DIVIDED SPHERES

GEODESICS & the ORDERLY SUBDIVISION of the SPHERE

SECOND EDITION

EDWARD S. POPKO
WITH CHRISTOPHER J. KITRICK

CRC Press
Taylor & Francis Group
AN A K PETERS BOOK

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To my loving family for all their support—Geraldine, Ellen, Gerald, and Amy.

Edward S. Popko

To my wife Tomoko, who has always been an integral part of the adventure.

Christopher J. Kitrick

Table of Contents

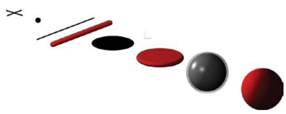
Foreword

I wonder

As one flips through each page from cover to cover of Edward Popko and Chris Kitrick's book *Divided Spheres: Geodesics and the Orderly Subdivision of the Sphere*, one will immediately realize the book will assume that rare location on your bookshelf reserved for books that will be visited most often. It has been wonderfully written for inquiring minds who are interested in discovering connections across many disciplines. It opens the mind to seeking answers to the ***I wonder*** questions all have from time to time. For those approaching the subject with little knowledge about *Divided Spheres*, introductory material is carefully explained when appropriate. For those seeking in-depth information on the different subjects and their connections, many threads run throughout the visual as well as the written text.

While growing up, one learns many lessons about spheres. There are evenings when one can point to twinkling points in the sky that seemed close enough to touch. Often those points of the Milky Way became indistinguishable from the points of light from the fireflies emerging from the ground. That point, it's *Venus*, a planet, or that one; it's a *photinus pyralis (Firefly),* that group of points is "*the Boys*," the Cherokee name for the *Pleides*. A geologist may explain the chemistry of the growth of the minerals introducing the atomic structure as related to crystal lattices of points of the atomic elements and lines of force. In the physical reality points are real, containable and measurable. The stars in the sky were spheres. The comets were described as lines as the spheres streaked across the sky. The spiral galaxies were explained as being planer made up of clusters of particles and gases. And, we live on a planet that is a volumetric spherical form. One would see them, touch them, and alter them. They exist. Those points, lines and planes are real. They are not imaginary.

However, in school it is taught that the point that was illustrated on the chalk board is imaginary and does not exist. It has zero dimensions. The teacher would place a series of points on the chalk board in a row and explain they represented a line that likewise did not exist (one dimension). Following that, the teacher would place a series of parallel lines on the chalk board and name it as a plane having two dimensions. It still did not exist. Finally, it was explained that by placing the non-existing planes on many parallel chalk boards, a solid having volume (three dimensions) would be generated.



Point, line, plane, and solid—metaphysical to physical

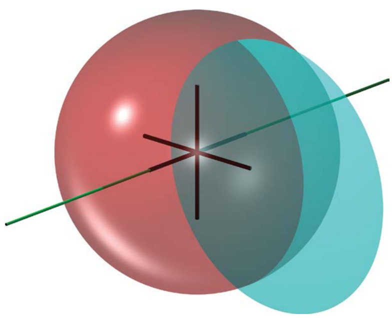
How is something made from nothing? It is all dependent on one's point-of- view which is constantly changing both individually and collectively.

A subtle lesson is learned throughout the book. As one looks more closely at the written text and the visual illustrations, separations of the two basic elements of Universe become apparent. One is the (metaphysical), the other is the physical. Both are required to make up the whole; Universe.

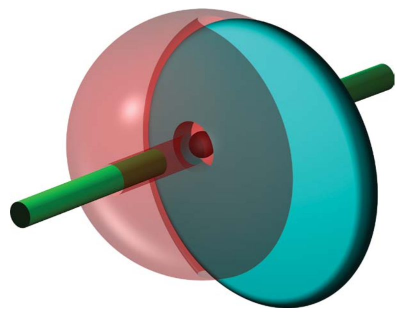
The points, lines, planes, and solids appear separated into two groups, one being abstract (metaphysical), the other real (physical).[1] Ways of communicating ideas are metaphysical and objects made up of concentrated energy are physical.

Therefore, the metaphysical points can be a location in space; lines can denote direction and distance, planes define area and solids volume. Lines, points, planes, and solids can all be at a common location at the same time.

However, physical lines, points, planes, and solids cannot be at a common location at the same time. They must accommodate each other by altering their physical form in order to share a common space.



Metaphysical at common location



Physical at common location

Early in the book is the introduction of Fuller's comprehensive *anticipatory design science* method, a unique process of combining metaphysical and physical with the scientific method.

Searching for principles operating in *Universe* crosses many disciplines but it is common to all. *Universe* is plural; the physical being *energy* and the metaphysical being *the rules of behavior.*

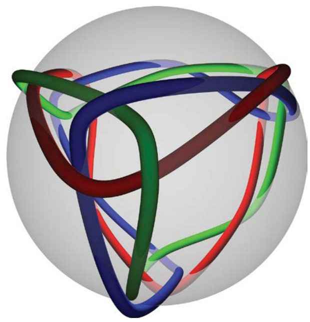
Without the whole the parts would not be.
    Without the parts the whole could not be.

As the physical and metaphysical make connection, their relationship initiates a transformation of energy to take on a form that can be measured.

The physical has need for the metaphysical for instruction.
    The metaphysical needs the physical to give instruction.

[1] All images and photographs courtesy of Joseph D. Clinton.

While exploring the *anticipatory design science* method, (discussed in Chapter 2,) through the eyes of atomic physics, an ***I Wonder*** view of the fractal patterns of atoms might present itself. Instead of the common atomic model used to describe the atom one may ask; is it possible that a *fractal-link/knot* model may explain it more completely? [2]



Three-link knot of a spherical tetrahedron

One of the exciting aspects of *Divided Spheres* is the complimentary use of metaphysical visual language and written language to describe physical reality.

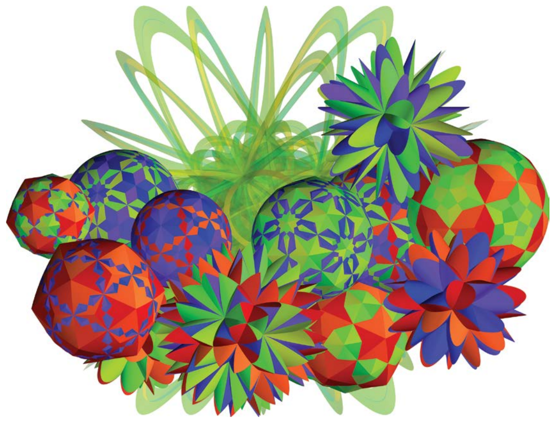
These two languages emerged from the need to explain individual points-of-view of the physical and metaphysical aspects of one's environment. The meaning of the elements of language evolved from a personal point-of-view to a collective point-of-view common to the understanding of the individual point-of-view. Yet the true meaning of a symbol used in our language today is still only known by the individuals using the word. And, this changing understanding comes from the experiences one draws from exposure to unique observations of one's environment plus one's understanding of others experiences. As Edwin A. Abbotts 1884 book *Flatland: A Romance of Many Dimensions* so aptly described. "*It all depends on one's-point-of-view*."[3]

After the initial spellbinding ***I wonder*** experience one is faced with the decision to read *Divided Spheres* from cover to cover or to follow the threads provided from the beginning. That is where the choice must be made. For those who are new to the subject, start at the beginning. But, don't be afraid to follow a subject thread and deviate from your original path. The ***I wonder*** path will be greatly rewarding. For those who wander through the book by starting in the middle, there will be a return to the beginning, looking for inspiration many more times in the future. *Divided Spheres* is a resource book that will help satisfy the yearning for answers to ***I wonder*** questions regardless of discipline or inquiry. The following examples may come to mind while traveling through the book.

In the early 1970s, an ***I wonder*** showed itself again. I began exploring polyhedral forms from the intersection of cylinders based on the symmetry axes of the regular and semi-regular polyhedra. During the same period, Charles E. Peck became interested in the forms.[4] After many discussions, we both expanded our ideas and began

[2] (Briddell 2012)
[3] (Abbott 1884)
[4] (Peck 1995)

creating computer drawings of new polyhedron forms. In December 2003 Paul Bourke published many of the same forms on his website.[5] In 2004, I published a paper expanding the family of forms to two additional classes: the dual forms and the stellated forms.[6] These forms have taken on the name *Polycylinderhedron*.



Class I, Class II, Class III polycylinderhedron

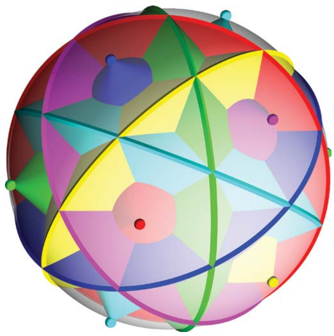
An art expression, like those discussed in Chapter 3, emerged from a series of thoughts revolving around the geometry of the *Polycylinderhedron*.

The *Polycylinderhedron* exemplifies the transformation from the metaphysical to the physical for the axis of the polyhedron chosen to become a *Polycylinderhedron.* The metaphysical axis, as a line, is given a diameter and becomes a physical reality, a cylinder. The metaphysical line of accommodation of two adjacent cylinders is an elliptical path of an energy transformation of adjacent metaphysical planes. The art form evolved after many iterations of applying metaphysical rules of behavior to the physical realities of energy and the "*point of origin of all things*" emerged as the *Radix Universum*.

Radix Universum sculpture

[5] paulbourke.net/geometry/.
[6] (Clinton 2004)

Divided Spheres, the first edition, was a primary ***I Wonder*** inspiration to add to the collection of new polyhedron forms: *Polyconehedrons*. The *Polyconehedrons* are an extension of the family of *Polycylinderhedrons.* The symmetry axis of any polyhedron can define a cone axis with its base circle being a great circle of the sphere. The vertex of the cone is located on the surface of the sphere. It is proposed that by using the same mathematical operations defined for the family of *Polycylinderhedrons,* three unique classes of forms will emerge. Gary Doskas describes them as *Polyconix*.[7]



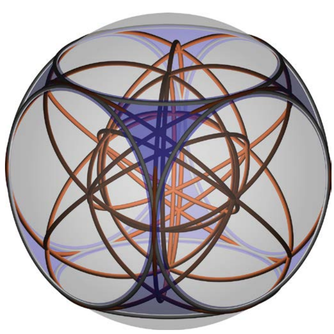
Twelve-cone polyconehedron

After discussing the regular and semi-polyhedron, an early ***I wonder*** project was introduced to a group of students in Mr. Boles class at Joplin Junior College, Joplin, Missouri in 1959. He assigned to the class to use the polyhedron as inspiration, to do something with the objects discussed.

Growing up in a mineral rich part of the Midwest, there was a fascination for the crystals collected and their polyhedron forms. Instead of decorating models of the polyhedron, a different approach was taken.

Weaving lesser circles of a sphere, using spring steel wire, the symmetry of the faces of any spherical form of polyhedron could be illustrated. By penetrating the sphere with a circle of the same material and diameter representing the edges of the polyhedron, something unusual occurred. The structures can collapse into a single circle.[8]

Over the years a family of new forms has emerged. *Divided Spheres* could become a source for continuous inspiration that could lead one to occasionally add new forms to the collection.



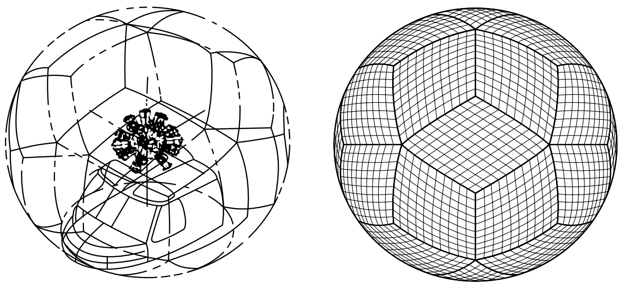
Twelve collapsible rings inside of a spherical Hexahedron

[7] (Doskas 2011), see Section 1.7 Other Ways to Use Polyhedra.

[8] (Clinton 1999)

Chapters 2 and 10 are major additions to this second edition of Edward Popko and Chris Kitrick's book *Divided Spheres*. They are excellent presentations, on Chris Kitrick's developments stemming from the following two ***I wonder*** thoughts.

In 1998, a change was taking place in the simulator field. The cathode ray tube (CRT) projectors were beginning to be phased out and replaced by solid state (LED) video projectors. In order to reshape an image for projection onto a spherical simulator screen with the CRT projectors, the image array was reshaped in the CRT. The image array was preserved. It was reshaped and there was little loss in image brightness and resolution. With the new LED projectors, however, the horizontal and vertical image array would be cropped with image reshaping. There was loss in the image array and both brightness and resolution was diminished. Being familiar with tessellations on a sphere, the ***I Wonder*** impulse was put to the test. Could a spherical surface be divided into diamonds having equal edges? If so, the image might be able to be reshaped optically for each area of the spherical projection screen. The result was a patent *A Method of Tessellating a Surface.*[9]



Rhombic Triacontahedron LED projector optics and projection screen

While studying Buckminster Fuller's geodesic domes and tensegrity structures, Dick Boyt had an ***I Wonder*** vision. As a professor at Crowder College, Neosho, Missouri, he wrote a paper describing his calculations for *Rotegrity* spheres.[10]

At the *Soft Energy Fair*[11] *in 1978,* while demonstrating his Rotegrity spheres, Dick Boyt was introduced to Fuller. The two began a discussion making Bucky thirty minutes late in giving his keynote lecture. Dick discovered the "*Rotegrity*" geometry based on Bucky's "*turbining*" effect at the tensegrity vertexes, see Figure 10.5. *Fuller's Central Angle Turbining.*

Dick Boyt and rotegrity sphere at the 1978 Soft Energy Fair

[9] (Clinton 2004).
[10] (Boyt 1991).
[11] Workshop and conference held at Amherst, MA, 1978.

The book is a remarkable collection of history, methods for tessellating the sphere and applications across disciplines. It is an inspiration to others to discover new forms and applications to add to the collection in the future editions. It is a must have, must use, must share addition to one's very special location on the bookshelf.

A sphere has no beginning, has no end. It is a duality. It is both an abstraction and a reality. It can be perceived as position lacking size, only location. Or, it is a totality as the whole of all things. It has a duality of inside and outside or both neither. If perceived as position, it must have another sphere or location cannot be measured. But to be measured, it must have size. If it has size it is not an abstraction, but a reality.

A sphere can be an imagination or thought as part of the world of abstractions where points are only positions in space. The lines can intersect at those positions in space, and planes that only exist to describe areas that can blossom into physical reality. Or, a sphere can take on form in the world of physical reality where micro points, lines, planes, and macro points all have dimensions where they are not allowed to share a common space, but they can only accommodate each other.

Joseph D. Clinton

Preface

This book summarizes the key spherical subdivision techniques that have evolved over 70 years to help today's designers, engineers, and scientists use them to solve new problems.

I became enthusiastic about geodesic domes in the mid-1960s, through my association with Buckminster Fuller and his colleagues. I was an architectural intern at Geometrics, Inc., a Cambridge, Massachusetts, firm whose principals, William Ahern and William Wainwright, had pioneered geodesic radome designs. During my internship, Ahern and Wainwright were collaborating with Fuller and Shoji Sadao on the US Pavilion dome at Expo'67 in Montreal. I was immediately drawn to the beauty and efficiency of designs based on geodesic principles.

I first met Fuller at Geometrics. He surprised me with his stature and energy, his easy rapport with audiences, and his discourse on design. It was, however, a struggle to understand him. He had his own language, which combined geometry with physics and design into a personal philosophy he called *Synergetics*. Later, I visited the original geodesics "*skunk works*"—Synergetics, Inc. in Raleigh, North Carolina. At Synergetics, Jim Fitzgibbon, T. C. Howard, and others showed me projects from their start-up years in the early 1950s, as well as their latest work. They were making history with their innovative spherical designs. Duncan Stuart, another early member of the firm, had made his own history in the early 1950s, when he invented a method (*triacon subdivision*) that made geodesic domes practical to build. The spherical grids this method produced had the fewest number of different parts of any subdivision technique. It is still one of the best gridding methods and it is detailed in this book.

More recently, I met Manuel Bromberg and Chizuko Kojima. In the late 1940s, Bromberg, along with Jim Fitzgibbon, Duncan Stuart, and other faculty members at North Carolina State University, formed the first geodesic start-up company, Carolina Skybreak. When architectural commissions materialized, the original company evolved into two others, Synergetics, Inc. and Geodesics, Inc., and Raleigh quickly became the epicenter of geodesic design. Kojima was a "*computer*" for Geodesics in the late-1950s. In those days, "*computer*" was a job title for someone who calculated. Kojima's job was to calculate hundreds of angles and grid coordinates for early dome projects—by hand.

Many years ago, I met Magnus Wenninger, a monk, teacher, mathematician, and polyhedral model builder *par excellence.* Wenninger has built thousands of models, and his classic books on polyhedral and spherical models have inspired generations of schoolchildren, artisans, and mathematicians. He showed me how to work with some lesser-known skewed spherical subdivisions; the main techniques are covered in this book.

My professional work with Computer Aided Design (CAD) systems for architects and engineers led me to more applications of geodesics. My hobbies in sailing and celestial navigation with a sextant sharpened my understanding of spherical trigonometry and improved my skills in geodesics. I had always been interested in world maps and their varied graphic projections as well as the delicate spherical structures found in microorganism exoskeletons. But what surprised me most were spherical designs in things like ocean-bobbing fish pens, panoramic photography, underground neutrino observatories, and virtual-reality simulators for the military—all of which used geodesic geometry. Geodesics appeared again in virus research, astronomy catalogs, weather forecasting, and kids' toys. But the most unexpected geodesic application I found was in the innovative design and layout of golf ball dimples. Dozens of different dimple patterns resembled small geodesic domes. These applications and many more are detailed in a later chapter.

Divided Spheres

For designers, the principles of spherical design are, at first, counterintuitive and somewhat obscured by a unique vocabulary. Chapter 1, "*Divided Spheres*," highlights the major challenges and approaches to spherical subdivision. The chapter states the design objectives that many designers use. We will meet them in later chapters. Key concepts and terms are introduced.

Buckminster Fuller's pioneering work in the late 1940s and the research and development of his colleagues in the 1950s led to many of the techniques we use today. Chapter 2, "*Bucky's Dome*," examines how Fuller's design cosmology, *Synergetic Geometry*, was first applied to cartography and then to geodesic domes. The interplay between Fuller and key associates who worked out practical solutions to geometry and construction problems is particularly important. Many of today's subdivision techniques were developed at this time. This chapter contains the first commentary on how geodesic domes were originally calculated.

Chapter 3, "*Putting Spheres to Work*," provides a brief glimpse of the wide diversity of today's spherical applications in fields like biology, astronomy, virtual-reality gaming, climate modeling, aquaculture, supercomputers, photography, children's games, and sports balls. If you are a golfer, you will enjoy seeing how manufacturers use spherical design and unique dimple patterns to maximize player performance.

Spherical geometry is quite different from Euclidean geometry, though they share common principles. Chapter 4, "*Circular Reasoning*," develops circular reasoning with points, circles, spherical arcs, and polygons. *Spherical triangles* are the most common *polygon* created when subdividing spheres. We look closely at their properties to establish an understanding of their areas, centers, type, and orientation.

It is easy to evenly divide the circumference of a circle on a computer to any practical level of precision. It's not so easy to evenly subdivide spheres, computer or not. Chapter 5, "*Distributing Points*," focuses on the challenge of evenly distributing points on a sphere and, in so doing, defines specific design conditions that this book will develop.

Spherical polyhedra offer a convenient starting point for subdivision. Chapter 6, "*Polyhedral Frameworks*," describes useful Platonic and Archimedean solids. We are particularly interested in their *symmetry* properties and their spherical versions.

Many readers will be surprised by the rich and varied spherical subdivisions that golf ball dimple patterns demonstrate. Chapter 7, "*Golf Ball Dimples*," references some of the famous US government patents, showing the golf ball industry's amazing diversity of designs and illustrating why a seemingly small thing, such as dimple patterns have become a key part of a multimillion-dollar sports ball industry.

Divided Spheres presents six classic subdivision techniques grouped into three classes. In Chapter 8, "*Subdivision Schemas*," each technique is presented in the same step-by-step format. These techniques are flexible and apply to many different spherical polyhedra. Depending on the designer's requirements, certain combinations of spherical polyhedra and subdivision techniques may be more appropriate to use than others. Thus, one objective of this chapter is to show how various techniques affect the final subdivision.

With so many design choices, it is natural to ask which combination is best. Of course, the answer depends on the application. Chapter 9, "*Comparing Results*," shows how to cut through all the design variables and select a combination that best fits your design requirements. This chapter relies extensively on graphics rather than taking a statistical approach. This book is also the first to use graphical analysis, such as Euler lines and stereographics, to highlight the subtle differences between subdivision techniques.

Chapter 10, authored by Christopher Kitrick, introduces a new concept in spherical subdivision—self-organizing grids. These grids include honeycomb hexagonal grids and grids that are a combination of hexagons and triangles called rotegrities or nexorades. The self-organizing concept is appealing because it can precisely solve some very challenging configurations that traditional subdivision techniques outlined in Chapter 8 cannot.

Three primers are included in the appendices: Stereographic Projection, Coordinate Rotations, and Geodesic Math. Stereographic projection is a graphical technique for making 2D drawings of the surface of the sphere. Stereograms appear in several sections of the book; we use them extensively in Chapter 9 to compare the results of subdivision methods in Chapter 8. Appendix A explains the theory behind this classic graphical technique. Each of the subdivision techniques explained in this book create grids or point distributions that covers only a small part of the sphere. This geometry must be replicated and rotated locally and then about the entire sphere to cover it without overlaps or gaps. Appendix B explains how this is done. Appendix C—Geodesic Math shows useful computer algorithms for each of the subdivision and optimization methods presented. Small computer code snippets suggest how they might be implemented.

Graphic Conventions

This book is about 3D spherical geometry. The foundations for the most uniform subdivisions are based on the Platonic and Archimedean solids—forms discovered by the Greeks and made from combinations of *regular polygons* (polygons in which all edges and angles are equal), such as equilateral triangles, squares, pentagons, or hexagons. These 3D forms have pure "*theoretic*" definitions, but we use a number of graphic conventions in this book to illustrate and explain their features.

Figure 1 shows six different graphic conventions for the common *dodecahedron* (12 faces, 30 edges, and 20 vertices). Each graphic is the same scale and is shown from the same viewpoint. If placed on top of one another, the vertices of all six figures would be coincident. Each convention emphasizes a different aspect of the solid.

The wireframe (a) and planar versions (d) are the most familiar and come closest to the pure definition of a planar dodecahedron. Its spherical cousin (b) makes the differences in volumes and face angles apparent. The holes in version (c) reveal the relationships of interior faces, show what lies behind the form, and draw attention to

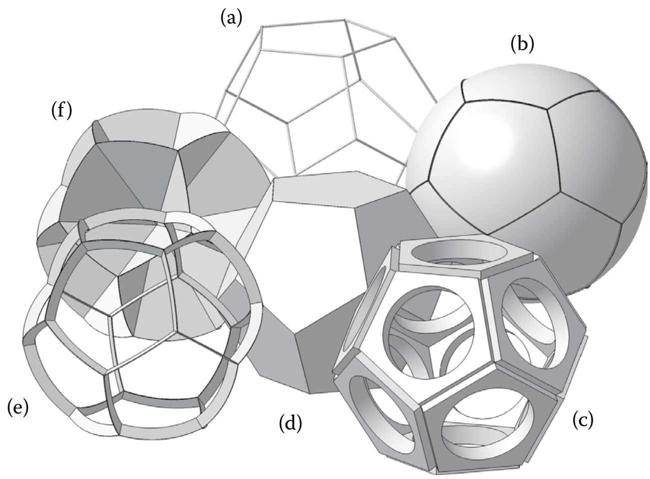


Figure 1. Polyhedron graphic conventions used in this book.

the centers of the faces (there is more than one type of center). Thick faces emphasize the angles (dihedral) between adjacent faces. Graphic convention (e) emphasizes the spherical arcs and planes of the dodecahedron's edges, while the graphic convention in (f) is useful when examining the surface angles where arcs meet. The planes and arcs are part of the volume encompassed by the form. These six graphic conventions appear throughout this book.

You have already seen another graphic convention used in this book - emphasized key words in italic and blue font. They appear near where they are defined and are important terms about spherical geometry and subdivision. You will find them throughout this book and in most of the references in the Bibliography.

I have covered the basics of what I have learned over the years in my own spherical work. I hope that *Divided Spheres* makes it easier for you to understand the principles of spherical subdivision and to develop your own designs. Spherical applications are as limitless as they are beautiful.

—Edward S. Popko
Woodstock, New York

Acknowledgments

Ed Popko would like to thank the following persons and institutions.

This book is a tribute to the work of many people. In a way, it does not seem fair that only two names are one on the cover. It is a pleasure to recognize those who contributed so much and supported me along the way.

I want to give a special thanks to Chris Kitrick for being part of this edition of Divided Spheres. In the years since the first edition, new fronters have opened for spherical subdivision and Chris is one of the pioneers. It's with great pride that he shares his research and pioneering work in self-organizing grids (Chapter 10).

No one has been more patient or taught me more about putting a book together than Jim Morrison. I cannot say enough about his enthusiastic support. Jim was my sounding board for new ideas, an instructor *par excellence* on PostScript graphics programming, and my mentor on manuscript preparation. His book, *The Astrolabe,* paved the way for mine, and I am grateful for all the advice and experience he shared with me.

I have been fortunate to have known and worked with experts in polyhedral geometry and spherical design. The late Father Magnus Wenninger, OSB, stands out. He has built thousands of models over the years and his books, including *Polyhedron Models, Spherical Models,* and *Dual Models,* have made these subjects accessible. I am grateful for his friendship and for the advice he offered throughout this project. The polyhedron chapter and skewed grid sections owe a great deal to him.

Buckminster Fuller's invention of the geodesic dome was the biggest stimulus for spherical subdivision research and development. This book samples some of his amazing work and the work he inspired in others. I was fortunate to have met him on several occasions and to have worked in one of his affiliate offices in the 1960s. Those experiences stayed with me and greatly enriched my life. I also met many of Fuller's associates and early geodesic pioneers. I am particularly indebted to Manuel Bromberg, Jim Fitzgibbon, T. C. Howard, Bruno Leon, Chizuko Kojima, Shoji Sadao, William Ahern, William Wainwright, and Duncan Stuart. All of them were part of early Fuller startups like Skybreak Carolina, Synergetics, Geodesics, and Geometrics. Each taught me different aspects of geodesics and the early history of the dome development. Manuel Bromberg and Bruno Leon shared their personal experiences with Fuller in the late

1940s early 1950s. These were the early days of research and development and Fuller's first major dome customer, the Marine Corps. I am fortunate to be able to include some of their accounts in this book. Duncan Stuart revolutionized spherical subdivision with his *triacon method*. He taught me how the method worked and encouraged me to write one of the first computer graphics programs based on it in the 1960s. A section of this book outlines Stuart's method and shows how it made domes practical to build. Stuart's method is used today in many applications unrelated to geodesic domes. Chizuko Kojima was a "*computer*" (a job title) at Geodesics in the 1950s. She gave me a sense of how exacting and tedious it was to calculate the geometry of Fuller's early domes *by hand.* William Ahern and William Wainwright pioneered radomes. They taught me the principles of *random subdivision.* Both have unique patented subdivision systems. I am grateful for every lesson.

Important records of Fuller's life and geodesic dome projects are cataloged in the R. Buckminster Fuller Papers collection at Stanford University's Department of Special Collections and University Archives. Staff librarians were particularly helpful in guiding me through this enormous collection and in securing rights to use and reference material. John Ferry, collections curator for the Estate of Buckminster Fuller, has been a constant supporter of my research. I am grateful for his help.

Joseph Clinton wrote many of the early primers on geodesic geometry and he developed several unique subdivision methods. He also created the class system we use today to characterize different spherical grid systems. I'm much indebted to him for his pioneering work and personal support in making this book happen. It's a privilege to have him present this work in the *Foreword*.

I'm grateful to Christopher Fearnley (aka CJ) for his help in content proofing Divided Spheres. His leadership of the *Synergetics Collaborative* (SNEC)[1], his activism in promoting the ideas of Buckminster Fuller and his professional consulting in Information Technology make him uniquely qualified to review every topic of *Divided Spheres*.

Mathematics is precise and clarity is important. I am grateful to Professors David W. Henderson, Cornell University Mathematics Department; Gregory Hartman, Virginia Military Institute Mathematics and Computer Science Department; and to Taylor and Francis reviewers for their advice and suggestions in clarifying terms and improving the text.

Many of the CAD images in this book were created with SOLIDWORKS and CATIA, both powerful 3D modelers from the French multinational Dassault Systèmes. Ricardo Gerardi taught me how to write CATIA macros that automatically create 3D geometry. Without his instruction, I would never have been able to create the number of detailed illustrations I have created for this book. Dassault Systèmes and Alain Houard facilitated my access to CATIA, without which I could not have created many of the illustrations.

Good editing clarifies a subject and makes it easier for readers to understand. This is particularly important in a technical primer like this one. Bernadette Hearne did an amazing job of editing. She tolerated many rough drafts and is an artist with scissors and paste pot. Valerie Havas and Lynne Morrison picked up where my high school English teacher left off and showed me the value of consistent style. Michael Christian provided invaluable advice about publishing. I greatly appreciate his positive support along the way.

Like most writers, I greatly underestimated how much time and effort it would take to write this book. At times, domestic projects and family were ignored. My family

[1] SNEC is a non-profit organization dedicated to bringing together a diverse group of people with an interest in Buckminster Fuller's Synergetics, See www.synergeticscollaborative.org

was patient and tactful; never once did they ask if I was done yet. My wife, Geraldine, daughter Ellen (aka Turtle), and son Gerald acted as a fan club that kept me going. Clearly, this book would not have happened without their support. Love to them all.

Christopher Kitrick would like to thank the following persons and institutions.

First and foremost, I want to thank Ed Popko for inviting me to contribute new material for the second edition of his wonderful book, *Divided Spheres*. I still remember being inspired when I first saw Ed's first book, *Geodesics*, in the late 1970s with its straightforward attention to design and construction details. Still a wonderful primer on the subject. The second edition of *Divided Spheres* has been a very fruitful collaboration and the results are a product of Ed's helpful feedback and editorial support. I first met Ed in the winter of 2014 at a Synergetics Collaborative conference in Rhode Island after having been personally invited by Joe Clinton to present some of my work and where Ed was also presenting so thanks to Joe for setting this in motion.

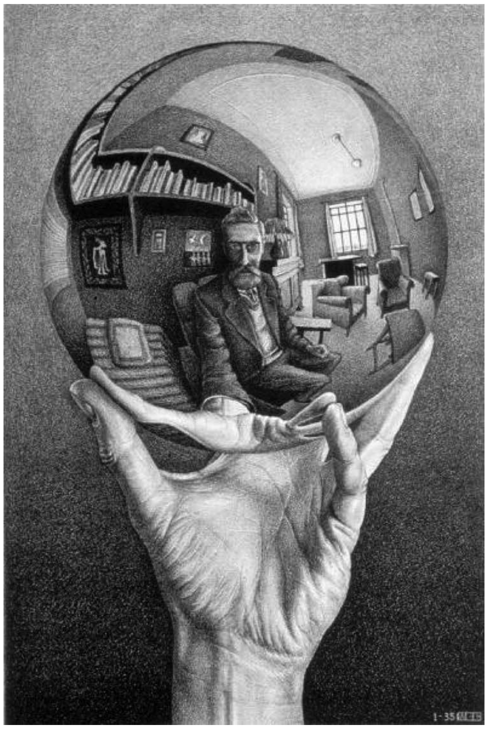
Buckminster Fuller was a pivotal influence on my work on spherical subdivision. Having had the rare opportunity to being a college intern for three years in his office in the late 1970s working on a myriad of projects and interacting directly with him for three years was all that was needed to keep me motivated for almost 40 years to research geometric challenges. Being in the presence of someone, already in their 80s, so motivated to continue to explore and explain all that he knew and learned cannot be overestimated. Being in that office exposed me to many who were in Bucky's circle and gave me the opportunity to develop long-term relationships with Joe Clinton, Fr. Magnus Wenninger, Prof. Koji Miyazaki, Shoji Sadao, Ed Applewhite, and Prof. Tibor Tarnai. In addition, I was able to work hand in hand with another intern Robert Grip, whose brilliance and tenacious model building provided a special working environment that forced me stay on my toes to learn and keep up.

While books like *Domebook* exposed the public to geodesic domes, it was Joe Clinton's NASA funded work that laid the foundation for detailed mathematical approaches to tessellations and gave me the impetus for expanding many of the original concepts that I built upon to further my own research. Joe's work over many years continues to provide a foundation for new ideas and explorations and I owe him a debt of gratitude. He has been kind for many years.

Fr. Magnus Wenninger and Tibor Tarnai are both important figures in my own development, both being key figures in modern work in polyhedra, geodesics, and structural topology. They had worked together on a number of projects and I was fortunate to have met them both separately and together. Fr. Magnus follows in the footsteps of so many monks who have throughout history done so much to enlarge our horizons on beauty of the elements of the real world. Prof. Tarnai has brought engineering and mathematical rigor to structural topology as well a keen historical perspective on polyhedra that and has been someone who has reviewed my work over the years.

My family has been steadfast in their support of my efforts both during the work on this book and for so many years before. I have been so fortunate not only to have met my wife, Tomoko, while working for Bucky, but to have had her meet Bucky and be directly involved with some of the projects that ensued. My children Francis, Ian, and Eileen have also been supportive of someone who continues quixotic pursuits in geometry.

The rendering software, *DisplaySphere*, used for all 3-dimensional images in Chapter 10, was developed specifically to view and capture images of spherical geometry, is available through GitHub as open source.



1 Divided Spheres

We owe the word *sphere* to the Greeks; *sphaira* means ball or globe. The Greeks saw the sphere as the purest expression of form, equal in all ways, and placed it at the pinnacle of their mathematics. Today, the sphere remains a focus of astronomers, mathematicians, artisans, and engineers because it is one of nature's most recurring forms, and it's one of man's most useful.

The sphere, so simple and yet so complex, is a paradox. A featureless, spinning sphere does not even appear to be rotating; all views remain the same. There is no top, front, or side view. All views look the same. The Dutch artist M. C. Escher (1898–1972) ably illustrated this truth in his 1935 lithograph where the artist holds a reflective sphere in his hand as he sits in his Rome studio.[1] It's easy to see that even if Escher were to rotate the sphere, the reflected image would not change. His centered eyes could never look elsewhere; the view of the artist and his studio would be unchanging. His sphere has no orientation, only a position in space.

The sphere is a surface and easy to define with the simplest of equations, yet this surface is difficult to manage.[2] A sphere is a closed surface with every location on its

[1] 1935 Lithograph, M. C. Escher's "Hand with Reflecting Sphere" © 2009 The M.C. Escher Company Holland. All rights reserved.

[2] The equation of a sphere is very simple. For a sphere whose three-dimensional Cartesian origin is (0, 0, 0), a point on the sphere must satisfy the equation $r = \sqrt{x^2 + y^2 + z^2}$.

surface equidistant from an infinitely small center point. A mathematician might go one step further and say the sphere is an unbounded surface with no singularities, which means there are no places where it cannot be defined. There are no exceptions.

The sphere is unusual, it has no edges, and it is *undevelopable*. By undevelopable, we mean that you cannot flatten it out onto a two-dimensional plane without stretching, tearing, squeezing, or otherwise distorting it.[3] You can test this yourself by trying to flatten an orange without distorting it. You can't without stretching or tearing it no matter how small or large the orange is.

Any plane through the center of the sphere intersects the sphere in a *great circle*. For any two points, not opposite each other on the sphere (opposite points are called *antipodal points*), the shortest path joining the points is the shorter arc of the great circle through the points, and this arc is called a *geodesic*. Geodesics are the straightest lines joining the points; they are the best we can do since we can't use straight lines or *chords* in three dimensions when we measure distance along the sphere. The length of a geodesic arc is defined as the distance between the two points on the sphere.[4]

Like the *equator* on a globe, any great circle separates the sphere into two *hemispheres*. A plane not passing through the origin that intersects the sphere either meets the sphere in a single point or it intersects the sphere in a *small circle*, also called a *lesser circle*. Small circles separate the sphere into two *caps*, one of which is smaller than a hemisphere and resembles a contact lens. We use spherical caps later in the book to compare spherical subdivisions.

Great circles play a dual role when subdividing spheres. Points define great circles and great circle intersections create more points. When we start to subdivide a sphere, we typically start with just a few points that define a relatively small number of great circles. We define more points by intersecting various combinations of the great circles. The new points derived from intersections can now be used to define more great circles, and the cycle repeats. You can see already that we are going to make great use of the dual role of great circles when we describe the various techniques and their resulting grids in Chapter 8. The difference in techniques is primarily how we define the initial set of points and great circles, and what combinations of great circles we select to intersect to define additional points.

Spherical polygons are polygons created on the surface of a sphere by segments of intersecting great circles. Spherical polygons demonstrate other differences between spherical and plane Euclidean geometry. The sides of spherical polygons are always *great circle arcs*. As a result, two-sided polygons are possible. Just look at a beach ball or slices of an orange or apple for examples. These two-sided polygons are called *lunes* or *bigons* (bi instead of polygons). Spherical triangles are also different. They can have one, two, or three right angles. And one of the oddest differences between spherical and Euclidean geometry is that there are no *similar triangles* on a sphere! They are either *congruent* or they are different. In plane Euclidean geometry, three angles define an infinite number of triangles differing only by the proportional length of their sides. But on a sphere, triangles cannot be similar unless they are actually congruent.

---

[3] The Swiss mathematician and physicist Leonhard Euler (1707–1783) did intense research on mathematical cartography (mapmaking). In a technical paper, "*On the geographic projection of the surface of a sphere*," published by the St. Petersburg Academy in 1777, he proved that it was not possible to represent a spherical surface exactly (preserving all distances and angles) on a plane. The surface of a sphere is undevelopable.

[4] The word geodesic comes from geodesy, the science or measuring the size and shape of the Earth. A geodesic was the shortest distance between two points on Earth's surface, but today, it is used in other contexts such as geodesic domes.

The sphere is a challenging but fascinating place to work. With all of these differences, how do we work with spheres and what Euclidean principles, if any, can we use?

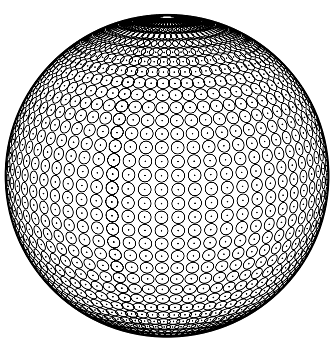
1.1 Working with Spheres

A sphere can be any size at all. Its radius, r, could be any distance and range from subatomic dimensions, to the size of a playground dome, to light-years across the observable universe. To make spheres easier to work with when their radius could be anything, we treat r as a positive real number and make it equal to one unit. A sphere with a *unit radius* is called a *unit sphere*. One what? Are we talking about 1 mile, 1 foot, 1 inch, or one anything? Yes, to all these questions. Unit spheres, ones where $r = 1$, are easy to calculate, and any spherical result is easily converted to an actual dimension such as miles, feet, inches, or whatever. Angles do not have to be converted; they are used as-is no matter the size of the sphere. However, for distances, lengths, and areas, we need to convert unit sphere dimension into the true radius of the sphere our application needs.

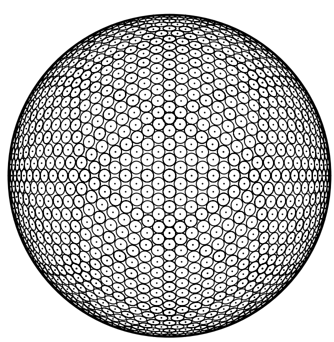
1.2 Making a Point

Although the sphere is an infinite set of points all equally distant from its center, practical design applications require us to locate specific points on the surface that relate to the design we have in mind. Locating points requires us to define a reference system and orientation for our work. In the simplest case, placing the sphere's center at the center of the *Cartesian axis system*, the familiar *xyz*-coordinate system we use most often, we have defined at least six special reference points on the sphere's surface, one for the positive and negative points where each coordinate axis intersects the sphere's surface. In so doing, we have also adopted standard design conventions where we can refer to a top, bottom, side, or front, if we need to. We are off to a good start. Out next challenge is to define points on the surface that help us with our design. So how do we do this?

It's natural to think of points on a sphere like points around a circle. While it is easy to *evenly* distribute any number of points around the circumference of a circle, doing so on the surface of a sphere is actually quite difficult and certain numbers of points are impossible. Figure 1.1 gives us a sense of the problem. In Figure 1.1(a), an equal number of points are arranged around rings, or lesser circles, similar to the lines of latitude on Earth. Small circles surround each point to give a visual indicator of their spacing. We see that as the lesser circles get closer and closer to the sphere's two poles,



(a)



(b)

Figure 1.1. Distributing points on a sphere.

the points around them are getting closer and closer together; the circles surrounding them are overlapping. At the two poles, dozens of points are nearly superimposed. In this subdivision, the points are not uniformly distributed at all.

In Figure 1.1(b), we see a dramatically different distribution for the same number of points as in (a). Clearly, they are more uniformly spaced with a consistent symmetry and appearance. Notice that while some circles touch, none overlap. A grid connecting each point with its nearest neighbors would produce consistently shaped triangles, whereas in the first layout they won't. These two layouts illustrate our geometric challenge when we subdivide a sphere—how do we define arrangements like the one in (b) that give us freedom to have more or fewer points that allow us to make different shapes on the sphere's surface that meet our design requirements?

We know that on a circle, we can evenly space an arbitrary number of points—say, 256, or 2,011, 9, or 37 points. In a sense, this logic is what was used in Figure 1.1(a). So why can't we evenly space an arbitrary number of points and get the result shown in (b)? The answer lies in the number of points we try to distribute and how symmetrical we want their arrangement to be. Let's look at each of these related design issues and how we intend to address them.

1.3 An Arbitrary Number

Physicists and mathematicians have theorized how to distribute an arbitrary number of points on a sphere for some time. And today, with the help of computers, there are some partial solutions. One approach, familiar to physicists, sees the point distribution problem much the way they see the way particles interact. One technique uses particle repulsion to distribute points. It beautifully illustrates the problem of trying to evenly space an arbitrary number of points on a sphere.

Particles with the same charge (positive or negative) repel each other with strength inversely proportional to the square of their distance apart. The closer they are together, the stronger they repulse each other. This natural law, called the *inverse-square law*, can be exploited to subdivide a sphere when particles represent points that can be connected to form spherical grids.[5]

If you cannot relate to charged particles, think of hermits instead of particles. Each hermit is standing on the Earth and each wants to get as far away from everyone else as they can.

Imagine a given number of particles randomly distributed over the surface of the sphere, say 252 of them. Figure 1.2(a) shows them surrounded by the same diameter sphere (white), each centered on the surface of an inner sphere (transparent gold). One particle's position is fixed, but all others can move on the surface relative to the fixed one. A computer program simulates the effect of the particles repulsing each other. For each particle, the strength and direction of repulsion of every other particle acting on it is found using the inverse-square law. After every particle has been evaluated, they are allowed to move a little in the direction found so that the repulsion forces acting on them are lessened. Although the particles move a little at this stage, they always remain on the surface of the sphere. The computer once again simulates the effect of the particles repulsing each other and again, they are allowed to reposition a little to decrease the forces acting on them. In each cycle, all the particles are closer and closer

[5] Inverse-square law—a physical quantity like the intensity of light on a surface or the repulsion strength of two like-charged particles is inversely proportional to the square of the distance from the source of that quantity. For example, if the distance to a source of light is doubled, only one quarter of the light now reaches the subject.

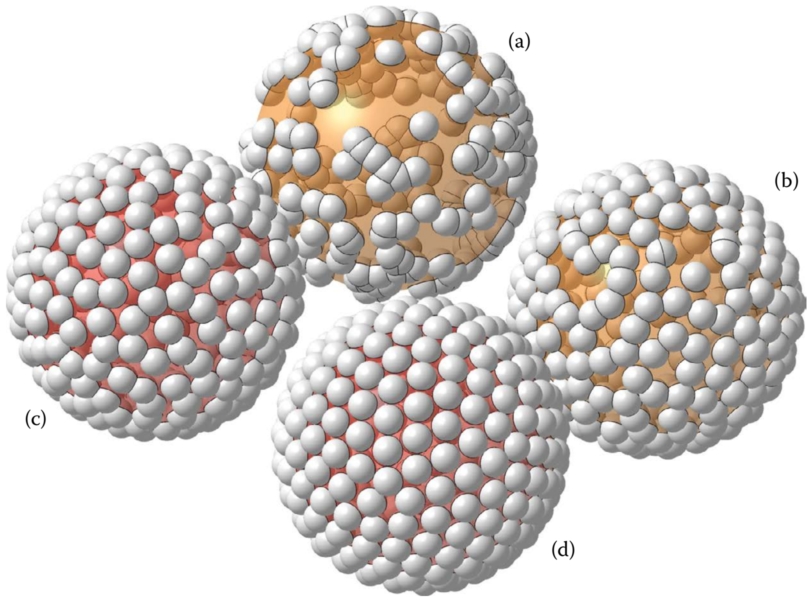


Figure 1.2. Distributing an arbitrary number of points on a sphere.

to maximizing their distance to their neighbors; that is, each one is trying to get away from its neighbors and yet remain on the sphere's surface.

Figure 1.2(b) and (c) shows their progress. At each stage, the particles are progressively more evenly distributed; the spaces between them are more uniform and fewer small spheres are intersected. After a number of iterations, a program parameter, the particles reach an equilibrium state. Figure 1.2(d) shows that within tolerances, another program parameter, they have established their final position on the sphere and the repositioning cycles end. None of the spheres surrounding the particles interfere now, though some touch each other. We have a visual check that the particles are as equally spaced from one another as they can be. A grid connecting neighboring particles (points) defines a triangular subdivision grid.[6]

The particle repulsion technique looks promising. The result in Figure 1.2(d) looks quite good and we have the advantage of distributing any number of points our design requires. So why not go with this approach? No need for great circles here!

As attractive as this approach is, there are some serious drawbacks that eliminate this approach for all but the most specialized applications. First, there are an infinite number of final arrangements for the same number of particles. Resprinkle the same number of them again, let them find their equilibrium positions again, and for sure, they will settle down in a different arrangement. This means you cannot repeat the process for the same number of points and get the same design outcome. Randomization and the sequence of rebalancing in each cycle guarantees that no two simulations for the same number of particles will produce the same end result. Second, for certain numbers of particles and initial placements, it is possible they will jostle around forever and never find their equilibrium state. Forever is not a good timeframe to wait if you are trying to design something. This is one reason we fixed one point's position before we started letting them rearrange themselves. Without one fixed point, each cycle would

[6] Particle repulsion simulation program Diffuse, courtesy of Jonathan D. Lettvin.

simply continue to move particles around and they would never balance. Third, and this is a very big drawback, there is no symmetry in the final arrangement of points. Except for an occasional lucky number of points, there may be no stable pattern. The final arrangement of points depends on the initial random distribution, how forces are simulated and the iterative effects of relaxation and re-positioning. There is no way to anticipate if multiple points will lie on the same great circle, form antipodal points (two points on opposites sides of the sphere), or create any symmetric arrangements at all. This also means there is no way to prove whether a final solution is unique or not.

None of the above outcomes are good, especially if the points are to be used in designs we intend to manufacture. What we are really looking for is a way to subdivide a sphere by

- distributing points evenly and define grids as course or fine as needed;
- minimizing the variation within the grid (chords and areas);
- creating grids where some members form continuous great circles for applications that require them;
- maximizing symmetry and reuse of local grids;
- defining coordinates that uniquely define any point on the grid;
- developing simple ways to convert from one coordinate system to another; and
- defining metrics for comparing one subdivision method with another.

So how can we do this? The Greeks will show us.

1.4 Symmetry and Polyhedral Designs

Like many things in geometry, the Greeks seem to have gotten there first. Although the five *Platonic solids* have been known since prehistoric times, the Greeks were the first to recognize their properties and relationships to one another.[7] Figure 1.3 shows

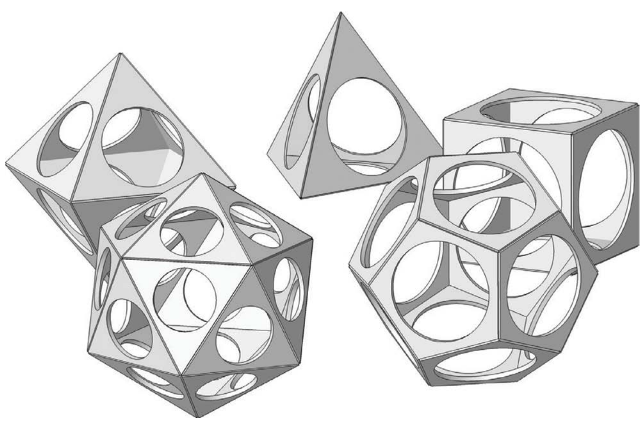


Figure 1.3. Platonic solids.

[7] Stone figures resembling polyhedra, found on the islands of northeastern Scotland, have been dated to Neolithic times, between 2000 and 3000 BC. These stone figures are about 2 inches in diameter and many are carved into rounded forms resembling regular polyhedra such as the cube, tetrahedron, octahedron, and dodecahedron. By 400 BC, the time of Plato, all five regular polyhedra were known.

them with holes in their faces to make it easier to see how their parts relate. Starting in the back and going clockwise, they are the tetrahedron, cube, dodecahedron, icosahedron, and octahedron. The Platonics are one family of polyhedra (there are dozens). *Polyhedra* are three-dimensional (3D) forms with flat faces and straight edges and they take on an amazing variety of forms, many of which are very beautiful in their symmetry and spatial design. They are one of the most intensely studied forms in mathematics.

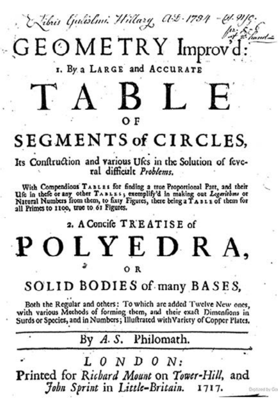
The Platonics are unique among polyhedra. Every Platonic solid has regular faces (equilateral triangles, squares, or pentagons) and every *edge* on a Platonic face has the same length. All *face angles* are the same and the angle between faces that meet at an edge are the same.

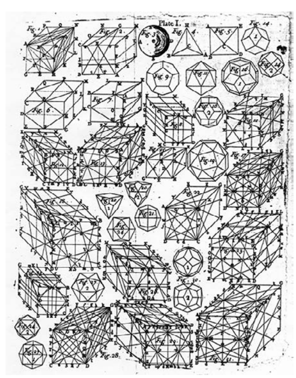
Every Platonic solid is highly symmetrical. This means that they can be orientated in many ways and still retain their appearance. Certain pairs of Platonics can be placed inside one another and this characteristic demonstrates how they are interrelated. The Platonics and the properties just mentioned are so important in subdividing spheres, we devote an entire chapter to them and explain them in detail.

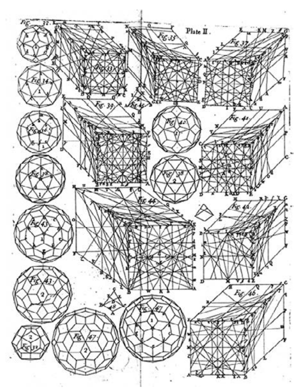
Abraham Sharp (1653–1742)

Sharp was an English mathematician, astronomer, and geometrist. He was assistant to the famous astronomer Flamsteed from 1689 to 1696. Demands for highly accurate sky references clearly influenced his work. In 1717, Sharp published a significant treatise on geometry and logarithms entitled *Geometry Improv'd*.[8]

Sharp's tables and polyhedron calculations were unsurpassed for accuracy, some to 15 and 20 decimal places. Numerous illustrations show polyhedra for which he calculated their geometric properties and the sequence of cuts needed to make 3D wooden models of them. For those engaged in spherical subdivision today, Sharp's illustrations resemble reference polyhedra for geodesic spheres where face areas that might be used for small-area grid development and projection to an enclosing circumsphere.[9]


GEOMETRY Improv'd:
1. By a LARGE and ACCURATE
TABLE
OF
SEGMENTS of CIRCLES,
Its Construction and various Uses in the Solution of several difficult Problems.
With Compendious TABLES for finding a true Proportional Part, and their Use in these or any other TABLES, exemplify'd in making out Logarithms or Natural Numbers from them, to sixty Figures, there being a TABLE of them for all Primes to 1100, true to 61 Figures.
2. A Concise TREATISE of
POLYEDRA,
OR
SOLID BODIES of many BASES,
Both the Regular and others: To which are added Twelve New ones, with various Methods of forming them, and their exact Dimensions in Surds or Species, and in Numbers; Illustrated with Variety of Copper Plates.
By A. S. Philomath.
LONDON:
Printed for Richard Mount on Tower-Hill, and John Sprint in Little-Britain. 1717.





[8] (Sharp 1717).
[9] In Section 1 of "Solid Bodies" p. 71, Sharp describes a "very elegant geometrical solid defined from the dodecahedron or icosahedron with planes upon all the edges of either." Today, this polyhedron is called the Rhombic Triacontahedron (32 vertices, 60 edges, and 30 faces) and it's a member of the Catalan family of polyhedra. It is also a dual to the icosahedron. A spherical version of it was used by Jeffrey Lindsay in 1950 as design reference for the triangular gridding system of the first full-scale geodesic dome, Weatherbreak, built in Montreal, Canada. See Section 2.7.3.

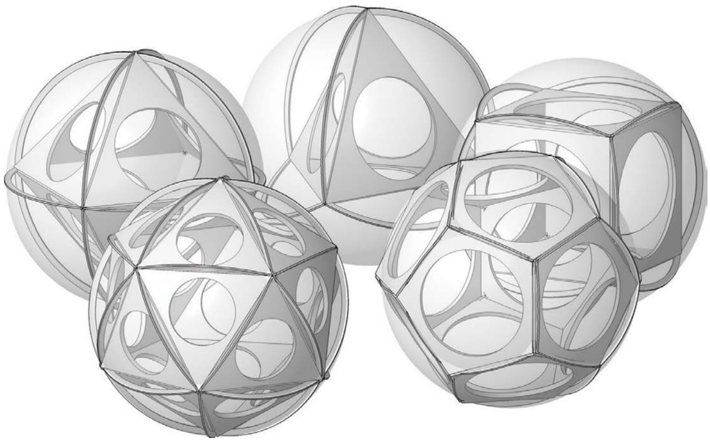


Figure 1.4. Spherical Platonics.

Another key property of the Platonics, particularly useful in spherical subdivision, is their vertices or points (we use the two words interchangeably). They are evenly spaced and lay on the surface of sphere that surrounds them and share the same center point as the polyhedron. This circumscribing sphere is called the polyhedron's *circumsphere*. Figure 1.4 shows the spherical versions of the five Platonics. The planar versions are shown inside to make the association between planar edges and spherical great circle arcs easier to see. Notice that in each spherical version, some pairs of vertices are on opposite sides of the sphere; they are antipodal points. Notice also that every spherical *face* in each Platonic is bounded by arcs of great circles and that a single Platonic has but one spherical face type (equilateral triangle, square, or pentagon). This is important because any design we develop on one face can be replicated to cover the others, thus covering the entire sphere with a pattern that has no overlaps or gaps.

1.5 Spherical Workbenches

Spherical Platonics give the designer a huge head start in subdividing the sphere because the polyhedron's vertices are already evenly distributed on its circumsphere and we can use them immediately to define great circles and reference points for further subdividing. Let's take a quick look at how we are going to develop a design starting with a simple polyhedron. Later chapters in the book will explain the details of how this is done.

Referring to Figure 1.5, we start the subdivision process by first selecting one of the Platonic solids that best meets our design requirements. In this example, we select the icosahedron (a). It is the most used in spherical work, by far, but we could have used any of the other Platonics. We define the icosahedron's spherical version (b); it's an easy step given the planar one. The spherical version is now our subdivision workbench. We pick a conveniently positioned face to work with, or some symmetrical area within it (c). In this example, a complete icosahedral face is selected; one of its vertices is at the sphere's zenith (top); this can simplify our calculations. Most of our efforts will be spent here, subdividing this single face. This face is not our only working area option, but it is an obvious choice. Within this face, we can use any one of several gridding techniques. In Chapter 8, we describe six techniques and Chapter 10 describes several more. Each technique produces a different grid and each grid's set of points offer its own benefits, depending on your application. Once the subdivision is

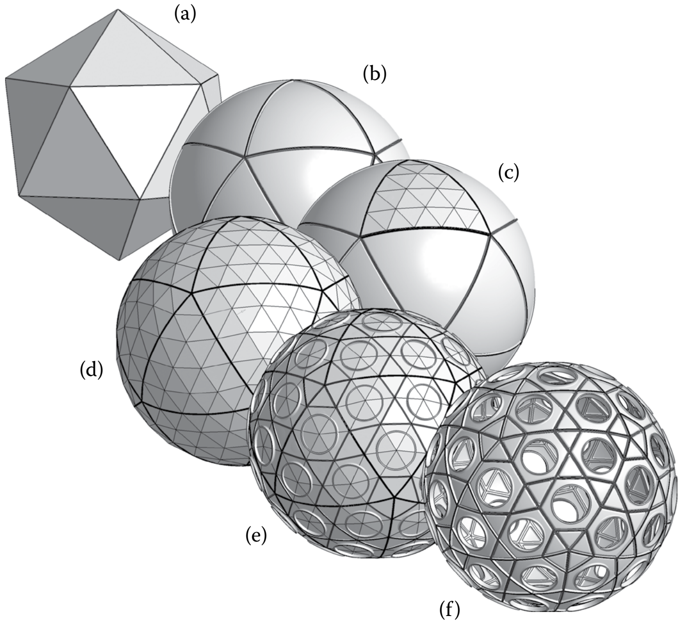


Figure 1.5. Progression of spherical subdivision to basic design application.

complete, we replicate the resulting points and grid to cover the rest of the sphere as shown in (d).

We have considerable flexibility in (a) through (c). We can use any of the five Platonics as our base, we can pick different standard areas to subdivide, and we can use different techniques to subdivide this area, making the grid as course or fine as we wish. Once we have a set of points, we have more flexibility in how we join combinations of points to make triangles, hexagons, pentagons (all shown in Figure 1.5(f)), diamonds, or any other spherical shape. The design possibilities are limitless. With so many choices, one might ask, which is best? The answer, of course, depends on the application you have in mind. In Chapter 9, we will describe a series of metrics that you can use to evaluate the appropriateness of one layout or another to your needs. Various metrics show the differences between subdivision methods and this helps you decide, which combination best fits your requirements.

For many users, the subdivision grid in Figure 1.5(d) is just the starting point for further refinement. It must be developed into a physical design. The grid might define locations for openings, panels, struts, or even positions of dimples on a golf ball, and not every subdivision point or grid member may be part of the final design.

Figure 1.5(e) shows one of many possible designs for this particular grid. Triangles might be combined into diamonds, hexagonal, or pentagonal patterns. Such refinements call for other programs such as Computer Aided Design (CAD) where points, chords, and face definitions generate, manually or automatically, prismatic shapes, structural elements, surfaces, or geometric elements, as shown in a simple design application (f). Grids can be combined or layered to create more intricate patterns or truss-like structures. If the design is for a geodesic dome, part of the subdivision may be cut off at the ground level to allow for foundations or supports. The design flexibility is limitless, and CAD provides powerful visualization and analysis capabilities as well. If the design is to be manufactured, the 3D geometry can be used to automate cutting, welding, molding, or bending equipment during production. CAD is an accessible technology and a powerful tool for the *spherist*.

1.6 Detailed Designs

The grid shown in Figure 1.5(d) is quite modest. The surface of the sphere is covered with 262 points. However, from the same starting point, the initial 12 vertices of the spherical icosahedron, shown in (b), we can distribute fewer or more points to make our grid. Figure 1.6 shows how. In each figure, the *frequency* of points, a term explained later, steadily increases from the spherical icosahedron's 12 vertices (top left figure) to 10,242 (bottom right figure) by progressively placing points between the points of the previous layout. We could continue the process indefinitely, but most spherical applications do not need more points. Notice also, as more points are added, we can generate more subdivision triangles by connecting points to their nearest neighbors.[10]

In this series, the number of points from left to right, top to bottom, is 12, 42, 162, 642, 2562, and 10,242, respectively. The radius of all accent spheres is the same in all illustrations. In the last illustration, some accent spheres touch, but none interfere. In later chapters, we will use accent spheres such as these to visualize and analyze the point distributions that result from the spherical subdivision techniques we present.

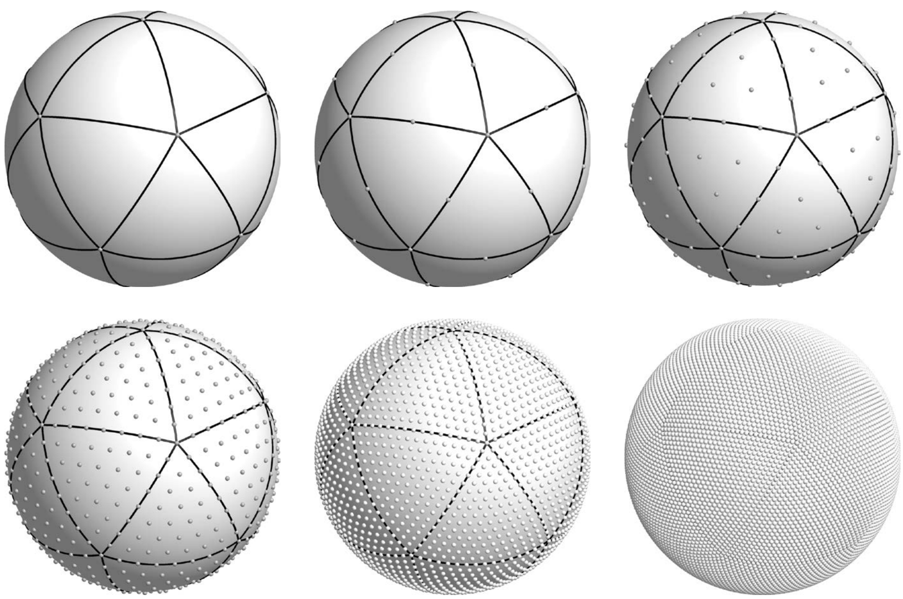


Figure 1.6. Increasing the number of points over the sphere.

[10] The total number of points in any one of these spherical triangles is $n(n+1)/2$. A single face in each of the images in Figure 1.6 contains 3, 6, 15, 45, 153, and 561 vertices, respectively. In Section 8.2, we will discuss the triangulation number formula which tells how many points result from any spherical subdivision.

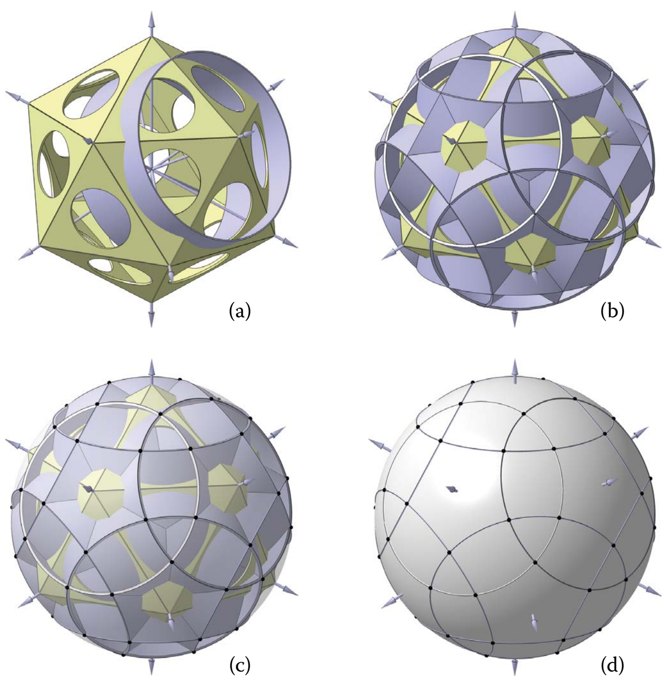
1.7 Other Ways to Use Polyhedra

The polyhedral subdivisions we have been discussing are based on points defined by intersecting great circles. Great circle techniques are the ones we emphasize in this book, but it is possible to use polyhedra to define points on a sphere by intersecting lesser circles instead. Here's how.

We will use the icosahedron again but instead of focusing on its spherical faces, we will use its vertex axes to develop our grid. Every one of the icosahedron's six vertex axes passes through its center and they are perpendicular to the polyhedron's circumsphere. A cylinder is positioned around each axis. Figure 1.7(a) represents the icosahedron's axes with arrows; a single cylinder is positioned around one of them. If the cylinder radius is large enough, every cylinder will intersect its neighbor's cylinder, as shown in (b). Each cylinder will also intersect the polyhedron's circumsphere and define a lesser circle, as shown in (c). A point can be defined at each lesser circle intersection, as shown in both (c) and (d).

The lesser circle arcs between the points in (d) are not geodesics. Only an arc of a great circle, the shortest distance, can be a geodesic. The lesser circle arcs, technically speaking, do not define spherical polygons either. Again, only great circle arcs define the sides of spherical polygons. It is possible, however, to define great circles between these points by passing a plane through pairs of them (and the origin).

There are many variations on this technique. For example, one can substitute cones for cylinders. The cone's apex is at the center of the sphere and the cone's surface intersects the sphere defining a lesser circle. Equations can systematically generate lesser circles on the sphere as well, and where circles intersect, subdivision grid points can be defined. Gary Doskas has done extensive research in this area; the patterns he explores



(a) (b) (c) (d)

Figure 1.7. Lesser circle spherical subdivisions.

and their symmetry are beautiful and many are utilitarian.[11] We mention lesser circle subdivision of spheres for completeness; they are not developed further in this book.[12]

1.8 Summary

In this chapter, we looked at the challenges of subdividing spheres and the fact that spheres have unique geometric properties. We have seen that it is impossible to evenly distribute an arbitrary number of points on a sphere and achieve symmetric and predictable results. Spherical polyhedra, particularly the Platonics, offer a symmetrical framework and jump start the subdivision process. The vertices of spherical Platonics are evenly spaced reference points on a sphere that surrounds the polyhedron. A small work area can be defined and grids developed there, as course or fine as a design requires. The grid can then be replicated to cover the rest of the sphere without overlaps or gaps.

Designers can use any of the spherical Platonics as a design framework, define working areas on its faces, and use any one of a number of different subdivision techniques to develop a grid over that face. The variety is limitless.

In the next chapter, we will look at the fascinating history of spherical subdivision. Buckminster Fuller, the inventor of the geodesic dome, was the first person to recognize the value of spherical polyhedra and subdivision grids to general architectural construction. Most of the subdivision techniques we use today were developed in the late 1940s and 1950s by Fuller and his associates to build geodesic domes. In their day, some domes were the largest free-span structures on Earth. Fuller's work is particularly important because he achieved highly creative results while leveraging manufacturing techniques. The spherical subdivision techniques that evolved from the 1950s are as relevant today as they were then and are applied in a wide range of science and industrial applications that have nothing to do with geodesic domes. We survey some of these applications in Chapter 3.

Additional Resources

Doskas, Gary. *Spherical Harmony—A Journey of Geometric Discovery*. LuLu Marketplace: Hedron Designs, 2011.

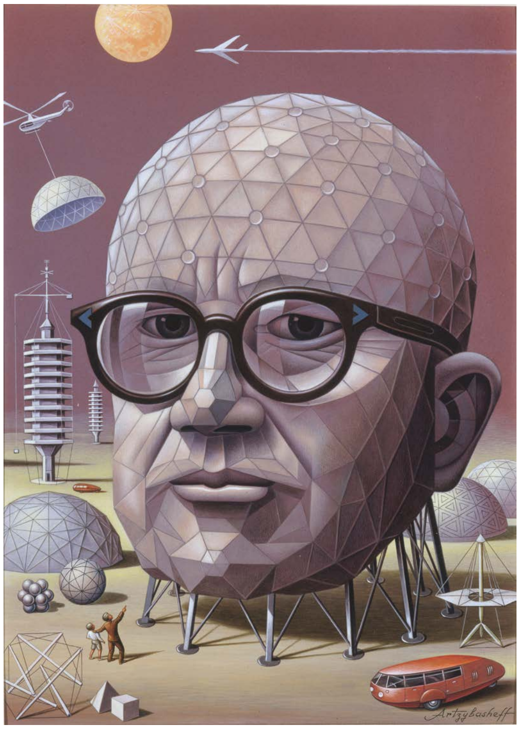
Kitrick, Christopher J. "Geodesic Domes." *Structural Topology* 11 (1985):15–20.

Messer, Peter W. "Polyhedra in Building." *Beyond the Cube: The Architecture of Space Frames and Polyhedra* ed. J. Francois Gabriel. New York: John Wiley & Sons, 1997.

Stuart, Duncan R. "The Orderly Subdivision of Spheres." *The Student Publications of the School of Design*. Raleigh, NC: North Carolina State University, 1963.

[11] See (Doskas 2011) for more information.

[12] Polyhedra created from cones and cylinders have been explored and a general typology developed. See (Pecks 1995) and (Clinton 2004).



2 Bucky's Dome

Buckminster Fuller (1895–1983), seen in the painting above,[1] was a true American polymath. Bucky, as he was called, was a philosopher, designer, engineer, architect, author, futurist, and prolific inventor. His earliest inventions include building blocks, temporary shelters, and an environmentally sensitive bathroom where you could take a "fog shower" utilizing pressurized mist and less than a gallon of water. He even invented a bullet-shaped, three-wheeled car that could turn complete circles in its own length. However, he is best remembered for the invention of the geodesic dome and designs that do more with less.[2]

Millions of visitors have seen the US Pavilion at Expo'67 in Montreal, Epcot Spaceship Earth at Disney World in Orlando, or the La Géode Theater in Paris. Countless others live, shop, or worship in geodesic domes. Geodesic domes enclose radar equipment at airports and air defense stations in remote polar regions. They corral fish in ocean-bobbing pens and record neutrinos from outer space, as they zip through underground spherical observatories. Geodesic tents shelter Boy Scouts and Mount

[1] Painting of inventor and philosopher, R. Buckminster Fuller, Jr., by Boris Artzybasheff (1899–1965). The media is tempera on board, 21.5 × 17 inch, circa 1964. *Time Magazine* cover, January 10, 1964, and gift of Time, Inc. to the National Portrait Gallery, Smithsonian Institution, Washington, DC, USA. Used with permission of the National Portrait Gallery and the Estate of R. Buckminster Fuller.

[2] Buckminster Fuller patents include Prefabricated Bathroom (Patent 2,220,482, November 5, 1940) and Motor Vehicle (Patent 2,101,057, December 7, 1937).

Everest climbers alike. A nearby playground might even have a geodesic jungle gym for kids to climb on. No single construction system has been built in so many sizes and of such diverse materials – wood, pipes, sheets of plastic and metal, foam panels, cardboard, plywood, bamboo, fiberglass, concrete, and even bicycle wheels and the tops of junked cars.[3]

These applications and many others use spherical techniques that Buckminster Fuller, or one of his associates, developed in the late 1940s and 1950s. They are as useful today, as they were then, because they continue to solve new problems in fields that have nothing to do with domes, such as astronomy, weather prediction, materials science, virology, product design, and PC game development. Even the dimple patterns on golf balls owe a debt to Buckminster Fuller.

The concept of geodesics is not entirely new. For mathematicians, they are the shortest path between two points on a curved surface, *any* curved surface, not just a sphere's. Spherical triangles and grids are hardly new, and neither is geodesic construction. The earliest example dates to 1922 when Walther Bauersfeld, an engineer for Carl Zeiss optical company, developed the world's first reinforced concrete dome in Jena, Germany,[4,5] for Zeiss' planetarium. The dome's steel reinforcing grid resembles the lattice we associate with today's geodesic dome.

Bauersfeld's structure was highly innovative at the time. However, unlike Fuller's domes, Bauersfeld's dome was never developed into a generalized construction system or used elsewhere. Fuller was the first to establish geodesics in a framework he called *energetic Synergetic Geometry,* or Synergetics for short. For sure, Fuller's relentless promotion of geodesics had a lot to do with the success of the geodesic dome, but this success was also the result of Synergetics. Synergetics acted as a vehicle for moving concepts in physics (spin, charge, attraction), mathematics (plane and spherical geometry), materials science (tension and compression), natural building processes (triangulate and space-filling forms), and design intent (conservation of energy, high strength-to-weight structures, and industrial processes) to different settings, such as mapping, long-span truss systems, and, most important, geodesic domes. The fact that these innovations were based on a broader framework for design increased their appeal and encouraged their application in new fields.

Fuller did not invent the geodesic dome in isolation. In the late 1940s, he taught at several colleges where he came into contact with highly creative and intelligent people. His style, somewhat non-academic for the times, mixed seminars, workshops, and hands-on projects, in which his ideas were freely associated with those of students and colleagues. He often made their ideas his own. This style, borrowing ideas, characterized his business life as well. There, designers, architects, engineers, and artisans, such as Kenneth Snelson, Duncan Stuart, Jeffrey Lindsay, Donald Richter, T. C. Howard,[6] William Wainwright, Jim Fitzgibbon, Shoji Sadao, and William Ahern, also had creative skills and energy. Some were inventors and would be awarded their own geodesic patents. They were often the designers and architects on projects that only Fuller received credit for. There is no doubt that Fuller's free-association inspired others,

[3] Drop City, a hippy commune in the 1960s, was an assemblage of geodesic panel domes made from the sheet metal of automobile roofs and other inexpensive materials.
[4] (Kahn 1989).
[5] (Fernández-Serrano 2019).
[6] Thomas C. Howard (aka T.C.) (1931) was principal designer, architect, and engineer for Synergetics, Inc. from 1955 until 2006 and lead designer on many award winning geodesic dome projects. The T. C. Howard Papers on Synergetics, housed at NC State University Libraries, archives significant architectural drawings and related documentation to worldwide projects and showcase geodesic domes, octetrusses, and Charter-Sphere Domes. Several of T. C. Howard's projects appear in the book.

increased the flow of ideas, and hastened the development of geodesics. And while most relationships were symbiotic, some were not. This book recognizes the contributions of others, as often as possible.

Buckminster Fuller is a fascinating study of innovation, entrepreneurship, self-belief, and opportunism. He was a prolific inventor and writer. While some writings are razor-sharp, others read like stream of consciousness. He was always concerned about being misunderstood and, given his elliptical style and self-made vocabulary, it was a reasonable worry. Many authors and historians have tried to place this complex personality and his inventions within the broader context of twentieth-century events. Anyone interested in the genesis of geodesic domes should attempt to understand the man, Buckminster Fuller. Those wanting to know more about him can consult the additional resources at the end of this chapter.

2.1 Synergetic Geometry

Volumes have been written by and about Fuller's Synergetic Geometry. This book examines only the very small part of it that built the foundation for dividing spheres. By the late-1940s, Fuller was shifting his attention from developing affordable housing to exploring new ways of thinking about design. He kept a sketchbook called *Noah's Ark II*, where he diagrammed geometric relationships and spherical grids.[7] Fuller had always been interested in natural forms: rocks, crystals, shells, and so forth. He sensed that nature always found the most efficient solution to problems and he was taken by the idea that nature is in constant motion and that motion itself is relative. He looked for a way to unify fundamental laws of physics (atoms, orbits, spin, energy, charges, and bonding) with geometry (polyhedra, space-filling lattices, great circles, symmetry, and maximum packing) into a design cosmology he would call Synergetic Geometry. Synergetics was Fuller's summary of natural phenomena and his framework for design. He drew his principles from natural systems and used mathematics, particularly solid geometry, to show relationships – *synergy* – between systems. "The whole is always more than the sum of its parts," he would say. In his cosmology, mathematics, observation, and the analysis of natural systems were all telling us how to make the most efficient design. Fuller personalizes Synergetics with his own language, which is often difficult to understand. But his basic dictum was clear: man is part of a natural system; he must learn from it, respect it, and use its principles in his work.[8]

He added, *"I would not suggest that it is the role of the individual to add something to the universe. Individuals can only discover the principles and then employ them to move forward to greater understanding."*

Fuller had a keen sense of how geometry increased strength and stability, and how some polyhedra could be packed together to fill space without leaving voids. He recognized the inherent stability and rigidity of the tetrahedron, octahedron, and icosahedron, three solids with only equilateral triangular faces. The cuboctahedron has both square and triangular faces, but can be constructed from just eight tetrahedra. As a result, it is highly stable and rigid. These solids were particularly important in Synergetics and would appear and reappear in different combinations in his future inventions.

[7] (Fuller 1950).
[8] (Edmondson 1987, 2007).

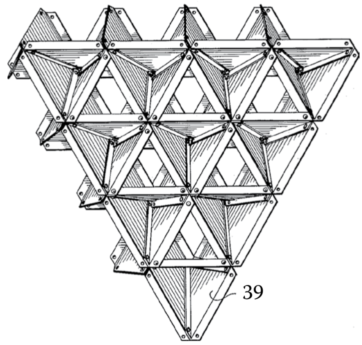
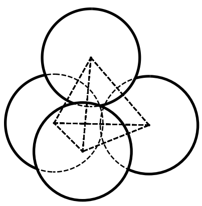
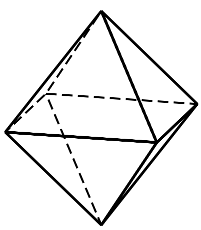


Figure 2.1. Synergetic building construction.

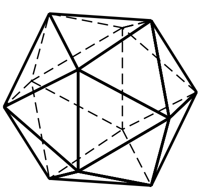


The tetrahedron (four equilateral triangular faces, six edges, and four vertices) is the only polyhedron where every vertex is equidistant from every other one. Fuller visualized its geometry by clustering three equal-diameter spheres into a triangle and nestling a forth on top, as seen in the figure to the left. All four spheres are packed in their closest three-dimensional (3D) configuration; their centers define the vertices of a tetrahedron. The tetrahedron is the most basic form in Fuller's Synergetic Geometry. It appears in numerous configurations in his later work.

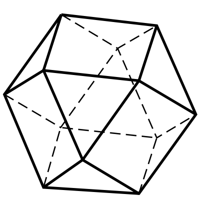


The octahedron (8 equilateral triangular faces, 12 edges, and 6 vertices) can be visualized as two back-to-back square pyramids. Octahedra and tetrahedra together can be placed side by side and perfectly fill 3D space. Figure 2.1 shows the octahedral-tetrahedral truss system Fuller later called the *octet truss*. You can see how these two solids pack together to fill all the space.[9]

Fuller describes the relationship of the tetrahedron-octahedron: "Nature's simplest structural system in the universe is the tetrahedron. The regular tetrahedron does not fill all space by itself. The octahedron and tetrahedron complement one another to fill all space. Together they produce the simplest, most powerful structural system in the universe."[10]



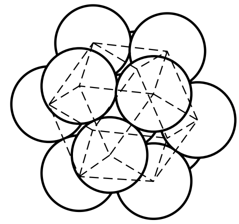
The icosahedron (20 equilateral triangular faces, 30 edges, and 12 vertices) has the highest number of identical regular faces of any regular polyhedron. Its faces can be subdivided into six right triangles; thus, the overall solid could be made from 120 right triangles. The subdivision of any one of them can be replicated over its surface without overlaps or gaps. It is the most used polyhedron for spherical work by far. We discuss the icosahedron in great detail later in the chapter.



The cuboctahedron (14 faces, 24 edges, and 12 vertices) is the only polyhedron out of hundreds where every edge is equal in length *and* this length is the distance between every vertex and the center of the polyhedron. You can visualize the cuboctahedron as a polyhedron made of eight tetrahedra, each sharing one vertex at the center of the polyhedron and their other vertices tangent to one another. Like the tetrahedron, the cuboctahedron can be created by close-packing spheres. A total of 12 spheres

[9] (Fuller 1961, sheet 7/7, fig. 14).
[10] (Fuller 1983, 168).

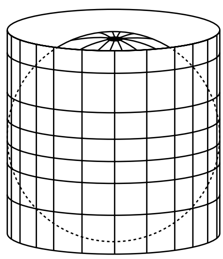
nested around a 13th, central sphere, as shown in the figure to the right. Fuller thought this configuration and angular relationship was so special he called it the *vector equilibrium* (VE), claiming that it was nature's coordinate system as opposed to normal *Cartesian coordinates* man uses.[11] Cuboctahedron and VE all refer to the same polyhedron and we use the terms interchangeably.

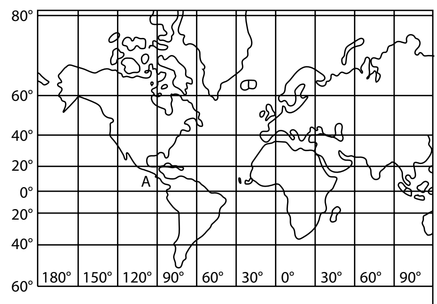


2.2 Dymaxion Projection

Perhaps due to his prior Navy service and the importance of charts and navigation, Fuller began his spherical work with cartography. Map projection, one of the longest standing geometric challenges, involves figuring out how to accurately represent features of a round Earth on a flat plane, such as a piece of paper. The surface of a sphere is undevelopable, which means it cannot be rolled out or flattened without distortion. This centuries-old problem has attracted scores of geographers and mathematicians and led to numerous projection schemes. Some projections and maps preserve the shape of spherical shapes and angles, while others preserve great circle distances or compass headings between points on the sphere; others represent the area of shapes accurately. No two-dimensional (2D) projection preserves all these characteristics. In the end, all 2D maps of a sphere have some form of distortion.

At the time, and it's still true today, the most common map projection was the *Mercator projection*. In this projection, the Earth is placed inside a cylinder (a curled-up map that wraps around the Earth), and rays from the center of the Earth paint the outline or land masses onto the cylinder. Only the equator, which is tangent to the cylinder, is accurately projected. The land masses distort more and more as the projection nears the poles. Greenland and the arctic regions appear huge in comparison to their true size. The poles cannot be projected at all, since rays from the Earth's center to the poles would be parallel to the surface of the cylinder and never project at all.

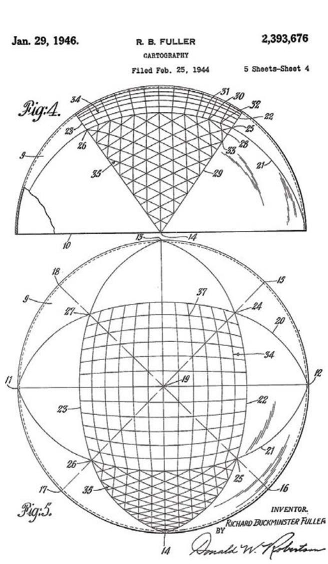




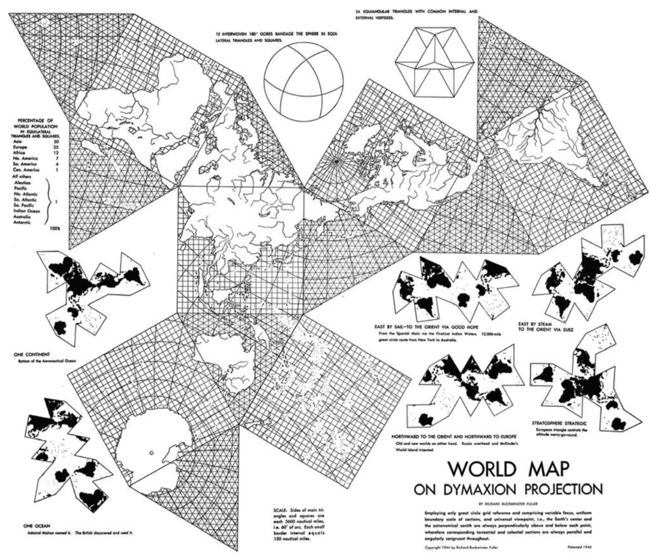
In 1944, Fuller took a totally different approach and based his map projection on the spherical subdivision of the cuboctahedron, shown in Figure 2.2(a). Essentially, Fuller developed a grid over the faces of the cuboctahedron and then projected the grid onto a sphere that surrounded the solid. The grid he developed essentially runs parallel to the edges of the cuboctahedron's regular square and triangular faces. The resulting spherical projection undeniably resembles a geodesic dome, though Fuller would not make this connection for a few more years.

In Figure 2.2(b), the projection is unfolded into a 2D view, sometimes called a *net*, and various combinations of land and ocean areas are displayed on a series of contiguous squares and triangles.[12] Some displays show a continuous ocean, others continuous land. Each serves its own purpose. Few thought it possible to invent anything new in a field so exhaustively studied and thoroughly developed as cartography

[11] The Cartesian coordinate system uses three mutually orthogonal axes; Fuller believed that nature's coordinate system is based on a 60° coordinate system. See also Williams (1979, 164).
[12] Map image courtesy of the Estate of R. Buckminster Fuller.



(a)



(b)

Figure 2.2. World maps with Dymaxion Projection.

but Fuller did just that. In his patent, he briefly describes the problem he solved: "The Earth is a spherical body, so the only true cartographic representation of its surface must be spherical. All flat surface maps are compromises with truth."[13]

Fuller called this configuration a *Dymaxion Projection*.[14] The projection is based on a subdivision grid that runs either parallel or perpendicular to the edges of the square and triangular faces.

After pointing out the disadvantages of current projections, including the popular Mercator projection, Fuller goes on to say: "Another expedient has been to resolve the Earth's surface into a polyhedron, projecting gnomonically to the facets of the polyhedron, the idea being that the sections of the polyhedron can be assembled on a flat surface to give a truer picture of the Earth's surface and of directions and distance."[15]

In essence, Fuller claims that a gnomonically projected[16] map placed onto the faces of a polyhedron, the cuboctahedron in this case, provides a truer representation of areas, boundaries, directions, and distances than any plane surface map heretofore known. By using the cuboctahedron as the base polyhedron for his projection, all vertices lie on one of four great circles.

Fuller makes three invention claims in his patent. The Dymaxion map is

- a projected map of square and triangular sections where edges are represented by projected great circles with a uniform cartographic scale;

[13] Fuller (1946, p. 1, col. 1, para. 1).

[14] In the 1930s, Fuller was advised to find a better name for his four-dimensional (4D) house invention, and with the help of an advertising wordsmith, Waldo Warren, the word "Dymaxion" was coined by selecting suitable syllables from Fuller's account of his design ideas. The roots of this name are "dynamism," "maximum," and "ions" (Marks 1973).

[15] Fuller (1946, p.1, col. 1, para. 2).

[16] Gnomonic or gnomic projection is a map projection obtained by projecting points on the surface of a sphere from a sphere's center to a plane that is tangent to the sphere or a projection from the center of a sphere to the surface that surrounds an object (such as a sphere around a polyhedron). In a gnomonic projection, great circles are mapped to straight lines.