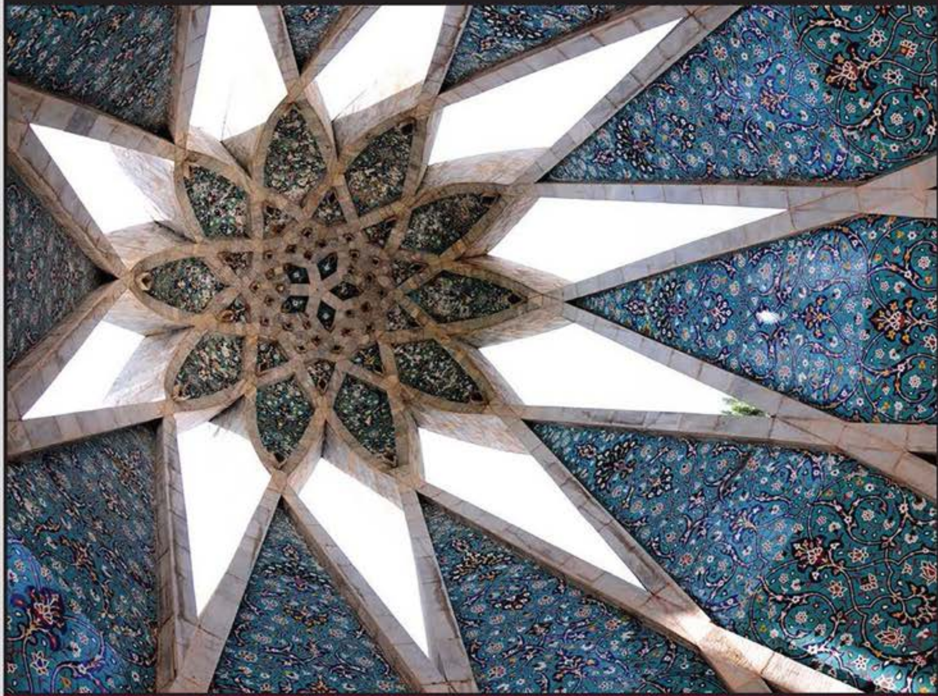

Lie Symmetry Analysis of Fractional Differential Equations

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CRC Press
Taylor & Francis Group
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Taylor & Francis
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http://taylorandfrancis.com

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CRC Press
Taylor & Francis Group
Boca Raton London New York

CRC Press is an imprint of the
Taylor & Francis Group, an **informa** business
A CHAPMAN & HALL BOOK

First edition published 2020
by CRC Press
6000 Broken Sound Parkway NW, Suite 300, Boca Raton, FL 33487-2742

and by CRC Press
2 Park Square, Milton Park, Abingdon, Oxon, OX14 4RN

Library of Congress Cataloging-in-Publication Data

Names: Hashemi, Mir Sajjad, author. | Baleanu, D. (Dumitru), author.
Title: Lie symmetry analysis of fractional differential equations / Mir Sajjad Hashemi and Dumitru Baleanu.
Description: First edition. | Boca Raton, FL : CRC Press, 2021. | Includes bibliographical references and index.
Identifiers: LCCN 2020014777 (print) | LCCN 2020014778 (ebook) | ISBN 9780367441869 (hardback) | ISBN 9781003008552 (ebook)
Subjects: LCSH: Fractional differential equations. | Lie groups. | Symmetry (Mathematics)
Classification: LCC QA372 .H37 2021 (print) | LCC QA372 (ebook) | DDC 515/.35--dc23
LC record available at https://lccn.loc.gov/2020014777
LC ebook record available at https://lccn.loc.gov/2020014778

ISBN: 978-0-367-44186-9 (hbk)
ISBN: 978-1-003-00855-2 (ebk)

Typeset in CMR
by Nova Techset Private Limited, Bengaluru & Chennai, India

Visit the Taylor & Francis Web site at
http://www.taylorandfrancis.com

and the CRC Press Web site at
http://www.crcpress.com

To ***Elham Darvishi*** *and* ***Mihaela Cristina*** *for their patience and support.*

Contents

Preface

The Lie method (the terminology "the Lie symmetry analysis" and "the group analysis" are also used) is based on finding Lie's symmetries of a given differential equation and using the symmetries obtained for the construction of exact solutions. The method was created by the prominent Norwegian mathematician Sophus Lie in the 1880s. It should be pointed out that Lie's works on application Lie groups for solving PDEs were almost forgotten during the first half of the 20th century. In the end of the 1950s, L.V. Ovsiannikov, inspired by Birkhoff's works devoted to application of Lie groups in hydrodynamics, rewrote Lie's theory using modern mathematical language and published a monograph in 1962, which was the first book (after Lie's works) devoted fully to this subject. The Lie method was essentially developed by L.V. Ovsiannikov, W.F. Ames, G. Bluman, W.I. Fushchych, N. Ibragimov, P. Olver, and other researchers in the 1960s–1980s. Several excellent textbooks devoted to the Lie method were published during the last 30 years; therefore one may claim that it is the well-established theory at the present time. Notwithstanding the method still attracts the attention of many researchers and new results are published on a regular basis. In particular, solving the so-called problem of group classification (Lie symmetry classification) still remains a highly nontrivial task and such problems are not solved for several classes of PDEs arising in real world applications.

Fractional calculus is an emerging field with ramifications and excellent applications in several fields of science and engineering. During the first attempt to think about what is derivative of order 1/2, stated by Leibniz in 1695, it was considered as a paradox as mentioned by L'Hopital. Since then the trajectory of the fractional calculus passed by several periods of intensive development both in pure and applied sciences. During the last few decades the fractional calculus has been associated with the power law effects and its various applications. It is a natural question to ask if the fractional calculus, as a non-local one, can produce new results within the well-established field of Lie symmetries and their applications. In fact the fractional calculus was associated with the dissipative phenomena; therefore it is a delicate question: can we have conservation laws for fractional differential equations associated to real world models?

In our book we try to answer to this vital question by analyzing, mainly, some different aspects of fractional Lie symmetries and related conservation laws. Also, finding the exact solutions of a given fractional partial differential equation is not an easy task but we present this issue in our book. The book includes also a generalization of Lie symmetries for fractional integro-differential equations. Nonclassical Lie symmetries are discussed for fractional differential equations. Moreover, the invariant subspace method is considered to find the exact solutions of some fractional differential equations. In the present book, we assume the reader to be familiar with preliminaries of Lie symmetries for integer order differential equations.

The structure of the book is as follows. The book consists of five chapters as it is given below. In order to make the readers understand easily the topic of Lie symmetries and their applications, in Chapter 1, we show briefly the classical, nonclassical symmetries and the conservation laws of some specific problems with integer order. Next, in Chapter 2, we discuss the Lie symmetries of fractional differential equations and exact solutions with invariant subspace methods. Chapter 3 focuses on Lie symmetries of fractional integro-differential equations. The nonclassical Lie symmetry analysis of fractional differential equations is described in Chapter 4. The self-adjointness and conservation laws of fractional differential equations are considered in Chapter 5.

We believe that our book will be useful for PhD and postdoc graduates as well as for all mathematicians and applied researchers who use the powerful concept of Lie symmetries.

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Chapter 1

Lie symmetry analysis of integer order differential equations

This chapter deals with the classical and nonclassical Lie symmetry analysis of some integer order differential equations. Finding the exact solutions of differential equations is an interesting field of many researchers. The Lie symmetry method is one of the most powerful and popular ones which can analyze different types of differential equations. In the last decade, various interesting textbooks have discussed the Lie symmetry analysis of integer order differential and integro-differential equations, e.g., [59, 94, 131, 150, 29]. Various classical concerns about Lie symmetries are discussed in these textbooks; so we avoid the preliminaries of the Lie symmetries. This chapter discusses the application of the Lie symmetry method and conservation laws for some integer order differential equations. However, some new and different approaches such as the Nucci's method [143, 89, 13, 129] are investigated. Among analytical methods for differential equations, the invariant subspace method is a very close one to the invariance theory, which plays an important role in the Lie symmetry analysis. We refer the interested readers to this topic in [60, 166, 80, 40, 12, 65, 122].

1.1 Classical Lie symmetry analysis

Various types of Lie symmetry method have been introduced up to now, e.g., classical [63, 120, 123, 42, 189, 153, 159, 158, 39, 91], nonclassical [32, 140, 139, 88] and approximate [58, 46, 93] Lie symmetries. Moreover, there are some numerical methods which are based upon Lie groups [76, 168, 10, 9, 82, 73, 4, 2, 70, 78, 79, 86, 74]. Briefly, a symmetry of a differential equation is a transformation which maps every solution of the differential equation to another solution of the same equation.

Here, we present some preliminaries of Lie Groups and Transformation Groups. The main ingredients for this section are the algebraic concept of a group and the differential-geometric notion of a smooth manifold. The term smooth constrains the overlap functions of any coordinate chart to be C^∞

functions. The following definition is the foundation of Lie symmetry methods for differential equations.

Definition 1 *(Lie group) A set G is called a Lie Group if there is given a structure on G satisfying the following three axioms.*
i) G is a group.
ii) G is a smooth manifold.
iii) The group operations

$$\underset{(g,h)\to g.h}{G \times G \to G}, \quad \underset{g\to g^{-1}}{G \to G},$$

are smooth functions.

When the dimension of G is r, we call this group an r-parameter Lie group.

Definition 2 *(Lie Transformation Groups) Let $\mathcal{M}$ be a n-dimensional smooth manifold and G a Lie group. An action T of the group G on $\mathcal{M}$ is a smooth mapping*

$$\underset{T(g,x)\equiv gx\to\bar{x}}{T : G \times \mathcal{M} \to \mathcal{M}}$$

with the following properties:

$$T(e, x) = x,\ T(a, T(b, x)) = T(ab, x)$$

for any $x \in \mathcal{M},\ g, a, b \in G,\ e \in G$ the unit element. Then G is called a Lie transformation group of the manifold $\mathcal{M}$.

It is well known that the applications of symmetry groups to differential equations include:

- mapping solutions to other solutions
- integration of ordinary differential equations in formulas
- constructing invariant (similarity) solutions, that is, solutions which are invariant under the action of a subgroup of the admitted group
- detection of linearizing transformations.

To carry out any of these, a true technique for finding symmetries of differential equations is needed. As a general idea, one could insert an arbitrary change of variables into the equation and then impose the new variables to satisfy the same differential equation. This earns a number of differential equations (determining equations) to be satisfied by the transformation. This direct approach is too drastic to be of much use: determining equations may be derived, but solving such a large system of nonlinear equations is usually out of the question. The crucial understanding of Lie was that this problem could prevail by considering the 'infinitesimal' action of the group. In order to define the infinitesimals, we defined a one-parameter Lie group of the form

$$\bar{x} = F(x; \epsilon), \tag{1.1}$$

where ϵ is the group parameter, which, without loss of generality, will be assumed to be defined in such a way that the identity element $\epsilon_0 = 0$. Hence

$$x = F(x; \epsilon)|_{\epsilon=0}. \tag{1.2}$$

Definition 3 *(Infinitesimal Transformation) Given a one parameter Lie group of transformation (1.1), we expand $\bar{x} = F(x; \epsilon)$ into its Taylor series in the parameter ϵ in a neighborhood of $\epsilon = 0$. Then, making use of the fact (1.2), we obtain what is called the infinitesimal transformations of the Lie group of transformation (1.1):*

$$\bar{x} = x + \epsilon\xi(x) + O(\epsilon^2), \tag{1.3}$$

where

$$\xi(x) = \frac{\partial \bar{x}}{\partial \epsilon}|_{\epsilon=0}. \tag{1.4}$$

The components of the vector $\xi(x) = (\xi_1(x), \xi_2(x), \ldots, \xi_n(x))$ are called the infinitesimals of (1.1).

Definition 4 *(Infinitesimal generator) The operator*

$$V = \sum_{i=1}^{n} \xi_i(x) \frac{\partial}{\partial x_i} \tag{1.5}$$

is called the infinitesimal generator (operator) of the one-parameter Lie group of transformations (1.1), where $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and $\xi(x) = (\xi_1(x), \xi_2(x), \ldots, \xi_n(x))$ are the infinitesimals of (1.1)

Besides, each constant in a one-parameter Lie group of transformations leads to a symmetry generator (which is a linear operator). These symmetry generators belong to a one-dimensional linear vector space in which any linear combination of generators is also a linear operator and the way of ordering generators is not important, that is, the symmetry group of transformation commutes, and this leads to the additional structure in the mentioned vector space called the commutator.

Definition 5 *Let G be the one-parameter Lie group of transformations (1.1) with the symmetry generators $V_i,\ i = 1, 2, \ldots, r$ given by (1.5). The commutator (Lie bracket) $[.,.]$ of two symmetry generators V_i, V_j is the first order operator generated as follows*

$$[V_i, V_j] = V_i V_j - V_j V_i.$$

Definition 6 *(Lie algebra) A Lie algebra $\mathcal{L}$ is a vector space over a field F with a given bilinear commutation law (the commutator) satisfying the properties*

1. *Closure:*
   For $X, Y \in \mathcal{L}$ it follows that $[X, Y] \in \mathcal{L}$.
2. *Bilinearity:*
   $[X, \alpha Y + \beta Z] = \alpha[X, Y] + \beta[X, Z], \quad \alpha, \beta \in F, \quad X, Y, Z \in \mathcal{L}.$
3. *Skew-symmetry:*
   $[X, Y] = -[Y, X].$
4. *Jacobi identity:*
   $[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$

Now, after brief preliminaries of the Lie symmetry method, we illustrate this technique by different integer order differential equations.

1.1.1 Lie symmetries of the Fornberg-Whitham equation

The Fornberg-Whitham equation (FWE)[53, 84],

$$u_t - u_{xxt} + u_x + uu_x = 3u_x u_{xx} + uu_{xxx} \ , \tag{1.6}$$

has appeared in the study of qualitative behaviors of wave breaking, which is a nonlinear dispersive wave equation. In 1978, Fornberg and Whitham obtained a peaked solution of the form $u(x,t) = A\exp\{\frac{-1}{2}|x - \frac{4}{3}t|\}$, where A is an arbitrary constant. Zhou et al. [190] have found the implicit form of a type of traveling wave solution called kink-like wave solutions and antikink-like wave solutions. After that, they found the explicit expressions for the exact traveling wave solutions, peakons and periodic cusp wave solutions for the FWE [191]. Tian et al. [173], under the periodic boundary conditions, have studied the global existence of solutions to the viscous FWE in L^2. The limit behavior of all periodic solutions as the parameters trend to some special values was studied in [186]. F. Abidi et al. [5] have successfully applied the homotopy analysis method to obtain the approximate solution of FWE and compared those to the solutions given by Adomian decomposition method.
The symmetry groups of the FWE will be generated by the vector field of the form

$$X = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \phi(t,x,u)\frac{\partial}{\partial u}. \tag{1.7}$$

The result shows that the symmetry of Eq. (1.6) is expressed by a finite three-dimensional point group containing translation in the independent variables and scaling transformations. The group parameters are denoted by k_i for $i = 1, 2, 3$ and characterize the symmetry of equation. Actually, we find that

Eq. (1.6) admits a three-dimensional Lie algebra $\mathcal{L}_3$ of its classical infinitesimal point symmetries generated by the following vector fields:

$$X_1 = \frac{\partial}{\partial t}, \quad X_2 = \frac{\partial}{\partial x}, \quad X_3 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Obviously, the Lie algebra of (1.6) is solvable and from the adjoint representation of the symmetry Lie algebra the optimal system of one-dimensional subalgebras corresponds to (1.6) which can be expressed by

$$X_1, \quad X_2, \quad \alpha X_1 + X_3,$$

where $\alpha \in \{-1, 0, 1\}$.

1.1.1.1 Similarity reductions and exact solutions

In order to reduce PDE (1.6) to a system of ODEs with one independent variable, we construct similarity variables and similarity forms of field variables. Using a straightforward analysis, the characteristic equations used to find similarity variables are:

$$\frac{dt}{\xi^1} = \frac{dx}{\xi^2} = \frac{du}{\phi}. \tag{1.8}$$

Integration of first order differential equations corresponding to pairs of equations involving only independent variables of (1.8) leads to similarity variables. We distinguish four cases:

Case 1: For the generator X_1, we have:

$$u(t,x) = S(\zeta), \qquad \zeta = x,$$

where $S(\zeta)$ satisfies the following ODE:

$$S' + SS' - 3S'S'' - SS^{(3)} = 0, \tag{1.9}$$

that admits the only Lie symmetry operator $\frac{\partial}{\partial \zeta}$. Instead of using the usual method based on invariants we apply a more direct method, namely the reduction method [143, 142, 145, 146, 128, 89]. Obtaining the first integrals of ODEs is often sophisticated work as shown in [137]. However, using the mentioned reduction method, the first integrals of the reduced ODEs are easily obtained. Equation (1.9) can be written as an autonomous system of three ODEs of first order, i.e.,

$$\begin{cases} w_1' = w_2, \\ w_2' = w_3, \\ w_3' = \dfrac{w_2 + w_1 w_2 - 3w_2 w_3}{w_1}, \end{cases} \tag{1.10}$$

using the obvious change of dependent variables

$$w_1(\zeta) = S(\zeta), \quad w_2(\zeta) = S'(\zeta), \quad w_3(\zeta) = S''(\zeta).$$

Since this system is autonomous, we can choose w_1 as a new independent variable. Then system (1.10) becomes the following nonautonomous system of two ODEs of first order:

$$\left\{ \begin{array}{l} \dfrac{dw_2}{dw_1} = \dfrac{w_3}{w_2}, \\ \dfrac{dw_3}{dw_1} = \dfrac{1 + w_1 - 3w_3}{w_1}. \end{array} \right. \tag{1.11}$$

We can integrate from the second equation:

$$w_3 = \frac{12a_1 + 3w_1^4 + 4w_1^3}{12w_1^3}, \tag{1.12}$$

with a_1 an arbitrary constant. This solution obviously corresponds to the following first integral of equation (1.9):

$$\frac{S(\zeta)^3}{12}\big(12S''(\zeta) - 3S(\zeta) - 4\big) = a_1.$$

Substituting (1.12) into (1.11) yields

$$\frac{dw_2}{dw_1} = \frac{12a_1 + 3w_1^4 + 4w_1^3}{12w_1^3 w_2};$$

that is a separable first order equation too. Therefore, the general solution is

$$w_2 = \sqrt{\frac{-12a_1 + 12a_2 w_1^2 + 3w_1^4 + 8w_1^3}{18w_1^2}}, \tag{1.13}$$

with a_2 an arbitrary constant. Replacing a_1 into this expression in terms of the original variables S and ζ yields another first integral of equation (1.9):

$$\frac{2S(\zeta)S''(\zeta) + 2\left(S'(\zeta)\right)^2 - S^2(\zeta) - 2S(\zeta)}{2} = a_2.$$

Finally, we replace (1.13) from (1.10) into the first equation of system (1.10) that becomes the following separable first-order equation

$$r_1' = p\sqrt{\frac{-2a_1 + a_2(p + q - 2a_1)r_1 - (p+q)r_1^2}{pr_1}},$$

and its general solution is implicitly expressed by

$$\int \sqrt{\frac{18w_1^2}{-12a_1 + 12a_2 w_1^2 + 3w_1^4 + 8w_1^3}}\, dw_1 = \zeta + a_3,$$

and replacing w_1 with $S(\zeta)$ yields the general solution of (1.9).

An explicit subclass of solutions can be obtained if one assumes $a_1 = 0$. Thus

$$u(t,x) = \frac{16 - 36a_2 + e^{\pm(x+a_3)} - 8e^{\pm\left(\frac{x+a_3}{2}\right)}}{6e^{\pm\left(\frac{x+a_3}{2}\right)}}.$$

Case 2: The solution obtained from generator X_2 is trivial. Thus, we find the traveling wave solution which is achievable from generator $X_1 + X_2$. The similarity variable related to $X_1 + X_2$ is

$$u(t,x) = S(\zeta), \quad \zeta = x - t,$$

where $S(\zeta)$ satisfies the following equation:

$$(1-S)\,S''' + SS' - 3S'S'' = 0. \tag{1.14}$$

Eq. (1.14) admits the only generator $\frac{\partial}{\partial \zeta}$. Therefore it is not possible to solve it by current Lie symmetry methods and we apply the reduction method. This equation transforms into the following autonomous system of first order equations, i.e.,

$$\begin{cases} w_1' = w_2, \\ w_2' = w_3, \\ w_3' = \dfrac{(3w_3 - w_1)w_2}{1 - w_1}, \end{cases} \tag{1.15}$$

by the change of dependent variables

$$w_1(\zeta) = S(\zeta), \quad w_2(\zeta) = S'(\zeta), \quad w_3(\zeta) = S''(\zeta).$$

Similar to Case 1, let us choose w_1 as the new independent variable. Then (1.15) yields:

$$\begin{cases} \dfrac{dw_2}{dw_1} = \dfrac{w_3}{w_2}, \\ \dfrac{dw_3}{dw_1} = \dfrac{(3w_3 - w_1)}{1 - w_1}. \end{cases} \tag{1.16}$$

The second equation of (1.16) is linear and therefore we have

$$w_3 = \frac{12a_1 + 3w_1^4 - 8w_1^8 + 6w_1^2}{12w_1^3 - 36w_1^2 + 36w_1 - 12}, \tag{1.17}$$

and substituting in the other equation of (1.16) yields:

$$\frac{dw_2}{dw_1} = \frac{12a_1 + 3w_1^4 - 8w_1^3 + 6w_1^2}{12w_2\left(w_1^3 - 3w_1^2 + 3w_1 - 1\right)}. \tag{1.18}$$

Replacing a_1 into this expression in terms of the original variables S and ζ yields a first integral of equation (1.14) as:

$$\frac{S''\left(12S^3 - 36S^2 + 36S - 12\right) - 3S^4 + 8S^3 - 6S^2}{12} = a_1.$$

Eq. (1.18) is separable and the solution is given by

$$w_2 = \sqrt{\frac{-12a_1 + 12a_2w_1^2 - 24a_2w_1 + 12a_2 + 3w_1^4 - 4w_1^3 - w_1^2 + 2w_1 - 1}{12\,(w_1 - 1)^2}}, \tag{1.19}$$

where a_2 is another first integral of equation (1.14) as following:

$$SS'' - S'' + (S')^2 + \frac{1 - 6S^2}{12} = a_2.$$

An implicit solution of Eq. (1.14) can be obtained from substituting (1.19) into the first equation of (1.15) and one time integration. However, in a special case, taking $a_1 = 0$ and $a_2 = \frac{1}{12}$ we have

$$\frac{S\left[\sqrt{3}(4 - 3S) + 4\sqrt{S(3S - 4)}\ln\left(6\sqrt{S} + 2\sqrt{9S - 12}\right)\right]}{\sqrt{S^3(3S - 4)}} = \zeta + a_3.$$

Back substitution of variables yields another solution of the Eq. (1.6).

Case 3: For the linear combination $X = \alpha X_1 + X_3$, we are just able to find the invariant solution with respect to $\alpha = 0$. Similarity variables of X_3 are:

$$u(t,x) = \frac{x}{t} + S(\zeta), \quad \zeta = t, \tag{1.20}$$

where $S(\zeta)$ admits the following equation:

$$\zeta S' + S + 1 = 0; \tag{1.21}$$

therefore,

$$S(\zeta) = -1 + \frac{c}{\zeta};$$

thus, we get

$$u(t,x) = \frac{x - t + c}{t}. \tag{1.22}$$

1.1.2 Lie symmetries of the modified generalized Vakhnenko equation

Now, we apply the Lie group analysis to the so-called modified generalized Vakhnenko equation (mGVE) [89]:

$$\frac{\partial}{\partial x}\left(\mathfrak{D}^2u + \frac{1}{2}pu^2 + \beta u\right) + q\mathfrak{D}u = 0, \qquad \left(\mathfrak{D} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right), \tag{1.23}$$

where p, q and β are arbitrary nonzero constants. This equation was introduced by Morrison and Parkes in 2003 [135]. There the N-soliton solution of the mGVE[1] was found if $p = 2q$.

[1] Actually Morrison and Parkes introduced equation (1.23) but they named it mGVE in the case $p = 2q$ only.