHERS
1982 Hillsdale, New Jersey

Lawrence Erlbaum Associates, Inc., Publishers
365 Broadway
Hillsdale, New Jersey 07642

Library of Congress Cataloging in Publication Data
Main entry under title:

Addition and subtraction.

Bibliography: p.
Includes index.
1. Addition. 2. Subtraction. 3. Mathematical ability
—Testing. I. Carpenter, Thomas P. II. Moser, James M.
III. Romberg, Thomas A.
QA115.A34 370.15′6 81-5392
ISBN 0-89859-171-6 AACR2

Contents

Preface

A hallmark of much of the recent research on children's thinking has been the focus on explicit content domains. Much of this research has been represented by an eclectic collection of studies sampled from a variety of disciplines and content areas. However, in the last few years, research in several content domains has begun to coalesce into a coherent body of knowledge. The chapters in this work represent one of the first attempts to bring together the perspectives of a variety of different researchers investigating a specific, well defined content domain.

This book presents theoretical views and research findings of a group of international scholars who are investigating the early acquisition of addition and subtraction skills by young children. Together, the contributors bring a blend of psychology, educational psychology, and mathematics education to this topic. Fields of interest such as information processing, artificial intelligence, early childhood, and classroom teaching and learning are included in this blend.

Following a brief introductory chapter, the book is separated into five parts. The first part, "The Structure of Addition and Subtraction Problems," presents four different, but complementary, approaches to understanding how children think and operate on verbal addition and subtraction problems by describing structural features of verbal problems and classifying different problem types by those features. Using empirical data, the authors demonstrate interesting differences in how children solve each of the various problems. These essays raise significant questions about the relationship between the semantic structure of various problem types, the strategies children actually use in solving such problems, the use of symbolic representations, and the utility of symbolic instruction.

The second part is entitled "The Role of Counting in Solution Processes." These essays describe current research on counting and how young children solve many addition and subtraction problems using counting procedures.

In the third part, "The Analysis of Error Patterns," two related papers examine the errors children make on the common "fair-trading" subtraction algorithm from an information processing perspective. These papers provide an explanation of how children's invention, apparent in their counting solutions to simple problems, can lead to learning incorrect procedures to solve problems requiring algorithmic solutions.

The fourth part is entitled "Alternate Theoretic Perspectives." Three essays argue for theoretical refocusing of research based on an information processing orientation for studies of children's arithmetical development, an examination of the structure of children's learned outcomes, and a consideration of levels of cognitive decision making.

The essays in the concluding part, "The Development of Addition and Subtraction Skills in Other Cultures," focus on the cultural and ideological context within in which addition and subtraction skills develop. Data from Africa and America suggest there are common strategies and patterns across cultures and classes. However, there are linguistic and cultural factors in the Japanese culture that contribute to a seemingly different developmental pattern.

The chapters are an outgrowth of a seminar held in November 1979 at the Wingspread Conference Center in Racine, Wisconsin, a facility which encouraged sustained discussion of the ideas presented. The first drafts of essays were shared prior to the seminar.

Three of the essays prepared for the book were substantially longer and more developed papers which were presented at the conference and have been subsequently published as separate projects papers by the Wisconsin Research and Development Center.

Collis, K. *Cognitive development, mathematics learning, information processing and a refocusing.* (Project Paper No. 81-1) Madison: Wisconsin Research and Development Center for Individualized Schooling, 1981.

Skemp, R. R. *Theories and methodologies.* (Project Paper No. 81-3) Madison: Wisconsin Research and Development Center for Individualized Schooling, 1981.

Weaver, J. F. *Addition, subtraction, and mathematical operations.* (Project Paper No. 79-7) Madison: Wisconsin Research and Development Center for Individualized Schooling, 1979.

A number of other papers were significantly rewritten for publication, and ideas presented in the original papers are not always included in the published versions. Karen Fuson's paper underwent substantial revision. Copies of her original paper and an expanded version of the chapter prepared for this book are available from the author. In addition, V. V. Davydov, after reading the papers on counting skills, brought with him an English translation of a recently published paper of his on that subject and shared it with the participants. Because of its inaccessibility in English, it too has been published as a project paper.

Davydov, V. V., & Andronov, V. P. *The psychological conditions for the origination of ideal* *actions*. (Project Paper No. 81-2) Madison: Wisconsin Research and Development Center for Individualized Schooling, 1981.

Copies of these four project papers are available from the ERIC Center.

The seminar was co-sponsored by the Wisconsin Research and Development Center for Individualized Schooling, National Institute of Education, and The Johnson Foundation of Racine. In addition to the editors and authors, a number of scholars participated in the seminar at Wingspread and helped to clarify ideas and issues in the problems discussed in the book. They are:

Constance Martin Anick
University of Wisconsin-Madison

Arthur J. Baroody
Keuka College

Glendon Blume
University of Iowa

Anne Buchanan
University of Wisconsin-Madison

Connie Seaman Cookson
University of Wisconsin-Madison

Gabriela Delgado
Universidad Nacional Autonoma de Mexico

Ed Esty
National Institute of Education

Jane Donnelly Gawronski
San Diego County Public Schools

James G. Greeno
University of Pittsburgh

James Hiebert
University of Kentucky

Joan I. Heller
University of Pittsburgh

David Klahr
Carnegie Mellon University

Vicky L. Kouba
University of Wisconsin-Madison

Richard Lesh
Northwestern University

Mary Montgomery Lindquist
National College of Education

Douglas McLeod
National Science Foundation

Gary G. Price
University of Wisconsin-Madison

Richard J. Shavelson
University of California at Los Angeles

Merl C. Wittrock
University of California at Los Angeles

We are indebted to them for the discussion, constructive comments, and conversation.

We particularly want to thank Mr. Henry Halstead and his staff at Wingspread for their gracious hospitality. Not only are the facilities ideal for a seminar, but the care and cordiality of the staff helped create the collegial atmosphere which prevailed.

We would also like to thank Constance Martin Anick for her skillful handling of many of the administrative details of the Conference.

Finally, in preparing the book, we would like to thank each author for their patience and willingness to react to our suggestions. We wish also to thank Mary Pulliam for her care in editing the manuscripts and checking references, and Louise Smalley and Dorothy Egener for typing and retyping each of the chapters.

Thomas P. Carpenter
James M. Moser
Thomas A. Romberg

1 An Emerging Paradigm for Research on Addition and Subtraction Skills

Thomas A. Romberg
University of Wisconsin-Madison

For several centuries, being able to find ''one's sums and differences'' has been considered one mark of a schooled person. Although today we may have expanded our expectations about what constitutes literacy, we still expect all children to efficiently carry out operations on whole numbers. Yet, in spite of these expectations about the skills of addition and subtraction, there has been little consensus about how such skills develop. Lack of consensus does not mean there has been little research. Recent reviews (Carpenter, Blume, Hiebert, Anick, & Pimm, 1981; Carpenter & Moser, in press; Riley, Greeno, & Heller, in press) have identified an extensive body of research on addition and subtraction. Some of these studies have been quite influential. For example, Thorndike's instructional suggestions in his *Psychology of Arithmetic* (1922) became the model of how to teach arithmetic for decades,[1] and Brownell's (1947) research on subtraction demonstrated the superiority of the ''decompositions'' subtraction algorithm over the ''equal additions'' algorithm when taught with rational explanation. This made the ''fair trading'' procedure central to contemporary instruction in subtraction. But, to a large extent, the many studies on addition and subtraction represent an eclectic morass. This copious literature has lacked an implicit body of intertwined theoretical and methodological beliefs that permit selection, evaluation, and criticism. However, today we believe a change is imminent. The research and theoretical positions set forth in this volume should be viewed as foreshadowing the emergence of a firm research consensus in this area.

[1]Chapter 3 in Cronbach and Suppes (1969) presents a convincing argument about Thorndike's influence on mathematics instruction.

To build this argument, I follow Thomas Kuhn's description of the "route of normal science." In his now classic treatise on the growth of science, *The Structure of Scientific Revolutions* (1979), Kuhn argues that a significant turning point in the history of science occurs when from the chaos of competing ideas about a problem area, a single paradigm emerges which implicitly defines for practitioners the legitimate problems and methods of research. A paradigm gains that status because it is more successful than others in solving a few problems a group of researchers have recognized as acute. In this sense Kuhn argues paradigms have two essential characteristics. First, the paradigm's achievement in solving the acute problems is sufficiently unprecedented to attract a group of adherents. Simultaneously, the paradigm is open-ended, leaving all sorts of problems for the redefined group of practitioners to resolve. Kuhn calls the research carried out by this new group "normal science." It consists of actualizing the promise of the paradigm, "extending the knowledge of those facts that the paradigm displays as particularly revealing, by increasing the extent of the match between those facts and the paradigm's predictions and by further articulation of the paradigm itself" [p. 24].

It would be both presumptuous and incorrect to argue that a paradigm for research on the development of addition and subtraction skills has emerged and that the papers in this volume reflect work within a normal science. Rather, the papers reflect growing agreement around a constellation of ideas with the potential to become such a paradigm. Current work mirrors what Kuhn discusses as the "route of normal science."

Historically, the road to a research consensus in any area is arduous. In the absence of a paradigm all facts that could possibly pertain to the development of a given science are likely to seem equally relevant. As a result, early fact-gathering is a nearly random activity. Furthermore, in the absence of a reason for seeking some particular form of information, early fact-gathering is usually restricted to the wealth of data that lie ready at hand. Thus facts accessible to casual observation and experiment are pooled together with data retrievable from reports of classroom teaching, curriculum development, or evaluation.

Although this sort of fact-collecting has been essential to the origin of many significant sciences, one somehow hesitates to call the resulting literature scientific. Similarly, it would be hard to describe early studies on addition and subtraction as scientific (Carpenter, et al., 1981). This is true because such studies juxtapose facts that will later prove revealing with others that will for some time remain too complex to be integrated with theory at all. In addition, since any descriptions must be partial, such a typical natural history often omits from its immensely circumstantial accounts just those details that will be sources of important illumination to later scientists. Because the casual fact-gatherer seldom possesses the time or the tools to be critical, natural histories often relate reasonable descriptions with others that we are now quite unable to confirm. This is the situation that creates the intellectual morass characterizing the early stages of a science's development, and as Kuhn (1979) states:

> No wonder, then, that in the early stages of the development of any science different men confronting the same range of phenomena, but not usually all the same particular phenomena, describe and interpret them in different ways [p. 16].

With regard to the development of addition and subtraction skills, a set of scholars is confronting the same range of phenomena from essentially similar perspectives, and is beginning to reach a consensus on the acute problems to be solved, and beginning to use the same language and research methods to attack these problems. The emerging general paradigm is to formulate precise models of the cognitive processes used by subjects when carrying out specific tasks and how those processes change over time.

Brown (1970) argues the origins of this paradigm stem from two primary sources-computer simulation of cognitive processes and the writings of Jean Piaget. The basic strategy for this simulation of human processing was sketched by Herbert Simon (1962).

> If we can construct an information processing system with rules of behavior that lead it to behave like the dynamic system we are trying to describe, then this system is a theory of the child at one stage of the development. Having described a particular stage by a program, we would then face the task of discovering what additional information processing mechanisms are needed to simulate developmental change-the transition from one stage to the next. That is, we would need to discover how the system could modify its own structure. Thus, the theory would have two parts-a program to describe performance at a particular stage and a learning program governing the transitions from stage to stage [pp. 154-155].

In order to specify rules of behavior and modifications of behavior it is necessary to characterize the child as an organism functioning under the control of a developing set of central processes. Some of Piaget's notions of child development, such as schema, assimilation, and accommodation, have gradually become the basis for creating dynamic models of children's cognitive processes in solving specific problems. The rapprochement between these two quite different conceptualizations has not been easy, as Klahr and Wallace (1976) have argued.[2] Yet, today agreement on some aspects is emerging. The developing paradigm has four elements upon which there is some consensus:

1. detailed descriptions of the contexts within which specific tasks are embedded;
2. analyses of all the behaviors associated with the subjects' responses to performing the task;
3. repeated assessment of performance behaviors over time; and

---

[2]Klahr and Wallace (1976) in the preface to their examination of cognitive development from an information processing view.

4. inferences about the cognitive processing mechanism which relates information about the task with performance, and about changes in this performance.

For several reasons, children's processing of addition and subtraction information–the topic of this book–is one area where this emerging paradigm has proved revealing. Addition and subtraction are the first set of mathematical ideas typically taught in schools. Children bring to such problems well developed counting procedures, some knowledge of numbers, and some understanding of physical operations, such as "joining" and "separating," on sets of objects. Thus, from this context researchers have a unique opportunity to examine variations in how children process information prior to, during, and after formal instruction. Identifying stages of development in strategies children use to solve such problems is the basic problem addressed in this book.

To solve a typical problem one first must understand its implied semantic meaning. Quantifying the elements of the problem comes next (e.g., choosing a unit and counting how many). Then, the implied semantics of the problem must be expressed in the syntax of addition and subtraction. Next the child must be able to carry out the procedural (algorithmic) steps of adding and subtracting. Finally, the results of these operations must be expressed.

As a group, the papers in this volume employ a variety of descriptions for the various cognitive processes or subprocesses children use on such problems. As yet, there is no agreement on terms for describing the problem contexts, the types of processes, or the processing mechanisms children use. Nevertheless, there is agreement that our aim is to formulate precise models that describe children's addition and subtraction skills and how those skills change over time.

The importance of specifying task context is reflected in the chapters by Thomas Carpenter and James Moser, Pearla Nesher, and Gérard Vergnaud in this volume. Because addition and subtraction sentences can be used to represent a wide variety of problems with different semantic structures, it is important for these authors to classify different types of verbal problems, and to study whether children can solve such problems prior to formal instruction. If children can, investigation of whether they use different strategies with problems having different semantic structures, and investigation of the changes in choice of strategies, is appropriate. Thus, the notions of verbal comprehension and the strategies used to quantify, represent, and calculate are acute problems of interest.

In this volume J. Fred Weaver and Vasily Davydov present arguments about the conceptualization of problem context from a mathematical perspective. Weaver stresses an alternative "unary operation" notion about addition and subtraction whereas Davydov embeds addition and subtraction in a broader mathematical perspective which stresses quantification processes before operational processes.

With few exceptions the authors in this book go well beyond tallying the number of correct and incorrect responses when describing children's behaviors in response to addition and subtraction problems. Identification of the actions and strategies children use on specific tasks is central to the papers by Carpenter and Moser; Vergnaud; Leslie Steffe, Patrick Thompson, and John Richards; Karen Fuson; and Prentice Starkey and Rochelle Gelman. Examining errors for prevalent patterns is a major emphasis in the investigation of both John Seely Brown and Kurt Van Lehn, and Lauren Resnick.

Inferences about cognitive processes used to produce responses and changes in responses over time are based on simulation models in both Brown and Van Lehn's model and Resnick's research; on notions of developmental stages in the work of Carpenter and Moser; Starkey and Gelman; Steffe et al.; and on instruction in Nesher's research. Cultural background and its influence on performance is stressed in both Giyoo Hatano's research and in Herbert Ginsburg's studies. It should be noted that the latter two authors are on opposite sides of the issue. Hatano argues that cultural background is important and Ginsburg cites evidence that it is not. Finally, in three broader theoretical papers, Robbie Case, Kevin Collis, and Richard Skemp stress different considerations for future models of cognitive processing. Case emphasizes children's developing memory capacity and how information is organized for storage, Collis agrees with Case but stresses learned outcomes, and Skemp argues for a theoretical formulation positing a "director system" at two levels.

All the papers build models to explain children's behaviors. For example, because children bring to typical verbal problems well developed counting skills and use those skills to quantify and often solve such problems, the study of the development of counting skills themselves is of particular interest. Steffe et al. and Fuson examine this topic.

Carpenter and Moser, Vergnaud, Nesher, and Starkey and Gelman examine the way children represent or use representations of various problems. The use of physical manipulatives, pictorial illustrations, and symbolic statements is modeled by this group of researchers.

As previously argued, one feature of an emerging consensus on a paradigm is agreement on methods of inquiry into the questions of critical importance. In the research presented in this volume, clinical interviewing of students is the predominant methodology. Carefully designed tasks and probing questions presented to a small sample of children are accepted as appropriate procedures. The papers by Carpenter and Moser, Resnick, Steffe et al, and Fuson, in particular, reflect this strategy. Davydov, Steffe et al., and Resnick use the "teaching experiment" extension of this procedure.

In most chapters, the data are generally presented descriptively with little use of statistics to bolster the arguments. In fact, because the concern is on formulation of models, attention is drawn to questions which may not be answerable with usual methods of statistical inference.

Underlying the book is a belief that by using this paradigm, we will eventually derive information that can be used to improve instruction. In particular, Case and Davydov draw inferences for teachers based on current knowledge.

The strength of this emerging scholarly consensus on how addition and subtraction skills develop lies in the fact that new research can begin where the last left off. Such research can concentrate on subtle or even esoteric aspects of the phenomena, assured that findings will add new information to a conceptual whole.

The weakness of consensus on any paradigm rests primarily in the fact that adherence to a single perspective makes questions appear insignificant which are deemed critical from other perspectives. For example, the question, "Who decides what subtraction algorithm should be taught?" is critical to the curriculum theorist interested in the structure of the content to be covered. For a behavioral psychologist, answering the question "What extrinsic motivational procedures are effective in getting children to add or subtract with low error rates?" is critical. For the instructional designer, it may be critical to answer the question "Which of two (or more) sequences of instruction is more efficient and effective?"

Since no perspective is all-encompassing, choosing a paradigm limits the variety of "acute" questions. In this book, the choice of a cognitive perspective limits the important questions to how children construct meanings for addition and subtraction situations and how those meanings change over time.

During the past decade, the authors of these chapters have carried out a great deal of significant work which is now coming together. This volume clearly reflects the emergence of a "normal science" approach to studying the development of addition and subtraction skills. What is important is to appreciate the growing consensus on the phenomena of interest, the acute problems to be studied, and the appropriate research methodology being used.

REFERENCES

Brown, R. Introduction. In Society for Research in Child Development (Ed.), *Cognitive development in children*. Chicago: University of Chicago Press, 1970.

Brownell, W. An experiment on "borrowing" in third-grade arithmetic. *Journal of Educational Research*, 1947, *41* (3), 161–171.

Carpenter, T., Blume, G., Hiebert, J., Anick, C., & Pimm, D. *The development of addition and subtraction concepts: A review*. (Theoretical Paper) Madison: Wisconsin Research and Development Center for Individualized Schooling, 1981.

Carpenter, T., & Moser, J. The acquisition of addition and subtraction concepts. In R. Lesh & M. Landau (Eds.), *Acquisition of mathematics on concepts and processes*. New York: Academic Press, in press.

Cronbach, L., & Suppes, P. (Eds.). *Research for tomorrow's schools: Disciplined inquiry for education*. New York: The Macmillan Company, 1979.

Klahr, D., & Wallace, J. *Cognitive development: An information-processing view*. Hillsdale, NJ: Lawrence Erlbaum Associates, Inc., Publishers, 1976.

Kuhn, T. *The structure of scientific revolutions* (2nd edition). Chicago: The University of Chicago Press, 1979.

Riley, M., Greeno, J., & Heller, J. The development of children's problem solving ability in arithmetic. In H. Ginsburg (Ed.), *The development of mathematical thinking*. New York: Academic Press, in press.

Simon, H. An information processing theory of intellectual development. *Monographs of the Society for Research in Child Development,* 1962, *27*(2, Serial No. 82).

Thorndike, E. *The psychology of arithmetic*. New York: The Macmillan Company, 1922.

2 The Development of Addition and Subtraction Problem-Solving Skills

Thomas P. Carpenter
James M. Moser
University of Wisconsin-Madison

A tacit assumption of most school mathematics programs is that addition and subtraction are best introduced through physical or pictorial representations of joining or separating sets of objects. Another common assumption is that verbal problems are difficult for children of all ages, and children must master addition and subtraction operations before they can solve even simple verbal problems. A growing body of research indicates that both assumptions may be false. The results presented in this chapter indicate that before children receive formal instruction in addition and subtraction, many of them can successfully solve basic addition and subtraction word problems. This suggests that verbal problems may give meaning to addition and subtraction and in this way could represent a viable alternative for developing addition and subtraction concepts in school.

Our interest is in the word problems commonly found in elementary mathematics textbooks, which can be solved by a single operation of addition or subtraction. This is not to suggest that children necessarily solve these problems by adding or subtracting. In fact, young children generally do not solve them by applying an arithmetic operation.

A substantial body of research indicates that young children solve addition and subtraction computation exercises by using several basic counting strategies (Groen & Parkman, 1972; Groen & Resnick, 1977; Suppes & Groen, 1967; Woods, Resnick, & Groen, 1975). The same basic strategies are used to solve simple word problems (Carpenter, Hiebert, & Moser, 1981; Carpenter & Moser, 1979; Gibb, 1956; Steffe, Spikes, & Hirstein, 1976). However, because a variety of semantically different word problems can be solved by addition or subtraction, the choice of strategy becomes somewhat more complex.

In this chapter, we describe the strategies children use to solve addition and subtraction word problems before they receive formal instruction in addition and subtraction, and how these strategies evolve during the first year of instruction. Children's strategies are strongly influenced by the semantic structure of the problem situation. Therefore, first it is necessary to characterize major differences between different addition and subtraction problems.

AN ANALYSIS OF VERBAL PROBLEMS

Previous research has taken several approaches to characterize verbal problems. One is to classify problems in terms of syntax, vocabulary level, number of words in a problem, etc. (Jerman, 1973; Suppes, Loftus, & Jerman, 1969). A second approach differentiates among problems in terms of the open sentences they represent (Grouws, 1972; Lindvall & Ibarra, 1980; Rosenthal & Resnick, 1974). We have chosen a third alternative that considers the semantic characteristics of the problem. Our analysis is generally consistent with other analyses based on problem structure (Gibb, 1956; Greeno, 1978; Nesher & Katriel, 1978; Vergnaud, this volume), although we distinguish among problem types somewhat differently.

We have identified several basic dimensions that characterize the actions or relationships involved in addition and subtraction word problems. The first dimension is based on whether an active or static relationship between sets or objects is implied in the problem. Some problems contain an explicit reference to a completed or contemplated action causing a change in the size of a quantity given in the problem. In other problems no action is implied; that is, there is a static relationship between quantities given in the problem.

The second dimension involves a set inclusion or set–subset relationship. In certain problems, two of the entities involved are necessarily a subset of the third. In other words, either the unknown quantity is made up of the two given quantities, or one of the given quantities is made up of the other given quantity and the unknown. In other situations one of the quantities involved in the problem is disjoint from the other two. In this case a comparison of the two disjoint quantities is implied.

For problems that involve action, there is a third dimension. The action described in a problem may result in an increase or decrease in the initial given quantity. Because the static problems do not involve changing the given quantities, this dimension does not apply to them. Altogether there are a total of six different classes of problems based on these distinctions. We have labeled these six classes Joining, Separating, Part–Part–Whole, Comparison, Equalizing–Add On, and Equalizing–Take Away.

Joining, Separating, and Equalizing problems all involve action, whereas Part–Part–Whole and Comparison problems describe static relationships between quantities. Equalizing problems are distinguished from Joining and Separating problems on the basis of set–subset relationships. A similar distinction differentiates Comparison and Part-Part-Whole problems. In other words, for Joining, Separating, and Part-Part-Whole problems two of the quantities are a subset of the third. Equalizing and Comparison problems involve comparing disjoint sets. The distinction between Joining and Separating problems and between the two Equalizing problems is based on whether the described action is an increase or a decrease. Joining and Equalizing-Add On involve an increase; Separating and Equalizing-Take Away involve a decrease.

Basically, Joining is the process of actively putting together two quantities. The problems generally give an initial quantity and a direct or implied action that causes an increase in that quantity. Separating problems have the same characteristics as Joining problems except that the action involves a decrease. In Separating problems a subset is removed from a given set. Part-Part-Whole problems describe a static relationship between an entity and its two parts. Problems in the Comparison class involve comparing two disjoint quantities. This includes problems in which the difference between two quantities is to be found as well as problems in which one of two quantities and the difference between them are given and the second quantity is the unknown. Equalizing problems involve the same sort of action that is found in Joining and Separating problems, but there is also a comparison involved. Basically, Equalizing involves changing one of two entities so that the two are equal on some attribute. Equalizing-Add On involves an increase in the smaller quantity; Equalizing-Take Away involves a decrease in the larger quantity.

This classification scheme characterizes the types of action or relationships that are represented by most addition and subtraction problems. However, a fourth variable must be taken into account to completely characterize addition and subtraction problems: the nature of the unknown. For each of the six classes of problems, there are as many as three distinct problem types, depending on which quantities are given and which is the unknown. Although the action or relationship involved in each class of problems is essentially the same, the problem types are very different and potentially involve different methods of solution. In fact, each of the six basic classes or problems includes both addition and subtraction problems. Furthermore, between the possible problems in a class there are significant differences in difficulty that are a function of which quantities are given and which is the unknown (Grouws, 1972; Lindvall & Ibarra, 1980). Examples of each of the 17 distinct problems generated by this scheme are presented in Table 2.1.

Limitations. The framework that we propose to characterize addition and subtraction word problems is limited to simple problems that are appropriate for primary age children. It is not as complete as the framework proposed by Vergnaud in this volume that extends to operations on integers. Although our framework does not unambiguously characterize all addition and subtraction word problems, it has been useful to help us clarify distinctions between problem