Dual Quaternions and Their Associated Clifford Algebras

Ronald Goldman

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Clifford algebra for dual quaternions has emerged recently as an alternative to standard matrix algebra as a computational framework for computer graphics. This book presents dual quaternions and their associated Clifford algebras in a new light, accessible to and geared toward the computer graphics community.

Collecting all the associated formulas and theorems in one place, this book provides an extensive and rigorous treatment of dual quaternions, as well as showing how two models of Clifford algebra emerge naturally from the theory of dual quaternions. Each section comes complete with a set of exercises to help readers sharpen and practice their understanding.

This book is accessible to anyone with a basic knowledge of quaternion algebra and is of particular use to forward-thinking members of the computer graphics community.

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To my most immediate family: Jackie, April, Cody, Lloyd, and Pinky.

He proves by algebra that Shakespeare's ghost is Hamlet's grandfather.

James Joyce, Ulysses

Contents

Preface

Dual quaternions are much more powerful, yet much more obscure than classical quaternions. Quaternions were introduced by Hamilton [14] to model rotations in 3-dimensions; dual quaternions were introduced by Clifford [1] to model rigid motions – translations and rotations —in 3 dimensions. Quaternion multiplication can be used to compute rotations, reflections, and perspective projections in 3-dimensions; dual quaternion multiplication can be used to compute translations as well as rotations and reflections along with perspective projections in 3-dimensions. Quaternions are vectors in 4-dimensions; dual quaternions are vectors in 8-dimensions. Quaternions can be represented by a pair of complex numbers; dual quaternions can be represented by a pair of quaternions.

Thus dual quaternions are both more powerful and more complicated than quaternions. These two factors are motivation enough for an extended rigorous study of dual quaternions. Moreover, recently several authors [5,11,12,13,25] have suggested that a Clifford algebra for dual quaternions is a more suitable framework for computer graphics than standard matrix algebra [8,15]. These claims too motivate a renewed interest in dual quaternions and their associated Clifford algebras.

This book is a sequel to my monograph *Rethinking Quaternions* [9], which also includes a discussion of a Clifford algebra for quaternions. Though this monograph on quaternions is not required reading for this sequel on dual quaternions, a thorough knowledge of quaternions is a prerequisite for understanding this book on dual quaternions.

Part I of this monograph is devoted to a thorough investigation of dual quaternions. Dual quaternions have already been written about by many authors, and I have borrowed extensively from the previous literature both classical and contemporary on this topic [1,3,4,10,18,19,20,23,24]. My goal here, however, is somewhat different: I want to present dual quaternions in a new light, accessible to and geared toward the computer graphics community. Therefore, in addition to presenting the algebra surrounding dual quaternions, I shall also show how to build on the intuitive geometric understanding of quaternions presented in [9] to develop a better intuitive understanding of dual quaternions. Moreover, in addition to deriving the formulas for rigid motions, I will also show here for the first time how to compute perspective and pseudo-perspective using dual quaternions.

Part II of this monograph is devoted to the study of two Clifford algebras associated to dual quaternions: the *plane model* and the *point model*. Although Part II begins with a brief review of the bare essentials of Clifford algebra, this initial chapter is not intended as a first introduction to Clifford algebra. Readers not at all familiar with Clifford algebra will be able to understand the subsequent material on Clifford algebras for dual quaternions, but for context readers would benefit immensely from some prior knowledge of Clifford algebra. For a good introduction to Clifford algebra, see for example [6,12,13].

The two Clifford algebras associated to dual quaternions have also been written about before [5,11,12,13,16,21,22,25], but some of the results are scattered throughout the literature; moreover, side-by-side comparisons are not readily available. My goals here are twofold: to collect in one place for easy learning and reference these two Clifford algebras, and to provide comparisons of the advantages and disadvantages of the strengths and weaknesses of each of these approaches to Clifford algebra for the dual quaternions. Rather than rehashing methods already in the literature, I will present a novel approach to these Clifford algebras. Once we have a thorough understanding of dual quaternions, I shall show how to derive easily, almost mechanically, geometric representations and algebraic formulas in these two Clifford algebras directly from representations and formulas of the corresponding geometry and algebra for the dual quaternions. This approach contrasts sharply with many standard presentations of these Clifford algebras, where the representations of geometry, though valid, are not well motivated, and the formulas in these algebras are derived *ab initio* without any direct reference to dual quaternions.

This book is divided into two parts: Part I deals exclusively with dual quaternions; Part II is devoted to the study of two Clifford algebras associated to dual quaternions. Readers interested only in dual quaternions can read Part I and avoid altogether the material on Clifford algebras presented in Part II. Exercises are provided after each section to help the reader master the mathematical techniques as well as to further illuminate the theorems by presenting alternative proofs.

Part I is organized in the following fashion. In order to place the dual quaternions in the context of the larger framework of mathematical algebras, Section 1 begins with a brief overview of some standard algebras and their corresponding dual algebras: real numbers and dual real number, complex numbers and dual complex numbers, quaternions and dual quaternions, octonions and dual octonions. Although the dual quaternions and octonions are both 8-dimensional algebras, the dual quaternions should not be confused with the octonions: the octonions are a nonassociative division algebra, whereas the dual quaternions are an associative algebra with zero divisors.

Section 2 commences the formal study of dual quaternions with the algebra underlying dual quaternions. Since dual quaternions are represented by pairs of quaternions, we begin with a brief review of quaternion algebra, just enough to understand the extension of quaternion algebra to dual quaternions. Then we introduce the basic formulas that form the foundation for the algebra of dual quaternions: the three conjugates and the dot product.

Geometry is next. Section 3 is devoted to representations for points, vectors, planes, and lines. Incidence relations, distance formulas, and intersection formulas are covered as well: some in the main body of the text, others in the exercises. The natural representations for lines in the space of dual quaternions are Plucker coordinates and dual Plucker coordinates, so for the uninitiated we review here as well these two representations for lines. Finally in Section 3, we discuss duality in the space of dual quaternions, a crucial topic for understanding our approach to deriving the formulas for perspective and pseudo-perspective in Section 7.

One of the primary goals of dual quaternions is to provide neat, alternative representations for some of basic transformations of computer graphics: translations, rotations, reflections, and perspective projections. Section 4 deals with how to apply unit dual quaternions to compute rigid motions: translations in arbitrary directions, rotations about arbitrary lines, and reflections in arbitrary planes. Along the way we prove Chasles' Theorem that every orientation preserving rigid motion is equivalent to a screw transformation, a single rotation about a fixed axis followed by a single translation in the direction of the axis of rotation.

Unit quaternions represent rotations in 4-dimensions. In Section 5, we show that unit dual quaternions represent rotations in 8-dimensions. We also show how to renormalize dual quaternions into unit dual quaternions to avoid distortions that could occur during rigid motions due to accumulated floating point errors.

One of the advantages of the quaternion representation for rotations over the matrix representation for rotations is the ease of interpolating between two rotations represented by unit quaternions using Spherical Linear Interpolation (SLERP). Similarly, Screw Linear Interpolation (ScLERP) can be applied to interpolate between two rigid motions represented by unit dual quaternions. Section 6 is devoted to deriving the formulas for Screw Linear Interpolation.

Perspective and pseudo-perspective are crucial in computer graphics for realistic rendering. In Section 7 duality is employed together with a dual quaternion representing translation to compute perspective and pseudo-perspective. Much of this material is new and is presented here for the first time.

Quaternions can be visualized by their projections into two mutually orthogonal planes in 4-dimensions. These projections can help us to visualize the results of multiplying arbitrary quaternions by unit quaternions. These visualizations in turn help to explain intuitively how quaternions can be used to compute rotations on vectors in 3-dimensions [8,9]. Similarly, dual quaternions can be visualized by their projections into four mutually orthogonal planes in 8-dimensions. In Section 8 we make dual quaternions visible by visualizing dual quaternions as projections into four mutually orthogonal planes in 8-dimensions. We then go on to explain intuitively why unit dual quaternions can be used to compute rigid motions in 3-dimensions. In particular we see that translations in 3-dimensions are represented by shears in 8-dimensions.

Matrices are typically used to represent transformations in computer graphics [8,15]. In Section 9 we compare and contrast the advantages and disadvantages of representations and computations with matrices to representations and computations with dual quaternions. We show that each method has some relative advantages and disadvantages. We also show how to convert between these two different representations for transformations.

In Section 10 we compare some properties of quaternions and dual quaternions and present as well some of the major insights that guided our intuition for the development of the theory of dual quaternions.

We close Part I in Section 11 with a summary for easy reference of our main formulas for algebra, geometry, duality, transformations, interpolation, and conversion between dual quaternions and matrices. Since we make occasional use in Part I of cross products, we also include an Appendix with some standard identities for cross products. These identities are derived in the Exercises of Part II, Chapter 1, Section 4.1, where we show how the cross product is related to the wedge product of Clifford algebra.

Part II contains three Chapters. Chapter 1 is a concise introduction to Clifford algebra, presenting only material needed to understand the Clifford algebras associated to dual quaternions. Chapter 2 presents what I call *the plane model of Clifford algebra*, and Chapter 3 presents what I call *the point model of Clifford algebra*. Chapter 3 ends with a side-by-side comparison of the strengths and weakness, the advantages and disadvantages of these two approaches to Clifford algebra for dual quaternions.

Chapter 1 is composed of four sections. Section 1 reviews the main goals of Clifford algebra, and Section 2 introduces the basic algebraic formalisms of Clifford algebra. Section 3 recalls the three main products in Clifford algebra: the Clifford product, the inner (dot) product, and the outer (wedge) product, and Section 4 presents duality (Hodge star), a major tool in our investigation of the Clifford algebras for dual quaternions.

Chapter 2 is devoted to the plane model of Clifford algebra. The basic algebraic formulas are presented in Section 1; geometry is described in Section 2. This model of Clifford algebra is called *the plane model* because planes are the basic geometric objects in this algebra (Section 2.1): lines are represented as the intersection of two planes (Section 2.3), points and vectors are represented as the intersection of three planes (Section 2.2). Rather than introduce the representations of planes, lines, points and vectors by fiat, we show how these representations arise naturally from the corresponding representations of planes, lines, points, and vectors in the space of dual quaternions. Incidence relations, distance formulas, and intersection formulas are covered as well in the same fashion, each derived from the corresponding formulas in the space of dual quaternions. Some of these formulas appear in the main body of the text, others in the exercises. Lines are naturally represented in this model as the intersection of two planes. But two points also determine a line. We introduce duality here (Section 2.4) to show how to represent lines as the join of two points.

One of the main powers of Clifford algebra is the computation of transformations represented by rotors and versors. In the plane model, rotors correspond to translations and rotations, versors correspond to reflections. Rather than start from first principles to derive the formulas for these transformations, we show in Section 3 how these formulas arise naturally and can be derived easily from the corresponding formulas in the space of dual quaternions. Perspective and pseudo-perspective using a translation rotor together with duality are presented in Section 3.4, and are again derived almost effortlessly from the corresponding formulas in the space of dual quaternions. Alternative derivations of all these transformations that do not rely on knowledge of the corresponding formulas for dual quaternions are provided in the exercises.

In Section 4 we list five of the major insights that guided our intuition for the study of the plane model of Clifford algebra, and in Section 5 we provide a summary for easy reference of our main formulas for the algebra, geometry, and transformations in the plane model of Clifford algebra.

We close Chapter 2 in Section 6 with a comparison of the corresponding formulas in the space of dual quaternions and in the plane model of Clifford algebra, and we compare as well the advantages and disadvantages of each of these two algebras.

Chapter 3 is devoted to the point model of Clifford algebra. Here we follow the same outline as in Chapter 2. The basic algebraic formulas are presented in Section 1; geometry is described in Section 2. This model of Clifford algebra is called *the point model* because points are the basic geometric objects in this algebra (Section 2.1): lines are represented as the join of two points (Section 2.4), planes are represented either as the join of three points or as the join of a point and two vectors (Section 2.2). Once again rather than introduce the definitions of points, vectors, planes, and lines by fiat, we show how these definitions arise naturally from the corresponding definitions of points, vectors, planes, and lines in the space of dual quaternions. Incidence relations, distance formulas, and intersection formulas are covered as well in the same fashion, each derived from the corresponding formulas in the space of dual quaternions. Again, some of these formulas appear in the main body of the text, others in the exercises.

Lines are naturally represented in this model as the join of two points. We introduce duality here (Section 2.3) to show how to represent lines in this model as the intersection of two planes.

Just as in the plane model, in the point model of Clifford algebra rotors correspond to translations and rotations, versors correspond reflections. Once again, we show in Section 3 how these formulas arise naturally and can be derived easily from the corresponding formulas in the space of dual quaternions. Perspective and pseudo-perspective using a translation rotor together with duality are presented in Section 3.4, and are derived almost effortlessly from the corresponding formulas in the space of dual quaternions. Alternative derivations of all these transformations that do not rely on knowledge of the corresponding formulas for dual quaternions are once again provided in the exercises.

In Section 4 we list seven of the major insights that guided our intuition for the study of the point model of Clifford algebra, and in Section 5 we provide a summary for easy reference of our main formulas for the algebra, geometry, and transformations in the plane model of Clifford algebra.

We close Chapter 3 in Section 6 with a comparison of the corresponding formulas in the point model and in the plane model of Clifford algebra, and we compare as well the advantages and disadvantages of each of these two Clifford algebras.

Why are there two models of Clifford algebra for dual quaternions? Neither model is completely satisfying. The plane model has a natural algebra, but an exotic geometry. Planes are the basic geometric elements in this model rather than points and vectors: points and vectors are generated from the intersections of three planes. In contrast, the point model has a natural geometry: points and vectors are the basic geometric elements, and planes are generated from the joins of points and vectors. But the algebra of the point model is exotic; an infinitesimal appears in the denominator of the square of one of the basis elements. Thus infinitesimals (or equivalently limits) often appear in computations involving the point model and must be discarded at the end of these computations. I leave it to the readers to decide which of these two Clifford algebras they prefer.

I did not come upon the ideas in this book in isolation; I learned from many other authors who preceded me. I initially learned about dual quaternions by reading the paper by Kavan et al. [19]. This paper first got me excited about the potential power of dual quaternions and eventually led me to write Part I of this monograph on dual quaternions. The material on ScLERP in Part I, Section 6 is largely an amplification of material in [18]. I learned about the plane model largely from [5,11,12,13,25], and I learned about the point model from [16,21,22]. I learned about the role and importance of duality in Clifford algebra from reading a preliminary version of [5]. I could not have written the sections on perspective and pseudo-perspective without an understanding of duality. Duality also plays a central role in the plane model for expressing a line as the join of two points, and in the point model for expressing a line as the intersection (meet) of two planes. I have borrowed many topics shamelessly from several other authors both past and present, most of whom are listed in the bibliography. My apologies to anyone I have inadvertently omitted. Of course, any errors in the text are solely my own. Signs are a particular tripping point in Clifford algebra; I trust I have finally gotten all of these many signs adjusted correctly.

To conclude: I would like to thank family and friends, colleagues and collaborators, students and teachers, novices and apprentices for their ubiquitous help and inspiration.

בָּרוּךְ אַתָּה יהוה, אֱלֹהֵינוּ מֶלֶךְ הָעוֹלָם, שֶׁהֶחֱיָנוּ וְקִיְּמָנוּ וְהִגִּיעָנוּ לַזְּמַן הַזֶּה.

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Truckee, California

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Ronald Goldman is a Professor of Computer Science at Rice University in Houston, Texas. Professor Goldman received his B.S. in Mathematics from the Massachusetts Institute of Technology in 1968 and his M.A. and Ph.D. in Mathematics from Johns Hopkins University in 1973.

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Dr. Goldman has published over two hundred research articles in journals, books, and conference proceedings. He has also authored two books on computer graphics and geometric modeling: *Pyramid Algorithms: A Dynamic Programming Approach to Curves and Surfaces for Geometric Modeling* and *An Integrated Introduction to Computer Graphics and Geometric Modeling*. In addition, he has written an extended monograph on quaternions: *Rethinking Quaternions: Theory and Computation.*

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Dr. Goldman left Control Data Corporation in 1987 to become an Associate Professor of Computer Science at the University of Waterloo in Ontario, Canada. He joined the faculty at Rice University in Houston, Texas as a Professor of Computer Science in July 1990.

Part I

Dual Quaternions

1.1

Algebras and Dual Algebras

Dual quaternions are a dual algebra. An *algebra* is a vector space together with a rule for multiplication. A *dual algebra* is similar to an algebra, but the coefficients of a dual algebra are not real numbers; rather the coefficients of a dual algebra are dual real numbers (see below).

To introduce the dual quaternions, we shall briefly review here for context and contrast four related algebras and their associated dual algebras: real numbers, complex numbers, quaternions, and octonions.

1. *Real Numbers*
   In 1-dimension we have the real numbers. Geometrically each real number is associated with a unique point on a line. Multiplication is associative, commutative, and distributes through addition.
2. *Complex Numbers*
   In 2-dimensions we have the complex numbers. Geometrically each complex number is associated with a unique point in the plane—that is, with a unique pair of real numbers:

   $$(x,y) \leftrightarrow z = x + yi \quad i^2 = -1.$$

   Algebraically the symbol i commutes with every real number and multiplication distributes through addition. These rules lead to the following formula for the product of two complex numbers:

   $$(a+bi)(x+yi) = (ax-by) + (ay+bx)i.$$

   Multiplication for complex numbers is both associative and commutative.
3. *Quaternions*
   In 4-dimensions we have the quaternions. Geometrically each quaternion is associated with a unique vector in 4-dimensions—that is, with a unique pair of complex numbers or equivalently with four real numbers:

   $$(z_1, z_2) \leftrightarrow q = z_1 + z_2 j = (x_1 + y_1 i) + (x_2 + y_2 i) j \quad j^2 = -1.$$

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Algebraically the symbol j commutes with every real number and anticommutes with the complex number i. To designate the product ij, we introduce a new parameter k to represent a new direction in space and set $k = ij$. Notice that

$$k^2 = (ij)(ij) = -i(j^2)i = i^2 = -1.$$

Thus any quaternion q can be written as

$$q = x_1 + y_1 i + x_2 j + y_2 k,$$

where

$$i^2 = j^2 = k^2 = -1.$$

Again multiplication distributes through addition, so using the identities $zj = jz^*$ and $jz = z^* j$, where $z^* = x - yi$ denotes the complex conjugate of $z = x + yi$, we have the following formula for the product of two quaternions (see Exercise 5):

$$(w_1 + w_2 j)(z_1 + z_2 j) = (w_1 z_1 - w_2 z_2^*) + (w_1 z_2 + w_2 z_1^*) j.$$

Alternatively, adopting the notation of [9], which we shall use throughout this text, we can express a quaternion as a sum $mO + v$, where m is a real number, O is the point at the origin in 3-dimensions and $v = v_1 i + v_2 j + v_3 k$ is vector in 3-dimensions. Now we have the following classical formula for the product of two quaternions [9]:

$$(mO + u)(nO + v) = (mn - u \cdot v)O + mv + nu + u \times v.$$

Notice that O is the identity for quaternion multiplication. Multiplication for quaternions is associative but not commutative.

4. *Octonions*

In 8-dimensions we have the octonions. Geometrically each octonion is associated with a unique vector in 8-dimensions—that is, with a unique pair of quaternions or equivalently with eight real numbers:

$$(q_1, q_2) \leftrightarrow q_1 + q_2 l = (a_1 + b_1 i + c_1 j + d_1 k) + (a_2 + b_2 i + c_2 j + d_2 k)l \quad l^2 = -1.$$

The new symbol l commutes with every real number and anticommutes with the quaternions i, j, k. Multiplication for octonions, like multiplication for quaternions, is not commutative, but multiplication does distribute through addition. One must be careful, however, because multiplication for octonions is not associative, so using the identities $ql = lq^*$ and $lq = q^*l$, where $q^* = a - bi - cj - dk$ denotes the quaternion conjugate of $q = a + bi + cj + dk$, is not enough to define multiplication for octonions (see Exercise 6). In fact, we have the following formula for the product of two octonions [2]:

$$(p_1 + p_2 l)(q_1 + q_2 l) = (p_1 q_1 - q_2^* p_2) + (q_2 p_1 + p_2 q_1^*)l.$$

Notice the similarity of octonion multiplication to quaternion multiplication. Nevertheless, multiplication for octonions is not associative (see Exercise 6).

The real numbers, complex numbers, quaternions, and octonions in dimensions $1, 2, 4, 8$ are *division algebras*: every nonzero element has a multiplicative inverse. This progression, however, stops here: no 16-dimensional division algebra can be constructed in this way, or in any other way.

Dual quaternions, like octonions, can be represented by a pair of quaternions. Thus, the dual quaternions, like the octonions, are an algebra for an 8-dimensional vector space. Nevertheless, the dual quaternions are not the same as the octonions. Octonions are hard to work with because octonion multiplication is not associative. We shall not have any further use for octonions in this text. For readers who want to learn more about octonions—their idiosyncratic algebra and their potential geometric applications—see [2]. In contrast to the multiplication for octonions, we shall see shortly that multiplication for dual quaternions is associative, although like quaternion multiplication the product rule for dual quaternions is not commutative.

The dual quaternions can be constructed in a way similar to the octonions, but with one important difference: the rule $l^2 = -1$ for the octonions is replaced by the rule $\varepsilon^2 = 0$ for dual quaternions, where $\varepsilon \neq 0$ is a new parameter.

Just like the rule $i^2 = -1$ leads to a progression from real numbers to complex numbers to quaternions to octonions, the rule $\varepsilon^2 = 0$ leads to a progression from dual real numbers to dual complex numbers to dual quaternions to dual octonions. Here is how this progression works.

5. *Dual Real Numbers*

In 2-dimensions we have the dual real numbers. Geometrically each dual real number, like each complex number, is associated with a unique point in the plane—that is, with pair of real numbers:

$$(x, y) \leftrightarrow d = x + y\varepsilon \quad \varepsilon^2 = 0.$$

Algebraically, the new symbol $\varepsilon \neq 0$ commutes with every real number and multiplication distributes through addition. But, in contrast to the complex numbers, we have the following formula for the product of two dual real numbers:

$$(a+b\varepsilon)(x+y\varepsilon) = ax + (ay+bx)\varepsilon.$$

Notice how this product formula mimics the addition formula for fractions:

$$b/a + y/x = (ay+bx)/ax.$$

Thus the dual real numbers are a mechanism for converting addition of fractions into multiplication of ordered pairs. Multiplication for dual real numbers, like addition for fractions, is both associative and commutative. Like complex numbers, the dual real number $x+y\varepsilon$ represents the point in the plane with coordinates (x,y), but multiplication by ε no longer represents rotation by 90° (see Exercise 1). We shall shortly see that these dual real numbers serve as the coefficients for the dual quaternions.

6. *Dual Complex Numbers*

In 4-dimensions we have the dual complex numbers. Geometrically each dual complex number is associated with a unique vector in 4-dimensions—that is, with a pair of complex numbers or equivalently with four real numbers:

$$(z_1, z_2) \leftrightarrow z_1 + z_2\varepsilon = (x_1 + y_1 i) + (x_2 + y_2 i)\varepsilon \quad \varepsilon^2 = 0.$$

Algebraically the new symbol $\varepsilon \neq 0$ commutes with every complex number and multiplication distributes through addition, so we have the following formula for the product of two dual complex numbers:

$$(w_1 + w_2\varepsilon)(z_1 + z_2\varepsilon) = w_1 z_1 + (w_1 z_2 + w_2 z_1)\varepsilon.$$

Notice again how this product formula mimics the addition formula for complex fractions:

$$w_2 / w_1 + z_2 / z_1 = (w_1 z_2 + w_2 z_1) / w_1 z_1.$$

Multiplication for dual complex numbers, like addition for complex fractions, is both associative and commutative, but we shall not make any use of these dual complex numbers here.

7. *Dual Quaternions*
In 8-dimensions we have the dual quaternions. Geometrically each dual quaternion is associated with a unique vector in 8-dimensions—that is, with a pair of quaternions or equivalently with eight real numbers or four dual real numbers:

$$\begin{aligned}(q_1, q_2) \leftrightarrow q_1 + q_2\varepsilon &= (a_1 + b_1 i + c_1 j + d_1 k) \\ &\quad + (a_2 + b_2 i + c_2 j + d_2 k)\varepsilon \\ &= (a_1 + a_2\varepsilon) + (b_1 + b_2\varepsilon)i + (c_1 + c_2\varepsilon)j \\ &\quad + (d_1 + d_2\varepsilon)k \quad \varepsilon^2 = 0.\end{aligned}$$

Algebraically the new symbol $\varepsilon \neq 0$ commutes with every quaternion and multiplication distributes through addition, so we have the following formula for the product of two dual quaternions:

$$(p_1 + p_2\varepsilon)(q_1 + q_2\varepsilon) = p_1q_1 + \left(p_1q_2 + p_2q_1\right)\varepsilon, \tag{1.1}$$

where the multiplication on the right-hand side is quaternion multiplication. This product formula seems to mimic a formula for addition for quaternion fractions, but since quaternion multiplication is not commutative, the notions of denominators and common denominators for quaternions are more subtle. Nevertheless, we have the following formula:

$$p_2p_1^{-1} + q_1^{-1}q_2 = p_2\left(p_1^{-1}q_1^{-1}\right)q_1 + p_1(p_1^{-1}q_1^{-1})q_2.$$

Unlike octonion multiplication, multiplication for the dual quaternions is associative, but like quaternion multiplication, multiplication for the dual quaternions is not commutative. Notice that the coefficients for the dual quaternions relative to the quaternions are the dual real numbers. Indeed let $d_1 = a_1 + b_1\varepsilon$ and $d_2 = a_2 + b_2\varepsilon$ be dual real numbers and let $q = q_1 + q_2\varepsilon$ be a dual quaternion. Then it is straightforward to verify that

$$(d_1d_2)q = d_1(d_2q) \quad (\text{see Exercise 7}).$$

Thus, we shall see in Section 3 that, unlike quaternions, for dual quaternions dot products and norms—angles and lengths—are not real numbers but rather are dual real numbers. Moreover, although the quaternions are contained as a subalgebra of the dual quaternions—the subalgebra where the coefficient of ε is