TEXTBOOKS in MATHEMATICS

Linear Algebra

A First Course with Applications

Larry E. Knop

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LINEAR ALGEBRA

A First Course
with Applications

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LINEAR ALGEBRA

A First Course with Applications

Larry E. Knop

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PREFACE FOR THE INSTRUCTOR

I started writing this book six years ago because I like linear algebra and I like talking about linear algebra with students, but I was not happy with my teaching results. Too many students ended their first course confused and uncertain, with no discernable improvement in their reasoning or writing skills, and none of the texts I tried seemed to help. So I spent six years writing, revising, and classroom-testing different ways of presenting the basics of linear algebra. Interesting times.

The intended audience is students who want to learn the fundamental ideas of linear algebra, either because they have to for their program of study or because they would like to. Calculus is not required, although almost all my test subjects had a background of at least one semester of calculus, and most had two or three. My definition of fundamental ideas includes: vector space, subspace, span, linear independence, basis, dimension, linear transformation, eigenvalue, and eigenvector. Elementary numerical methods and awareness of the computational problems of linear algebra are also on my fundamentals list. Implementations of efficient numerical methods are left to a later course. Applications are also fundamental to linear algebra, in my opinion, and a variety of applications are included, from inventories to graphics to Google's PageRank. All the applications in the text include student exercises.

For better or for worse, *Linear Algebra: A First Course with Applications* employs more words than is usual. Mathematics is an extremely compressed language, and linear algebra students are novices at reading the language of mathematics. To make the book more readable by *students* I have not compressed the language to professional standards. I have also employed a significant number of words to discuss mathematical motivation and strategy. Linear algebra students have lots of experience at following directions—e.g., to differentiate a product, take the first times the derivative of the second and add the second times the derivative of the first—but they have little experience in constructing mathematical structures, and some guidance seems appropriate.

The approach throughout the book is to go from the specific to the general. My experience is that people generalize naturally (witness the cautions against jumping to conclusions and the unfortunate abundance of negative stereotypes), but that people do not particularize nearly so easily. Consequently, the text typically presents one or more examples as motivation prior to introducing a definition or theorem. This approach also has the advantage of introducing students to the question *before* presenting an answer to it.

The formal mathematics in this book is written as carefully as I know how. I vividly remember looking at a linear textbook my department was considering for adoption some years ago, and reading a "proof" in that text. The proof started normally, morphed into a numerical example, cited other results, and said "QED." Yes, I could fill in the gaps and make a convincing argument out of what was written, but I do not think that students could. To me, one of the most important goals of an introductory linear algebra course is to improve students' ability to reason logically. Facts can always be referenced; it is modes of thought that need to be learned. The mathematical development in *A First Course* is done carefully and done within the body of the text. Places where the rigor is not self-contained are labeled as such. I do not think that saying "trust me, this is true" does anything toward developing students' reasoning abilities, and I have avoided that whenever possible.

A First Course is written to be read by students. I know that "students don't read math texts," but that is a terrible indictment of mathematics textbooks. As they say, if you give a person an answer then you have satisfied that person for a day; if you teach a person to read mathematics then you have satisfied that person for a lifetime. You read math books, I read math books, and our students are capable of reading math books—and need to learn to do so. Unfortunately, most mathematics textbooks are not very interesting reading. The mathematicians I know are passionate about their discipline and their work, but there is little passion in elementary textbooks. Some texts use history to add a bit of humanity to their pages. Although I stand in awe of the accomplishments of the masters, I do not identify with their lives and neither do most of the students I know.

In *A First Course* I have tried to engage students by drawing on their life experiences and using those experiences to help them learn. Poincare said, "Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change." Poincare astutely identified the goal toward which mathematicians strive, but by implication he also pointed out the foundation from which mathematics arises. Mathematics arises from context and content, not from spontaneous generation. I think we are working against ourselves by showing novices only the finished mathematical product, shorn of meaning and divorced from life. Why not use the connections between mathematics and life to teach the mathematics? If there is an analogy between a seed and a spanning set, why not use it? If Monty Python, Woody Allen, and Madonna made comments that are relevant to mathematics, why not quote them? If an Abraham Lincoln story offers a graphic illustration of the meaning of "well-defined," why not tell the story?

The writing in the discussion parts of *A First Course* is decidedly informal, but none of the writing is for "show." There are serious intentions behind all the pieces. For instance, my colleagues suggested that I divide the exercise sets into two parts: one part to emphasize calculations and the other part to contain more open-ended questions. I liked the idea and adopted it, but then I had to decide what to call the two parts. Part A and Part B was a good start, but I also wanted descriptive names to distinguish them. The name "Computational" for Part A was a no-brainer, but I had difficulty coming up with a descriptive name for Part B. I considered "Theoretical" and discarded it because I dislike the dichotomy it implies between theory and computation. I also considered "Written" but discarded it both because it was inaccurate (computational problems involve writing) and because of the negative associations some math-science students have with anything that says "writing." I eventually came up with "Conversational." The alliteration, "Computational and Conversational," makes the terms easy to remember, and the word "conversational" conveys an attitude that mathematics is not just something one does but is also something to be conversed about with friends. Truth be told, I still wince a bit at the label "conversational" but students don't, and they absorb a bit of the attitude I wanted to convey so I use it.

Coverage: I recommend covering one section of the text per (50 minute) class. No, 50 minutes is not always enough time to talk about every part of a section, but what's not covered in lecture can be covered in the reading. Seriously, if we want students to read mathematics then it is necessary to give them mathematics to read—and expect them to read it. Overall, the text continually builds on previous work, so it is difficult to skip sections—with two exceptions. The exceptions are the first chapter on logic and the section (3.3) on technology.

The first chapter of the text is called Chapter 0 because it is a background primer on mathematical logic. Coverage of Chapter 0 is not required for the rest of the text, but I strongly recommend covering it, either at the beginning of the course or whenever is convenient. My experience is that most students are seriously deficient in the fundamentals of logic. A "proof by contradiction" is an adventure when students cannot negate an *if–then* statement, and discussing span and linear independence is a challenge when students don't understand quantifiers. Covering the chapter does not magically eliminate students' logical deficiencies but it is a start, and it provides a reference whenever problems of logic arise.

Technology: No one gets paid for doing linear algebra by hand, but there is also no consensus as to the role of technology in a first course in linear algebra. Consequently *A First Course* is structured so that you may use technology as little or as much as you wish.

1. If you feel that computer-assisted calculations are a distraction in an introductory linear algebra course, then skip Section 3.3 and the Technology Appendices to Sections 6.1, 6.4, 7.2, and 8.2. All of the exercises in the text, except those in 3.3 of course, can be done by hand (although a few are arguably more arduous than is reasonable).
2. If you wish to use technology, support for Maple, MATLAB, and the TI-83 Plus is embedded in the text in a "just-in-time and just-enough" approach. The text uses technology in two ways. One use is to reinforce student learning of standard procedures by mimicking pencil and paper. Each of the tools will go through procedures step by step in response to student commands, and doing so allows students to focus on learning the process without the mind-clogging distractions of the arithmetic. The other use is to significantly reduce the manual labor associated with many types of exercises. For instance, by the time students are wrestling with the problem of determining linear independence they have already done so many row reductions that further practice is almost counter-productive. Relegating row

> reduction to a few keystrokes allows students to focus on the problem at hand, and to do more problems in the same amount of time. The text is *not* an instruction manual for Maple, or for MATLAB, or for the TI-83 Plus. The text presents just enough instruction to allow students to use the tools to do linear algebra exercises. The primary role of technology in *A First Course* is to enhance the linear algebra learning experience. Giving students a simple, hands-on, and enjoyable (compared to the alternative of pages of hand arithmetic) experience with modern mathematical tools is a side benefit (albeit a substantial one).

Personally, I am on the heavy tool user end of the spectrum. I have no preference between Maple, MATLAB, and the TI-83 Plus (provided the TI-83 is a TI-89). My college makes Maple available to students, so that is what I use in classes. I devote one (1) day in linear algebra to technology instruction. My classrooms have computer projection equipment, so when Section 3.3 comes up I spend a class showing students how to access Maple, how to get started using Maple, and how to have a bit of fun with it. Students are then encouraged to, and do, use Maple on homework assignments, and are required to use Maple on my exams. I see no reason why MATLAB or the TI-83 Plus would require additional time or confer fewer benefits.

Additional goodies such as technology updates, application supplements, and maybe even a chapter on orthogonality will be posted, as they are written, on the Internet at:

knoplinearalgebra.com

Cheers,

Larry Knop

Postscript: I began this Preface by explaining why I wrote *A First Course*. Has the book helped my teaching results? Cause and effect in teaching is always hard to ascertain, but these days my students are more confident in their abilities and more excited about linear algebra. My students are happier, the percentage who take further courses in mathematics is up, and I know students are reading the text from the unsolicited comments I get during office hours about aspects of the book *that I have not talked about in class*. So, yes, I'm convinced that a book can make a difference, and that conviction inspired me to go through the agony and the ecstasy (mainly agony) of preparing this book for publication.

Acknowledgments

First and foremost I want to express my gratitude to my wife Shirley, without whom this book would not exist (and without whom I would probably be a homeless derelict). Shirley not only provided support and encouragement, but she also proofread the book for both style and content despite the fact that her mathematical training and proofreading experience were years behind her. The archetypical experience occurred when I asked Shirley if she would do me a favor, and she replied “yes” and asked what I wanted. I said, “Would you please learn to do linear algebra on a TI-83 calculator?” and handed her the instructions I had just written and a TI-83 Plus. And she did, most helpfully, although she gave me a very funny look when I asked.

The book was, at times, a family project. Oldest son Evan and his wife Emily were particularly helpful in developing my understanding of Google’s PageRank. Middle son Darren worked on the exercises and gave me some very helpful reality checks. Youngest son Travis did a lot of necessary style and format correction.

I also owe a debt of gratitude to my colleagues, particularly Richard Bedient and also Rob Kantrowitz and Sally Cockburn. I greatly appreciate the opportunity that Hamilton College gave me to pursue this project and bring it to completion.

I am particularly indebted to my students at Hamilton College—all my students. I know I hold the title of “teacher,” but I think I have learned as much from my students as they have learned from me. I am particularly indebted to Thao Nguyen Nguyen for work on solutions to the exercises in the text; her work was exemplary. Other students are referenced in the exercises; unfortunately, I was not systematic in my choices so if I missed you in this edition I may be able to remedy that in the future.

Finally, I wish to thank my editor Bob Stern both for taking a chance on me and for being patient while I wrestled through stress, writer’s block, and hubris. I hope I have justified your decision.

And, now, on to the good stuff!

Larry E. Knop

INTRODUCTION: LANGUAGE, LOGIC, AND MATHEMATICS

"When I use a word," Humpty Dumpty said in a rather scornful tone,
"it means just what I choose it to mean—neither more nor less."

Lewis Carroll

Mathematics is the universal language, right?

If mathematics is the universal language, then why is this book written in English?

The short answer is that this book is not written in English. This book is written in mathematics; its appearance is deceiving. Mathematics assimilates popular languages. There is an English dialect of mathematics (the language of this book), and there is a French dialect of mathematics, and a German dialect, and a Mandarin dialect, and so on.

There are similarities between mathematics and the Borg of Star Trek fame. (*You will be assimilated. Resistance is futile.*) In both cases existing things are recruited and adapted to serve another purpose; the Borg recruit people and adapt them to serve a machine intelligence, while mathematics takes over languages and adapts them to serve the cause of logic. The difference between mathematics and the Borg is that incorporation into the Borg diminishes humanity while mathematics has the opposite effect.

Mathematics does not have to be built on a popular language, but it is extremely convenient to do so. The little (English) word "or" illustrates the reasons, and the drawbacks. Given two sentences P and Q, in any language, we might want to assert that at least one of the sentences is true. So we want a way of combining sentences to form a new sentence such that the combination is true when at least one of the components is true, and false when both are false. One way to proceed is to make up an artificial word, such as "gerp," and say:

Definition: Let P and Q be propositions. The **gerp** of P with Q is the compound sentence "P **gerp** Q." The truth value of "P gerp Q" is given by the following table:

P	Q	P gerp Q
T	T	T
T	F	T
F	T	T
F	F	F

⌘

We could then say things like: "For any natural number n, n is even gerp n is odd," and know that what we are saying has a well-defined meaning. The drawback to a language-neutral approach is that we must learn a nonsense string of symbols for each new idea and, even worse, remember the string with no clues or associations to help us remember.

A language neutral approach is sometimes employed in advanced studies in logic, but for nearly all of mathematics the cost is too great. Why make up an artificial word when the English language already has the word "or" in it, a word that means in English *almost* what we want it to mean mathematically? So we do not make up a new word. We borrow (assimilate) the English word "or" and assign to it the mathematical meaning given to "gerp" above.

Unfortunately, there is a downside to the practice of borrowing words. The downside is that the popular meaning of a word and the mathematical meaning given to the word are usually somewhat different. Mathematical words mean exactly what they are defined to mean, neither more nor less. English words have multiple meanings (which is absolutely forbidden in mathematics), and none of the meanings may correspond exactly to the mathematical definition. For instance, the English "or" may be properly used in the inclusive ("Cream or sugar?") or the exclusive ("You behave or you'll be in big trouble.") sense. The mathematical "or" is always inclusive; if two propositions are true then the compound proposition formed by connecting the two with an "or" is also true. When reading, writing, and speaking mathematics, it is always the mathematical "or" that is used.

Another example that you know, but may not have noticed, is the difference between English and math in the use of "open" and "closed." In English, a door may be open or a door may be closed, but a door is never open and closed at the same time. In mathematics, an interval of real numbers is open if it contains none of its endpoints and closed if it contains all of its endpoints. So the interval $(1, 3) = \{x|1 < x < 3\}$ is an open interval and the interval $[-1, 2] = \{x| -1 \leq x \leq 2\}$ is closed. Now the entire real number line, $\mathbb{R}^1$, is an interval. $\mathbb{R}^1$ contains none of its endpoints (there are no endpoints to $\mathbb{R}^1$) so $\mathbb{R}^1$ is open. $\mathbb{R}^1$ also contains all of its endpoints (all 0 of them) so $\mathbb{R}^1$ is closed. The statement "$\mathbb{R}^1$ *is both open and closed*" is a true statement in the language of mathematics, whereas in English the statement is nonsense. And if you think of the meanings of "open" and "closed" that you know (the English meanings) when you read the mathematical statement, then you will confuse the heck out of yourself.

An example that will arise in the near future is that of linear equations. A linear equation in three variables is defined to be any equation of the form $ax + by + cz = d$ where a, b, c, and d are fixed real numbers. Thus $2x + 1y + 3z = 7$, $1x + 0y - \pi z = 12.3$, $0x + 0y + 0z = 1$, and $0x + 0y + 0z = 0$ are all linear equations in three variables. Given the name *linear* equation, there is an understandable tendency to think that a linear equation has something to do with lines and a linear equation does—but not in three variables. In three variables a linear equation describes a plane in 3-space, or all of 3-space, or the empty set—and that's all.

To communicate in the language of mathematics you must know the vocabulary. You get no points in mathematics for misinterpretations, no matter how creative. In particular, you must have a formal knowledge of the language, which means knowing the precise definitions of the words we use. You must also have an intuitive understanding of the ideas embodied in the words. What is prescribed here is only common sense. You cannot play the game if you do not know the rules, and you cannot speak the language if you do not know the words. The game of mathematics is ruled by logic. The words of the mathematics language are created as we go along, but they are always specified precisely. Do learn them.

CHAPTER **0**

A Little Logic

SECTION 0.1: LOGICAL FOUNDATIONS

"Contrariwise," continued Tweedledee, "if it was so, it might be; and if it were so, it would be; but as it isn't, it ain't. That's logic."

Lewis Carroll

English sentences come in many different types: declarative, exclamatory, interrogative, and imperative are the usual classifications.

Mathematics is spoken in declarative sentences.

Definition 1: A **proposition** (or **statement**) is a declarative sentence that is either true (T) or false (F), but not both. ⌘

An example of a proposition is "$1+1=2$." The example is written as a string of symbols, but the symbols are shorthand for the English sentence "*One plus one is equal to two*." The example is a proper declarative sentence with a truth value of T, and hence is a proposition. Another example of a proposition is "$1+1=3$." Again we have a declarative sentence with a definite truth value (false).

The content of propositions is not restricted to arithmetic. "*John F. Kennedy was President of the United States in 1961*" is a (true) statement. "*Helium is the first element in the periodic table*" is a (false) proposition.

Of course many sentences are not propositions. "*Are we having fun yet?*" is an interrogative rather than a declarative sentence, and so cannot be a proposition. Similarly, "*Make love, not war!*" is an imperative rather than a declarative sentence, and so is not a proposition. The claim that "*29 angels can dance on the head of a pin*" is a declarative sentence, but the truth value cannot be determined because of the difficulty in counting dancing angels. Hence the sentence is not a proposition. "*This sentence is false*" is a sentence that cannot be true and cannot be false, and so cannot be a statement. "*Would you like to party with me tonight?*" is also not a mathematical(!) proposition.

In mathematics, propositions have both form and content. The study of logic is primarily the study of form. To strip away content and focus on form, we will talk about arbitrary propositions and denote them by capital letters such as P or Q. Simply replacing statements by letters gains us little in the way of insight however. We gain understanding by taking a complex structure and reducing it to simple (understandable) pieces, or by taking simple (understandable) pieces and joining them to make a complex product. We will use both approaches here.

Definition 2: Sentences can be connected by words such as "and" and "or," and a sentence can be modified by a word like "not." Such words are called **logical connectives** or **logical operators**. Any proposition that is the result of a logical operator applied to one or more propositions is called a **compound proposition**. ⌘

A fundamental assumption of logic is that

The truth value of a compound proposition is determined by the truth values of its component propositions and by the logical connectives.

If you know the truth values of the pieces and if you know how the pieces are connected then you know the truth value of the statement.

* * *

The three basic logical operators are "and," "or," and "not." In each case, the mathematical meaning corresponds reasonably well to common English usage.

Definition 3: Let P and Q be propositions. The **conjunction** of P with Q is the compound sentence "***P* and *Q***." The truth value of "P and Q" is given by the following table:

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

"P and Q" is written $\boldsymbol{P \wedge Q}$. ⌘

Example 1a: To illustrate these ideas consider the following sentence:

"*Monarch butterflies migrate and crocodiles lay eggs*."

Let P be the sentence "*Monarch butterflies migrate*" and Q be the sentence "*Crocodiles lay eggs*." Our compound sentence has the form $P \wedge Q$. Because both P and Q are true propositions, line 1 of the conjunction truth table applies. Our sentence is a true statement.

Example 1b: Another example is the sentence

> "*Los Angeles is in Florida, and Pasadena is in California*."

Let P be the sentence "*Los Angeles is in Florida*" and Q be the sentence "*Pasadena is in California*." Again our sentence has the form $P \wedge Q$, but this time P is false while Q is true. Line 3 of the conjunction truth table is the line that applies to the example and, consequently, our sentence is a false statement.

Example 1c: You should be aware that standard English, at times, is almost perversely dedicated to obscuring logical form. Consider the sentence

> "*Chicago and New Orleans are cities in Illinois.*"

The sentence is a compound sentence even though it contains only one verb. The given sentence is shorthand for the sentence:

> "*Chicago is a city in Illinois and New Orleans is a city in Illinois.*"

The logical form of the sentence is $P \wedge Q$, where P is the sentence "*Chicago is a city in Illinois,*" and Q is the sentence "*New Orleans is a city in Illinois*." Since P is true and Q is false, line 2 of the conjunction truth table is the relevant line. By our agreement as to the meaning of "and," the given sentence is a false statement.

Example 1d: One further complication arises when sense is combined with nonsense. Consider the sentence

> "*$1+1=3$ and 29 angels can dance on the head of a pin.*"

This sentence has no truth value and is not a proposition. (!) The first sentence, "$1+1=3$," is a proposition, but the second sentence has no truth value, and hence is not a proposition. *The logical "and" only connects propositions*. There is no line in the table for the case where P is false and Q is nonsense. Definition 3 does not apply, and the "and" in the sentence above is not the logical "and" (even though it is spelled the same). The sentence may be proper English, but it is not a proper sentence in the language of logic.

Definition 4: Let P and Q be propositions. The **disjunction** of P with Q is the compound sentence "***P* or *Q***." The truth value of "P or Q" is given by the following table:

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

"P or Q" is written $\boldsymbol{P \vee Q}$. ⌘

Example 2a: An example of sentences connected by a disjunction is the compound sentence

''*Romeo loved Juliet or MTV is a home shopping channel*.''

Let P be ''*Romeo loved Juliet*'' and Q be ''*MTV is a home shopping channel*.'' Our example has the symbolic form $P \vee Q$. Since P is true while Q is false, line 2 of the disjunction truth table applies and hence our compound proposition is true.

Example 2b: Another example is the sentence

''*Elvis Presley is alive or* $10^{215} + 1$ *is an even number*.''

The sentence has the form $P \vee Q$, provided we take P to be ''*Elvis Presley is alive*'' and Q to be ''$10^{215} + 1$ *is an even number*.'' Because both P and Q are false, the compound statement is false by the definition of disjunction.

Definition 5: Let P be a proposition. The **negation** of P is the compound sentence "**not *P***." The truth value of "not P" is given by the following table:

P	not P
T	F
F	T

"Not P" is written $\sim\boldsymbol{P}$. The "not" may also be read as "it is false that." ⌘

Example 3a: An example of a negation of a sentence is the sentence

''*The U.S. Supreme Court does not have nine justice positions*.''

The sentence has the form $\sim P$, where P is the statement ''*The U.S. Supreme Court has nine justice positions*.'' Because P is a true statement, the statement $\sim P$ is false. An alternative way to write the sentence $\sim P$ is

''*It is false that the U.S. Supreme Court has nine justice positions*.''

Example 3b: Another example of a negation is the sentence

''*Julius Caesar was not a sumo wrestler*.''

Let P be the sentence ''*Julius Caesar was a sumo wrestler*.'' Our example has the form $\sim P$, and $\sim P$ is a true statement because P is false.

* * *

Example 4a: The previous examples used only one or two simple statements and a single logical operator. Life can be more complicated, and so can compound propositions. For instance consider the logical form and the truth value of the proposition:

''Elephants have wings and 6 is divisible by 2, or,
6 is not divisible by 2 and Beijing is in China.''

Let P, Q, and R be the sentences

P: *''Elephants have wings.''* Q: *''6 is divisible by 2.''* R: *''Beijing is in China.''*

The symbolic form of the proposition is $(P \wedge Q) \vee (\sim Q \wedge R)$. Since Q and R are true while P is false, $P \wedge Q$ must be false and so must $\sim Q \wedge R$. Thus by the definition of disjunction, the compound statement $(P \wedge Q) \vee (\sim Q \wedge R)$ must be false.

Example 4b: Another example of a complicated proposition is the sentence

''Einstein was not a dancer, or, it is false that, the sun is not a star and Einstein was a dancer.''

Let P and Q be the sentences

P: *''Einstein was a dancer.''* Q: *''The sun is a star.''*

For this choice of P and Q, the proposition has the logical form $(\sim P) \vee \sim ((\sim Q) \wedge P)$. The statement P is false, while the statement Q is true. Thus we have $\underbrace{(\sim P)}_{T} \vee \underbrace{\sim \underbrace{((\sim Q) \wedge P)}_{F}}_{T}$, and so the truth value of the entire statement is true.

* * *

English does not necessarily follow the rules of logic, but mathematics always does.

In mathematics, the words "and," "or," and "not" *always* have the meanings given above. The same is not true of English. In English, for example, "or" can be used in the inclusive sense of meaning one or the other or both of the connected propositions are true (which corresponds to mathematical usage). Or "or" can be used in the exclusive sense of meaning one or the other but not both. When you visit a friend's home, your friend may ask you: "Coffee or tea?" The "or" in this case is the exclusive "or." You may have coffee or you may have tea, but you are not being offered both. If you choose one then your friend may say "Cream or sugar?" The "or" in this case is now the inclusive "or", and you can indeed have both cream and sugar if you so choose. Needless to say, English can get quite confusing. Such ambiguity and confusion is contrary to the spirit of mathematics.

To reiterate a very important point: In mathematics, the words "and," "or," and "not" *always* have the meanings given them by the definitions above.

* * *

The logical operators "and," "or," and "not" are all the operators you will ever need. There are however, two additional logical connectives that occur so frequently that they are given special names. One of these connectives is the conditional, "if P then Q."

The conditional connective is the least intuitive (least "English-like") of all the logical operators. The unease that many people feel when first encountering the conditional comes from the fact that our basic principle requires us to specify the effect of the conditional in *all possible* cases. The truth value of a compound proposition must be determined by the truth values of its component propositions and by the logical connectives. An "if P then Q" statement has two component propositions and, consequently, the truth table has four lines. We must specify the truth value of "if P then Q" in all four possible cases.

There is general agreement as to the truth value of the following statements:

(1) "If $1 = 1$ then $2 = 2$." The sentence is a T statement.
(2) "If $1 = 1$ then $2 = 3$." The sentence is a F statement.

There is often an initial confusion and disagreement as to the truth values of the following statements:

(3) "If $1 = 2$ then $3 = 3$." The sentence is a ??? statement.
(4) "If $1 = 2$ then $2 = 3$." The sentence is a ??? statement.

Statements (3) and (4) must be either true or false; to decide otherwise is not an option. The choice made by mathematicians is specified in the following definition.

Definition 6: Let P and Q be propositions. The **conditional** of P with Q is the compound sentence "**if *P* then *Q*.**" The truth value of "if P then Q" is given by the following table:

P	Q	If P then Q
T	T	T
T	F	F
F	T	T
F	F	T

"If P then Q" is written as $\boldsymbol{P \Rightarrow Q}$. ⌘

Example 5: By definition, the sentences *"If $1 = 2$ then $3 = 3$"* and *"If $1 = 2$ then $2 = 3$"* are both true statements. To further illustrate the consequences of the definition, the sentence *"If blue is a color then sweet is a taste"* is a true statement. The sentence *"If bridges can vibrate then tapioca is a metal"* is a false statement. The sentence *"If computers always work properly then a daffodil is a flower"* is a true statement. The sentence *"If pigs can fly then your uncle is a camel"* is a true statement.

The truth values in the "if P then Q" truth table are arbitrary in the sense that other choices could be made. You may or may not like the choices embodied in Definition 6, but those are the choices that mathematicians have agreed upon. *All* of mathematics uses the

Definition 6 meaning of "if P then Q," although you may not have been consciously aware of that in your previous mathematical work. To communicate in the language of mathematics you too must use "if P then Q" in the Definition 6 sense.

While the choices are arbitrary, the choices are not random. It is a legitimate question to ask "why make those particular choices?" and you should ask "why?" One answer lies in the following situations: Your physics teacher says to you: "If you release the ball then the ball will fall." Your parents say to you: "If you are not home by midnight then you will not get to use the car this weekend." Your advisor says to you: "If you do not satisfy the Phys Ed requirement then you will not graduate." Now suppose that you do not release the ball, that you get home before midnight, and that you meet the Phys Ed requirement. Do your actions mean that you made liars out of your physics teacher, your parents, and your advisor? I think not. Your physics teacher said: "IF you release the ball then the ball will fall." The only way to show that your physics teacher spoke falsely is to show that you can release the ball and the ball does not fall. The fact that you did not release the ball does not make your physics teacher a liar. To say that your parents spoke falsely you must get home after midnight and still be allowed to use the car on the weekend. To say that your advisor lied you must fail to satisfy the Phys Ed requirement and still be allowed to graduate.

A mathematical version of the same rationale is provided by the Pythagorean Theorem. The Pythagorean Theorem states: If R is a right triangle with sides a, b, and c, where c is the longest side then $a^2 + b^2 = c^2$. Now a theorem is a statement that is always true and, even if we do not remember a proof, we have been told the Pythagorean Theorem is true often enough that we certainly believe it to be a theorem. So take a triangle S, where S is *not* a right triangle, and apply the Pythagorean Theorem. The Pythagorean Theorem has the logical form "if P then Q," and in the case we are considering the statement P is false. So we are either on Line (3) or Line (4) of the conditional truth table, and for the Pythagorean Theorem to be a theorem the truth value *must* be true. Hence we *must* require statements of the form "if P then Q" to be true whenever P is false.

The requirement is *necessary* for the Pythagorean Theorem to be a theorem. An unavoidable consequence of the requirement is that the sentence "*If elephants can tap dance, then porcupines can juggle*" is a true statement.

* * *

There is much additional terminology associated with conditional statements. For instance, the component propositions that make up a conditional statement have their own names.

$$\text{If } \underbrace{\quad P \quad}_{\substack{\text{hypothesis} \\ \text{sufficient condition}}} \text{ then } \underbrace{\quad Q \quad}_{\substack{\text{conclusion} \\ \text{necessary condition}}} .$$

Consider the conditional proposition: "*If the U.S. is a democracy then U.S. citizens have the right to vote.*" The *hypothesis* is the statement "*the U.S. is a democracy*," and the *conclusion* is the statement "*U.S. citizens have the right to vote.*"

The "sufficient" and "necessary" terminology comes from logical argumentation. Suppose the statement "if P then Q" is true. Under this supposition, to guarantee Q is true it is *sufficient* to know that P is true. Similarly, under this supposition, if we also know that P is true then it is *necessary* that Q be true.

Different ways of writing a conditional statement include the following:

If P then Q.
P implies Q.
P only if Q.
P is a sufficient condition for Q.
Q is a necessary condition for P.

Each of the sentence forms above is an English translation of $P \Rightarrow Q$. It is not clear why we need so many different ways of saying the same thing. This text will primarily use the "if P then Q" form when making conditional statements.

Given a conditional statement "if P then Q," many variations of the statement can be formed by altering the order in which P and Q appear and by inserting negations. Several of these variations have their own names.

Statement Form	**Name**
If P then Q	(Original statement)
If $(\sim Q)$ then $(\sim P)$	Contrapositive
If Q then P	Converse
If $(\sim P)$ then $(\sim Q)$	Inverse

To illustrate the terminology, suppose "if P then Q" is the statement "*If rabbits eat wolves then rabbits are carnivorous.*" The *contrapositive* is the statement "*If rabbits are not carnivorous then rabbits do not eat wolves.*" The *converse* is the statement "*If rabbits are carnivorous then rabbits eat wolves.*" The *inverse* is the statement "*If rabbits do not eat wolves then rabbits are not carnivorous.*"

Please note that the four statements (the original, the contrapositive, the converse, and the inverse) are four different statements. There are logical relationships between some of the statements and we shall explore those relationships shortly, but the statements themselves are *different* statements.

* * *

There is one more common logical connective to investigate, and that is the biconditional.

Definition 7: Let P and Q be propositions. The **biconditional** of P with Q is the compound sentence "***P* if and only if *Q*.**" The truth value of "P if and only if Q" is given by

P	Q	P if and only if Q
T	T	T
T	F	F
F	T	F
F	F	T

"P if and only if Q" is written $\boldsymbol{P \Leftrightarrow Q}$. ⌘

Example 6a: As an example of the biconditional, let us determine the truth value of the sentence

"A bee is an insect if and only if a watermelon is a reptile."

The sentence has the logical form $P \Leftrightarrow Q$ where P is the proposition "*A bee is an insect*" and Q is the proposition "*A watermelon is a reptile.*" As P is true and Q is false, Line 2 of the biconditional truth table applies, and the compound statement is false.

Example 6b: Another example of a biconditional statement is the sentence

"$4 < 2$ if and only if $5 < 3$."

The sentence has the form $P \Leftrightarrow Q$, and both P and Q are false propositions. Thus Line 4 of the biconditional truth table applies, and the biconditional statement is true.

* * *

To illustrate the logical connectives, suppose we have three cards as shown in Figure 1.

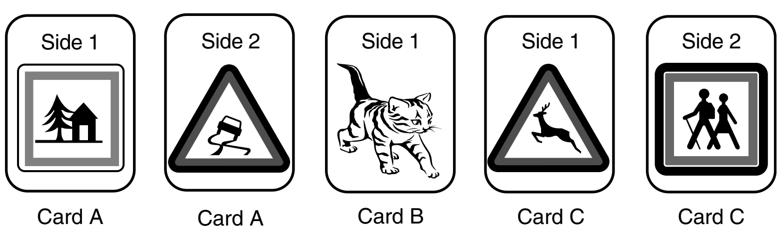

FIGURE 1

Assume that the cards are all two-sided, with the sides labeled Side 1 and Side 2, and both sides have pictures. Because we can only see the sides of the cards shown in the figure, we do not know what is on Side 2 of Card B; we only know that there is a picture. Let P and Q be the sentences:

P: "*Side 1 has a picture of an information sign (a square).*"
Q: "*Side 2 has a picture of a road sign (a triangle).*"

Under our assumptions, the sentences P and Q are propositions for each of the cards. Sentence P is true if we are talking about Card A, and false if we are talking about either Card B or Card C. Sentence Q is true for Card A, false for Card C, and the truth value of Q is indeterminate for Card B, because we have incomplete information.

Example 7a: For the cards in Figure 1, consider the sentence $P \wedge Q$: *"Side 1 has a picture of an information sign (a square) and Side 2 has a picture of a road sign (a triangle)."*

The sentence $P \wedge Q$ is true when the object of discussion is Card A. The sentence $P \wedge Q$ is false when the subject is Card C. The sentence $P \wedge Q$ is also false for Card B; we do not know the truth value of Q for Card B, but we do know P is false and that (together with the knowledge that Q has a truth value) is enough to guarantee the falsity of $P \wedge Q$.

Example 7b: For the cards in Figure 1, consider the sentence $P \vee Q$: *"Side 1 has a picture of an information sign (a square) or Side 2 has a picture of a road sign (a triangle)."*

The sentence $P \vee Q$ is true when the subject is Card A and false when the subject is Card C. The truth value of $P \vee Q$ is indeterminate for Card B. Because P is false for Card B, the truth value of $P \vee Q$ will be determined by the truth value of Q, and we need to see the other side of Card B to determine that truth value.

Example 7c: For the cards in Figure 1, consider the sentence $\sim P$: *"It is false that Side 1 has a picture of an information sign (a square)."* (Alternatively, *"Side 1 does not have a picture of an information sign (a square)."*)

The proposition $\sim P$ is true when the subject is Card B or Card C, and $\sim P$ is false when the subject is Card A.

Example 7d: For the cards in Figure 1, consider the sentence $P \Rightarrow Q$: *"If Side 1 has a picture of an information sign (a square) then Side 2 has a picture of a road sign (a triangle)."*

For Card A, the truth value of $P \Rightarrow Q$ is true, because the hypothesis P is true and the conclusion Q is true. For Card B, the truth value of $P \Rightarrow Q$ is also true. The hypothesis P is false and so long as the conclusion Q has a truth value (which it does) then the compound sentence will be true. Finally, for Card C, we have another instance in which the truth value of $P \Rightarrow Q$ is true. For C, both the hypothesis P and the conclusion Q are known to be false so the compound sentence is defined to be true.

Example 7e: For the cards in Figure 1, consider the sentence $P \Leftrightarrow Q$: *"Side 1 has a picture of an information sign (a square) if and only if Side 2 has a picture of a road sign (a triangle)."*

The truth value of $P \Leftrightarrow Q$ is true for each of cards A and C because the truth values of P and Q are the same for each card. The truth value is indeterminate for Card B. To determine the truth value of $P \Leftrightarrow Q$ we need to know whether or not the truth values associated with the two propositions are a match, and to do that we need to know what is on both sides of the card.

* * *

A compound proposition can involve more than a single logical operator of course. Fortunately, common decency and human limitations generally inhibit the logical complexity of sentences used in communications, although there are notable exceptions in areas such as literature and philosophy.

Example 8: What is the logical form and the truth value of the following proposition?

"If horses have tails or peanuts grow on trees
then $1+1=2$ if and only if frogs do not play chess."

Let P, Q, R, and S be the sentences

P: *"Horses have tails."* Q: *"Peanuts grow on trees."*
R: *"$1+1=2$."* S: *"Frogs play chess."*

For this choice of P, Q, R, and S, the compound proposition has the logical form

$$(P \vee Q) \Rightarrow (R \Leftrightarrow \sim S).$$

Now the sentences P and R are true, while the sentences Q and S are false. Thus $P \vee Q$ is true by the definition of disjunction, and $R \Leftrightarrow \sim S$ is also true by the definitions of negation and biconditional. Because an implication is true if both the hypothesis and the conclusion are true, we can conclude that the overall statement is true.

The content of the previous example is silly and was chosen that way to highlight the logical structure of the sentence. The logical forms we are studying underlie all of linear algebra (all of mathematics for that matter), and for now we will concentrate on form. There is no shortage of serious content later.

* * *

EXERCISES

Exercises with boldface italicized numbers have answers and/or hints in the back.

Part A: Computational

1. Let P, Q, R, and S be the sentences:
   P: Penguins can fly. Q: Math is a 4-letter word.
   R: TX stands for Texas. S: Tofu is a fruit.
   Using P, Q, R, and S, find the logical form of each of the following sentences:
   a) Penguins can fly and TX stands for Texas.
   b) Math is not a 4-letter word or penguins cannot fly.
   c) If tofu is a fruit then, TX stands for Texas and penguins cannot fly.
   d) Math is a 4-letter word if and only if tofu is a fruit, and, penguins can fly if and only if TX does not stand for Texas.
   e) If math is a 4-letter word then math is not a 4-letter word, or, if math is not a 4-letter word then math is a 4-letter word.
   f) If penguins can fly and tofu is a fruit, then, TX does not stand for Texas and penguins cannot fly.

g) It is false that, math is a 4-letter word and tofu is not a fruit.

h) If penguins can fly then tofu is a fruit, if and only if, TX stands for Texas or math is not a 4-letter word.

2. Let P, Q, R, and S be the sentences:

   P: IA grows corn. Q: ID grows potatoes.

   R: CA grows grapes. S: GA grows peanuts.

   Using P, Q, R, and S, translate the following logical forms into English sentences.

   a) $R \wedge S$
   b) $\sim(P \vee \sim Q)$
   c) $(P \Rightarrow S) \wedge (R \Rightarrow Q)$
   d) $(P \vee R) \wedge (\sim P \vee \sim R)$
   e) $R \Rightarrow (Q \wedge \sim S)$
   f) $P \wedge (Q \vee R \vee S)$
   g) $(\sim R \vee \sim P) \Leftrightarrow (S \wedge Q)$
   h) $P \Rightarrow (R \Rightarrow S)$

3. Find the truth value of each of the following propositions:

   a) It is false that $1+1$ is equal to 3.
   b) (2)(3) is even and 3^6 is odd.
   c) $2+6=26$ or $2^{\pi} > \pi^2$.
   d) If $1+2=4$ then $1+2+3=7$.
   e) If $1+2=4$ then $1+2+3=6$.
   f) If $1+2+3=7$ then $1+2=4$.
   g) If $1+2+3=6$ then $1+2=4$.

4. For each of the following conditional sentences, identify the hypothesis and the conclusion:

   a) If Ray Charles was a blind singer then Bob Dylan plays the harmonica.
   b) If John is a marathon runner and John has bad footwear then John will suffer the agony of de-feet.
   c) If the definition of "lymph" is "to walk with a lisp," then, the definition of "to abdicate" is "to give up all hope of ever having a flat stomach" and the letters DNA stand for the National Dyslexics Association.
   d) If elephants are called pachyderms and skin doctors are called dermatologists then elephant skin doctors are called pachydermatologists.

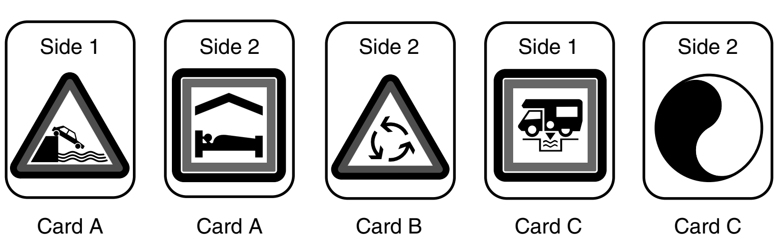

FIGURE 2

Suppose we have three cards as shown in Figure 2. Assume that the cards are all two-sided with the sides labeled Side 1 and Side 2, and that both sides have pictures. Assume further that we can only see the sides of the cards shown in the figure. Let P and Q be the sentences:

P: "*Side 1 has a picture of an information sign (a square).*"
Q: "*Side 2 has a picture of a road sign (a triangle).*"

5. For Card A, based on the information in Figure 2, determine the truth value of the given sentence if possible. Is the sentence true, false, or indeterminate?
   a) Q b) $P \wedge Q$ c) $P \vee Q$ d) $\sim P$ e) $P \Rightarrow Q$ f) $Q \Rightarrow P$ g) $P \Leftrightarrow Q$

6. For Card B, based on the information in Figure 2, determine the truth value of the given sentence if possible. Is the sentence true, false, or indeterminate?
   a) Q b) $P \wedge Q$ c) $P \vee Q$ d) $\sim P$ e) $P \Rightarrow Q$ f) $Q \Rightarrow P$ g) $P \Leftrightarrow Q$

7. For Card C, based on the information in Figure 2, determine the truth value of the given sentence if possible. Is the sentence true, false, or indeterminate?
   a) Q b) $P \wedge Q$ c) $P \vee Q$ d) $\sim P$ e) $P \Rightarrow Q$ f) $Q \Rightarrow P$ g) $P \Leftrightarrow Q$

Part B: Conversational

1. For each of the following, decide whether or not the sentence is a proposition and justify your decision.
   a) Iowa is a state.
   b) New York is the best state.
   c) Alabama is a larger state than Mississippi in terms of area.
   d) Chicago is a city and Illinois is a country.
   e) Nebraska is a state or Nebraska is a planet.
   f) If Florida is a state and Miami is a city then the U.S. is a country or Australia is easy to find on a map.
   g) "I would not live forever, because we should not live forever, because if we were supposed to live forever then we would live forever, but we cannot live forever, which is why I would not live forever."
   – A competitor in the 1994 Miss USA contest

2. For each of the following sentences, first write the contrapositive and then determine the truth values of both the original sentence and the contrapositive.
   a) If pigs can fly then alligators can dance ballet.
   b) If the *Wall Street Journal* is a newspaper then the Dow Jones Industrial Average is a number.
   c) If chocolate is a flower then corn is a vegetable.
   d) If tennis is a sport then football is a drink.

3. For each of the following sentences, first write the converse and then determine the truth values of both the original sentence and the converse.
   a) If pigs can fly then alligators can dance ballet.
   b) If the *Wall Street Journal* is a newspaper then the Dow Jones Industrial Average is a number.
   c) If chocolate is a flower then corn is a vegetable.
   d) If tennis is a sport then football is a drink.

4. If possible, for each of the following symbolic forms give an example of a *true* English sentence that has the specified form. If a true sentence is not possible, please explain why not.
   a) $(P \wedge Q) \Rightarrow (\sim Q)$
   b) $(P \vee \sim P) \Rightarrow (P \wedge \sim P)$
   c) $(P \vee Q) \wedge (\sim P \vee \sim Q)$
   d) $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

5. Is "$3 < \pi < 7$" a compound statement? If so, what are the simple statements that make up the sentence, and what is the logical connective?

6. Suppose P and Q are propositions for which "$P \Rightarrow Q$" is true and "P" is true. What can you say about the truth value of Q, and why?

7. Suppose P and Q are propositions for which "$P \vee Q$" is true and "Q" is false. What can you say about the truth value of P, and why?

8. Suppose P and Q are propositions for which "$P \wedge Q$" is false and "$P \vee Q$" is true. What can you say about the truth values of P and Q, and why?

SECTION 0.2: LOGICAL EQUIVALENCES

Either we are alone in the universe, or we are not.
Either way, it's a mighty sobering thought.

Pogo

We will now look at logical forms in general rather than looking at specific sentences. For motivation consider the following two sentences:

"*Proving three theorems before breakfast is a significant achievement.*"
"*Proving three theorems before breakfast is a not insignificant achievement.*"

These are different sentences with different logical forms. If the first sentence is P then the second sentence has the form $\sim(\sim P)$. At the same time however, these sentences express the same meaning. In fact we are taught in English, no less, that a double negative "is" a positive and that, in the interest of good writing, we should avoid double negatives.

It is this idea of "different but same" that we wish to formalize.

* * *

Definition 1: Two sentence forms are **logically equivalent** if and only if the forms have the same truth value in every case. Logical equivalence is written $\equiv$. ⌘

Theorem 1: $\sim(\sim P) \equiv P$

Proof: To prove the claim we examine the following truth table.

$\boldsymbol{P}$	$\sim P$	$\sim(\sim \boldsymbol{P})$
T	F	**T**
F	T	**F**

The first and third columns are the same, so P and $\sim(\sim P)$ have the same truth value in every case. Definition 1 is satisfied, so $\sim(\sim P) \equiv P$. ⌁

Note: The following story is almost certainly a myth, but a myth with a point.

A linguistics professor was lecturing to her class one day. "In English," she said, "A double negative forms a positive. In some languages though, such as Russian, a double negative is still a negative. However, there is no language wherein a double positive can form a negative." At which point a voice from the back of the room piped up, "Yeah, right."

In the English of Shakespeare's day a double negative was also an emphatic negative and not a positive, but English changed over time in this respect. In the language of mathematics, $\sim(\sim P)$ is *always* logically equivalent to P no matter what native language you use to express the mathematics. If "$\sim$" is an operator that changes the truth value of a sentence then applying the operator twice must change the truth value twice, logically speaking. Popular languages are not always logical, but mathematics always is.

* * *

There are many other logical equivalences, and not all of them are completely obvious.

Theorem 2: (DeMorgan's Laws): $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$ and
$\sim(P \vee Q) \equiv (\sim P) \wedge (\sim Q)$

Proof: To prove the first of the laws we construct the following truth table:

P	Q	$P \wedge Q$	$\sim(\boldsymbol{P} \wedge \boldsymbol{Q})$	$\sim P$	$\sim Q$	$(\sim\boldsymbol{P}) \vee (\sim\boldsymbol{Q})$
T	T	T	**F**	F	F	**F**
T	F	F	**T**	F	T	**T**
F	T	F	**T**	T	F	**T**
F	F	F	**T**	T	T	**T**

The fourth and seventh columns are the same, so $\sim(P \wedge Q)$ and $(\sim P) \vee (\sim Q)$ have the same truth value in every case. Hence $\sim(P \wedge Q) \equiv (\sim P) \vee (\sim Q)$. ⌿

The proof of the second law is similar.

Example 1: To illustrate the use of DeMorgan's Laws, we can ask: What is the negation of

"Roses are red and violets are blue."?

A trivial and somewhat unsatisfying answer is that the negation is

"It is false that, roses are red and violets are blue."

The negation has the logical form $\sim(P \wedge Q)$, and by DeMorgan's Laws this form is logically equivalent to $(\sim P) \vee (\sim Q)$. Thus an equivalent way of expressing the negation is

"Roses are not red OR *violets are not blue."*

Please be careful to change "and" to "or," and "or" to "and," when negating compound sentences.

* * *

There are interesting parallels between properties of logical equivalence and properties of arithmetic, but here we will only observe and not explore the similarities.

Theorem 3:

a) $P \wedge Q \equiv Q \wedge P$, $P \vee Q \equiv Q \vee P$ (Commutative Property)
b) $P \wedge (Q \wedge R) \equiv (P \wedge Q) \wedge R$ (Associative Property)
b′) $P \vee (Q \vee R) \equiv (P \vee Q) \vee R$

c) $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$ (Distributive Property)
c′) $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

All of these logical equivalences can be established by constructing an appropriate truth table.

* * *

In Section 0.1 your author stated that "the logical operators 'and,' 'or,' and 'not' are all the operators you will ever need"—and then the text went on to introduce *another* logical operator, the conditional. Is the conditional operator logically equivalent to some combination of "and," "or," and "not"? The answer is: yes.

Theorem 4: $(P \Rightarrow Q) \equiv ((\sim P) \vee Q)$

Proof: The proof of the claim is straightforward, but the result is quite important.

P	Q	$\mathbf{P \Rightarrow Q}$	$\sim P$	$(\sim \mathbf{P}) \vee \mathbf{Q}$
T	T	**T**	F	**T**
T	F	**F**	F	**F**
F	T	**T**	T	**T**
F	F	**T**	T	**T**

The third and fifth columns are the same, so $(P \Rightarrow Q) \equiv ((\sim P) \vee Q)$ by the definition of logical equivalence. ↯

Example 2: An example of the equivalence is the sentence

"If I see my parents this weekend then I will be happy,"

and the sentence

"I will not see my parents this weekend or I will be happy."

The two sentences are different; they use different words and they have different logical forms. The two sentences are related in that they *always* have the same truth value. The fact that they must be both true or both false is a consequence of their logical forms and is independent of their content.

The logical equivalence between $P \Rightarrow Q$ and $(\sim P) \vee Q$ should be (or should become) not only believable but obvious. One way to think about the equivalence is to think about what it means for a sentence of the form $P \Rightarrow Q$ to be true. Saying "P implies Q" is saying either we do not have P or (we do have P and hence) we must have Q—i.e., saying "P implies Q" is saying "not P or Q."

* * *

The conditional logical form is critically important in mathematics because mathematics is the study of relationships, and very few relationships are true in all possible circumstances. So in mathematics we must constantly deal with sentences of the form: "*If* such and so circumstances hold *then* this particular outcome must follow." We must be completely comfortable with the conditional form, and part of understanding any logical form is knowing the negative of the form.

Example 3: To begin with an example consider again the sentence

"If I see my parents this weekend then I will be happy."

The negation of the sentence is

"It is false that, if I see my parents this weekend then I will be happy."

Unfortunately this form is not very enlightening. We need a form that is logically equivalent to the negation, but whose meaning is more transparent. We can deduce an equivalent version of the negation by considering the circumstances that would make the original sentence false. For the sentence *"If I see my parents this weekend then I will be happy"* to be false, it must be the case that I see my parents this weekend *and* that I am not happy. Consequently an equivalent version of the negation of the original sentence is the sentence

"I will see my parents this weekend and I will not be happy."

Example 3 is a specific instance of the following general property.

Theorem 5: $\sim(P \Rightarrow Q) \equiv (P \wedge (\sim Q))$

Proof: The validity of the claim can be established by a truth table, but here we can also use other means. By Theorem 4, we know that $\sim(P \Rightarrow Q) \equiv \sim((\sim P) \vee Q)$. By DeMorgan's Laws (Theorem 2), we know that $\sim((\sim P) \vee Q) \equiv \sim(\sim P) \wedge (\sim Q)$. Since $\sim(\sim P) \equiv P$ by Theorem 1, we have $\sim((\sim P) \vee Q) \equiv P \wedge (\sim Q)$. Putting everything together we see that $\sim(P \Rightarrow Q) \equiv (P \wedge (\sim Q))$, as claimed. ⌁

Example 4: To further illustrate the relationship proven in Theorem 5:

"It is false that, if $1 = 2$ then $2 = 3$"

must have the same truth value as

"$1 = 2$ and $2 \neq 3$"

because their forms are logically equivalent.

The sentence

"It is false that, if computers always work properly then a daffodil is a flower"

must have the same truth value as

"Computers always work properly and a daffodil is not a flower"

because their forms are logically equivalent.

The logical form of the sentence

"It is false that, if pigs can fly then your uncle is a camel"

is logically equivalent to the logical form of the sentence

"Pigs can fly and your uncle is not a camel."

Hence the two sentences must have the same truth value.

Returning to a point made in Section 0.1, the statement "$1=2$" is clearly false. Hence the compound statement "$1=2$ *and* $2 \neq 3$" must also be false. The statement "$1=2$ *and* $2 \neq 3$" is logically equivalent to the negation of the statement "*If* $1=2$ *then* $2=3$." When the negation of a statement is false however, the original statement must be true. Hence, as we said before, "*If* $1=2$ *then* $2=3$" must be a true statement.

* * *

Let us now consider an equivalent form for the biconditional. There can be no suspense here, because the most important biconditional equivalence is given by the language of the biconditional. Taking apart the sentence form "*P if* and ONLY IF *Q*," we get "*P if Q*" and "*P* ONLY IF *Q*." The second piece is just "if P then Q" and the first piece can be written as "*if Q* then *P*." In other words, we have the following logical equivalence.

Theorem 6: $(P \Leftrightarrow Q) \equiv ((P \Rightarrow Q) \wedge (Q \Rightarrow P))$

Proof: To prove the claim we need only examine the following truth table:

P	Q	$\boldsymbol{P \Leftrightarrow Q}$	$P \Rightarrow Q$	$Q \Rightarrow P$	$\boldsymbol{(P \Rightarrow Q) \wedge (Q \Rightarrow P)}$
T	T	**T**	T	T	**T**
T	F	**F**	F	T	**F**
F	T	**F**	T	F	**F**
F	F	**T**	T	T	**T**

The third and sixth columns are the same, so $P \Leftrightarrow Q$ and $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ have the same truth value in every case. ⌿

Example 5: As an example of Theorem 6, consider the following sentence:

"$4 < 2$ if and only if $5 < 3$."

A different sentence, but one that has the same truth value because it has an equivalent logical form, is the sentence

"If $4 < 2$ then $5 < 3$, and, if $5 < 3$ then $4 < 2$."

Theorem 6 is an important result. All definitions, for instance, are biconditional statements. Suppose we make the definition: "*A system of equations $AX = B$ is homogeneous if and only if $B = 0$.*" This means, by Theorem 6, that as soon as we identify a system of equations $AX = B$ as homogeneous then we may immediately claim that $B = 0$ (because $P \Rightarrow Q$). Conversely, if we have a linear system $AX = B$ and learn that $B = 0$ then we may state that $AX = B$ is a homogeneous system of equations (because $Q \Rightarrow P$).

In addition, many theorems are biconditional statements and Theorem 6 is a blueprint for proving such theorems. For instance, suppose we wanted to prove, for any natural number n,

"*n is an even number if and only if n^2 is an even number.*"

Theorem 6 tells us how to proceed. We first prove that "*if n is an even number then n^2 is an even number.*" Next we prove "*if n^2 is an even number then n is an even number.*" We have then shown that $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$ is true for this P and this Q, and hence we know that $P \Leftrightarrow Q$ is true.

* * *

The final problem in this section is to sort out the relationships between a conditional statement and its contrapositive, its converse, and its inverse. First of all, there is a simple relationship between a conditional statement and its contrapositive.

Theorem 7: $(P \Rightarrow Q) \equiv ((\sim Q) \Rightarrow (\sim P))$

Proof: Once again we use a truth table to prove the claim.

P	Q	$\boldsymbol{P \Rightarrow Q}$	$\sim Q$	$\sim P$	$(\sim \boldsymbol{Q}) \Rightarrow (\sim \boldsymbol{P})$
T	T	**T**	F	F	**T**
T	F	**F**	T	F	**F**
F	T	**T**	F	T	**T**
F	F	**T**	T	T	**T**

The third and sixth columns are the same, so $P \Rightarrow Q$ and $(\sim Q) \Rightarrow (\sim P)$ have the same truth value in every situation. ⌁

Example 6: To illustrate Theorem 7, consider the sentence

"If I have chocolate cake for breakfast then I have a happy day."

A different sentence, but one that has a logically equivalent form and hence the same truth value, is the contrapositive

"If I do not have a happy day then I did not have chocolate cake for breakfast."

At this point you may be asking (and *should* be asking): Why do we care about the contrapositive? A definitive answer will come later when we use it. As a preview of one way we will use the contrapositive, suppose we wish to prove that some statement of the form "if P then Q" is always true (i.e., we wish to prove that the statement is a theorem). A standard approach is to assume that the hypothesis P is true and use the content of P to show that the conclusion Q must follow. In this way we show that the second line of the conditional truth table can never happen—i.e., that in all *possible* situations the statement is true.

An alternative approach is to restate the proposition in contrapositive form, "if $\sim Q$ then $\sim P$." We can then assume $\sim Q$ is true and use the content of $\sim Q$ to show that $\sim P$ must follow. Such a proof would show that "if $\sim Q$ then $\sim P$" is always true, and hence "if P then Q" must always be true also. For many theorems the standard approach is sufficient, but for some results the contrapositive approach is much easier.

There is also a simple relationship between the converse and the inverse of a conditional statement and, in fact, we have already established it:

$$(P \Rightarrow Q) \equiv ((\sim Q) \Rightarrow (\sim P))$$
$$(Q \Rightarrow P) \equiv ((\sim P) \Rightarrow (\sim Q))$$

The first line of symbols is Theorem 7 above, and the second line of symbols is also Theorem 7 but with P replaced by Q and Q replaced by P. Since the left half of the second line is the converse and the right half the inverse, the converse is logically equivalent to the inverse.

The final relationship to consider is that between a conditional statement and its converse, and the relationship here is a non-relationship.

Theorem 8: $(P \Rightarrow Q)$ is NOT logically equivalent to $(Q \Rightarrow P)$.

Proof: Consider the following truth table:

P	Q	$\boldsymbol{P \Rightarrow Q}$	$\boldsymbol{Q \Rightarrow P}$	
T	T	T	T	
T	F	**F**	**T**	◁
F	T	**T**	**F**	◁
F	F	T	T	

The truth values of $(P \Rightarrow Q)$ and $(Q \Rightarrow P)$ are not the same in all cases, so we do not have logical equivalence. ⌿

Example 7: As an example of the non-equivalence, consider the sentence

"If a person is a doctor then the person knows where the heart is located in a human."

The converse of the sentence is the sentence

"If a person knows where the heart is located in a human then the person is a doctor."

The two sentences do not have the same truth value. Doctors do know where the human heart is located, but knowing where the heart is located does not make a person a doctor.

The converse is a particularly easy and natural way to generate new mathematics. Whenever you prove an "if P then Q" relationship, you should always ask if the converse relationship is also true. The converse does not follow automatically, and either a proof of the converse or an example to show the converse fails is often as mathematically interesting as the initial result.

* * *

ADDENDUM

As with chocolate, alcohol, and trust, logic can also be abused.

"Theorem": $2 = 1$

"Proof": Suppose $2 = 1$. Then we have

$$2 - 1 = 1 - 1, \text{ so}$$
$$1 = 0.$$
$$\therefore\ (0)1 = (0)0, \text{ and so}$$
$$0 = 0.$$

Since $0 = 0$ is obviously true, we have proved that $2 = 1$. ⌿

Obviously, the "Theorem" is false, and the "Proof" is garbage. Because the "Proof" *assumed* $2 = 1$, all the "Proof" shows is that "*If* $2 = 1$ *then* $0 = 0$." Because we already know that $0 = 0$ and we already know that a conditional statement is true whenever the conclusion is true, we do not need a "Proof" to say that "*If* $2 = 1$ *then* $0 = 0$" is true. The statement "*If* $2 = 1$ *then* $0 = 0$" also has nothing to say about the truth or falsity of the statement "$2 = 1$."

More generally, the technique of "assume what you want to prove and show that a true statement must follow" is simply not a valid proof technique. *In a proof, you must never, ever assume what you are trying to prove.* If you assume what you want to prove is true then you have no place to go. You cannot prove the statement is true because you have already assumed that it is. Yet you do not *know* the statement is true because you have not proved anything.

There is a subtle distinction here. *In a proof* you may not assume what you are trying to prove, but in figuring out a proof anything goes. *To create a proof* you may employ meditation, prayer, or an out-of-body experience if you wish. One very popular and very productive strategy is a "working backwards" technique. When "working backwards," the first step is to write down the last line of the proof—which makes it look like you are assuming what you want to prove. Then given the last line, you try to figure out what a next to the last line might be, and so on. The process stops when you reach a place that is known to be true, from which you can start the proof. At the conclusion of the "working backwards" procedure however, you do *not* have a proof; you have a guide, in reverse, for writing a proof.

The problem with the "proof" above is that it is not a proof in reverse because the argument is not reversible. Of course $0 = 0$, and from that it follows that $(0)1 = (0)0$. We cannot cancel the 0's however (division by 0 is seriously illegal), so we cannot get from $(0)1 = (0)0$ to $1 = 0$, and hence we are stuck.

Some mathematicians, in their writing, only show the "working backwards" procedure and leave it to the reader to construct a valid proof from the outline they provide. That style seems inappropriate for an introductory textbook. This book will, at times, show the thinking that went into creating a proof, and hence this book will, at times, show a "working backwards" strategy. A discussion of how a proof was created will not be labeled a proof and, at the risk of being redundant, a valid proof will accompany the discussion.

* * *

EXERCISES

Mathematicians do not study objects, but relations among objects; they are indifferent to the replacement of objects by others as long as relations do not change. Matter is not important, only form interests them.

Henri Poincaré

Part A: Computational

1. Construct a truth table for each of the following logical forms:
   a) $(\sim P) \Rightarrow Q$
   b) $\sim(P \wedge (\sim Q))$
   c) $P \Rightarrow (P \wedge Q)$
   d) $(P \wedge Q) \Leftrightarrow (P \vee Q)$
   e) $(\sim P \wedge Q) \vee (P \wedge (\sim Q))$
   f) $(P \Rightarrow Q) \vee (Q \Rightarrow P)$

2. Use DeMorgan's Laws to rewrite the following propositions in a logically equivalent form, and determine the truth values of the propositions:
   a) It is false that, $1 + 1 = 2$ and $2 + 2 = 5$.
   b) It is false that, $2^{\pi} > 8$ or $\pi^2 < 9$.
   c) 5 is not a prime or 144 is not a square.
   d) $\sqrt{2}$ is not a rational number and π is not a real number.
   e) It is false that, triangles have 4 sides or rectangles have 5 sides.
   f) $5! \neq 60$ or 10^6 is not a million.

3. Rewrite each of the following sentences in a logically equivalent form, using the given logical equivalence:
   a) Roses are red and violets are blue. (Commutative Property)
   b) Hats go on heads or feet go in shoes. (Commutative Property)
   c) Tony's Pizza needs cash, or, Tony's Pizza needs goodwill and Tony's Pizza needs a line of credit. (Distributive Property)
   d) Plants need sunlight, and, plants need rain or plants need watering. (Distributive Property)
   e) I need love and I need respect, or, I need love and I need money. (Distributive Property)
   f) I will solve math problems or I will write a paper, and, I will solve math problems or I will bake a cake. (Distributive Property)
4. Rewrite each of the following sentences in a logically equivalent form, using the logical equivalence proved in Theorem 4:
   a) If n is an even integer then n^2 is an even integer.
   b) If you hit me first then it is OK for me to hit you.
   c) If n is not an odd natural number then n is an even natural number.
   d) If x is less than -2 then x^2 is greater than 4.
   e) If x^2 is greater than 4 then x is less than -2.
   f) You will gain weight if you eat the entire cake.
5. Rewrite each of the following sentences in a logically equivalent form, using the logical equivalence proved in Theorem 5:
   a) It is false that, if $2+2=4$ then $3+3=9$.
   b) $2+3 \neq 7$ and $5+3 \neq 10$.
   c) It is false that, if the sun is shining then rain is not falling.
   d) It is false that, if interest rates rise then the inflation rate falls.
   e) Mount St. Helens is an active volcano and Mount Rainier is not an active volcano.
   f) It is false that, if Abbie knows Lubi and Lubi knows Yuliya then Abbie knows Yuliya.
6. Rewrite each of the following sentences in a logically equivalent form, using the logical equivalence proved in Theorem 6:
   a) n is an even integer if and only if $n=2k$ for some integer k.
   b) Jack is Jill's brother if and only if Jill is Jack's sister.
   c) Switch is closed if and only if the bulb is lit.
   d) Action causes no pain if and only if the action results in no gain.
   e) If I think then I am, and, if I am then I think.
   f) If $|x|=a$ then $x=a$ or $x=-a$, and, if $x=a$ or $x=-a$ then $|x|=a$.
7. Rewrite each of the following sentences in a logically equivalent form, using the logical equivalence proved in Theorem 7 (i.e., write the contrapositive of the given sentence):
   a) If hummingbirds hum then mockingbirds mock.
   b) If today is Tuesday then I am in Belgium.
   c) If two lines are not parallel then the two lines intersect in a single point.

d) If the animal is a zebra then the animal is striped.
e) If the earth is a sphere then Mars is a sphere.
f) You will gain weight if you eat the entire cake.

8. Write the converse of the given sentence, and determine the truth values of the original and of the converse.
   a) If $1+1=2$ then $2+2=4$.
   b) If $1+1=2$ then $2+2=5$.
   c) If $1+1=3$ then $2+2=4$.
   d) If $1+1=3$ then $2+2=5$.
   e) If the graph of $y=x$ is a line in the plane then the graph of $y^2=x^2$ is a circle in the plane.
   f) If the graph of $y=x$ is a line in the plane then the graph of $y^2=-x^2$ is a point in the plane.
9. Prove $({\sim}P \wedge Q) \vee P \equiv (P \vee Q)$ by means of a truth table.
10. Prove $(P \Leftrightarrow Q) \vee (Q \wedge \sim P) \equiv (P \Rightarrow Q)$ by means of a truth table.

Part B: Conversational

1. Prove the second of the DeMorgan's Laws, namely ${\sim}(P \vee Q) \equiv ({\sim}P) \wedge ({\sim}Q)$, by means of a truth table.
2. Prove the Distributive Law, $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$, by means of a truth table. (*Hint:* How many lines are needed in the truth table?)
3. Prove $(({\sim}P) \Rightarrow Q) \equiv (({\sim}Q) \Rightarrow P)$ in two different ways. Which way do you prefer, and why?
4. Rewrite the following sentences in a form that is logically equivalent to the original and that contains no conditional operators:
   a) If today is Friday then we will have catfish for dinner.
   b) If at first you don't succeed, skydiving is not for you.
   c) If wishes were horses then beggars would ride.
   d) If you can't walk the walk then you shouldn't talk the talk.
   e) One if by land and two if by sea. (More formally: If the British come by land then you will light one lantern, and, if the British come by sea then you will light two lanterns. Even American history involves conditional operators.)
5. Simplify the following logical forms, where "simplified" means that you have left no negations undistributed and there are no redundant parts.
   a) $\sim (({\sim}P) \Rightarrow Q)$
   b) $\sim (({\sim}Q) \Rightarrow ({\sim}P))$
   c) $\sim (P \Rightarrow (Q \vee R))$
   d) $\sim ((P \wedge Q) \Rightarrow ({\sim}R))$
6. Find a simplified form that is logically equivalent to $(P \wedge Q) \vee (P \wedge ({\sim}Q))$, and justify your answer.

7. Find a simplified form that is logically equivalent to $(P \vee Q) \Rightarrow (P \wedge (\sim Q))$, and justify your answer.

8. Give examples of English sentences that have the logical form (a) $P \Rightarrow Q$, (b) $(\sim Q) \Rightarrow (\sim P)$, and (c) $(\sim P) \vee Q$. Is it possible to find sentences P and Q such that at least one of (a), (b), (c) is true and at the same time at least one of (a), (b), (c) is false? Please explain.

9. Is the biconditional operator logically equivalent to some combination of "and," "or," and "not"? If so, find one combination and justify the logical equivalence.

10. Complete the following sentences:
    a) "The negation of the disjunction of two propositions is the..."
    b) "The negation of the conjunction of two propositions is the..."

11. Kathryn became interested in the Distributive Law and conjectured:

$$\begin{array}{ll} \text{A)} & (P \Rightarrow (Q \vee R)) \equiv ((P \Rightarrow Q) \vee (P \Rightarrow R)). \\ \text{B)} & (P \Rightarrow (Q \wedge R)) \equiv ((P \Rightarrow Q) \wedge (P \Rightarrow R)). \end{array}$$

    a) Prove or disprove Conjecture (A).
    b) Prove or disprove Conjecture (B).

12. The following questions are about the number of lines that are needed in a truth table.
    a) How many lines are needed in a truth table if the form has 1 variable P_1?
    b) How many lines are needed in a truth table if the form has 2 variables P_1 and P_2?
    c) How many lines are needed in a truth table if the form has 3 variables P_1, P_2, and P_3?
    d) What is your conjecture as to the number of lines needed in a truth table if the form has 37 variables $P_1, P_2, \ldots,$ and P_{37}?
    e) Approximately how many digits are there in the answer to Part (d)?
    f) What is your conjecture as to the number of lines needed in a truth table if the form has n variables $P_1, P_2, \ldots,$ and P_n?

13. A **tautology** is a logical form that only takes on the value T (true). A **contradiction** is a logical form that only takes on the value F (false).
    a) Show that the logical form $P \vee (\sim P)$ is a tautology.
    b) Are there other logical forms that are tautologies?
    c) Show that the logical form $P \wedge (\sim P)$ is a contradiction.
    d) Are there other logical forms that are contradictions?
    e) Suppose you find that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ is true. What is the truth value of Q, and why?

Note: Part (e) is the basis for what is called the "indirect method of proof" or "proof by contradiction." In a "proof by contradiction," one assumes the negation of what one wants to prove, and shows that the assumption leads to a contradiction. So the goal of a "proof by contradiction" is to show that $(\sim Q) \Rightarrow (P \wedge (\sim P))$ is true, and from that the truth of Q follows.

SECTION 0.3: SETS AND SET NOTATION

"There's nothing like eating hay when you're faint."
... "I didn't say there was nothing better," the King continued,
"I said there was nothing like it."

Lewis Carroll

Mathematicians are very, very careful about specifying the meaning of almost (!) all the words they use. Just as "extent of tool using" distinguishes humans from other animal species, so does "precision of language" distinguish mathematics from other disciplines. That being the case, why is it that mathematicians do not define *all* the words they use? There is a good answer to that: Words must be defined in terms of other words. If all words are defined then you either get circular references or an infinite regress, neither of which is productive or conducive to understanding.

Note: An example of a circular reference is defining "go" as "leave," defining "leave" as "depart," and defining "depart" to mean "go." Circular references may set your head to spinning, but they do little to straighten out your thinking. Serious examples of circular references can be found in any dictionary. An example of an infinite regress is defining a "bo" as half a "bobo," defining a "bobo" as half a "bobobobo," and so on.

The strategy used by mathematicians is to start with a small collection of words that are deliberately left undefined, and use the undefined terms as the foundation for all that follows. "*Set*" and "*member of a set*" (or "*element of a set*") are undefined terms (for us).

Sets are usually denoted by capital letters, and a simple way to specify the members of a set is to list them using set brackets "{" and "}" to mark the start and the end of the list. For example, let S be the set that has as its members the numbers 1, 2, and 3. We can write S as {1,2,3}. A symbol that indicates membership in a set is "$\in$." The collection of symbols "$1 \in S$" is a very condensed way of writing the sentence "1 is a member of the set S." It is also useful to be able to note when an object is not a member of a set. The appropriate symbol is "$\notin$." We can write "$4 \notin S$," and the sentence is read as "4 is not an element of S."

An alternative way to specify the members of a set is to use a sentence with one or more variables. In particular, suppose $P(x)$ is a sentence that is true if and only if x is a member of the set under discussion. The members of the set can then be written as $\{\boldsymbol{x} \mid \boldsymbol{P}(\boldsymbol{x})\}$. The collection of symbols is read: "the set of elements x such that $P(x)$." You should observe and remember that $\{x \mid P(x)\}$ is an object and *not* a sentence (there is no verb).

To illustrate this way of specifying the members of a set, let S again be the set whose members are 1, 2, and 3. The sentence "$x = 1$ or $x = 2$ or $x = 3$" is true if and only if x is in the set S. Thus we can write S as $\{x \mid x = 1 \text{ or } x = 2 \text{ or } x = 3\}$. There are, of course, many other ways to specify the elements of S. We can also write S as $\{x \mid (x-1)\,(x-2)\,(x-3) = 0\}$ or as $\{x \mid x^3 - 6x^2 + 11x - 6 = 0\}$, for instance.

The ability to specify the members of a set by means of a property rather than a list is quite useful. Many sets have too many members to list them all. For instance the set of all positive real numbers can be written as $\{x \mid x \text{ is a positive real number}\}$, but there are simply too many positive real numbers to put in a list.

* * *

We now have the foundation words; next we will expand our vocabulary.

Definition 1: S is a **subset** of V if and only if, for any object x, if $x \in S$ then $x \in V$.

In symbols, "S is a subset of V" is written as $\mathbf{S \subseteq V}$, or as $\mathbf{V \supseteq S}$. The sentence "$S \subseteq V$" is also read as "S is contained *in* V," while the sentence "$V \supseteq S$" can be read as "V *contains* S." The fact that S is not a subset of V is written as "$\mathbf{S \not\subseteq V}$." ⌘

Example 1: To illustrate the subset idea let S be the set {1,2,3} and V be the set {1,2,3,7}. The set S is a subset of V because each member of S is an element of V, and hence we can write $S \subseteq V$. On the other hand, V is not a subset of S because there is an element in V, namely 7, such that $7 \in V$ but $7 \notin S$. We can write the fact that V is not a subset of S as $V \not\subseteq S$.

For another illustration, let E be the set $\{2,4,6,\ldots\}$ and W be the set $\{1,2,3,4,\ldots\}$. We have $E \subseteq W$ because every even natural number is a natural number. We also have that $W \not\subseteq E$ because $1 \in W$, but $1 \notin E$. For a third illustration, let X be the set $\{x \mid 1 \leq x < 3\}$ and let Y be the set $\{y \mid 0 < y < 3\}$. The different letters inside the set brackets do not matter. X is the set of all numbers between 1 and 3, including 1 but not including 3, whereas Y is the set of all numbers strictly between 0 and 3. Each member of X is a member of Y, so $X \subseteq Y$. The number 0.5 is in Y, but 0.5 is not in X, so $Y \not\subseteq X$.

Definition 2: **S is equal to V** if and only if $S \subseteq V$ and $V \subseteq S$.

"S is equal to V" is written as $\mathbf{S = V}$, of course. ⌘

Example 2: Let $C = \{1,2\}$ and $D = \left\{x \in \mathbb{R} \middle| x^3 - 4x^2 + 5x - 2 = 0\right\}$. The relationship between C and D may or may not be obvious. Once we figure out the relationship, we need to prove it.

Claim 1: $C \subseteq D$.

Proof: We have $(1)^3 - 4(1)^2 + 5(1) - 2 = 0$, so $1 \in D$. Also, $(2)^3 - 4(2)^2 + 5(2) - 2 = 0$, so $2 \in D$. Each element of C is an element of D and so, by Definition 1, $C \subseteq D$.

Claim 2: $D \subseteq C$.

Proof: We have $x^3 - 4x^2 + 5x - 2 = (x-1)\ (x^2 - 3x + 2) = (x-1)\ (x-2)\ (x-1)$. Hence if $x^3 - 4x^2 + 5x - 2 = 0$ then $x = 1$ or $x = 2$. Thus each element of D is an element of C, and so $D \subseteq C$.

Because both $C \subseteq D$ and $D \subseteq C$ are true, we have $C = D$ by Definition 2. ⌿

Caution: Please use the language carefully! If $S = \{1,2,5,7,8\}$ then saying "$1 \in S$" is a proper (and true) statement. You may *not* replace the symbol "$\in$" with a subset symbol however. You cannot talk about the object 1 being a subset (or not being a subset) of S because the object 1 is not a set. "Subset" is a relationship between sets—only! You can (properly) say: "$\{1\} \subseteq S$." You cannot talk about {1} being an element of S however, and it certainly makes no sense to connect 1 and {1} with an equal sign. A string of symbols, each meaningful by itself, may be garbage when combined. Do be careful in your writing—and in your thinking.

* * *

Counting was fun when we first learned it. Sometime after we learned to count however, we learned to add, and to multiply, and to subtract, and so on. Life certainly got much more interesting after that. We currently can make up and write down sets. As mathematicians, the question we must ask ourselves is: Are there ways to combine two sets to give another set? The answer is: Yes, there are many ways, and some are quite useful.

Definition 3: The **union** of the sets S, V is the set of all elements x such that $x \in S$ *or* $x \in V$.
The union of S with V is written as $\mathbf{S \cup V}$. ⌘

Definition 4: The **intersection** of the sets S, V is the set of all elements x such that $x \in S$ *and* $x \in V$.
The intersection of S with V is written as $\mathbf{S \cap V}$. ⌘

Definition 5: The **complement of S, relative to V**, is the set of all elements x such that $x \in V$ and $x \notin S$.
The complement of S relative to V is written as $\mathbf{V - S}$; it is also written as $\mathbf{S'}$ when the set V is understood. ⌘

Example 3: Let $S = \{1,2,5,7,8\}$ and let $V = \{2,3,5,7,9\}$. The sets S and V can be pictured as shown in Figure 1.

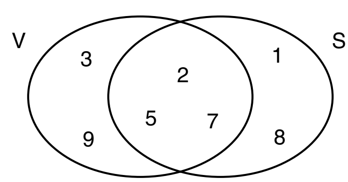

FIGURE 1

The union of S with V is the set consisting of all elements that are either in S or in V (or in both because our "or" is inclusive). Thus the union of S with V is the set $S \cup V = \{1,2,3,5,7,8,9\}$. The intersection of S with V is the set consisting of all elements that are (both) in S and in V. Hence the intersection of S with V is the set $S \cap V = \{2,5,7\}$. The complement of S relative to V is the set consisting of all elements that are in V but are not in S. So the complement of S, relative to V, is the set $V - S = \{3,9\}$. Of course we can also turn the tables and talk about the complement of V, relative to S. The complement of V, relative to S, is the set of all elements that are in S but are not in V. Thus $S - V = \{1,8\}$.

Example 4: To further illustrate the idea of intersection let L_1 be the set of all points in the plane that lie on the line $x + y = 3$. So $L_1 = \{(x, y) \mid x + y = 3\}$. Let L_2 be the set of all points in the plane that lie on the line $2x - y = 0$. So $L_2 = \{(x, y) \mid 2x - y = 0\}$. A picture of the two sets is shown in Figure 2.

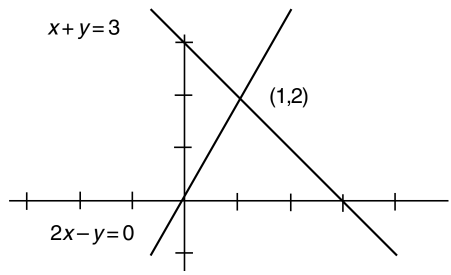

FIGURE 2

The intersection of L_1 with L_2 can be written as $L_1 \cap L_2 = \{(x, y) \mid x+y=3 \text{ and } 2x-y=0\}$. This description is not very informative however. In particular we are talking about nonparallel lines in the plane, and Euclid said such lines should intersect in a single point. If we solve the two equations simultaneously we find there is indeed a single solution, namely $x=1$ and $y=2$. Hence the point (1,2) is the only point that is an element of both L_1 and L_2. Thus $L_1 \cap L_2 = \{(1,2)\}$.

* * *

Definition 6: The set with no members is called the **empty set**, or **null set**. The empty set is written as $\emptyset$ or { }. ⌘

The empty set can come in many disguises. For instance,

$$\{(x,y)|2x + 3y = 1 \text{ and } 2x + 3y = 5\}$$

is just a lengthy way of writing { }, as can be seen both geometrically and algebraically. Algebraically, there can be no pair of numbers (x,y) such that $2x+3y$ is equal to both 1 and 5. Geometrically $2x+3y=1$ is a line and $2x+3y=5$ is a parallel line. Because the two lines are not identical (coincident), the intersection of the lines must be empty.

Although some readers may feel that spending even this much space on the empty set is making much ado about nothing, the empty set is everywhere and, in fact, the empty set is contained in every set. Seriously.

Theorem 1: For every set S, $\emptyset \subseteq S$.

Proof: Consider the sentence "*If* $x \in \emptyset$ *then* $x \in S$." The sentence has a conditional form, with the hypothesis "$x \in \emptyset$." Now the empty set has no members, so the statement "$x \in \emptyset$" is false for every x. Since the hypothesis is always false, the statement "*If* $x \in \emptyset$ *then* $x \in S$" is true for every x. Thus the definition of subset is satisfied, and $\emptyset \subseteq S$. ⁄⁄

You may or may not like the result in Theorem 1, but if you cannot find a flaw in the logic then you must accept it.

Definition 7: The sets S, V are **disjoint** if and only if $S \cap V = \emptyset$. ⌘

Let $E = \{2,4,6,8,\ldots\}$, $O = \{1,3,5,7,\ldots\}$, and $T = \{3,6,9,12,\ldots\}$. Because $E \cap O = \emptyset$, the sets E, O are disjoint. Since $E \cap T \neq \emptyset$, the sets E, T are not disjoint.

* * *

There is much more that could be said about sets, but our focus is on linear algebra so after one more definition we shall move on.

Definition 8: Let S and V be *nonempty* sets. The **Cartesian product (or the cross product) of S and V** is $\{(x,y) | x \in S \text{ and } y \in V\}$.
The Cartesian product of S and V is written as $\mathbf{S} \times \mathbf{V}$. ⌘

Example 5: For a little example of the Cartesian product of two sets, let $S = \{1,2,3\}$ and $V = \{e,\pi\}$. Then the set $S \times V = \{(1,e),(2,e),(3,e),(1,\pi),(2,\pi),(3,\pi)\}$.

For a larger example of the cross product of two sets, let $X = [0,1] = \{x | 0 \leq x \leq 1\}$ and $Y = [0,1]$. The sets X and Y are unit intervals on the real number line. The cross product of X and Y is the set

$$X \times Y = \{(x,y) | 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1\}.$$

The cross product is pictured in Figure 3; $X \times Y$ is just the unit square in the plane.

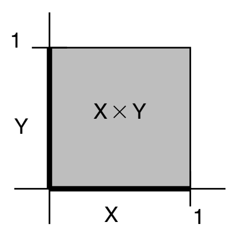

FIGURE 3

The Cartesian product is yet another way to construct new sets from known sets, and many of our most useful sets are Cartesian products of other sets. A case in point is Euclidean space.

Example 6: Let $\mathbb{R}$ be the set of all real numbers. The Euclidean plane, where we graph functions, do Euclidean geometry, and paint pictures, is the set of ordered pairs (x,y), where $x,y \in \mathbb{R}$, together with a measure of the distance between points. In other words, the set of points that make up the Euclidean plane is just $\mathbb{R} \times \mathbb{R}$, the Cartesian product of $\mathbb{R}$ with $\mathbb{R}$. The set $\mathbb{R} \times \mathbb{R}$ is commonly written as $\mathbb{R}^2$.

Euclidean 3-space, where we graph surfaces, do Euclidean geometry, and create sculptures, is the set of all ordered triples (x,y,z), where $x,y,z \in \mathbb{R}$, together with a measure of the distance between points. In other words, the set of points that make up Euclidean 3-space is $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$. The set $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ is commonly written as $\mathbb{R}^3$.

Note: The set $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ could mean $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R}$ or it could mean $\mathbb{R} \times (\mathbb{R} \times \mathbb{R})$. Technically, the set $(\mathbb{R} \times \mathbb{R}) \times \mathbb{R} = \{((x,y),z) | x,y,z \text{ are real numbers}\}$ is *not* the same set as $\mathbb{R} \times (\mathbb{R} \times \mathbb{R}) = \{(x,(y,z)) | x,y,z$ are real numbers$\}$. The technicalities are not important to our study however, so we shall ignore them.

Of course mathematicians never stop with the number 3, but at this point we do not have time to go beyond 3-space.

* * *

Besides not stopping at 3, mathematicians also do not sit around passively admiring a mathematical structure; that would be boring. Mathematicians like to mix and match and compare and transform. A fundamental tool for mixing and matching and comparing and transforming is what we call a "function."

Definition 9: Let A and B be nonempty sets. A **function f from set A to set B** is any correspondence that assigns to each x in A a unique object $f(x)$ in B. A function is also called a **mapping** or a **transformation**. A function f from A to B is denoted by $\boldsymbol{f}$: $\mathbf{A} \rightarrow \mathbf{B}$. The **domain** of the function f is the set A. The **codomain** of f is the set B. The **range** of the function f is $\{f(x) \in \mathrm{B} | x \text{ is in A}\}$, and is written as $\boldsymbol{f}(\mathbf{A})$ or **Range(f)**. ⌘

The definition of function should be familiar—which may be unfortunate. Knowledge is usually not a negative, but the initial study of functions is often restricted to functions from the real numbers to the real numbers and is usually focused on those very, very special functions that can be expressed by formulas. The situation is akin to that of the blind man who felt an elephant's trunk, and concluded that an elephant is very like a snake. There is more to an elephant than the trunk, and there are more to functions than real number formulas. We will, nevertheless, start out talking about functions that have formulas because that is what we know best, but we will not stay there.

Example 7: Functions can go from any (nonempty) set to any (nonempty) set. Since we have just discussed $\mathbb{R}^2$ and $\mathbb{R}^3$, let us make up a function $f: \mathbb{R}^2 \rightarrow \mathbb{R}^3$. An element of $\mathbb{R}^2$ is a pair of real numbers, and an element of $\mathbb{R}^3$ is a triple of real numbers, so our function must look like $f((x,y)) = (\alpha,\beta,\gamma)$, where we can make α, β, and γ anything we want (as long as they are always real numbers). So let us set $\alpha = x + y$, $\beta = x - y$, and $\gamma = 2x + 19y$. We have

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}^3, \text{ defined by } f((x,y)) = (x+y, x-y, 2x+19y).$$

So $f((1,1)) = (2,0,21)$, $f((5,0)) = (5,5,10)$, $f((-2,\pi)) = (-2+\pi, -2-\pi, -4+19\pi)$, and so on. The correspondence f is a function because for each point (x,y) in $\mathbb{R}^2$ there corresponds a unique point $(x+y, x-y, 2x+19y)$ in $\mathbb{R}^3$. The domain of f is $\mathbb{R}^2$. The codomain of f is $\mathbb{R}^3$. The range of f is $\{(x+y, x-y, 2x+19y) | x,y \text{ are real}\}$. (The description of Range(f) is not very satisfying in this example, but we will be able to do better with it later.) In this case the function f transforms the plane $\mathbb{R}^2$ into 3-space $\mathbb{R}^3$.

Example 8: In the previous example we moved away from the real numbers $\mathbb{R}$, but we still used formulas in specifying the function. We do not have to use formulas. Let S = {Ali, Jorge, Mary, Walt}. Suppose each person in the set writes a proof and gives his or her proof to another person in the set to read and critique. The correspondence g (for *gives*) is shown in Figure 4.

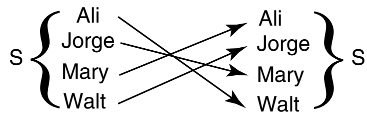

FIGURE 4

So g (Ali) = Walt, g(Jorge) = Mary, g(Mary) = Ali, and g(Walt) = Jorge. The correspondence g is a function from S to S, because for each person x in S there corresponds a unique person $g(x)$ in S. The domain of g is S, the codomain of g is S, and Range(g) = S. The function g both mixes and matches the elements of S, and does so without a formula in sight.

There are several cautions about functions that need to be heeded. For instance, recall that a rational number is any number that can be written in the form p/q, where p and q are integers with $q \neq 0$, and that an irrational number is any real number that cannot be so written. Define a correspondence $h: \mathbb{R} \to \{0,1\}$ by setting $h(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$. We have $h(9/4) = 1$, $h(17) = 1$, and $h(0.1111\ldots) = 1$, while $h(\pi) = 0$, $h(\pi^2) = 0$, and $h\left(\sqrt{2}\right) = 0$. The correspondence h is a function because each real number is either rational or irrational, and no real number is both. The caution here comes from the fact that the function property is a one-way requirement. For each element in the domain there must be a unique corresponding element in the codomain. An element of the codomain does not have to be the image of a single element from the domain however. For the function h there are many real numbers in $\mathbb{R}$ that map to 1 and also many that map to 0.

A caution for functions that is sometimes ignored even by people who know better is the requirement that for *each* element of the domain there must be a corresponding element in the codomain. For instance you may be able to find textbooks that claim the formula $f(x) = 1/x$ defines a function from the real numbers to the real numbers. The formula does *not* define a function from $\mathbb{R}$ to $\mathbb{R}$, because $0 \in \mathbb{R}$ but there is no corresponding $f(0)$. We can say (truthfully) that the correspondence $f: (\mathbb{R} - \{0\}) \to \mathbb{R}$ defined by $f(x) = 1/x$ is a function, but for $1/x$ to make sense we must require x to be nonzero.

A caution that is usually not needed is that $f(x)$ must have a single value for each x if f is to be a function. If we think of a function as a "black box" machine that turns x's into $f(x)$'s, then each input into a function machine must result in a single outcome. For each thing you drop in, you get ONE thing out. While the desirability of the "single value" requirement may seem obvious, problems do still arise. Let $\mathbb{R}^+$ be the set of all nonnegative real numbers and define the correspondence $f: \mathbb{R}^+ \to \mathbb{R}$ by $f(x) = \sqrt{x}$. The number 9 is in $\mathbb{R}^+$, so the corresponding value is $f(9) = \sqrt{9} =$? There is a temptation to say that $\sqrt{9}$ is ± 3 because both $3^2 = 9$ and $(-3)^2 = 9$. Please resist that temptation. Saying that $\sqrt{9}$ is ± 3 is saying that the "square root" is not a function, and that in turn says that all the neat things from calculus such as limits and derivatives and integrals do not apply to square roots. If the "square root" is not a function then there are dire consequences for our mathematics, and we do not want that. Consequently, $\sqrt{x}$ is defined to be the *nonnegative* real number s such that $s^2 = x$. So $\sqrt{9} = +3$, only, and the "square root" is a function.

Note 1: The common misconception that $\sqrt{9}$ is ± 3 seems to come from the very human tendency to jump to conclusions. If we want to solve $x^2 = 9$ then we take the square root of each side and that leads to the answer of $x = \pm 3$. Hence if taking square roots gives us:

$$x^2 = 9$$
$$\therefore x = \pm 3,$$

then (obviously?) $\sqrt{x^2}$ must be x and $\sqrt{9}$ must be ± 3—but that is not the way it is. The square root is required to be nonnegative, so $\sqrt{x^2}$ cannot be x because x may be a negative number. What is true is that $\sqrt{x^2} = |x|$. A less misleading sequence of steps is

$$x^2 = 9$$
$$|x| = 3$$
$$\therefore x = \pm 3.$$

Note 2: The necessity for leaving a few words undefined was discussed at the beginning of this section. Of course fewer is better when it comes to undefined words. In the definition of function however, a new undefined term, "correspondence," was snuck into our vocabulary. Functions can, in fact, be defined in terms of sets and no new undefined words are needed. One way is to identify a function with its graph. The work required to present functions in terms of sets is not particularly difficult or subtle, but nevertheless that is a discussion best left to another course.

* * *

EXERCISES

Throughout the following exercises, $\mathbb{R}$ is the set of all real numbers and $\mathbb{N}$ is the set of all natural numbers.

Part A: Computational

1. Write each of the following sets in set notation:
   a) The set consisting of the first three letters in the English language.
   b) The set consisting of the numbers 5, 12, and 33.
   c) The set of one-digit numbers.
   d) The set of odd natural numbers between 20 and 24.
   e) The set of all natural numbers that are both even and odd.
   f) The set of all natural numbers that are multiples of 3.
   g) The set of even natural numbers between 44.5 and 45.5.
   h) The set of all real numbers greater than 12.
   i) The set of all real numbers between -1 and 3, including -1 but not including 3.
   j) The set of all real numbers that are either less than -5 or greater than $+5$.
   k) The set of all points in $\mathbb{R} \times \mathbb{R}$ that lie to the right of the y-axis.
   l) The set of points in $\mathbb{R} \times \mathbb{R}$ that lie inside the circle of radius 1 centered at the origin.

m) The set of points in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ that make up the xy-plane.
n) The set of points in $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$ that make up the xz-plane.

2. Translate each of the following symbolic sentences into English:
   a) $A = \{1,2,3\}$.
   b) $B = \{m,a,t,h\}$.
   c) $I \in \{I, like, math\}$.
   d) $\{5\} \subseteq \{1,5,9\}$.
   e) $\{2,\pi\} = \{\pi,2\}$.
   f) $S = \{x \in \mathbb{R} | x > 0\}$.
   g) $T = \{y \in \mathbb{N} | y^2 < 14\}$.
   h) $1 \notin \{x \in \mathbb{R} | x^2 > x\}$.
   i) $L = \{(x,y) \in \mathbb{R} \times \mathbb{R} | 2x + 3y = 0\}$.
   j) $\{(-3,2),(6,-4)\} \subseteq L$.
   k) $\{1,2\} = \{x \in \mathbb{R} | (x-1)(x-2) = 0\}$.
   l) $(1,2) \in \{(x,y) \in \mathbb{R} \times \mathbb{R} | y > 0\}$.

3. Let $A = \{1,2,3,4,5\}$, $B = \{3,4,5,6,7,8\}$, and $C = \{2,4,6,8,10\}$. List the elements in each of the following sets:
   a) $P = A \cap B$ b) $Q = B \cup C$ c) $S = (A \cup B) \cap C$
   d) $T = (A - B) \cup (B - A)$ e) $U = (A \cap B) - C$ f) $W = (A - B) \cup (A - C)$

4. Let $D = \{1,3,5,7\}$, $E = \{3,4,5\}$, and $G = \{2,4,5,8\}$. List the elements in each of the following sets:
   a) $P = D \cup E$ b) $Q = D \cap E \cap G$ c) $S = (D \cap E) \cup G$
   d) $T = (D \cup G) - D$ e) $U = (D \cap G) - E$ f) $W = (E \cap D) \cup (E \cap G)$

5. Label each of the following sentences as T(true), F(false), or N(no sense):
   a) $8 \in \{2,4,6,8\}$.
   b) $3 \notin \{x \in \mathbb{R} | x > 0\}$.
   c) $\{1\} \neq \{x \in \mathbb{R} | x^2 - 2x + 1 = 0\}$.
   d) $(1,1) \notin \{(x,y) \in \mathbb{R} \times \mathbb{R} | 2x + 3y = 0\}$.
   e) $9 \subseteq \{1,3,9,27\}$.
   f) $\{x | x \text{ is a unicorn}\} \subseteq \mathbb{N}$.
   g) $\{x \in \mathbb{R} | x < 1\} \subseteq \{x \in \mathbb{R} | x^2 < 4\}$.
   h) $\{1,2,3\} \subseteq \{1,2,3\}$.
   i) $\{4\} \in \{n \in \mathbb{N} | n^2 < 30\}$.
   j) $\{1,5,7,11,14\} \cap \{2,5,11,17\} \subseteq \{4,5,6,8,11\}$.

6. For each of the given functions, calculate the specified value:
   a) f: $\mathbb{R} \to \mathbb{R}$ is defined by $f(x) = x^3 - 2x + 1$. What is $f(2)$?
   b) g: $\mathbb{R} \to \mathbb{R}^2$ is defined by $g(x) = (5x + 1, 4 - 3x)$. What is $g(-1)$?
   c) h: $\mathbb{N} \to \mathbb{R}$ is defined by $h(y) = y^{-3}$. What is $h(2)$?
   d) k: $\mathbb{N} \to \mathbb{R}$ is defined by $k(n) = 2^n$. What is $k(5)$?
   e) F: $\mathbb{N} \to \{0,1\}$ is defined by $F(n) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$. What is $F(17)$?

f) G: $\mathbb{R}^2 \rightarrow \{0,1\}$ is defined by $G((x,y)) = \begin{cases} 0 & \text{if } x \geq y \\ 1 & \text{if } x < y \end{cases}$. What is $G((3,2))$?

g) H: $\mathbb{R}^3 \rightarrow \mathbb{R}$ is defined by $H((x,y,z)) = 2x + y - 4z$. What is $H((2,4,5))$?

h) K: $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ is defined by $K((x,y)) = (3x - y, 2x + y)$. What is $K((1,-1))$?

7. Suppose P is the amount of money you have to invest, r is the rate of interest per year you will earn on your investment, and t is the length of time in years that you will invest your money. If the interest on your investment is compounded continuously then the amount of money you have at time t is a function $f: \mathbb{R} \rightarrow \mathbb{R}$ given by $f(t) = Pe^{rt}$ (where e is Euler's constant; $e \approx 2.718$). If you have \$10,000 to invest at 6% ($r = 0.06$), how much money do you have at $t = 0$, $t = 1$, $t = 5$, and $t = 10$? How many years will it take to double your money?

Part B: Conversational

1. For each of the following sets, list the elements in the set if a list is possible. If a list is not possible, explain why.
   a) S $= \{x \in \mathbb{R} | x^2 - 2 = 0\}$.
   b) U $= \left\{(x,y) \in \mathbb{R}^2 \middle| x - 5y = 0\right\}$.
   c) V $= \{n \in \mathbb{N} | 9 < n < 13\}$.
   d) W $= \{x \in \mathbb{R} | 9 < x < 13\}$.
   e) Y $= \{(n,m) \in \mathbb{N} \times \mathbb{N} | n^2 + m^2 < 2\}$.
   f) T $= \left\{(x,y) \in \mathbb{R}^2 \middle| 2x - y = 1 \text{ and } x + y = 2\right\}$.

2. For each of the following, give an example of a function with the specified domain, codomain, and range:
   a) f: $\{1,2,3\} \rightarrow \{a,b,c\}$ with range $\{a,c\}$.
   b) g: $\mathbb{R} \rightarrow \mathbb{R}$ with range $\{x \in \mathbb{R} | x \geq 0\}$.
   c) h: $\{1,2\} \rightarrow \{a,b,c\}$ with range $\{b,c\}$.
   d) k: $\mathbb{R} \rightarrow \mathbb{R}$ with range $\mathbb{R}$.
   e) F: $\mathbb{R}^2 \rightarrow \mathbb{R}$ with range $\mathbb{R}$.
   f) G: $\{1,2,3,4,5\} \rightarrow \{\clubsuit,\blacklozenge,\heartsuit,\spadesuit\}$ with range $\{\heartsuit,\spadesuit\}$.
   g) H: $\mathbb{N} \rightarrow \mathbb{N}$ with range $\{1,2,3\}$.
   h) T: $\mathbb{R} \rightarrow \mathbb{R}$ with range $\{x \in \mathbb{R} | -1 \leq x \leq 1\}$.

3. How many functions are there from the set A $= \{1,2,3\}$ to the set B $= \{8,9\}$? Why?

4. How many functions are there from the set B $= \{8,9\}$ to the set A $= \{1,2,3\}$? Why?

5. Is $\{\emptyset\}$ another name for the empty set?

6. Let A $= \{1,2\}$ and $B = \{x \in \mathbb{R} | x^3 - 3x^2 + 2x = 0\}$.
   a) Is A $\subseteq$ B? Why?
   b) Is B $\subseteq$ A? Why?
   c) Is A $=$ B? Why?

7. Let $C = \{1\}$ and $D = \{x \in \mathbb{R} | x^3 - x^2 + x - 1 = 0\}$.
   a) Is $C \subseteq D$? Why?
   b) Is $D \subseteq C$? Why?
   c) Is $C = D$? Why?

8. For any sets A and B, prove that the sets $A \cap B$ and $A - B$ are disjoint. (*Hint:* Try a proof by contradiction. In such a proof you assume there is some element that $A \cap B$ and $A - B$ have in common, and show the assumption leads to a contradiction.)

9. The following questions are about the number of subsets of a finite set.
   a) How many subsets of $\{1\}$ are there, and what are they?
   b) How many subsets of $\{1,2\}$ are there, and what are they?
   c) How many subsets of $\{1,2,3\}$ are there, and what are they?
   d) What is your conjecture as to the number of subsets of $\{1,2,\ldots,80\}$? (Please do *not* write them all out.)
   e) Approximately how many digits will there be in your answer to Part (d)? Which is larger, your answer to Part (d) or the U.S. national debt?
   f) What is your conjecture as to the number of subsets of $\{1,2,3,\ldots,n\}$?
   g) Is your conjecture true for the empty set?

10. Define a function $f: \mathbb{N} \to \mathbb{N}$ by setting $f(n)$ equal to the n-th prime number. Find the values of $f(1), f(2), f(3), f(4), f(5), f(6)$, and $f(7)$.

    Note: It would be nice to have a formula for f, but since no one on this earth knows such a formula it seems unfair to ask you to find one.

11. Let $C = \{\text{countries in the world}\}$. One interesting function $f: C \to \mathbb{R}$ is $f(X) =$ the per capita consumption of cigarettes in 1995 by the citizens of country X. So for instance, $f(\text{Finland}) = 1351$ cigarettes per person in 1995, $f(\text{India}) = 114$, and $f(\text{China}) = 1889$.

    Describe two (or more) additional functions on the set C that are of interest. (You need not look up values of your function.)

SECTION 0.4: QUANTIFICATION

...and no one has the right to say that no water-babies exist
till they have seen no water-babies existing;
which is quite a different thing, mind, from not seeing water-babies.

Charles Kingsley

The idea of a sentence with a variable in it was casually introduced in the previous section. Such sentences pervade mathematics and may be so familiar as to be beneath notice. Despite the familiarity there are aspects to such sentences that require understanding and agreement before we can proceed. Consider, for instance, the sentence "$x^2 = 1$." This collection of symbols is a sentence: it has a subject, an object, and a verb. "$x^2 = 1$" is, in fact, a declarative sentence, but it is *not* a statement. The sentence "$x^2 = 1$" is neither true nor false. "$x^2 = 1$" is true if we replace x with 1 or -1, it is false if we replace x with 0 or 2 or $-\pi/2$, and it is nonsense if we replace x with a horse.

We have two problems to resolve. One problem is how to restrict the substitutions that are allowed in a sentence variable. We cannot substitute a horse into "$x^2 = 1$" because the operation of squaring a horse is undefined. We can substitute whole numbers and fractions and real numbers and even complex numbers for x in the sentence "$x^2 = 1$." Should all these substitutions be allowed, in all situations?

The second problem is how to deal with the ambiguous status of a variable-containing sentence. The sentence "$x^2 = 1$" does not have a truth value until we make a substitution for x, so how can we use the sentence "$x^2 = 1$"—or any sentence like it—in a mathematical argument? Mathematics is supposed to be about truth. How can we draw valid conclusions using sentences that are neither true nor false?

The substitution problem can be resolved by specifying a set of allowable substitutions, and we will address that problem first. We begin by defining exactly what we mean by "a sentence with a variable."

Definition 1: An **open sentence** or **predicate** about the set S is a function from S to the set {T,F}. Open sentences are written as $\boldsymbol{P}(x)$. ⌘

Definition 1 is a bit startling if you equate function with formula. Functions may be expressed by formulas, but *a function does not have to have a formula*. A function has to set up a correspondence so that to each element in the domain there corresponds an element in the range, but the correspondence may be given by a picture, or by a diagram, or by words. With respect to functions, whatever works (in the sense of clearly defining a correspondence) is fine.

Example 1: As an example of an open sentence, let $\mathbb{R}$ be the set of all real numbers and for each real number x let $P(x)$ be the truth value of the sentence "$x^2 = 1$." Then $P(1)$ is the truth value of

"$1^2 = 1$," which is T (true). $P(2)$ is the truth value of "$2^2 = 1$," which is F (false), and so on. For each real number $x \in \mathbb{R}$ there is a corresponding value T or F, and so $P(x)$ is an open sentence about $\mathbb{R}$.

Because $P(x)$ is the truth value of the sentence "$x^2 = 1$," it is a common practice to assume that the truth value part is understood and simply say $P(x)$ *is* the sentence "$x^2 = 1$." We will adopt this convention. The name "open sentence" for $P(x)$ is a consequence of this practice. We say "open" because $P(x)$ has an open (unassigned) variable, and we say "sentence" because $P(x)$ *is* a sentence.

Example 2: For another example of an open sentence, let $S = \{1,2,3\}$ and let $P(x)$ be the sentence "*x is an odd number*." Then $P(1)$ is T because the sentence "*1 is an odd number*" is true. $P(2)$ is F because "*2 is an odd number*" is false, and $P(3)$ is T because "*3 is an odd number*" is true. We do not talk about $P(x)$ for anything else because $P(x)$ is an open sentence about the set S (only!). In particular, we may not substitute 5 or π or a horse for x.

Example 3: For a somewhat different example of an open sentence let the plane, $\{(x,y)|x,y \text{ are real}\}$, be the set and let $P((x,y))$ be the sentence "$3x + 2y = 7$." The elements of the plane that make the sentence true are exactly those points that lie on the line $3x + 2y = 7$, and any point of the plane not on the line makes the sentence false. Thus $P((1,2))$ is true as is $P((7/3,0))$ and $P((3,-1))$. The statements $P((1,1))$, $P((0,-2))$, and $P((\pi,2\pi))$ are all false.

The open sentence terminology may be new, but the concept of an open sentence is a familiar idea. Every equation with a variable in it is an example of an open sentence about some set of possible solutions. For instance, "$x^3 - x = 0$" can be an open sentence about the set of all real numbers. The numbers 0, 1, and -1 make the sentence true, and all other real numbers make the sentence false. "Solving an equation" means finding those elements that make the open sentence true.

Another use of open sentences occurs in computer programming. An unexceptional line of Pascal computer code is

if $i + j - 1 < s1.length$ **then** $limit := i + j - 1$;

The line of code is an open sentence. The code is neither true nor false, and it does nothing until the line is executed. When this line of code executes, the values stored in the memory locations i, j, and *s1.length* are substituted into the open sentence $i + j - 1 < s1.length$. If the resulting truth value is true then $i + j - 1$ is stored in the memory location *limit*, and if the truth value is false then nothing is done.

A third use of open sentences is in the conduct of Internet searches. If you type "mathematics education" into the search box of your Internet search engine, for instance, you are creating the open sentence: "Net site x has the words 'mathematics education' in its description." The search program then checks the truth or falsity of the sentence for each site x that is accessible to the program, and displays those sites for which the sentence is true.

* * *

Now we must resolve the ambiguity problem that open sentences pose. We really cannot allow sentences that have no truth value into a mathematical argument. How can we draw true conclusions from sentences that have no truth value? Our logic applies only to statements, that is, to declarative sentences that are either true or false. If we use sentences that have no truth value then we cannot use logic, and how can we do mathematics without logic? That would, indeed, be il-logical.

We can resolve the problem by asking what do we really want to know about open sentences? Basically, for an open sentence $P(x)$ about a set S, we want to know how often $P(x)$ is true. We cannot talk about whether $P(x)$ is true or not, because $P(x)$ has the unknown x in it, but we can and will make statements about how often $P(x)$ is true. In other words we will quantify our open sentences, and the quantification will change our open sentences into (true or false) statements. There are two common quantifiers, the universal and the existential. We will consider the universal quantifier first.

Definition 2: Let $P(x)$ be an open sentence about the set S. The **universal quantifier** is the phrase "for all x." The sentence "**For all x in S, $P(x)$**" is true if and only if F (false) is not in the range of $P(x)$. The statement "For all x in S, $P(x)$" is written: $\forall \boldsymbol{x} \in \mathbf{S}, \boldsymbol{P}(\boldsymbol{x})$. ⌘

The language of Definition 2 may seem a bit convoluted, but the language does say what is meant. When we claim "For all x in S, $P(x)$," what we mean is that there is nothing in S that makes $P(x)$ false.

The universal quantifier may also be read as "for each x" or as "for every x." You may feel that there are different shades of meaning among the phrases "for all x," "for each x," and "for every x," and that may be true—in English. In mathematics the three phrases are used interchangeably. You should use whichever phrase seems most appropriate and most meaningful to you in the situation you are discussing.

Example 4: Let us now look at some examples of universally quantified statements. For starters, let $S = \{1,2,3\}$ and $P(x)$ be the open sentence "$x < 10$." The corresponding universally quantified statement, in symbols, is "$\forall x \in S$, $P(x)$" or "$\forall x \in S$, $x < 10$." In words the statement is "*for every x in S, x is less than 10*." The statement $P(1)$ is "$1 < 10$," and the truth value of $P(1)$ is T. Similarly, $P(2)$ is "$2 < 10$" and is T, while $P(3)$ is "$3 < 10$" and $P(3)$ is also T. So F is not in the range of the open sentence $P(x)$, and thus the universally quantified statement "*For all x in S, x < 10*" is true. If however, $Q(x)$ is the open sentence "$x \leq 2$," then the universally quantified statement "$\forall x \in S$, $Q(x)$" is false. We have $Q(1)$ is "$1 \leq 2$" so $Q(1)$ is T, $Q(2)$ is "$2 \leq 2$" so $Q(2)$ is T, but $Q(3)$ is "$3 \leq 2$" and thus $Q(3)$ is F. Hence there is an x in S, namely $x = 3$, such that $Q(3)$ is false, which means that F is in the range of $Q(x)$. Thus by Definition 2, the statement "*For all $x \in S$, $x \leq 2$*" is false.

Example 5: For a second example, let $\mathbb{R}$ be the set of all real numbers and let $P(x)$ be the open sentence "$x^2 \geq 0$." The corresponding universally quantified statement, in symbols, is "$\forall x \in \mathbb{R}$, $P(x)$" or "$\forall x \in \mathbb{R}, x^2 \geq 0$." The statement in words is "*For all x in $\mathbb{R}$, $x^2 \geq 0$*" or "*For every real number x, x^2 is greater than or equal to zero*." The universally quantified statement is true, because there are no real numbers whose square is negative.

Example 6: For another example, let $\mathbb{N}$ be the set of natural numbers (so $\mathbb{N} = \{1,2,3,\ldots\}$) and $P(n)$ be the open sentence "*if n is odd then* $n^2 - 10n + 21 \geq 0$." The corresponding universally quantified statement, in symbols, is "$\forall n \in \mathbb{N}, P(n)$" or

$$\text{“}\forall n \in \mathbb{N}, \big((n \text{ is odd}) \Rightarrow (n^2 - 10n + 21 \geq 0)\big).\text{”}$$

The statement in words is "*For each n in* $\mathbb{N}$, *if n is odd then* $n^2 - 10n + 21 \geq 0$." The universally quantified statement is false. The statement is false because 5 is an odd natural number and $(5)^2 - 10(5) + 21 = -4$ which is less than 0. Thus $P(5)$ is false, hence F is in the range of $P(n)$, and so the universally quantified statement is false.

Example 7: Finally, let S be the set of all pink elephants and let $P(x)$ be the open sentence "*x can fly*." The corresponding universally quantified statement, in symbols, is "$\forall x \in S, P(x)$." The statement in words is "*For all x in the set of pink elephants, x can fly*" or simply "*All pink elephants can fly*." The universally quantified statement is true. (!) The truth of the statement is a consequence of the fact that there are no pink elephants. Since there are no pink elephants, there are no elements in S that make $P(x)$ false. If there are no elements in S that make $P(x)$ false, then the universally quantified statement is true by Definition 2. So it is true that "*All pink elephants can fly*." Try explaining that to friends and family. Your mathematician friends will simply nod their heads and say "*Yes, of course*."

The pink elephant example is a bit silly, but it makes an important point. Saying that "*For all x in S, P*(x)" is true does *not* say there is anything in S. It simply says there is nothing in S that makes $P(x)$ false.

Stereotypes are further examples of universal statements—universal statements gone awry. One stereotype is that "*Blondes have more fun*." Letting S be the set of all people, the sentence says: "*For all x in S, if x has blond hair then x has more fun*." Of course "fun" is not defined, nor is there a way of measuring "fun" so that comparisons can be made between the amounts of "fun" that different people experience. The truth of this universal statement is highly doubtful, if it has a truth value at all; nevertheless blond hair coloring is a popular product.

Another stereotype is expressed by the sentence "*Math majors are smart*." Letting S again be the set of all people, the stereotype is that "*For all x in S, if x is a mathematics major then x is smart*." Of course the statement, if it has a truth value at all, is false—except that the stereotype is widely perceived to be true, and there is often an advantage in being thought of as smart, so that being a math major is a smart thing to do, which does indeed make math majors smart.

* * *

The second quantifier is the existential quantifier.

Definition 3: Let $P(x)$ be an open sentence about the set S. The **existential quantifier** is the phrase "there exists an x (in S) such that." The sentence "**There exists an *x* in S such that *P*(*x*)**" is true if and only if T (true) is in the range of $P(x)$. The statement "There exists an x in S such that $P(x)$" is written: $\exists\, \boldsymbol{x} \in \mathbf{S}, \boldsymbol{P(x)}$. ⌘

The language of Definition 3 is reasonably straightforward. When we claim "There exists an x in S such that $P(x)$," what we mean is that there is something in S that makes $P(x)$ true. Unlike a universal statement, an existential statement does require the existence of an object in order for it to be true.

The existential quantifier may also be read as "for some x" or as "there is an x such that." You may again feel that there are different shades of meaning among the phrases "there exists an x such that," "for some x," and "there is an x such that" and again that may be so in English. In mathematics the three phrases are interchangeable. You may use whichever one you prefer, and your choice makes no difference in the mathematical meaning.

Example 8: For an example of an existentially quantified statement, let $S = \{1,2,3\}$ and let $P(x)$ be the open sentence "$x^2 = 4$." The corresponding existentially quantified statement, in symbols, is "$\exists x \in S, P(x)$" or "$\exists x \in S, x^2 = 4$." In words, the statement is "There exists an x in S such that x squared is equal to 4." The statement $P(1)$ is "$1^2 = 4$" and the truth value of $P(1)$ is false. Similarly, $P(2)$ is "$2^2 = 4$" and is true, while $P(3)$ is "$3^2 = 4$" and $P(3)$ is again false. Thus T is in the range of the open sentence $P(x)$, and so the existentially quantified statement "$\exists x \in S, x^2 = 4$" is true. On the other hand, if $Q(x)$ is the open sentence "$x = 2.5$," then the existentially quantified statement "$\exists x \in S, Q(x)$" is false. $Q(1)$ is "$1 = 2.5$" so $Q(1)$ is false, $Q(2)$ is "$2 = 2.5$" so $Q(2)$ is false, and $Q(3)$ is "$3 = 2.5$" so $Q(3)$ is also false. Hence T is not in the range of $Q(x)$, so by Definition 3 the existential statement "$\exists x \in S, Q(x)$" is false.

Example 9: For a second example, let $\mathbb{R}$ be the set of all real numbers and let $P(x)$ be the open sentence "$x^2 + 2x - 63 = 0$." The corresponding existentially quantified statement, in symbols, is "$\exists x \in \mathbb{R}, P(x)$" or "$\exists x \in \mathbb{R}, x^2 + 2x - 63 = 0$." The statement in words is "There exists an x in $\mathbb{R}$ such that $x^2 + 2x - 63 = 0$." The existentially quantified statement is true. To show the truth of the statement, we only need to observe that $(7)^2 + 2(7) - 63 = 0$, and so $P(7)$ is true. Thus there does exist an x in $\mathbb{R}$, namely $x = 7$, such that $P(x)$ is true.

Note: $P(-9)$ is also true, but to establish the truth of the existential statement all we need is *one* instance.

Example 10: For a third example, let $\mathbb{R}$ again be the set of all real numbers and let $P(x)$ be the open sentence "$x^2 = -1$." The corresponding existentially quantified statement, in symbols, is "$\exists x \in \mathbb{R}, P(x)$" or "$\exists x \in \mathbb{R}, x^2 = -1$." The statement in words is "There exists an x in $\mathbb{R}$ such that $x^2 = -1$." The statement is false. For each real number x we know that $x^2 \geq 0$, so $P(x)$ is false for every element x in $\mathbb{R}$. Thus T is not in the range of $P(x)$ and so, by definition, the existential statement is false.

* * *

We have several examples of quantified statements that are false. For instance the universal statement "$\forall n \in \mathbb{N}$, if n is odd then $n^2 - 10n + 21 \geq 0$" is false, as is the existential statement "$\exists x \in \mathbb{R}, x^2 = -1$." Because these statements are false, the negations of these statements must be true. That observation raises the question as to what is the logical form of the negation of a quantified statement.

Theorem 1: $\sim(\forall x \in S, P(x)) \equiv \exists x \in S, (\sim P(x))$

Proof: The only way "$\sim(\forall x \in S, P(x))$" is true is for "$\forall x \in S, P(x)$" to be false. But saying "$\forall x \in S, P(x)$" is false is saying, by definition, that F is in the range of $P(x)$. For F to be in the range of $P(x)$, there must be an x in S such that $P(x)$ is false (i.e., such that $\sim P(x)$ is true). Thus the statement "$\exists x \in S, (\sim P(x))$" must also be true.

On the other hand, when "$\sim(\forall x \in S, P(x))$" is false we must have that "$\forall x \in S, P(x)$" is true. So by definition $P(x)$ is true for all x in S, and hence $\sim P(x)$ is false for every x in S. Thus the statement "there is an x in S such that $\sim P(x)$ holds" must be false. So when "$\sim(\forall x \in S, P(x))$" is false, it follows that "$\exists x \in S, (\sim P(x))$" is false.

In all cases, "$\sim(\forall x \in S, P(x))$" and "$\exists x \in S, (\sim P(x))$" always have the same truth value. Hence the sentence forms are equivalent. ⌿

In words, Theorem 1 says that to negate a universal statement we must change the quantifier from universal to existential and negate the following open sentence.

Example 11: By Theorem 1, the negation of the statement

"$\forall n \in \mathbb{N}$, *if* n *is odd then* $n^2 - 10n + 21 \geq 0$"

is logically equivalent to the statement

"$\exists n \in \mathbb{N}$ *such that* $\sim$ (*if* n *is odd then* $n^2 - 10n + 21 \geq 0$)."

Recalling that $\sim(Y \Rightarrow Z) \equiv Y \wedge \sim Z$, we can rewrite the negation as

"$\exists n \in \mathbb{N}$ *such that*, n *is odd and* $n^2 - 10n + 21 < 0$."

Because the original statement is false, the statement just above must be true.

Example 12: For another example, consider the statement "*For every integer n, the cube root of n is an irrational number*." By Theorem 1, the negation of this statement is logically equivalent to "*There exists an integer n such that the cube root of n is not an irrational number*." The negation is true, because 8 is an integer and the cube root of 8 is the very rational number 2. The original statement must therefore be false.

Now let us consider the negation of an existentially quantified statement. The pattern is very similar.

Theorem 2: $\sim(\exists x \in S, P(x)) \equiv \forall x \in S, (\sim P(x))$

Proof: The logical equivalence can be shown by an argument similar to that in Theorem 1. The equivalence also follows directly from the statement of Theorem 1. In particular, Theorem 1 is true for all open sentences, and $(\sim P(x))$ is an open sentence, so we may replace $P(x)$ by $(\sim P(x))$ in Theorem 1. The statement then becomes

$$\sim(\forall x \in S,(\sim P(x))) \equiv \exists x \in S, \sim(\sim P(x))$$
$$\equiv \exists x \in S, P(x).$$

Taking the negation of each side results in

$$\sim(\sim(\forall x \in S,\ (\sim P(x)))) \equiv \sim(\exists x \in S, P(x)).$$

Eliminating the double negation and reversing the order gives the result:

$$\sim(\exists x \in S, P(x)) \equiv \forall x \in S,\ (\sim P(x)).$$ ✓

In words, Theorem 2 says that to negate an existential statement we must change the quantifier from existential to universal and negate the following sentence.

Example 13: To illustrate Theorem 2 let us find the negation of the statement "$\exists x \in \mathbb{R}, x^2 = -1$." By the Theorem, the negation is logically equivalent to the statement "$\forall x \in \mathbb{R}, \sim(x^2 = -1)$," which in turn is logically equivalent to "$\forall x \in \mathbb{R}, x^2 \neq -1$." In words, saying "*it is false that there exists an x in $\mathbb{R}$ such that $x^2 = -1$*" is logically equivalent to saying "for every x in $\mathbb{R}$, $x^2 \neq -1$." Because the original statement is false, the negation is true.

Example 14: As a second example, consider the statement "*there exists a real number x such that $x > 0$ and $x^2 + 3x = -2$.*" By Theorem 2 the negation of the statement is logically equivalent to the statement "*for each real number x, it is false that, $x > 0$ and $x^2 + 3x = -2$.*" Applying DeMorgan's Law (Theorem 2 of Section 0.2) we can then say that the negation is logically equivalent to "*for each real number x, $x \leq 0$ or $x^2 + 3x \neq -2$.*" Of the two statements, the original and its negation, the negation is true and the original is false. To see that the negation is true, take any real number x. Then $x \leq 0$, or, $x > 0$ in which case $x^2 + 3x > 0$ and so $x^2 + 3x \neq -2$. Thus for each x, either $x \leq 0$ or $x^2 + 3x \neq -2$.

* * *

We have used the word "theorem" many times already and we shall use it many times more, but we have not defined the word. It would not hurt to leave "theorem" as an undefined term, but the word does have a common meaning to mathematicians.

Definition 4: A **theorem** is a statement that is true—and interesting. ⌘

A personal note from the author: While I have seen many statements labeled "Theorem," I do not recall ever seeing a definition of the word. So I made up the preceding definition, and it is a terrible definition. If I had inserted a period and stopped after "true," as I was tempted to do, then "theorem" would be a sharply defined word; it would be a synonym for a true statement. That definition would not be faithful to the mathematical meaning of "theorem" however. The addition of "and interesting" makes the definition fuzzy, subjective, and more in keeping with how mathematicians use the word.

A consequence of the definition is that there are two parts to showing that a statement is a theorem. One part is showing the statement is true, and that part is commonly labeled "proof." The other part is showing that the statement is interesting, and that part is commonly labeled "discussion." A proof is a demonstration of the truth of the statement, and the demonstration must be *universally* accepted by all mathematicians. The amount of interest in a statement is a personal value judgment, and unanimity of opinion is neither expected nor feasible. To complete the claim that a statement is a theorem however, reasons should be given as to why the statement is interesting and there should be some agreement among mathematicians.

Two words closely related to "theorem" are "lemma" and "corollary."

Definition 5: A **lemma** is a true statement that is not very interesting by itself, but that is useful in proving an interesting result and so is semi-interesting. A **corollary** is a true statement that is interesting, but which follows immediately from another result so that its proof is not very interesting. ⌘

Theorems are not true because of their logical form. If you say "For all natural numbers n, either n is even or n is not even" then you are speaking truth but you are not saying much. Theorems are true because of their content. For any theorem, the logical form will be such that the theorem could, conceivably, be false. The purpose of a proof is to show that the case or cases in which the theorem could be false simply do not occur. Thus the logic of proofs is quite simple:

I: If a theorem is a simple statement then a proof must show that the statement is true.

II: If a theorem has the form of a disjunction then a proof must show that one or the other of the simple statements must be true.

III: If a theorem has the form of a conjunction then a proof must show that both of the two simple statements must be true.

IV: If a theorem has the form of an implication then a proof must show that the situation in which the hypothesis is true and the conclusion is false cannot happen. The direct approach is to assume that the hypothesis is true and show that the conclusion must follow. Alternatively one can assume the conclusion is false and show it must follow that the hypothesis is false.

V: If a theorem has the form of a biconditional then a proof must show that the two simple statements always have the same truth value. The standard approach is to look at the biconditional as a conjunction of two conditional statements, and to do separate proofs of each of the conditionals.

And that is all there is to proofs. (Or rather, almost all.)

* * *

An example of a theorem is the following statement. Following the statement are reasons, proof, and discussion as to why the statement is a theorem.

Theorem 3: (The Cancellation Law for Multiplication) For any real numbers a, b, and c, if $a \neq 0$ and $ab = ac$ then $b = c$.

The first criterion for being a theorem is that the statement is true. Before jumping into a proof of the statement, let us examine the logical form. Overall, the proposed theorem is a universally quantified sentence. If we let $P(x) = P((a,b,c))$ be the open sentence "if $a \neq 0$ and $ab = ac$ then $b = c$" then the sentence has the form: "For all $x = (a,b,c)$ in $\mathbb{R}^3$, $P(x)$."

To show that a universally quantified statement is true, we must show that false is not in the range of $P(x)$. To do this, we *must begin* our proof by taking an *arbitrary* element $x = (a,b,c)$ in $\mathbb{R}^3$ and then show that $P(x)$ is true for this arbitrary element. Now $P(x)$ is an implication, so the only way $P(x)$ can be false is for the hypothesis to be true and the conclusion to be false. One way to rule out the possibility that $P(x)$ is false is to show that whenever the hypothesis is true then the conclusion must also be true. Hence to prove an implication, we may *assume* the hypothesis of the sentence is true. Our task then is to show the conclusion must follow, and that is what we will do here.

Proof: Let a, b, and c be any real numbers. Suppose $a \neq 0$ and $ab = ac$. Because $a \neq 0$, there is a real number a^{-1} such that $(a^{-1})(a) = 1$. Since $ab = ac$, we must have $(a^{-1})(ab) = (a^{-1})(ac)$. Because real numbers can be re-associated, it follows that $(a^{-1}a)b = (a^{-1}a)c$. Hence $(1)b = (1)c$, and so $b = c$. Thus if $a \neq 0$ and $ab = ac$ then $b = c$ must be true for all real numbers a, b, and c. ⌁

To finish establishing that our purported theorem really is a theorem, we need to say something about why the statement is interesting and one reason is in the title. The result allows us to cancel common factors and simplify equations, which is very important in solving equations. The *proof* of the result also points out a relationship between cancellation and the existence of multiplicative inverses, which is interesting.

* * *

Another example of a theorem is given below, but to appreciate the theorem we need a precise remembrance of numerical distinctions. The word "number" is a very inclusive term, and to organize our thoughts we separate numbers into various categories. Knowledge of the following categories is assumed to be part of your background, and some of the terms have already been used without explanation. Just to make sure we are speaking a common language however, we make the following definition.

Definition 6: A **natural number** is any member of the set $\{1,2,3,\dots\}$ and the set of natural numbers is commonly denoted by $\mathbb{N}$. An **integer** is any member of the set $\{\dots -3,-2,-1,0,1,2,3,\dots\}$ and the set of integers is commonly denoted by $\mathbb{Z}$. A **rational number** is any number that can be written in the form p/q, where p and q are integers with $q \neq 0$. The set of all rational numbers is commonly denoted by $\mathbb{Q}$. A **real number** is any number that can be written as a decimal. The set of all real numbers is denoted by $\mathbb{R}$, and a geometric representation of $\mathbb{R}$ is given by the number line. An **irrational number** is a real number that is not a rational number. The set of irrational numbers can be represented as $\mathbb{R} - \mathbb{Q}$, and does not have its own symbol. ⌘

Natural numbers are "natural" in the sense that they are the counting numbers and so come from "nature." The integers are a more inclusive category of numbers than the natural numbers. A still more inclusive category is that of rational numbers. The name of this category does not refer to the "logical" meaning of rational, but rather to the "ratio-nal" meaning. An alternative and equivalent description is to say that a number is a rational number if and only if the decimal representation of the number eventually repeats in blocks of equal size. The largest set of numbers we will consider at this time is the set of real numbers. For the categories of numbers described thus far, we have $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$. Continuing to catalog our menagerie of numbers, irrational numbers are those numbers needed to fill in holes in the real number line. Irrational numbers do exist. For instance the number $0.10100100010\ldots$ never repeats in blocks of equal size, and so is not a rational number. For a given real number however, deciding whether the number is rational or irrational can be challenging, and that brings us to the next theorem.

Theorem 4: $\sqrt{2}$ is an irrational number.

To prove the statement we need to show that $\sqrt{2}$ can *not* be written in p/q form. A direct proof of the statement would require us to show that, for *every* pair of integers p and q with $q \neq 0$, we have $p/q \neq \sqrt{2}$. A method for constructing such a proof is not obvious.

Since mathematicians are supposed to be smart, let us not belabor the difficult. We shall adopt a different approach—the indirect approach. We shall *assume* that the statement is false and our strategy will be to show this assumption leads us to a contradiction—that is, to a statement of the form $(P \wedge {\sim}P)$. In logical terms, if Q is the statement we wish to prove then our goal is to show that $({\sim}Q) \Rightarrow (P \wedge {\sim}P)$ is true for some statement P. The key fact is that contradictions such as $(P \wedge {\sim}P)$ are always false. From the conditional truth table, if an implication is true and the conclusion is false then the hypothesis *must* also be false. So if $({\sim}Q) \Rightarrow (P \wedge {\sim}P)$ is true then ${\sim}Q$ must be false, and hence Q must be true.

Proof: Suppose the Theorem is false. Then $\sqrt{2}$ is a rational number, and hence by the definition of rational number there exist integers p, q with $q \neq 0$ such that $\sqrt{2} = p/q$. From this we can say that there exist integers c, d with $d \neq 0$ such that $\sqrt{2} = c/d$ and either c or d is odd. We can say this because we can cancel common factors of 2 from p and q, without changing the ratio, until all factors of 2 are eliminated from at least one of the numbers. The reduced quotient then gives us c and d.

If $\sqrt{2} = c/d$ where either c or d is odd then $\left(\sqrt{2}\right)^2 = (c/d)^2$ and so $2 = c^2/d^2$. Hence $2d^2 = c^2$ and so c^2 is an even number. The only way c^2 is even however, is for c to be even. Thus $c = 2k$ for some integer k. This in turn means that $2d^2 = c^2 = (2k)^2 = 4k^2$. By cancellation, we get $d^2 = 2k^2$ which means that d^2 must be even. Since the only way d^2 can be even is for d to be even, we have just shown that both c and d are even.

We have a contradiction. Under the assumption that $\sqrt{2}$ is a rational number, we have shown that there are integers c and d such that either c or d is odd *and* both c and d are even. The assumption that the theorem is wrong leads to the conclusion that a statement of the form $(P \wedge \sim P)$ is true. The only way this can happen is that our assumption must be false. Hence it must be true that $\sqrt{2}$ is irrational. ↯

Why is our statement interesting? The square root of 2 is a common number. If we draw a square with sides of length 1 then the length of the diagonal is $\sqrt{2}$—and therein lies a surprise. The sides of the square are of unit length, but the diagonal length is not even a ratio of integers. Such incompatibility is not "rational." Our result also means that the decimal expansion of $\sqrt{2}$ has a randomness that is not obvious. In particular, the decimal expansion of $\sqrt{2}$ will never repeat in blocks of equal size no matter how far out we start.

A folk story is that the ancient Greeks, who discovered this theorem, regarded the result as an insult to the gods and a flaw in the perfection of the universe that humans should not have noticed. The severity of the insult is demonstrated by the fact that the mathematicians who first communicated this result to the common folk were killed (by the anger of the gods) in a shipwreck, or so the story goes. The truth of the story is not at all well established, but it is an interesting story—and an interesting result.

* * *

There is a final topic in logic that we need to touch upon before moving on to the Linear Algebra heart of this book, and the topic arises in a way that we will encounter over and over and over. An open sentence of the form $P(x)$ has one (1!) variable. Mathematicians do not stop at 1. You, dear reader, have not stopped at 1. Consider the sentence "$x - 2y = 0$" where x and y may be any real numbers. "$x - 2y = 0$" is indeed a sentence; it has a subject, a verb, and an object. "$x - 2y = 0$" is an open sentence because it takes pairs of real numbers and transforms them into T (when the pair satisfies the equation) or F (when the pair makes the equation false). In symbols we must represent "$x - 2y = 0$" as $P(x,y)$ because there are two variables. $P(2,1)$ is the sentence "$(2) - 2(1) = 0$," which is T (true). $P(-8,-4)$ is the sentence "$(-8) - 2(-4) = 0$," which is also true. $P(1,3)$ is the sentence "$(1) - 2(3) = 0$," which is false, and so on.

Open sentences with two variables have the same problem as open sentences with one variable; open sentences are neither true nor false, and mathematicians have a commitment to speaking truth. There are two ways of dealing with the problem.

Sometimes we will be interested in knowing exactly which values make $P(x,y)$ true, and in this case we embed the open sentence in set brackets. Using the notation from Section 0.3, the set $\{(x,y)|P(x,y)\}$ is the set of all values (x,y) that make $P(x,y)$ true and is called the **solution set** of the open sentence $P(x,y)$. For our open sentence "$x - 2y = 0$," the solution set is the set $\mathrm{L} = \left\{(x,y) \in \mathbb{R}^2 \middle| x - 2y = 0\right\}$. L consists of all those points in the plane that lie on the line through the origin with slope 1/2, and only those points. Solution sets sidestep the true/false dilemma because for solution sets the question of truth or falsity does not arise. Sets are objects, not sentences. The labels "true" and "false" only apply to sentences.

Sometimes however, we will be content with less detailed knowledge. As we did for open sentences with one variable, we can quantify two-variable open sentences—using the same quantifiers, with one quantifier for each variable. For instance, we could say $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x - 2y = 0$. (In English: For each x in $\mathbb{R}$, (*and*) for each y in $\mathbb{R}$, x minus $2y$ equals 0.) The quantified open sentence is false, because $1 \in \mathbb{R}$ and $3 \in \mathbb{R}$ but $(1)-2(3)=0$ is false. We could also say $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x - 2y = 0$. (In English: There exists an x in $\mathbb{R}$ for which there exists a y in $\mathbb{R}$ such that x minus $2y$ equals 0.) This quantified open sentence is true, because $2 \in \mathbb{R}$ and $1 \in \mathbb{R}$, and $(2)-2(1)=0$ is true. When all the quantifiers are the same, the language is often condensed and the two variables x and y are treated as a single ordered pair. So "$\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x - 2y = 0$" is the same as "$\forall (x,y) \in \mathbb{R}^2, x - 2y = 0$," and the sentence "$\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x - 2y = 0$" is the same as "$\exists (x,y) \in \mathbb{R}^2, x - 2y = 0$."

Of course the quantifiers for x and y do not have to be the same, in which case caution is advised. Order matters! Consider the quantified open sentence: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x - 2y = 0$. (In English: For each x in $\mathbb{R}$ there exists a y in $\mathbb{R}$ such that x minus $2y$ equals 0.) The sentence is true because, for any x you choose, there will always be some y (namely $y=(1/2)x$) that will pair with the x to make $x-2y=0$ true. On the other hand, consider the quantified open sentence: $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x - 2y = 0$. (In English: There exists an x in $\mathbb{R}$ such that for all y in $\mathbb{R}$, x minus $2y$ equals 0.) This sentence is false. The sentence asserts that there is some real number x such that, for this particular x, $x-2y=0$ for every possible value of y. Since there is no such x, the sentence is false. As these sentences demonstrate, reversing the order of the quantifiers can change the truth value of the sentence.

And that is the story on open sentences with two variables.

Oh, mathematicians do not stop at 2 either.

* * *

EXERCISES

Logically speaking, when trapped in a horror movie,
Never read a book of demon-summoning aloud, even as a joke.
As a general rule, don't solve puzzles that open portals to Hell.
When trying to escape from a serial killer, never run UPstairs.

Part A: Computational

1. Let $S=\{1,2,3\}$ and $P(x)$,$Q(x)$, $R(x)$, and $V(x)$ be the open sentences:
   $P(x)$: x is an odd number. $Q(x)$: x^2 is an odd number.
   $R(x)$: $x^2-3x+2=0$. $V(x)$: $x^2-9=0$.

   Translate the following sentences into symbolic form, using the symbols defined above:

a) For all x in S, x is an odd number.
b) There exists an x in S such that $x^2-9=0$.
c) For all x in S, $x^2-3x+2=0$ or $x^2-9=0$.
d) For every x in S, x^2 is not an odd number.
e) There is an x in S such that if x is an odd number then $x^2-3x+2=0$.
f) For some x in S, x is an odd number and x^2 is an odd number.
g) For each x in S, $x^2-3x+2\neq 0$.
h) For every x in the set S, $x^2-3x+2=0$ if and only if $x^2-9\neq 0$.
i) There is at least one x in S such that x is an odd number and $x^2-9\neq 0$.
j) There exists some x in S such that if x^2 is not an odd number then $x^2-3x+2=0$.

2. Let $S=\{2,5,9\}$ and $P(x)$, $Q(x)$, $R(x)$, and $V(x)$ be the open sentences:
$P(x)$: x is a prime. $Q(x)$: x^2 is less than 8.
$R(x)$: $\sqrt{x}$ is a rational number. $V(x)$: x is an even integer.
Translate the following symbolic forms into English, using the given sentences:
a) $\forall x\in S, P(x)$.
b) $\exists x\in S, Q(x)$.
c) $\forall x\in S, V(x)\wedge Q(x)$.
d) $\exists x\in S, \sim R(x)$.
e) $\exists x\in S, P(x)\wedge V(x)$.
f) $\forall x\in S, Q(x)\Rightarrow R(x)$.
g) $\forall x\in S, P(x)\Leftrightarrow \sim R(x)$.
h) $\forall x\in S, R(x)\Rightarrow \sim P(x)$.
i) $\exists x\in S, \sim P(x)\vee Q(x)$.
j) $\forall x\in S, P(x)\vee Q(x)\vee V(x)$.

3. Let $S=\{-1,0,1\}$. Determine the truth value of each of the following sentences:
a) For all x in S, $x^2=1$.
b) For all x in S, $x^2\neq 1$.
c) There exists an x in S such that $x^2\neq 1$.
d) There exists an x in S such that $x^2=1$.
e) For some x in S, $x^2>x$.
f) For each x in S, $x^3-x=0$.
g) For every x in the set S, $x^2-3x+2>0$.
h) For all x in the set S, $x^4=x^2$.
i) There is at least one x in S such that $2^x<1$.
j) There exists some x in S such that $x^2-3x+2<0$.

4. For each of the following sentences:
(i) Write the negations of the following sentences, using the equivalences shown in Theorems 1 and 2.
(ii) Determine which is true, the original sentence or the negation.
a) For all n in $\mathbb{N}$, n is a rational number.
b) For all n in $\mathbb{N}$, $2n+1$ is odd.

c) There exists an x in $\mathbb{R}$ such that $x^2 = 3$.
d) There exists an x in $\mathbb{R}$ such that $x^2 = -1$.
e) For every z in the set $\mathbb{Z}$, z is a rational number and z is a real number.
f) There is an x in the set of all real numbers such that $\cos(x) = 1.3$.
g) There is some x in $\mathbb{Q}$ such that $x^2 = 2$.
h) For each n in the set $\mathbb{N}$, if n is even then $n/2$ is even.
i) There is at least one real number x such that $1 < x < 2$.
j) Each member of the set of unicorns was hatched from an egg.

5. Let $S = \{1,2\}$, $T = \{4,5,6\}$, and $P(x,y)$ be the open sentence "$x + y$ is even," where x is an element of S and y is an element of T.
a) Find is the solution set of $P(x,y)$—i.e., find $\{(x,y) \mid x + y \text{ is even}\}$.
b) What is the truth value of the sentence: $\forall x \in S$, $\forall y \in T$, $x + y$ is even.
c) What is the truth value of the sentence: $\exists x \in S$, $\exists y \in T$, $x + y$ is even.
d) What is the truth value of the sentence: $\forall x \in S$, $\exists y \in T$, $x + y$ is even.
e) What is the truth value of the sentence: $\exists y \in T$, $\forall x \in S$, $x + y$ is even.

6. Let $U = \{1,3\}$, $V = \{4,5\}$, and $Q(x,y)$ be the open sentence "xy is odd," where x is an element of U and y is an element of V.
a) Find the solution set of $Q(x,y)$—i.e., find $\{(x,y) \mid xy \text{ is odd}\}$.
b) What is the truth value of the sentence: $\forall x \in U$, $\forall y \in V$, xy is odd.
c) What is the truth value of the sentence: $\exists x \in U$, $\exists y \in V$, xy is odd.
d) What is the truth value of the sentence: $\forall x \in U$, $\exists y \in V$, xy is odd.
e) What is the truth value of the sentence: $\exists y \in V$, $\forall x \in U$, xy is odd.

Part B: Conversational

1. For each set S, give two examples of English sentences of the form "$\forall x \in S$, $P(x)$." One example should be true and the other should be false.
a) S_a is the set of natural numbers.
b) S_b is the set of all butterflies.
c) S_c is the set of all people in your Linear Algebra class.
d) S_d is the set of all Fortune 500 companies.

2. For each set U, give two examples of English sentences of the form "$\exists x \in U$, $P(x)$." One example should be true and the other should be false.
a) U_a is the set of all real numbers.
b) U_b is the set of all doctors.
c) U_c is the set of all kangaroos.
d) U_d is the set of all U.S. Senators.

3. For each of the following statements, determine whether the given sentence is true or false and explain why:
a) For all natural numbers n, $n^2 - 6n + 9 > 0$.
b) There exists a natural number n such that $2n < 1$.

c) For all real numbers x, if $x^2 < 0$ then x is a rational number.
d) There exists a real number x such that $x^2 = 17$.
e) For every real number x, either $x^2 > 0$ or $\cos(x) = 1$.
f) For some real number y, $y^{16} = 2$ and $y < 0$.

4. Write each of the following statements in symbolic form using variables and quantifiers. Be sure to clearly define the simple open sentences you are using and the set of objects to which the sentence refers.
a) Every person has a mother.
b) Something is rotten in the state of Denmark.
c) For each real number x, there exists a real number y such that $2x + 3y = 4$.
d) You can fool all of the people some of the time.
e) For each $\varepsilon > 0$ there is a $\delta > 0$ such that for all real numbers x, if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

(**Note:** Yes, Part (e) is triply quantified. You may have encountered Part (e) in calculus; Part (e) is the formal definition of $\lim_{x \to a} f(x) = L$. Linear algebra is logically nicer than calculus.)

5. Simplify the following logical forms, where "simplified" means that you have left no negations undistributed and there are no redundant parts:
a) $\sim(\exists x \in S, P(x) \vee Q(x))$.
b) $\sim(\forall x \in S, P(x) \Rightarrow Q(x))$.
c) $\sim(\forall x \in S, (\sim P(x)) \wedge Q(x))$.
d) $\sim(\exists x \in S, P(x) \Rightarrow (\sim Q(x)))$.
e) $\sim(\forall x \in S, \exists y \in T, P(x,y))$.
f) $\sim(\exists x \in S, \forall y \in T, Q(x,y))$.
g) $\sim(\forall x \in S, \forall y \in T, P(x,y) \vee Q(x,y))$.
h) $\sim(\forall x \in S, \exists y \in T, P(x,y) \Rightarrow Q(x,y))$.

6. Prove the following sentence is true: For any natural number n, the number $(n)(n + 1)$ is even. Then explain why the result might be called a theorem.

7. Prove the following sentence is true: For all real numbers x, if $x^\pi > 4$ then $x^\pi + 1 > 5$. Then explain why the result should not be called a theorem.

8. Consider the following argument:
I am nobody.
Nobody is perfect.
Therefore I am perfect.
The argument appears to have the form: "*A* is *B*. *B* is *C*. Therefore *A* is *C*." *Assuming* the first two sentences are true propositions (rather than being sentences with subjective truth values), does the conclusion follow?
(*Hint:* Is the "nobody" in line 1 the same as the "nobody" in line 2?)

CHAPTER **1**

An Introduction to Vector Spaces

SECTION 1.1: THE VECTOR SPACE $\mathbb{R}^2$—THE BASICS

A little learning is a dangerous thing; drink deep, or taste not the Pierian spring:
There shallow draughts intoxicate the brain, and drinking largely sobers us again.

Alexander Pope

Although we claim to be 3-dimensional beings, much of our mathematical lives has been lived in the plane. The set of ordered pairs of real numbers, $\{(x, y) | x, y \text{ are real}\}$, is our favored mathematical hangout (thus far anyway) for many good reasons. A visual representation of the plane is given by the Cartesian (or rectangular) coordinate system, and allows us to picture what we do. The Cartesian coordinate system is illustrated in Figure 1.

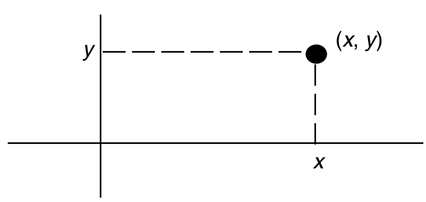

FIGURE 1

Actual physical representations of planes are everywhere, from the sheet of paper on which these words are printed, to the board on which your teacher writes, to the floor beneath your feet.

The plane is also a good place to work because it is relatively small and friendly, yet at the same time it has enough complexity to be interesting. If we define the (Euclidean) distance between two points (x_1, y_1) and (x_2, y_2) to be the number

$$dist((x_1, y_1), (x_2, y_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

then we can take the measure of all manner of things. Within the plane we can also investigate (Euclidean) geometry, graph functions such as $y = x^3$, and plot relationships between physical variables. In short, the plane is a delightful (and delightfully comfortable) setting for doing mathematics.

There has been something missing from our good times in the plane however, and what's missing is arithmetic. Your immediate reaction may be to say: "Yes, and good riddance!" Please restrain that impulse. The *doing* of arithmetic can be drudgery or worse, but the *ability* to combine numbers to get other numbers is a tool that can work magic. How can you build a house, determine the inflation rate, or find the amount of a drug needed to save a life without taking measurements and doing addition (and subtraction and multiplication and division)? Our civilization wouldn't exist without arithmetic. And if there is so much value in adding numbers, might there not be value in adding points?

* * *

The notion of adding points may seem a bit odd at first. Addition is for numbers, not points. Yet every number can be thought of as a point on the real number line, and hence even the addition of ordinary numbers can legitimately be regarded as an addition of points. Furthermore, points in the plane are represented by coordinates—by pairs of numbers—and we can do arithmetic with numbers. For starters, we will limit our new arithmetic to the addition of points in the plane and to the multiplication of a point in the plane by a real number.

To emphasize that we are not dealing just with points, but rather with points that have an arithmetic, we will adopt different notation. Instead of writing a point in the plane as (x, y), whenever we wish to allow even the possibility of doing arithmetic we will write the point in an upright position, as $\begin{bmatrix} x \\ y \end{bmatrix}$. The upright notation takes a little more space on the page, but it is particularly well suited to our needs. The "natural" way to add two points is to add their respective coordinates. So if we wish to add $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ -5 \end{bmatrix}$, for instance, then we simply add along each row: $\begin{bmatrix} \mathit{1} \\ \mathbf{3} \end{bmatrix} + \begin{bmatrix} \mathit{2} \\ \mathbf{-5} \end{bmatrix} = \begin{bmatrix} \mathit{1+2} \\ \mathbf{3+(-5)} \end{bmatrix} = \begin{bmatrix} \mathit{3} \\ \mathbf{-2} \end{bmatrix}$. What could be easier? We have combined two points in the plane and the result is a new point in the plane. We will require multiplication of a point by a real number to be similarly simple. To multiply a point by a real number we will multiply each coordinate by the number. So if we want to multiply $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ by 4 then the multiplication gives us the point $4\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 12 \end{bmatrix}$.

We must, of course, formalize the foregoing discussion and specify exactly what we are talking about.

Definition 1: The **vector space** $\mathbb{R}^2$ is the set $\mathrm{V} = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} \middle| x, y \text{ are real} \right\}$ together with the operations of vector addition and scalar multiplication, where vector addition in $\mathbb{R}^2$ is defined by