Alan Brace

Engineering BGM

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Dedicated to the memory of my father George James Brace

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Preface

Over the past several years the author has found himself frequently asked to give explanatory talks on BGM, some of which extended into one- or two-week workshops with detailed head-to-head technology transfer. The main interest came from small groups of quants either in banks or in software companies wanting to implement the model without wasting too much time decoding papers to find a suitable approach, and also academics and students wanting to get into the subject. This book is therefore naturally targeted at such people, who generally have several years experience around finance and a good grounding in the relevant mathematics.

The stimulus to begin writing was an invitation to join the Quantitative Finance Research Centre (QFRC) at the University of Technology Sydney (UTS) as an Adjunct Professor, and give a series of lectures on BGM for an audience of academics, students and industry quants over the course of a couple of semesters during 2006. This book grew out of those lectures, but the starting point was some eleven years of notes on various aspects of BGM, that were all prepared either for implementers writing production code, or as formal documentation to accompany production code, or in response to consulting tasks. Thus most of the techniques and methods described in this book originate in practical problems needing a solution and address real requirements. Moreover, many of them have been implemented, tried and tested either in an R&D environment like MatLab, or in production code.

A reader from a mathematics, physics or engineering background (or the quantitative end of another science) with a decent knowledge of analysis, optimization, probability and stochastic calculus (that is, familiar with *Ito* and *Girsanov* at the very least) should find this book fairly self-contained and thus hopefully a suitable resource and guide to implementing some version of the model. Indeed, part of the reason why the author has tried to keep the book relatively short is to make it easy to slip into one's briefcase and use as a ready reference; the other part is a pathological fear of catching *blitherer's disease*, which in extreme cases seems to dilute ideas to one per page!

The book starts with the standard lognormal flat BGM, and then focuses on the shifted (or displaced diffusion) version to develop basic ideas about construction, change of measure, correlation, calibration, simulation, timeslicing (like lattices), pricing, delta hedging, vega hedging, callable exotics and barriers. Further chapters cover cross-economy BGM, adaption of the HJM inflation model to the BGM framework, a simple tractable stochastic volatility version of BGM, and financial instruments in Brazil, which have evolved in a unique way and are amenable to BGM analysis.

Because shifted BGM can fit a cap or swaption implied volatility skew (but not a smile) and has the advantage of being just as tractable as flat BGM, it seems the right framework to present basic techniques. The stochastic volatility version aims to add a measure of convexity to the skew version, but we do not go so far as trying to calibrate to a full smile, which is a complex task appropriate to a cutting edge specialist. Overall the author can't help feeling that shifted BGM with the stochastic volatility extension as described here is about right for both the Mortgage Backed Security world, and also second tier banks wanting a robust framework in which to manage structured products sold into their customer base, without having to worry too much about being arbitraged.

To sum up, the reader is presented with several, progressively more sophisticated, versions of BGM, and a range of methods and recipes that (after some expansion and articulation) can be programmed into production code, and is free to choose an implementation to suit his requirements. Thus the book attempts to be an *implementer's handbook* offering straightforward models suitable for more conservative institutions who want a robust, safe and stable environment for calibrating, simulating, pricing and hedging interest rate instruments. Advanced versions for market makers, hedge funds or leading international banks are left to their top quants, though their newer quants might conveniently learn about market models from this book and then do better.

Many people contributed in some way to this book. In particular, it was a pleasure working with Marek Musiela through the early '90s at Citibank, where Mike Hawker in Sydney and Pratap Sondhi in Hong Kong provided support and a framework to do much of the original work. Since then, innumerable conversations with colleagues, reading and decoding many excellent papers, attendance at wide ranging professional conferences and some foolish mistakes have added enormously to the author's basic knowledge.

In direct preparation of this manuscript Chapman and Hall were patient and encouraging, Marek Rutkowski gave me a copy of his extensive bibliography greatly simplifying the task of preparing references, and my thanks to Carl Ang, Peter Buchen, Andrew Campbell, Daniel Campos, Tim Glass, Ben Goldys, Ivan Guo, Steve McCarthy, Frank Merino, Paul O'Brien, Erik Schlogl and Rob Womersley for helping check different parts of the book. Further thanks are due to both National Australia Bank[1] and UTS for their material support in terms of time and infrastructure over the past couple of years, and also to *MY* for encouragement at some difficult moments.

A word on the title 'Engineering BGM'. The background is that Miltersen, Sandmann and Sondermann (MSS), see [78], were the first to get a 'kosher'

[1] All views expressed in this book are the author's and in no way reflect NAB policy, philosophy or technology.

Black caplet formula out of HJM, but unfortunately they did not establish existence, which is an essential feature of a model (along with, the author feels, the technology to price complex options). We, that is Brace, Gatarek and Musiela (BGM), see [30], grasped the intuition behind the model, proved existence, derived swaption formulae, calibrated to the market and constructed simulation technology for pricing.

So generally speaking the model has more-or-less become known as ‘BGM’ in the industry and the ‘Libor Market Model’ in academic circles. My preference for the title ‘Engineering BGM’ over the alternative ‘Engineering the Libor Market Model’, is partly because this book is aimed at industry quants and traders and partly because it is shorter and more punchy. But unequivocally, MSS made the first breakthrough in this area, and we referenced their work in our paper [30] describing it as a ‘key piece of information’.

Finally, if that nightmare for a single author ‘the bad stupid mistake’ should materialize, it is soley the author’s fault and he apologizes in advance. Of course, all information about any, hopefully more minor, mistakes found by readers would be gratefully received (at any one of the author’s email addressess on the title page), as would any suggestions for inclusions, exclusions and better ways of doing things (in case there should ever be a second edition of this book).

Alan Brace

(Sydney 25 September 2007)

Chapter 1

Introduction

Modern interest rate modelling began[1] with Ho & Lee's (HL) important 1986 paper [54], and matured into the Heath, Jarrow and Morton (HJM) model [52], which was circulating in 1988, and which became the standard framework for interest rates in the early '90s. Initial work on the market models was done within that framework, so to set the scene, the single-currency *domestic* version of HJM is reviewed in Section-1.1.

When the volatility function is deterministic, HJM is Gaussian, extremely tractable, and includes versions like Hull and White [58] and many other models. But until the advent of the *market models* [30], [66], [78] and [79] around 1994-97, the market's use of the *Black caplet* and *Black swaption* formulae (which priced assuming that forwards and swaprates were lognormal) was regarded as an aberration which could not be reconciled with HJM. A further problem was that HJM *exploded* when the instantaneous forward rates were made lognormal. The author can recall comments at conferences in the early '90s along the lines that 'the market is foolish and should adopt some arbitrage free Gaussian HJM model as a standard'.

To avoid explosions, attention shifted to modelling the cash forwards, and in 1994 Miltersen, Sandmann and Sondermann [78] found a PDE method, described in Section-1.2 below, to derive the Black caplet formula within the arbitrage free HJM framework. Knowing that was possible, and that the Black caplet formula was not an aberration, was a key piece of information.

The author's main contribution to events was to grasp the intuition, described in Section-1.3 below, that the cash forwards *want to be lognormal*, but under the forward measure at the *end of their interval*. With that realization, the derivation of the Black caplet formula became trivial, and led to the so called *forward construction* of BGM detailed in [30], by Musiela, Gatarek and the author, which established existence of the model, derived approximate analytic swaption formulae, calibrated to the market, and provided suitable simulation technology for pricing exotics.

[1] Though intriguingly, the previous long standing actuarial practice of hedging bonds by matching duration turned out to be equivalent to delta hedging within the HL model.

1.1 Background HJM

REMARK 1.1 Before beginning, a word on our '$*$' notation for transposes. Throughout this book we will generally be dealing with multi-factor models involving an n-dimensional vector volatility function, say $\xi : \mathbb{R} \to \mathbb{R}^n$ and a corresponding multi-dimensional Brownian motion $W(t) \in \mathbb{R}^n$. Usually they (or similar expressions as in (1.3) below) appear together as inner products, so we use the '$*$' notation to indicate transpose and write

$$\xi^*(t)\, dW(t) \equiv \langle \xi(t), dW(t) \rangle$$

for that inner product. Of course, in single factor models $\xi(t)\, dW(t)$ would simply mean the product of two scalor quantities. Note that many authors today adopt the practice (which is beginning to appeal to the author) of simply writing $\xi(t)\, dW(t)$ and leaving the reader to work out from the context if an inner product is implied. $\square$

The *ingredients* of the HJM domestic interest rate model are:

1. An *instantaneous* at t *forward rate* $f(t,T)$ for maturity T, with SDE

$$df(t,T) = \alpha(t,T)\; dt + \sigma^*(t,T)\, dW_0(t) \tag{1.1}$$

   where the stochastic driving variable $W_0(t)$ is multi-dimensional Brownian motion *(BM)* under the *arbitrage-free measure* $\mathbb{P}_0$, and $\sigma(t,T)$ is a possibly stochastic vector *volatility function* for $f(t,T)$.

2. A *spot rate* $r(t) = f(t,t)$ and *numeraire bank account* to accumulate it

$$\beta(t) = \exp\left(\int_0^t r(s)\, ds\right),$$

3. *Assets* in the form of a spectrum of time T maturing *zero coupon* bonds

$$B(t,T) = \exp\left(-\int_t^T f(t,u)\, du\right),$$

   paying 1 at their maturity T.

To be arbitrage free, the zeros discounted by the bank account as numeraire

$$Z(t,T) = \frac{B(t,T)}{\beta(t)} = \exp\left(-\int_0^t r(s)\, ds - \int_t^T f(t,u)\, du\right), \tag{1.2}$$

must be $\mathbb{P}_0$-martingales for all T. Because

$$d\int_t^T f(t,u)\,du = \int_t^T df(t,u)\,du - f(t,t)\,dt,$$
$$= -r(t)\,dt + \left(\int_t^T \alpha(t,u)\,du\right)dt + \left(\int_t^T \sigma^*(t,u)\,du\right)dW_0(t),$$

applying Ito to (1.2), the SDE for $Z(t,T)$ is

$$\frac{dZ(t,T)}{Z(t,T)} = \left\{\begin{array}{c} -r(t)\,dt + r(t)\,dt - \left(\int_t^T \alpha(t,u)\,du\right)\,dt \\ -\left(\int_t^T \sigma^*(t,u)\,du\right)dW_0(t) + \frac{1}{2}\left|\int_t^T \sigma(t,u)\,du\right|^2 dt \end{array}\right\},$$
$$= \left\{\begin{array}{c} -\left[\int_t^T \alpha(t,u)\,du - \frac{1}{2}\left|\int_t^T \sigma(t,u)\,du\right|^2\right]\,dt \\ -\int_t^T \sigma^*(t,u)\,du\,dW_0(t) \end{array}\right\}.$$

For this to be a $\mathbb{P}_0$ martingale the drift must vanish, so

$$\alpha(t,T) = \sigma^*(t,T)\int_t^T \sigma(t,u)\,du,$$

and the SDE for the instantaneous forwards is

$$df(t,T) = \sigma^*(t,T)\int_t^T \sigma(t,u)\,du\,dt + \sigma^*(t,T)\,dW_0(t). \tag{1.3}$$

Differentiating $B(t,T) = \beta(t)\,Z(t,T)$, the corresponding SDE for the zero coupon bond is

$$\frac{dB(t,T)}{B(t,T)} = r(t)\,dt - \int_t^T \sigma^*(t,u)\,dudW_0(t). \tag{1.4}$$

REMARK 1.2 The HJM approach therefore implies that the volatility

$$b(t,T) = -\int_t^T \sigma(t,u)\,du, \tag{1.5}$$

of each zero coupon bond $B(t,T)$ is continuous in T, a restriction ruling out piecewise constant bond volatilities. □

Because assets discounted by the bank account numeraire are $\mathbb{P}_0$-martingales, the *present value* of a cashflow $X(T)$ occurring at time T is

$$X(t) = \mathbf{E}_0\left(\frac{\beta(t)}{\beta(T)}X(T)\bigg|\,\mathcal{F}_t\right), \tag{1.6}$$

where $\mathbf{E}_0$ is expectation under $\mathbb{P}_0$, and $\mathcal{F}_t$ is the underlying filtration (total accumulated information up to t). In particular, because a zero coupon pays 1 at maturity

$$B(t,T) = \mathbf{E}_0\left(\left.\frac{\beta(t)}{\beta(T)}\,1\right|\mathcal{F}_t\right) = \mathbf{E}_0\left(\left.\exp\left(-\int_t^T r(s)\,ds\right)\right|\mathcal{F}_t\right). \tag{1.7}$$

A *forward contract* $F_T(t,T_1)$ on a zero-coupon bond $B(t,T_1)$ maturing at T_1, exchanges at time T the zero coupon $B(T,T_1)$ for $F_T(t,T_1)$. The present value of the exchange must be zero, hence $F_T(t,T_1)$ must satisfy

$$\mathbf{E}_0\left\{\left.\frac{\beta(t)}{\beta(T)}\left[F_T(t,T_1) - B(T,T_1)\right]\right|\mathcal{F}_t\right\} = 0$$

giving the following *model free result* for forward contracts

$$F_T(t,T_1) = \frac{B(t,T_1)}{B(t,T)}. \tag{1.8}$$

When $T_1 = T+\delta$, the *cash forward* $K(t,T)$ over the interval $(T,T_1]$ is defined in terms of the forward contract $F_T(t,T_1)$ by

$$F_T(t,T_1) = \frac{B(t,T_1)}{B(t,T)} = \frac{1}{1+\delta K(t,T)}. \tag{1.9}$$

REMARK 1.3 In the following equation (1.10), please note that the one variable Radon-Nikodym derivative $Z(t) = \mathbf{E}_0\{Z(T)|\mathcal{F}_t\}$ is not the two variable discounted zero coupon function $Z(t,T) = \frac{B(t,T)}{\beta(t)}$. ▯

Being a strictly positive process, the bank account $\beta(t)$ induces a *forward measure* $\mathbb{P}_T$ (expectation $\mathbf{E}_T$) at any maturity T through

$$\mathbb{P}_T = Z_T\,\mathbb{P}_0 \quad or \quad \mathbf{E}_T\{\cdot\} = \mathbf{E}\{\cdot\ Z_T\} \tag{1.10}$$
$$Z(T) = \frac{1}{\beta(T)\,B(0,T)}.$$

It follows, from the conditional change of measure result of Appendix-A.3.5, that

$$\mathbf{E}_T(X(T)|\mathcal{F}_t) = \frac{\mathbf{E}_0(X(T)\,Z(T)|\mathcal{F}_t)}{\mathbf{E}_0(Z(T)|\mathcal{F}_t)} = \frac{\mathbf{E}_0\left(\left.\frac{\beta(t)}{\beta(T)}X(T)\right|\mathcal{F}_t\right)}{\mathbf{E}_0\left(\left.\frac{\beta(t)}{\beta(T)}\right|\mathcal{F}_t\right)} = \frac{X(t)}{B(t,T)},$$

which simplifies the present value equation (1.6) to

$$X(t) = \mathbf{E}_0\left(\left.\frac{\beta(t)}{\beta(T)}X(T)\right|\mathcal{F}_t\right) = B(t,T)\,\mathbf{E}_T(X(T)|\mathcal{F}_t). \tag{1.11}$$

Also $X(t)$ discounted by $B(t,T)$ is a martingale under the forward measure $\mathbb{P}_T$ because for $s<t$

$$\mathbf{E}_T\left(\left.\frac{X(t)}{B(t,T)}\right|\mathcal{F}_s\right)=\mathbf{E}_T\left(\mathbf{E}_T\left(X(T)|\,\mathcal{F}_t\right)|\,\mathcal{F}_s\right)=\mathbf{E}_T\left(X(T)|\,\mathcal{F}_s\right)=\frac{X(s)}{B(s,T)}.$$

Integrating (1.4) over $[0,T]$ identifies $Z(T)$ because

$$\begin{aligned} B(T,T)=1=B(0,T)\,\beta(T)\,\mathcal{E}\left(-\int_0^T\int_t^T\sigma^*(t,u)\,du\,dW_0(t)\right), \\ \Rightarrow \qquad Z(T)=\mathcal{E}\left\{-\int_0^T\int_t^T\sigma^*(t,u)\,du\,dW_0(t)\right\}, \end{aligned}$$

showing, from the Girsanov Theorem of Section-A.3.5, that $W_T(t)$, given by

$$dW_T(t)=dW_0(t)+\int_t^T\sigma(t,u)\,du\ dt, \tag{1.12}$$

is $\mathbb{P}_T$-BM. Subtracting from a similar expression for $W_{T_1}(t)$, a $\mathbb{P}_{T_1}$-BM,

$$dW_{T_1}(t)=dW_T(t)+\int_T^{T_1}\sigma(t,u)\,du\ dt. \tag{1.13}$$

From equations (1.4), (1.9) and the result in the Appendix A.3.3, the SDE for the forward contract $F_T(t,T_1)$ is

$$\begin{aligned} \frac{dF_T(t,T_1)}{F_T(t,T_1)} &= \left\{\begin{array}{c} r(t)\,dt-r(t)\,dt \\ -\int_T^{T_1}\sigma^*(t,u)\,du\left[dW_0(t)+\int_t^T\sigma(t,u)\,du\ dt\right] \end{array}\right\}, \\ &= -\int_T^{T_1}\sigma^*(t,u)\,du\,dW_T(t), \end{aligned} \tag{1.14}$$

while the SDE for its reciprocal is

$$\begin{aligned} \frac{d\left(\frac{1}{F_T(t,T_1)}\right)}{\left(\frac{1}{F_T(t,T_1)}\right)} &= \left\{\begin{array}{c} r(t)\,dt-r(t)\,dt \\ +\int_T^{T_1}\sigma^*(t,u)\,du\left[dW_0(t)+\int_t^{T_1}\sigma(t,u)\,du\ dt\right] \end{array}\right\}, \\ &= \int_T^{T_1}\sigma^*(t,u)\,du\,dW_{T_1}(t). \end{aligned} \tag{1.15}$$

Hence $F_T(t,T_1)$ is a $\mathbb{P}_T$-martingale while, more importantly as we will see, its reciprocal $\frac{1}{F_T(t,T_1)}$ is a $\mathbb{P}_{T_1}$-martingale.

1.2 The first 'correct' Black caplet

Miltersen, Sandmann and Sondermann [78] started with the assumption that under the T-forward measure $\mathbb{P}_T$ the cash forward $K(t,T)$ over $[T,T_1]$ was of *lognormal type* with deterministic volatility γ (which we here set constant for easy exposition), that is, they assumed the SDE for $K(t,T)$ has form

$$dK(t,T) = (\mathit{drift})\, dt + K(t,T)\ \gamma\, dW_T(t)\,, \tag{1.16}$$

and then worked with the corresponding forward contract $F_T(t,T_1)$ (because it is a $\mathbb{P}_T$-martingale). Differentiating (1.9) using (1.14), and then comparing the stochastic term with that of (1.16), gives an SDE for $F_T(t,T_1)$:

$$\begin{aligned}
dK(t,T) &= \frac{1}{\delta} d\left(\frac{1}{F_T(t,T_1)} - 1\right) \qquad (1.17)\\
&= (\mathit{drift})\, dt + \frac{1}{\delta F_T(t,T_1)} \int_T^{T_1} \sigma(t,u)\, du\; dW_T(t)\\
\Rightarrow \qquad & \int_T^{T_1} \sigma(t,u)\, du = K(t,T)\,\gamma\delta F_T(t,T_1) = [1 - F_T(t,T_1)]\,\gamma\\
\Rightarrow \qquad & dF_T(t,T_1) = -F_T(t,T_1)\,[1 - F_T(t,T_1)]\ \gamma\, dW_T(t)\,.
\end{aligned}$$

The time t value of a Black caplet struck at κ, fixed at T and paid at T_1, is

$$\begin{aligned}
\mathrm{cpl}(t) &= \mathbf{E}_0\left\{ \frac{1}{\beta(T_1)} \delta\,[K(T,T) - \kappa]^+ \middle| \mathcal{F}_t \right\}\\
&= \mathbf{E}_0\left\{ \frac{B(T,T_1)}{\beta(T)} \delta\,[K(T,T) - \kappa]^+ \middle| \mathcal{F}_t \right\},\\
&= B(t,T)\,\mathbf{E}_T\left\{ F_T(T,T_1) \left[\frac{1}{F_T(T,T_1)} - 1 - \delta\kappa\right]^+ \middle| \mathcal{F}_t \right\},\\
&= B(t,T)\,\mathbf{E}_T\left\{ [1 - (1+\delta\kappa)\, F_T(T,T_1)]^+ \middle| \mathcal{F}_t \right\}.
\end{aligned}$$

Applying Ito, Miltersen et al then set to zero the drift of the $\mathbb{P}_T$-martingale

$$v(t, F_T(t,T_1)) = \frac{\mathrm{cpl}(t)}{B(t,T)},$$

so that $\mathrm{cpl}(t)$ is given by the solution $v(t, F_T(0,T_1))$ to the non-linear PDE

$$\frac{\partial v}{\partial t} + \tfrac{1}{2}\gamma^2 x^2 (1-x)^2 \frac{\partial^2 v}{\partial x^2} = 0 \qquad \textit{with} \qquad v(T,x) = [1 - (1+\delta\kappa)\, x]^+\,.$$

This converts to a *heat equation* problem with the transformations

$$s = \gamma^2 (T - t), \quad z = \ln \frac{x}{1-x}, \tag{1.18}$$

$$v(t,x) = \frac{e^{-\frac{s}{8}}}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} u(s,z) \Rightarrow \quad \frac{\partial u}{\partial s} = \frac{1}{2}\frac{\partial^2 u}{\partial z^2}$$

$$\textit{with} \quad u(0,z) = \left\{e^{\frac{z}{2}} + e^{-\frac{z}{2}}\right\}\left[1 - \frac{(1+\delta\kappa)}{1+e^{-z}}\right]^+,$$

which has the solution (substitute in the PDE and integrate by parts)

$$\begin{aligned} u(s,z) &= \int_{-\infty}^{\infty} u\left(0, z + v\sqrt{s}\right) \mathbf{N}_1(v)\, dv \\ &= \int_{-\infty}^{\Upsilon} \left[\exp\left(-\tfrac{1}{2}\left[z + v\sqrt{s}\right]\right) - \delta\kappa \exp\left(\tfrac{1}{2}\left[z + v\sqrt{s}\right]\right)\right] \mathbf{N}_1(v)\, dv, \\ &= \exp\left(-\frac{z}{2} + \frac{s}{8}\right)\mathbf{N}\left(\Upsilon + \frac{\sqrt{s}}{2}\right) - \delta\kappa \exp\left(\frac{z}{2} + \frac{s}{8}\right)\mathbf{N}\left(\Upsilon - \frac{\sqrt{s}}{2}\right), \\ &\textit{in which} \quad \Upsilon = -\frac{1}{\sqrt{s}}(z + \ln \delta\kappa). \end{aligned}$$

Inverting the transforms (1.18) to go from $u(s,z)$ back to $v(t,x)$

$$v(t,x) = x\left\{\frac{[1-x]}{x}\mathbf{N}(h) - \delta\kappa\mathbf{N}\left(h - \tfrac{1}{2}\gamma\sqrt{T-t}\right)\right\},$$

$$h = \frac{1}{\gamma\sqrt{T-t}}\left\{\ln\left(\frac{1-x}{x}\frac{1}{\delta\kappa}\right) + \tfrac{1}{2}\gamma^2(T-t)\right\}$$

the caplet price $\operatorname{cpl}(t)$ follows from $v(t,x)$ on using

$$x = F_T(t,T_1) = \frac{B(t,T_1)}{B(t,T)}, \qquad \delta K(t,T) = \frac{1-x}{x},$$

$$\operatorname{cpl}(t) = B(t,T)\, v(t,x) = \delta B(t,T_1)\ \mathbf{B}\left(K(t,T),\ \kappa,\ \gamma\sqrt{T-t}\right).$$

where $\mathbf{B}(\cdot)$ is the Black formula, see Appendix-A.2.3.

A probabilistic proof of this result obtained by the author while trying to articulate the insight of MSS, runs as follows. Simplify notation by setting $\mathbb{P}_T = \mathbb{P}$, $F_T(t,T_1) = F_t$, $K_t = K(t,T)$ and $W_T(t) = W_t$. From the SDE (1.17) for F_t, if

$$Z_t = \ln \frac{F_t}{1-F_t} \quad \textit{or} \quad F_t = \frac{1}{1+\exp(-Z_t)}, \quad \textit{then}$$

$$dZ_t = -\gamma\left[dW_t - \tfrac{1}{2}\gamma \tanh\left(\tfrac{1}{2}Z_t\right) dt\right], \quad \exp(-Z_t) = \delta K_t.$$