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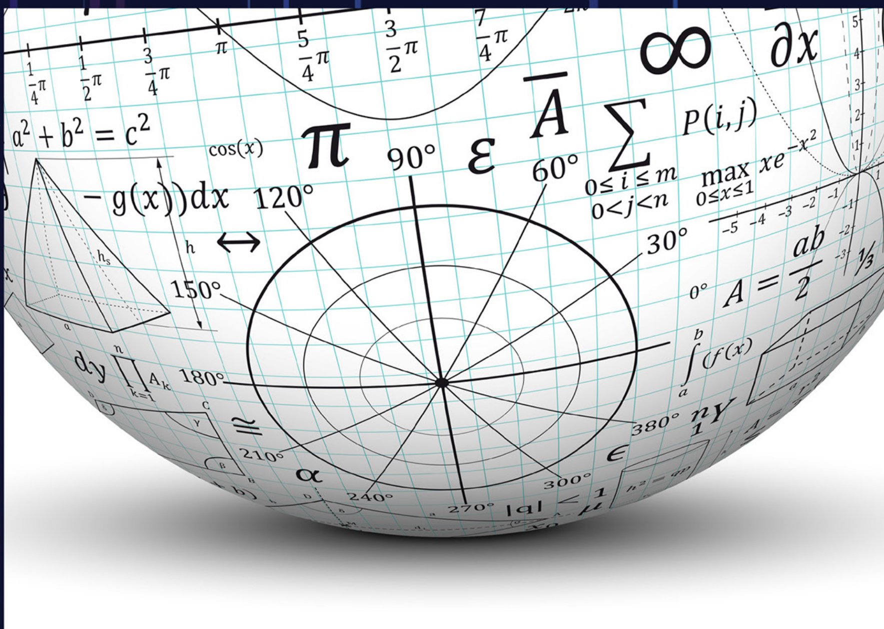

Mohammad Mursaleen
Feyzi Başar

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Sequence Spaces

Topics in Modern Summability Theory

Mohammad Mursaleen
Department of Mathematics
Aligarh Muslim University
Aligarh, India

Feyzi Başar
Department of Mathematical Education
İnönü University
Central Campus
Malatya, Turkey

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Foreword

Summability theory is more than a century old. It began with a paper in 1890 by E. Cesàro dealing with multiplication of series. The main aim of summability in its early days was the development of summability methods for divergent series and divergent integrals. The topic then developed its own identity far beyond its beginnings. An important and central theme in summability was the introduction of matrix methods such as Cesàro, Abel, Hölder, Riesz, Hausdorff, Nörlund and others. Summability theory relied initially on classical analysis, and as such it was considered a branch of Classical Analysis. The book by Hardy [97] marks the highlight of that era. The use of functional analysis methods began with the seminal research by Karl Zeller and his colleagues (see [239]) and continued with the fundamental contributions of A. Willansky and others (see [228]). It is gratifying to note that the topic has found its way into introductory textbooks on functional analysis (see [147] and [227]).

Over the past century there have been many landmarks in the theory and applications of summability theory, both in the contexts of classical analysis and functional analysis. For example, Tauberian theory, one of the classical topics in the theory, compares summability methods for series and integrals with the aim of deciding which of these methods converge and providing asymptotic estimates. There are profound and celebrated results in this area, such as the Hardy-Littlewood theorems and Norbert Wiener's breakthroughs and his simple proof of one of those theorems based on Fourier analysis (see the charming book by Korevaar [131], which traces a century of developments on Tauberian theorems). There are also applications of various Tauberian methods to prime number theory. Closer to the content of the present monograph, there have been remarkable applications of functional analysis methods in summability to iterative methods of linear and nonlinear operator equations in Hilbert and Banach spaces, in addition to the applications covered in this monograph. Summability theory in return has led to introduction of new classes of matrices and many interesting spaces of summable sequences and double sequences.

Professors M. Mursaleen and F. Başar are two of the renowned researchers in the field of summability in the last two decades. They have cultivated a research school on summability in their respective countries, India and Turkey. They have mentored two generations of students and researchers on various aspects of summability theory, sequence spaces, different notions of convergence and other topics. They have also collaborated on many joint research papers. This monograph reflects their achievements in these endeavors. The book is written for graduate students and researchers with an interest in sequence spaces, matrix transformations in the context of summability, various spaces of summable sequences and other topics mentioned in the preface. The book is a welcome addition to the literature. I look forward to adding it to my bookshelf as a companion to the other books [32, 52, 97, 131, 147, 168, 227, 228] and [239].

M. Zuhair Nashed
University of Central Florida,
Orlando, Florida

Preface

This book is intended for graduate students and researchers with a (special) interest in sequence spaces, matrix transformations and related topics. Besides a preface and index, the book consists of six chapters with abstracts and is organized as follows:

In Chapter 1; we present some basic definitions, notations and various basic ideas that will be required throughout the book. In this chapter, we state and prove Hahn-Banach, Baire's category, Banach-Steinhaus, bounded inverse, closed graph and open mapping theorems together with uniform boundedness principle, which are basic for functional analysis.

In Chapter 2, we investigate the geometric properties of normed Euler sequence spaces and Cesàro sequence space $ces(p)$, and some sequence spaces involving lacunary sequence space equipped with the Luxemburg norm besides topological, and some other usual properties.

Chapter 3 is devoted to some classes of matrix transformations with establishing the necessary and sufficient conditions on the elements of a matrix to map a sequence space X into a sequence space Y. This is a natural generalization of the problem to characterize all summability methods given by infinite matrices that preserve convergence.

In Chapter 4, we study the notion of almost convergence and the related matrix transformations with their some applications.

In Chapter 5, after giving some elementary examples following Yeşilkayagil and Başar [234], Dündar and Başar [75], Başar and Karaisa [38], and Srivastava and Kumar [205], we determine the spectrum and the fine spectrum of the lambda matrix Λ, the upper triangle double band matrix Δ^+, the generalized difference operator defined by a double sequential band matrix $B(\widetilde{r}, \widetilde{s})$ and the generalized difference operator Δ_{uv} acting on the sequence spaces c_0, c; ℓ_p and ℓ_1 with respect to Goldberg's classification, where $1 < p < \infty$.

In Chapter 6, we summarize the literature on some sets of fuzzy valued sequences and series. Talo and Başar [213] have extended the main results of Başar and Altay [35] to fuzzy numbers and defined the α-, β- and γ-duals of a set of fuzzy valued sequences, and gave the duals of the classical sets of fuzzy valued sequences together with the characterization of the classes of infinite matrices of fuzzy numbers transforming one of the classical sets into another one. Also, Kadak and Başar [104, 105] have recently studied the power series of fuzzy numbers and examined the alternating and binomial series of fuzzy numbers and some sets of fuzzy-valued functions with the level sets, and gave some properties of the level sets together with some inclusion relations, in [103, 108]. Finally, following Talo and Başar [215]; we introduce the classes $\ell_\infty(F)$, $c(F)$, $c_0(F)$ and $\ell_p(F)$ consisting of all bounded, convergent, null and absolutely p-summable fuzzy valued sequences with the level sets and the sets $\ell_\infty(F;f)$, $c(F;f)$, $c_0(F;f)$ and $\ell(F;f)$ of fuzzy valued sequences defined by a modulus function.

Mohammad Mursaleen & Feyzi Başar

Aligarh & İstanbul

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Mohammad Mursaleen & Feyzi Başar
Aligarh & İstanbul
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Authors

Mohammad Mursaleen is currently a Principal Investigator for a SERB Core Research Grant at the Department of Mathematics, Aligarh Muslim University. He has published more than 330 research papers in the field of summability, sequence spaces, approximation theory, fixed point theory, measures of noncompactness. He has also published eight books and completed several national and international projects, in several countries. Besides several master's students, he has guided twenty Ph.D. students, and served as a reviewer for various international scientific journals. He is also member of editorial boards, for many international scientific journals. Recently, he has been recognized as Highly Cited Researcher 2019 by Web of Science.

Feyzi Başar is a Professor Emeritus since July 2016, at İnönü University, Turkey. He has published an e-book for graduate students and researchers and more than 150 scientific papers in the field of summability theory, sequence spaces, FK-spaces, Schauder bases, dual spaces, matrix transformations, spectrums of certain linear operators represented by a triangle matrix over some sequence space, the alpha-, beta- and gamma-duals and some topological properties of the domains of some double and four-dimensional triangles in the certain spaces of single and double sequences, sets of the sequences of fuzzy numbers, multiplicative calculus. He has guided 17 master's and 10 Ph.D. students and served as a referee for 121 international scientific journals. He is a member of an editorial board of 21 scientific journals. Feyzi Başar is also a member of scientific committees of 17 mathematics conferences, gave talks at 14 different universities as invited speaker and participated in more than 70 mathematics symposiums with a paper.

List of Abbreviations and Symbols

$\overline{A}$	:	the closure of a set A
A^0	:	the interior of a set A
A^C	:	the complement of a set A
$P(A)$	:	the collection of all subsets of a set A
$\mathbb{N}_0$	:	set of natural numbers, i.e., $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$
$\mathbb{N}_k$	:	set of integers which are greater than or equal to $k \in \mathbb{N}_0$
$\mathbb{Z}$	:	set of integers, i.e., $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{Q}$	:	set of rational numbers
$\mathbb{R}$	:	set of real numbers, the real field
$\mathbb{R}^+$	:	set of non-negative real numbers
$\mathbb{C}$	:	set of complex numbers, the complex field
$\mathbb{K}$	:	either of the fields of $\mathbb{R}$ or $\mathbb{C}$
$Re[z]$	:	real part of $z \in \mathbb{C}$
$Im[z]$	:	imaginer part of $z \in \mathbb{C}$
$\mathbb{R}^2$	:	set of all pairs of real numbers
$\mathbb{R}^n$	:	n-dimensional Euclidean space
$\mathbb{C}^n$	:	n-dimensional complex Euclidean space
$[a]$	:	integer part of a number a
$\mathcal{F}$	:	collection of all finite subsets of $\mathbb{N}_0$
$P[0, 1]$	:	set of all polynomials defined on the interval $[0, 1]$
$C[0, 1]$	:	space of all continuous real or complex valued functions on the interval $[0, 1]$
$C[a, b]$	:	space of all continuous real or complex valued functions on the interval $[a, b]$
$C_B[a, b]$	:	space of all continuous and bounded functions on $[a, b]$
$C_F[a, b]$	:	set of all continuous fuzzy-valued functions on the interval $[a, b]$
$B_F[a, b]$	:	set of all bounded fuzzy-valued functions on the interval $[a, b]$
$L_p(X)$	:	collection of all complex measurable functions on X with $1 \leq p < \infty$
e	:	sequence whose elements are equal to 1
$e^{(k)}$	:	sequences whose only non-zero term is a 1 in k^{th} place for each $k \in \mathbb{N}_0$
δ_{ij}	:	Kronecker delta which is $= 1$ if $i = j$ and $= 0$ if $i \neq j$
ω	:	space of all sequences over the complex field
$\omega(F)$	:	set of all sequences of fuzzy numbers
$C(F)$	:	set of all fuzzy valued Cauchy sequences

ϕ	:	set of all finitely non-zero sequences
ℓ_∞	:	space of bounded sequences over the complex field
$\ell_\infty(F)$	:	set of bounded sequences of fuzzy numbers
$\ell_\infty(F;f)$	:	set of bounded sequences of fuzzy numbers defined by a modulus function
f	:	space of almost convergent sequences over the complex field
$f(X)$	:	set of all almost convergent sequences in X
$wf(X)$	:	set of all weakly almost convergent sequences in X
$[f]$	:	space of all strongly almost convergent sequences over the complex field
f_0	:	space of almost null sequences over the complex field
c	:	space of convergent sequences over the complex field
$c(F)$	:	set of convergent sequences of fuzzy numbers
$c(F;f)$	:	set of convergent sequences of fuzzy numbers defined by a modulus function
c_0	:	space of null sequences over the complex field
$c_0(F)$	:	set of null sequences of fuzzy numbers
$c_0(F;f)$	:	set of null sequences of fuzzy numbers defined by a modulus function
ℓ_1	:	space of absolutely summable sequences over the complex field
$\widehat{\ell}$	:	set of all absolutely almost convergent sequences
ℓ_p	:	space of absolutely p-summable sequences over the complex field
$\widehat{\ell_p}$	:	set of all absolutely p-almost convergent sequences
$\ell_p(F)$	:	set of absolutely p-summable sequences of fuzzy numbers
$\ell_p(F;f)$	:	set of absolutely p-summable sequences of fuzzy numbers defined by a modulus function
ℓ_Φ	:	Orlicz sequence space
fs	:	space of almost convergent series over the complex field
bs	:	space of bounded series over the complex field
$bs(F)$	:	set of bounded series of fuzzy numbers
fs_0	:	space of series almost converging to zero over the complex field
cs	:	space of convergent series over the complex field
$cs(F)$	:	set of convergent series of fuzzy numbers
cs_0	:	space of series converging to zero over the complex field
$cs_0(F)$	:	set of series of fuzzy numbers converging to zero
bv	:	space of sequences of bounded variation over the complex field
bv_σ	:	space of sequences of σ-bounded variation over the complex field
$bv(F)$	:	set of sequences of bounded variation of fuzzy numbers
bv_0	:	space of sequences of both bounded variation and null over the complex field
bv_p	:	space of sequences of p-bounded variation over the complex field
$bv_p(F)$	:	set of p-bounded variation sequences of fuzzy numbers
$\ell_\infty(p)$	:	space of all sequences (x_k) such that $\sup_{k\in\mathbb{N}_0}|x_k|^{p_k}<\infty$
$c(p)$	:	space of all sequences (x_k) such that $|x_k-l|^{p_k}\to 0$, as $k\to\infty$
$c_0(p)$	:	space of all sequences (x_k) such that $|x_k|^{p_k}\to 0$, as $k\to\infty$

$\ell(p)$	:	space of all sequences (x_k) such that $\sum_k \lvert x_k\rvert^{p_k} < \infty$
X^α	:	alpha dual of a sequence space X
X^β	:	beta dual of a sequence space X
X^γ	:	gamma dual of a sequence space X
λ^*	:	continuous dual of a sequence space λ
λ^f	:	f-dual of a sequence space λ
$x^{[n]}$	:	n^{th} section of a sequence $x = (x_k)$
$\{(Ax)_n\}$	:	A-transform of a sequence x
$\mathcal{A}x$	:	$\{(Ax)_n^i\}_{i,n=0}^\infty$
c_A	:	convergence domain of a matrix A
$\chi(A)$	:	characteristic of a matrix A
L	:	Banach limit
S	:	shift operator
C_1	:	Cesàro mean of order one
$\triangle$	:	forward difference operator, i.e., $(\triangle x)_k = x_k - x_{k+1}$ and $(\triangle^2 x)_k = \triangle(x_k - x_{k+1})$
Δ	:	backward difference operator, i.e., $(\Delta x)_k = x_k - x_{k-1}$
$\lvert C_1\rvert$	:	absolute summability of Cesàro mean of order one
C_r	:	Cesàro mean of order r
E_1	:	original Euler matrix
E_q	:	Euler mean of order q
E^r	:	Euler-Knopp matrix of order r
T^r	:	Taylor matrix
R^t	:	Riesz mean with respect to the sequence $t = (t_k)$
N^t	:	Nörlund mean with respect to the sequence $t = (t_k)$
θ	:	zero vector in a linear space X
$L(\mathbb{R})$	:	set of all fuzzy numbers on $\mathbb{R}$
$L(\mathbb{R})^+$	:	set of all non-negative fuzzy numbers on $\mathbb{R}$
$L(\mathbb{R})^-$	:	set of all non-positive fuzzy numbers on $\mathbb{R}$
W	:	set of all closed bounded intervals A with endpoints $\underline{A}$ and $\overline{A}$
$[u]_\alpha$	:	$\alpha-$level set of $u \in L(\mathbb{R})$
$supp(u)$	:	set of real numbers t such that $u(t) > 0$
$u \not\sim v$	:	neither $u \preceq v$ nor $v \preceq u$
BSFN	:	a bounded sequence of fuzzy numbers
$x_k \sim \infty$	:	$x = (x_k)$ is definitely divergent
$\mathcal{A}$	:	sequence of infinite matrices $A^i = \{a_{nk}(i)\}$
$\mathcal{B}x$	:	$\{(Bx)_m^i\}_{i,m=0}^\infty$
$\lvert K\rvert$	:	cardinality of K
D_∞	:	Hausdorff metric on the set $\ell_\infty(F)$
D_p	:	Hausdorff metric on the set $\ell_p(F)$
$(\lambda : \mu)$	:	class of all matrices A such that $A : \lambda \to \mu$
$(c : c)$	:	class of conservative matrices
$(c : c; p)$	:	class of Teoplitz matrices
$(c : c)_{reg}$	:	class of regular matrices
$(cs : c; p)$	:	class of series to sequence regular matrices

$(c : v_\sigma)$	:	class of sequence to sequence sigma-conservative matrices
$(c : f)$	:	class of almost conservative matrices
$(c : f)_{reg}$	:	class of almost regular matrices
$(f : c)$	:	class of strongly conservative matrices
$(f : c; p)$	:	class of strongly regular matrices
$(\ell_\infty : c)$	:	class of Schur (coercive) matrices
$(\ell_\infty : f)$	:	class of sequence to sequence almost coercive matrices
$(\ell_\infty : fs)$	:	class of sequence to series almost coercive matrices
$(bs : f)$	:	class of series to sequence almost coercive matrices
$(bs : fs)$	:	class of series to series almost coercive matrices
$\emptyset$	:	empty set
$(AB)_{ij}$	:	i^{th} row and j^{th} column entry of the matrix product AB
I	:	unit matrix
$G(A)$	:	graph of a continuous operator A
$D(T)$	:	domain of a linear operator T
$R(T)$	:	range of a linear operator T
$Ker(T)$	:	kernel or null space of a linear operator T
$r_\sigma(T)$	:	spectral radius of an operator $T \in B(X)$
T^*	:	adjoint of a bounded linear operator T
T_α	:	resolvent operator of T with each $\alpha \in \mathbb{C}$
$B(x_0; r)$	:	open ball of radius r with center x_0
$S(x_0; r)$	:	sphere of radius r with center x_0
$S[\theta, \delta]$	:	closed sphere of radius δ with center origin $\theta = (0, 0, 0, \ldots)$
S_X	:	the unit sphere in X
$\mathcal{L}(X)$	:	set of all linear and continuous operators on a space X into itself
$\mathcal{L}(X : Y)$	:	set of all linear and continuous operators $T : X \to Y$
$\mathcal{B}(X)$	:	set of all bounded linear operators on a space X into itself
$\mathcal{B}(X : Y)$	:	set of all bounded linear operators $T : X \to Y$
$\mathcal{C}(X : Y)$	:	set of all compact operators $T : X \to Y$
$\mathcal{F}(X : Y)$	:	set of all finite rank operators $T : X \to Y$
X'	:	set of bounded linear functionals on a space X
X^*	:	continuous dual of a space X
$\sigma(T, X)$	:	spectrum of a linear operator T on a space X
$\rho(T, X)$	:	resolvent set of a linear operator T on a space X
$\sigma_e(T, X)$	:	eigenspace of a linear operator T corresponding to the eigenvalue α
$\sigma_a(T, X)$	:	approximate spectrum of a linear operator T on a space X
$\sigma_p(T, X)$	:	point (discrete) spectrum of a linear operator T on a space X
$\sigma_c(T, X)$	:	continuous spectrum of a linear operator T on a space X
$\sigma_r(T, X)$	:	residual spectrum of a linear operator T on a space X
$\sigma_{ap}(T, X)$	:	approximate point spectrum of a linear operator T on a space X
$\sigma_\delta(T, X)$	:	defect spectrum of a linear operator T on a space X
$\sigma_{co}(T, X)$	:	compression spectrum of a linear operator T on a space X

Chapter 1

Basic Functional Analysis

Keywords. *Metric sequence spaces, normed linear spaces, bounded linear operators, Köthe-Toeplitz duals, Hahn-Banach theorem, Baire category theorem, uniform boundedness principle, Banach-Steinhaus theorem, bounded inverse theorem, closed graph theorem, open mapping theorem, compact operators, Schauder basis, separability, reflexivity, weak convergence, Hilbert spaces, topological vector spaces, FK-spaces.*

1.1 Metric Spaces

In $\mathbb{R}$, the set of all real numbers or in $\mathbb{C}$, the set of all complex numbers, the concept of absolute value plays an important role in defining two basic concepts, i.e., the concepts of convergence and continuity, on which the whole theory of real (or complex) variables depends. Metric space is a generalization of $\mathbb{R}$ (or $\mathbb{C}$), insofar as it is a space with a metric or a distance function. In the theory of metric spaces, the concept of distance is generalized by replacing $\mathbb{R}$ (or $\mathbb{C}$) with an arbitrary non-empty set X in such a way that one can have a notion of convergence and continuity in a more general setting.

Definition 1.1.1. *A metric space is a set X together with a function d, called a metric or distance function, which assigns a real number $d(x, y)$ to every pair x, y belonging to X satisfying the following axioms:*

(M1) (positive): $d(x, y) \geq 0$ for all x, y in X.

(M2) (strictly positive): $d(x, y) = 0$ iff $x = y$ for all x, y in X.

(M3) (symmetry): $d(x, y) = d(y, x)$ for all x, y in X.

(M4) (triangle inequality): $d(x, z) \leq d(x, y) + d(y, z)$ for all x, y, z in X.

Definition 1.1.2. *Let X be a non-empty set. Define d for $x, y \in X$ by*

$$d(x, y) = \begin{cases} 0 & , \quad x = y, \\ 1 & , \quad x \neq y. \end{cases} \tag{1.1.1}$$

The metric d given by (1.1.1) is called the trivial metric or discrete metric on X. The metric space (X,d) is called discrete metric space and is denoted by X_d.

Examples 1.1.3. *We have the following:*

(a) *The usual distance $d(x,y)=|x-y|$ is a metric for the set $\mathbb{R}$ of all real numbers.*

(b) *On the plane $\mathbb{R}^2$, the metric d_1 is defined by $d_1[(x_1,y_1),(x_2,y_2)]=|x_1-x_2|+|y_1-y_2|$. Another metric d_2 on $\mathbb{R}^2$ is the "usual distance" (measured using Pythagoras's theorem):*

$$d_2[(x_1,y_1),(x_2,y_2)]=\sqrt{(x_1-x_2)^2+(y_1-y_2)^2}.$$

Note that a non-empty set X may have more than one metric.

(c) *On the set $\mathbb{C}$ of all complex numbers, the metric d is defined by $d(z,w)=|z-w|$, where $|\cdot|$ represents the modulus of the complex number rather than the absolute value of a real number.*

(d) *On the plane $\mathbb{R}^2$, another metric d_∞ is defined with the supremum or maximum as*

$$d_\infty[(x_1,y_1),(x_2,y_2)]=\max\{|x_1-x_2|,|y_1-y_2|\}.$$

(e) *Let $C[0,1]$ be the set of all continuous real-valued functions on the interval $[0,1]$. We define the metrics d_1, d_2 and d_∞ on $C[0,1]$ by analogy to the above examples:*

$d_1(f,g)=\int_0^1|f(x)-g(x)|dx.$

$d_2(f,g)=\sqrt{\int_0^1[f(x)-g(x)]^2dx}.$

$d_\infty(f,g)=\max\limits_{0\le x\le 1}|f(x)-g(x)|.$

Definition 1.1.4. *A sequence (x_n) in a metric space (X,d) is said to be convergent to x in X if for every $\varepsilon>0$ there is $N>0$ such that $d(x,x_n)<\varepsilon$ whenever $n\ge N$; it is said to be Cauchy if $d(x_m,x_n)<\varepsilon$ whenever $n,m\ge N$. A metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent in X.*

Now, we may give the definition of closure and the interior of a set.

Definition 1.1.5. *Let (X,d) be a metric space and let $S\subset X$. A point $x_0\in X$ is a closure point of S if, for every $\varepsilon>0$, there is a point $x\in S$ with $d(x_0,x)<\varepsilon$. The closure $\overline{S}$ of S is the set of all closure points of S. We call x_0 an interior point of a set $S\subset X$ if S is a neighborhood of x_0. The interior S° of S is the set of all interior points of S. S° is open and is the largest open set in S.*

Definition 1.1.6. *A subset S of a metric space (X, d) is said to be dense in X iff $\overline{S} = X$. S is said to be nowhere dense in X if $(\overline{S})^0 = \emptyset$.*

A metric space (X, d) is said to be separable if it contains a countable dense subset.

Examples 1.1.7. *We give the following examples for separable/non-separable spaces:*

(i) The set of rational numbers $\mathbb{Q}$ dense in $\mathbb{R}$, hence $\mathbb{R}$ is separable.

(ii) The set of all rational polynomials $P[0, 1]$ is dense in $C[0, 1]$ with sup-norm $\|\cdot\|_\infty$ as well with integral norm $\|\cdot\|_p$, $(1 \le p < \infty)$, hence $C[0, 1]$ is separable.

(iii) ϕ is dense in the spaces c_0 and ℓ_p with the norms $\|\cdot\|_\infty$ and $\|\cdot\|_p$, respectively, i.e., c_0 and ℓ_p are separable, where $1 \le p < \infty$ and ϕ denotes the set of all finetely non-zero sequences.

(iv) Finite sets, $\mathbb{N}_0$ and $\mathbb{Z}$ are nowhere dense in $\mathbb{R}$.

(v) ℓ_∞ is not separable.

Proof. We prove here only (v). It is easy to see that the set $E := \{x = (x_j) \in \ell_\infty : x_j \in \{0, 1\}, j \in \mathbb{N}_0\} \subset \ell_\infty$ is uncountable, and for every distinct $x, y \in E$, $\|x - y\|_\infty = 1$. We have to show that E is not dense in ℓ_∞. Let if possible, E be dense in ℓ_∞. Then, there exists $z \in \ell_\infty$ such that $\|x - z\|_\infty < 1/4(= \epsilon)$ for $x \in E$. Now,

$$1 = \|x - y\|_\infty \le \|x - z\|_\infty + \|z - y\|_\infty < \frac{1}{4} + \|z - y\|_\infty$$

for all $y \in E$. This implies that $\|z - y\|_\infty > 3/4$, i.e., E is not dense in ℓ_∞. Hence, ℓ_∞ cannot be separable. □

Definition 1.1.8. *Let M and S be two subsets of a metric space (X, d) and $\epsilon > 0$. Then, the set S is called ϵ-net of M if for any $x \in M$ there exists $s \in S$ such that $d(x, s) < \epsilon$. If the set S is finite, then the ϵ-net S of M is called finite ϵ-net.*

Definition 1.1.9. *The set M is said to be totally bounded if it has a finite ϵ-net for every $\epsilon > 0$.*

Definition 1.1.10. *A subset M of a metric space X is compact if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of that subsequence is in M.*

Definition 1.1.11. *The set M is said to be relatively compact if the closure $\overline{M}$ of M is a compact set.*

If the set M is relatively compact, then M is totally bounded. If the metric space (X, d) is complete, then the set M is relatively compact if and only if it is totally bounded. It is easy to prove that a subset M of a metric space X is relatively compact if and only if every sequence (x_n) in M has a convergent subsequence; in that case, the limit of that subsequence need not be in M.

1.2 Metric Sequence Spaces

The space bv is the space of all sequences of bounded variation, that is, consisting of all sequences (x_k) such that $(x_k - x_{k+1})$ in ℓ_1, and $bv_0 = bv \cap c_0$. Let $e = (1, 1, \ldots)$ and $e^{(k)} = (0, 0, \ldots, 0, 1(\text{kth place}), 0, \ldots)$.

Examples 1.2.1. *We give the following examples for metric sequence spaces:*

(i) The most popular metric d_ω which is known as the Frèchet metric on the space ω of all real or complex valued sequences is defined by

$$d_\omega(x, y) = \sum_k \frac{|x_k - y_k|}{2^k(1 + |x_k - y_k|)}; \quad x = (x_k),\ y = (y_k) \in \omega.$$

For simplicity in notation, here and in what follows, the summation without limits runs from 0 to ∞, and use the convention that any term with negative subscript is equal to zero.

(ii) The space of bounded sequences is denoted by ℓ_∞, i.e.,

$$\ell_\infty := \left\{ x = (x_k) \in \omega : \sup_{k \in \mathbb{N}_0} |x_k| < \infty \right\}.$$

The natural metric on the space ℓ_∞ known as the sup-metric is defined by

$$d_\infty(x, y) = \sup_{k \in \mathbb{N}_0} |x_k - y_k|; \quad x = (x_k),\ y = (y_k) \in \ell_\infty.$$

(iii) The spaces of convergent and null sequences are denoted by c and c_0, that is,

$$\begin{aligned} c &:= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ \textit{such that} \ \lim_{k \to \infty} |x_k - l| = 0 \right\}, \\ c_0 &:= \left\{ x = (x_k) \in \omega : \lim_{k \to \infty} x_k = 0 \right\}. \end{aligned}$$

The metric d_∞ is also a metric for the spaces c and c_0.

(iv) *The space of absolutely convergent series is denoted by* ℓ_1*, i.e.,*

$$\ell_1 := \left\{ x = (x_k) \in \omega : \sum_k |x_k| < \infty \right\}.$$

The natural metric on the space ℓ_1 *is defined by*

$$d_1(x, y) = \sum_k |x_k - y_k|; \quad x = (x_k),\ y = (y_k) \in \ell_1.$$

(v) *The space of absolutely* p*-summable sequences is denoted by* ℓ_p*, that is,*

$$\ell_p := \left\{ x = (x_k) \in \omega : \sum_k |x_k|^p < \infty \right\},\ (0 < p < \infty).$$

In the case $1 < p < \infty$*, the metric* d_p *on the space* ℓ_p *is given by*

$$d_p(x, y) = \left(\sum_k |x_k - y_k|^p \right)^{1/p}; \quad x = (x_k),\ y = (y_k) \in \ell_p.$$

Also in the case $0 < p < 1$*, the metric* $\widetilde{d}_p$ *on the space* ℓ_p *is given by*

$$\widetilde{d}_p(x, y) = \sum_k |x_k - y_k|^p; \quad x = (x_k),\ y = (y_k) \in \ell_p.$$

(vi) *The space of bounded series is denoted by* bs*, i.e.,*

$$bs := \left\{ x = (x_k) \in \omega : \sup_{n \in \mathbb{N}_0} \left| \sum_{k=0}^{n} x_k \right| < \infty \right\}.$$

The natural metric on the space bs *is defined by*

$$d(x, y) = \sup_{n \in \mathbb{N}_0} \left| \sum_{k=0}^{n} (x_k - y_k) \right|; \quad x = (x_k),\ y = (y_k) \in bs. \qquad (1.2.1)$$

(vii) *The space of convergent series and the space of the series converging to zero are denoted by* cs *and* cs_0*, respectively, that is,*

$$\begin{aligned} cs &:= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \ \ \textit{such that} \ \ \lim_{n \to \infty} \left| \sum_{k=0}^{n} x_k - l \right| = 0 \right\}, \\ cs_0 &:= \left\{ x = (x_k) \in \omega : \lim_{n \to \infty} \left| \sum_{k=0}^{n} x_k \right| = 0 \right\}. \end{aligned}$$

The relation d *defined by (1.2.1) is the natural metric on the spaces* cs *and* cs_0*.*

(viii) The space of sequences of bounded variation is denoted by bv, i.e.,

$$bv := \left\{ x = (x_k) \in \omega : \sum_k |x_k - x_{k+1}| < \infty \right\}.$$

Define the forward difference sequence $\triangle u = \{(\triangle u)_k\}$ *by* $(\triangle u)_k = u_k - u_{k+1}$ *for all* $k \in \mathbb{N}_0$*. The natural metric on the space bv is defined by*

$$d(x,y) = \sum_k \left| \big(\triangle(x-y)\big)_k \right|; \quad x = (x_k),\ y = (y_k) \in bv.$$

Let $p = (p_k)_{k\in\mathbb{N}_0}$ be a bounded sequence of positive real numbers with $\sup_{k\in\mathbb{N}_0} p_k = H$ and $M = \max\{1, H\}$. The following spaces were introduced and studied by Lascarides and Maddox [137], and Simons [197]:

$$\begin{aligned}
\ell_\infty(p) &:= \left\{ x = (x_k) \in \omega : \sup_{k\in\mathbb{N}_0} |x_k|^{p_k} < \infty \right\}, \\
c(p) &:= \left\{ x = (x_k) \in \omega : \exists l \in \mathbb{C} \text{ such that } \lim_{k\to\infty} |x_k - l|^{p_k} = 0 \right\}, \\
c_0(p) &:= \left\{ x = (x_k) \in \omega : \lim_{k\to\infty} |x_k|^{p_k} = 0 \right\}, \\
\ell(p) &:= \left\{ x = (x_k) \in \omega : \sum_k |x_k|^{p_k} < \infty \right\}.
\end{aligned}$$

If $p_k = p$ for all $k \in \mathbb{N}_0$ for some constant $p > 0$, then these sets are reduced to ℓ_∞, c, c_0 and ℓ_p, respectively. The metrics d_∞ and d_p on the spaces $\ell_\infty(p)$, $c(p)$, $c_0(p)$ and $\ell(p)$ are defined by

$$\begin{aligned}
d_\infty(x,y) &= \sup_{k\in\mathbb{N}_0} |x_k - y_k|^{p_k}, \\
d_p(x,y) &= \left(\sum_k |x_k - y_k|^{p_k} \right)^{1/M};
\end{aligned}$$

respectively, where $0 \le p_k < \sup_{k\in\mathbb{N}_0} p_k < \infty$.

1.3 Normed Linear Spaces

The Euclidean distance between two points $x = (x_1, x_2)$ and $y = (y_1, y_2)$ belonging to two-dimensional Euclidean space $\mathbb{R}^2$ is given by

$$\|x - y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}.$$

In this way, we can view $\|\cdot\|$ as a real valued-function defined on the real Euclidean plane, we desire to extend this concept to a linear space, in general, which leads us to seek a conception of "norm."

Definition 1.3.1. *Let X be a linear space over the field $\mathbb{K}$ of real or complex numbers. A function $\|\cdot\| : X \to \mathbb{R}$ is said to be a norm on X if the following axioms hold for arbitrary points $x, y \in X$ and any scalar α:*

(N1) (positive definiteness): $\|x\| = 0$, if and only if $x = \theta$, where θ denotes the zero vector.

(N2) (absolute homogeneity): $\|\alpha x\| = |\alpha|\|x\|$.

(N3) (triangle inequality): $\|x + y\| \leq \|x\| + \|y\|$.

A normed linear space is a pair $(X, \|\cdot\|)$, where X is a linear space and $\|\cdot\|$ is a norm defined on X. When no confusion is likely we denote $(X, \|\cdot\|)$ by X.

Note that $\|\cdot\|$ is always non-negative: By (N2) and (N3), we have $0 = \|\theta\| = \|x - x\| \leq \|x\| + \|-x\| = 2\|x\|$, i.e., $\|x\| \geq 0$.

We have the following important relation between a metric space and a normed linear space:

Remark 1.3.2. *Each norm $\|\cdot\|$ of X defines a metric d on X given by $d(x, y) = \|x - y\|$ for all $x, y \in X$ and is called as induced metric. But it is known that not every metric on a linear space can be obtained from a norm.*

It is easy to check the first part. For the second part, let us consider the linear space ω; the metric d_ω cannot be obtained from a norm. Indeed, if $d_\omega(x, y) = \|x - y\|$ then we have

$$\begin{aligned} d_\omega(\alpha x, \alpha y) &= \|\alpha x - \alpha y\| \\ &= \sum_k \frac{|\alpha||x_k - y_k|}{2^k(1 + |\alpha||x_k - y_k|)} \neq |\alpha| d_\omega(x, y), \end{aligned}$$

that is, $\|\alpha(x - y)\| \neq |\alpha|\|x - y\|$. This means that the space ω is not a normed linear space.

Definition 1.3.3. *A seminorm ν on a linear space X is a function $\nu : X \to \mathbb{R}$ such that*

(i) $\nu(\alpha x) = |\alpha|\nu(x)$ for all $\alpha \in \mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and all $x \in X$ (absolute homogeneity).

(ii) $\nu(x + y) \leq \nu(x) + \nu(y)$ for all $x, y \in X$ (subadditivity).

Note that by (i), we have $\nu(0x) = 0 \cdot \nu(x) = 0$.

Note that every norm is a seminorm but not conversely. For converse, define $\nu(x) = |\lim_{n\to\infty} x_n|$ on c. Take $x_n = 1/(n + 1)$ for all $n \in \mathbb{N}_0$. Then, $\nu(x) = 0$ while $x \neq \theta$. Hence, ν is not a norm while it is a seminorm on c.

Definition 1.3.4. *A normed linear space X is complete if every Cauchy sequence in X converges in X, that is, if $\|x_m - x_n\| \to 0$, as $m, n \to \infty$; where $x_n \in X$, then there exists $x \in X$ such that $\|x_n - x\| \to 0$, as $n \to \infty$. A complete normed linear space is said to be a Banach space.*

Definition 1.3.5. *Let X be a normed linear space. We say that the series $\sum_k x_k$ with $x_k \in X$, converges to $s \in X$ if and only if the sequence of partial sums $(s_n) = (\sum_{k=0}^n x_k)_{n\in\mathbb{N}_0}$, converges to s, that is, $\|s_n - s\| \to 0$, as $n \to \infty$, and we write $\sum_k x_k = s$. A series $\sum_k x_k$ in X is said to be absolutely convergent if $\sum_k \|x_k\| < \infty$.*

Remark 1.3.6. *In $\mathbb{R}$ or $\mathbb{C}$, every absolutely convergent series is convergent. This is a direct consequence of the completeness of $\mathbb{R}$ or $\mathbb{C}$. But, in general, an absolutely convergent series need not be convergent in a normed space. For example, consider the space $X = P[0,1]$ with respect to $\|f\|_\infty = \sup_{t\in[0,1]} |f(t)|$. Then, the series $\sum_n x^n/n!$ is not convergent in X, since*

$$\sum_n \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x \notin P[0,1].$$

On the other hand, it is absolutely convergent. Since

$$\sum_n \left\|\frac{x^n}{n!}\right\| = \sum_n \frac{1}{n!} \quad \textit{for} \quad x = 1,$$

which is convergent by the ratio test and for $x = 0$, $\sum_n \|x^n/n!\| = 0$ for $|x| < 1$, $\sum_n \|x^n\|$ is convergent and $(1/n!)$ is a positive monotone decreasing sequence, then the series $\sum_n \|x^n/n!\|$ is also convergent by Abel's test.

Theorem 1.3.7. *If a Cauchy sequence has a convergent subsequence, then the whole sequence is convergent.*

Proof. Let (x_n) be a Cauchy in a normed linear space X and (x_{n_k}) be a subsequence of (x_n) converging to $x \in X$, say. Then, (x_{n_k}) is also Cauchy. Therefore, for every $\epsilon > 0$ there exists a $n_0 \in \mathbb{N}_0$ such that

$$\|x_n - x\| \le \|x_n - x_{n_k}\| + \|x_{n_k} - x\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

for all $n_k \ge n_0$. Hence, (x_n) converges to x. □

Remark 1.3.8. *If a subsequence of a sequence in a normed linear space X is convergent then the sequence itself need not be convergent. For example, consider the sequence $(x_n) = \{(-1)^n\}$ in the usual normed linear space $\mathbb{R}$. It is trivial that (x_n) is not convergent, but its subsequence $(x_{2n}) = (1, 1, 1, \ldots)$ converges to 1.*

Theorem 1.3.9. *A normed linear space X is complete if and only if every absolutely convergent series is convergent.*

Proof. Let X be complete and $\sum_n x_n$ be an absolutely convergent series. Then, since $\sum_k \|x_k\| < \infty$ it is immediate that

$$\lim_{m\to\infty} \|s_n - s_m\| = \lim_{m\to\infty} \left\| \sum_{k=m+1}^{n} x_k \right\| \leq \lim_{m\to\infty} \sum_{k=m+1}^{n} \|x_k\| = 0$$

for $n > m$. Hence, (s_n) is a Cauchy sequence in X and is convergent since X is complete, that is, $\sum_n x_n$ is convergent.

Conversely, let every absolutely convergent series be convergent and (x_n) be a Cauchy sequence in X. Then, we can find an increasing sequence $(n_k)_{k\in\mathbb{N}_0}$ of natural numbers such that

$$\left\|x_{n_{k+1}} - x_{n_k}\right\| < \frac{1}{2^k} \quad \text{for all} \quad k \in \mathbb{N}_0.$$

Therefore, $\sum_k \|x_{n_{k+1}} - x_{n_k}\| < \infty$. It follows that $\sum_k (x_{n_{k+1}} - x_{n_k})$ converges. Therefore, there is $x \in X$ such that $\sum_{k=0}^{m} (x_{n_{k+1}} - x_{n_k}) \to x$, say, as $m \to \infty$, that is, $x_{n_{m+1}} - x_{n_1} \to x$ implies $x_{n_{m+1}} \to x + x_{n_1}$, as $m \to \infty$. Hence, (x_{n_k}) converges. That is, the Cauchy sequence (x_n) has a convergent subsequence (x_{n_k}) and so, by Theorem 1.3.7, the whole sequence (x_n) is convergent. Therefore, X is complete. □

Examples 1.3.10. *We have the following examples:*

(i) c_0*,* c *and* ℓ_∞ *are Banach spaces with the sup-norm* $\|x\|_\infty = \sup_{k\in\mathbb{N}_0} |x_k|$*.*

We consider only the space c*. Let* $\left\{x^{(m)}\right\}$ *be a Cauchy sequence in* c*, we have*

$$\lim_{m,n\to\infty} \left\|x^{(n)} - x^{(m)}\right\|_\infty = 0.$$

Now, for each $\epsilon > 0$*, there exists* N *such that* $\|x^{(n)} - x^{(m)}\|_\infty < \epsilon$ *for all* $n, m \geq N$*, i.e.,*

$$\sup_{i\in\mathbb{N}_0} \left|x_i^{(n)} - x_i^{(m)}\right| < \frac{\epsilon}{3}; \ m, n \geq N$$

and hence,

$$\left|x_i^{(n)} - x_i^{(m)}\right| < \frac{\epsilon}{3} \quad \textit{for} \quad i \in \mathbb{N}_0 \quad \textit{and for all} \quad n, m \geq N. \tag{1.3.1}$$

Hence, for each i *the sequence of real numbers* $\left\{x_i^{(n)}\right\} = \left\{x_i^{(0)}, x_i^{(1)}, \ldots\right\}$ *is a Cauchy sequence in* $\mathbb{R}$ *and hence convergent, say, to* x_i*, i.e.,* $\left|x_i^{(m)} - x_i\right| \to 0$*, as* $m \to \infty$*, for each* $i \in \mathbb{N}_0$*.*

Now, fix $n \geq N$ *and letting* $m \to \infty$ *in (1.3.1), we get*

$$\left|x_i^{(n)} - x_i\right| < \frac{\epsilon}{3} \quad \textit{for each} \quad i \in \mathbb{N}_0. \tag{1.3.2}$$

So that

$$\sup_{i\in\mathbb{N}_0}\left|x_i^{(n)}-x_i\right|<\frac{\epsilon}{3} \quad \textit{for all } n\geq N,$$

that is, $\|x^{(n)}-x\|_\infty \to 0$*, as* $n\to\infty$*; where* $x=(x_i)$*. Hence,* $x^{(n)}\to x$*, as* $n\to\infty$*, i.e., a Cauchy sequence* $\{x^{(n)}\}$ *converges to* x*. Now, we have to show that* $x\in c$*.*

Now, the sequence $\left\{x_i^{(N)}\right\}\in c$ *and is a Cauchy sequence, hence*

$$\left|x_i^{(N)}-x_j^{(N)}\right|<\frac{\epsilon}{3} \quad \textit{for all} \quad i,j\geq M. \tag{1.3.3}$$

Consequently by (1.3.2) and (1.3.3), we have

$$\begin{aligned}|x_i-x_j| &= \left|x_i-x_i^{(N)}+x_i^{(N)}-x_j^{(N)}+x_j^{(N)}-x_j\right|\\ &\leq \left|x_i-x_i^{(N)}\right|+\left|x_i^{(N)}-x_j^{(N)}\right|+\left|x_j^{(N)}-x_j\right|\\ &< \frac{\epsilon}{3}+\frac{\epsilon}{3}+\frac{\epsilon}{3}=\epsilon.\end{aligned}$$

Therefore, $x=(x_i)$ *is a Cauchy sequence in* $\mathbb{R}$ *and hence convergent, i.e.,* $x\in c$*. That is,* c *is a Banach space.*

(ii) Let $1\leq p<\infty$*. Then,* ℓ_p *is complete with* $\|x\|_p=\left(\sum_k|x_k|^p\right)^{1/p}$*.*

Let $\{x_k^{(m)}\}_{m\in\mathbb{N}_0}$ *be a Cauchy sequence in* ℓ_p*. Then, there is* $N\in\mathbb{N}_0$ *such that for all* $r,s\geq N$*,* $\|x^{(r)}-x^{(s)}\|_p<\epsilon$*. Hence,*

$$\sum_k\left|x_k^{(r)}-x_k^{(s)}\right|^p<\epsilon^p, \tag{1.3.4}$$

which implies that

$$\left|x_k^{(r)}-x_k^{(s)}\right|<\epsilon \quad \textit{for each} \quad k \textit{ and for all} \quad r,s\geq N.$$

Hence, $\{x_k^{(m)}\}$ *is a Cauchy sequence in* $\mathbb{R}$ *and so is convergent to* x_k *in* $\mathbb{R}$*. Define* $x=(x_k)_{k\in\mathbb{N}_0}$*. We show that* $x\in\ell_p$*. From (1.3.4), we have*

$$\sum_k\left|x_k^{(r)}-x_k^{(s)}\right|^p<\epsilon^p \quad \textit{for all} \quad r,s\geq N. \tag{1.3.5}$$

Therefore, we get by letting $s\to\infty$ *in (1.3.5) that*

$$\sum_k\left|x_k^{(r)}-x_k\right|^p<\epsilon^p \quad \textit{for all } r\geq N. \tag{1.3.6}$$

This implies that the sequence $\left\{x_k^{(r)} - x_k\right\}_k \in \ell_p$ *for each* r. *Also,* $x^{(r)} \in \ell_p$ *by hypothesis. Hence, by Minkowski's inequality*

$$\sum_k |x_k|^p = \sum_k \left|x_k^{(r)} - x_k^{(r)} + x_k\right|^p \leq \sum_k \left|x_k^{(r)}\right|^p + \sum_k \left|x_k^{(r)} - x_k\right|^p.$$

That is, $x = (x_k) \in \ell_p$. *Finally, by (1.3.6), we have*

$$\left\|x^{(m)} - x\right\|_p = \left[\sum_k \left|x_k^{(m)} - x_k\right|^p\right]^{1/p} < \epsilon$$

for all $m \geq N$, *i.e.,* $x^{(m)} \to x$, *as* $m \to \infty$, *in* ℓ_p. *Hence,* ℓ_p *is complete.*

(iii) *The space* ϕ *is a normed linear space but not a Banach space with respect to any norm.*

Examples 1.3.11. *We have the following examples:*

(i) *The space* $C[a,b]$ *of all continuous functions on* $[a,b]$ *is complete normed linear space with* $\|f\|_\infty = \sup_{t\in[a,b]} |f(t)|$.

Let (f_n) *be a Cauchy sequence in* $C[a,b]$ *for each* $t \in [a,b]$. *Then, there is* $N \in \mathbb{N}_0$ *such that*

$$\begin{aligned} \|f_n - f_m\| &= \max_{t\in[a,b]} |f_n(t) - f_m(t)| < \epsilon \ \ \textit{for all} \ \ n,m \geq N \\ &\Rightarrow \ |f_n(t) - f_m(t)| < \epsilon \ \ \textit{for all} \ \ n,m \geq N. \end{aligned}$$

Hence, for a fixed $t_0 \in [a,b]$, $|f_n(t_0) - f_m(t_0)| < \epsilon$ *for all* $n,m \geq N$.

$\Rightarrow$: $\{f_n(t_0)\}$ *is a Cauchy sequence in* $\mathbb{R}$, *hence convergent, say to* $f(t_0)$, *i.e.,* $f_n(t_0) \to f(t_0)$, *as* $n \to \infty$, *which is the pointwise convergent to* f. *We have to show that it is uniformly convergent. For given* $\epsilon > 0$, *choose* N *such that* $|f_n(t) - f(t)|$ *for all* $n,m \geq N$. *Then,*

$$\begin{aligned} |f_n(t) - f(t)| &= |f_n(t) - f(t) + f_m(t) - f_m(t)| \\ &\leq |f_n(t) - f_m(t)| + |f_m(t) - f(t)| \\ &\leq \sup_{t\in[a,b]} |f_n(t) - f_m(t)| + |f_m(t) - f(t)| \\ &= \|f_n - f_m\| + |f_m(t) - f(t)|. \end{aligned}$$

By choosing m *sufficiently large, we can make each term on the right-hand side less than* $\epsilon/2$. *Hence,*

$$M_n = \sup_{t\in[a,b]} |f_n(t) - f(t)| < \epsilon \ \ \textit{for all} \ \ n \geq N,$$

i.e., $M_n \to 0$, *as* $n \to \infty$. *Therefore,* $f_n(t) \to f(t)$, *as* $n \to \infty$, *uniformly on* $[a,b]$. *Since* (f_n) *is a sequence of continuous functions which converge to* f *uniformly on* $[a,b]$, *we have that* $f \in C[a,b]$.