UNDERSTANDING MATHEMATICS FOR YOUNG CHILDREN

DEREK HAYLOCK AND ANNE D. COCKBURN

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5TH EDITION

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A GUIDE FOR TEACHERS OF CHILDREN 3-7

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CONTENTS

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Derek Haylock is an education consultant, speaker and writer in the field of mathematics education. He is a Senior Fellow in Education at the University of East Anglia, Norwich, where he worked for over 30 years in primary teacher training and in education research. He is a major author in education, with a large number of published books and chapters. His best-selling *Mathematics Explained for Primary Teachers* (SAGE) has been the leader in the field for many years, with a fifth edition published in 2014. Derek's work is characterized by a commitment to explain mathematics in simple and accessible ways and to enable the subject to be taught and learnt with understanding and confidence.

Anne Cockburn is Professor in Early Years Education at the University of East Anglia, Norwich. Her research interests focus on mathematics education in the primary school and teacher well-being, recruitment and retention. She has completed a range of research projects, including *Understanding the Primary Mathematics Classroom Part 2* (Economic and Social Research Council Award) with Paola Iannone; and *Mathematical Misconceptions in the Primary Years* (British Academy) with international partners. Her most recent project – *Challenging Colleagues: Primary Head Teachers Managing Mediocre Practitioners* – was funded by the Nuffield Foundation. Professor Cockburn chaired the International Meeting of the Psychology of Mathematics Education Group in 2002 and was an Associate Editor of *The Journal of Mathematics Teacher Educators*.

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We again express our appreciation to those teachers who attended our in-service course at the University of East Anglia prior to the writing of the first version of this book and those many teachers and trainees who have met with us since to share their experience of teaching and learning mathematics in the age range 3–7 years. Special thanks go to Catherine Bates, Emma Yerby-Smith, Sarah Johnston and Kate Oakley, who have allowed us to spend time in their classes gathering material for various editions of this book and shared with us their insights; and to Joe Haydn for the drawings in Figure 1.5.

INTRODUCTION

This book is about teaching and learning mathematics in the age range 3–7 years. The first book that we wrote together, way back in 1989, was called *Understanding Early Years Mathematics*. It covered the age range 4–8 years. After its publication, however, the phrase ‘early years’ began to be used solely for the age range up to 5 years. So the next three incarnations in 1997 and 2003, more substantial and comprehensive, were called *Understanding Mathematics in the Lower Primary Years*. We then found that in the educational set-up in England the phrase ‘lower primary’ no longer sent out the right signals for a book that has as much focus on the Early Years Foundation Stage (3–5 years) as Years 1 to 3 (5–8 years). So, in producing the versions in 2008 and 2013 we changed the title yet again, to *Understanding Mathematics for Young Children*. We have been able to retain that title for this new edition, but have decided to limit the scope of the book to teaching children aged 3–7 years, the age range that in England covers the Foundation Stage and Key Stage 1. This has enabled us to ensure that we cover all the mathematics in the current English National Curriculum (Department for Education, 2013) for Key Stage 1. We are pleased therefore to have this opportunity to ensure that the book remains up-to-date and relevant to those teaching or being trained to teach children in this age range.

THE AIMS OF THE BOOK

The book is written for those who teach or who are preparing to teach mathematics to children in the age range 3–7 years and who wish to have a clearer understanding of the mathematical ideas behind the material they deal with in the classroom.

Although it draws mainly on the curriculum and classroom context of schools in England, we are confident that, as has proved to be the case with earlier versions, the book will be useful and relevant to colleagues involved in elementary or primary education in other countries in the United Kingdom and, indeed, around the world.

Our interest in writing this book arose from our concern about the long-term effects on children's confidence in mathematics resulting from their teachers' own mathematical misconceptions and limited understanding of the subject. This is no doubt often because they and their teachers before them were taught mathematics by drill, as a set of rules and recipes. Understanding as a goal may have played little part in their mathematics education. Our aims are therefore that this book will help prospective and experienced teachers who work in this age range to understand:

- the mathematical ideas that underpin what they teach
- what understanding of that mathematics entails
- how children can be helped to construct that understanding for themselves.

We hope that we can demonstrate that understanding mathematics and mathematical pedagogy need not be the sole prerogative of those who call themselves mathematicians.

This is not a book of superficial tips for teachers and we guarantee that you will not be able to canter through it. It is a book that should make you think and reassess your own understanding of basic mathematical ideas. We hope, of course, that it will make you a better teacher of mathematics to young children, by increasing your confidence and dispelling some of the fears and anxieties that you might have accumulated over the years (see Chapter 1 of Haylock, 2014). It is essential that teachers themselves should have a thorough understanding of the basic mathematical concepts and principles that underpin the mathematics that is taught and learnt as a young child embarks on their educational journey. Readers should note that, to help them develop this understanding, we find that it is necessary here and there to discuss some material that goes a little beyond the 3–7 age range.

INPUT FROM TEACHERS

To explore the mathematical understanding of Foundation Stage and Key Stage 1 teachers, we have met at the University of East Anglia in Norwich with various groups who work with this age range. They have shared with us honestly their experience of teaching the subject to young children and the aspects of mathematics where their own understanding needed boosting. We have also been privileged to visit a number of nurseries and classes to observe children learning mathematics. We quote generously and gratefully from these conversations and observations throughout the book.

INPUT FROM RESEARCH

We have included a Research Focus in each chapter, to indicate how a range of research studies in the field of mathematics education relate to the content of the chapter. We begin some of these with reference to seminal or classic studies, which are often the foundation on which more recent work is built. In these research sections we have also drawn on Anne's extensive contacts with the international mathematics education community. These research sections are not intended to be a comprehensive survey of the field, rather, we have chosen a few particularly interesting studies on which to focus, in order to whet the reader's appetite for exploring the field further. All the details of the publications referred to in these Research Focuses (and in our Suggestions for Further Reading) are given in the References at the end of the book.

PAUSES TO REFLECT

We have also included in each chapter one or two Pauses to Reflect. These, we hope, may intrigue you, amuse you, or help you to relate the content of the chapter to your own experience or that of young children. We have in mind that these might best be discussed with a colleague, so that you can share each other's perspectives and observations.

CLASSROOM ACTIVITIES

Although the book focuses mainly on the understanding of mathematical ideas rather than on how to teach mathematics, pedagogical implications are considered throughout. At the end of each chapter we include a few examples of activities that might be used with children in the nursery or lower primary years. The activities are all aimed at developing what we might recognize as understanding. They are not intended to cover comprehensively the material in the chapter, but we hope that they may provide you with some indication of ways in which the ideas we have outlined in the preceding pages might influence your practice. Many of these activities can be used with a variety of ages of children, adapting them as appropriate. For each activity we have given an indication of the ages (3–4, 4–5, 5–6, or 6–7 years) for which we expect it to be most suitable, but this is certainly not intended to be prescriptive.

Derek Haylock and Anne Cockburn

A NOTE ON TERMINOLOGY

In this book the following terminology is used as it relates to education in England:

Nursery: The year group of children who have their fourth birthday within the school year.

Reception class: The year group of children who have their fifth birthday within the school year.

Foundation Stage: The two years of schooling comprising nursery and reception.

Year 1, Year 2: The year groups of children who within the school year have their sixth and seventh birthdays respectively.

Key Stage 1 (KS 1): The period of schooling comprising Years 1 and 2.

Key Stage 2 (KS 2): The period of schooling comprising Years 3–6.

Primary school: A school with children up to 11 years of age.

1

UNDERSTANDING MATHEMATICS

A REVEALING CONVERSATION

Gemma, aged 6 years, had no problems with questions like $2 + 3 = \square$ and even $8 + \square = 9$. Her teacher said that she thought Gemma had a good understanding of the equals sign. But then the teacher asked Gemma how she did $2 + \square = 6$. Gemma replied, 'I said to myself, two (then counting on her fingers), three, four, five, six, and so the answer is four. Sometimes I do them the other way round, but it doesn't make any difference.' She pointed to $1 + \square = 10$: 'For this one I did ten and one, and that's eleven.'

This conversation prompts us to ask the following questions:

- How does Gemma show here that she has some understanding of the concept of addition?
- What about her understanding of the concept represented by the equals sign?
- How would you analyse the misunderstanding shown at the end of this conversation?

IN THIS CHAPTER

In this chapter we discuss the importance of teaching mathematics in a way that promotes the deep understanding that is necessary for mastery of the subject. So we aim to help the reader understand what constitutes understanding in mathematics. Our main theme is that understanding involves establishing connections. For young children learning about number, connections often have to be made between four key components of children's experience of doing mathematics: symbols, pictures, concrete situations and language. We also introduce two other key aspects of understanding that will run through this book: equivalence and transformation.

LEARNING AND TEACHING MATHEMATICS WITH UNDERSTANDING

This book is about *understanding* mathematics. The example given above of Gemma doing some written mathematics was provided by a Key Stage 1 teacher in one of our groups. It illustrates some key ideas about understanding. First, we can recognize that Gemma does show some degree of understanding of addition, because she makes connections between the symbol for addition and the process of counting on, using her fingers. We discuss later the particular difficulties of understanding the equals sign that are illustrated by Gemma's response towards the end of this conversation. But we note here that, as seems to be the case for many children, she appears at this point to perceive the numerical task as a matter of moving symbols around, apparently at random and using an arbitrary collection of rules.

LEARNING WITH UNDERSTANDING

Of course, mathematics does involve the manipulation of symbols. But the learning of recipes for manipulating symbols in order to answer various types of questions is not the basis of understanding in mathematics. All our experience and what we learn from research indicate that learning based on understanding is more enduring, more psychologically satisfying and more useful in practice than learning based mainly on the rehearsal of recipes and routines low in meaningfulness.

For a teacher committed to promoting understanding in their children's learning of mathematics, the challenge is to identify the most significant ways of thinking mathematically that are characteristic of understanding in this subject. These are the key cognitive processes by means of which learners organize and internalize the information they receive from the external world and construct meaning. We shall see that this involves exploring the relationship between mathematical symbols and the other components of children's experience of mathematics, such as formal mathematical and everyday language, concrete or real-life situations, and various kinds of pictures. To help in this we will offer a framework for discussing children's understanding of number and number operations. This framework is based on the principle that the development of understanding involves building up *connections* in the mind of the learner.

Two other key processes that contribute to children learning mathematics with understanding are *equivalence* and *transformation*. These processes also enable children to organize and make sense of their observations and their practical engagement with mathematical objects and symbols. These two fundamental processes are what children engage in when they recognize what is the same about a number of mathematical objects (equivalence) and what is different or what has changed (transformation).

TEACHING WITH UNDERSTANDING

This book has arisen from an attempt to help teachers to understand some of the mathematical ideas that children handle in the early years of schooling. It is based on our experience that many teachers and trainees in nursery and primary schools are helped significantly in their teaching of mathematics by a shift in their perception of the subject away from the learning of a collection of recipes and rules towards the development of understanding of mathematical concepts, principles and processes. So our emphasis on understanding applies not just to children learning, but also to teachers teaching, in two senses: first, it is important that teachers of young children teach mathematics in a way that promotes understanding, that helps children to make key connections, and that recognizes opportunities to develop key processes such as forming equivalences and identifying transformations; second, in order to be able to do this the teachers must themselves understand clearly the mathematical concepts, principles and processes they are teaching. Our experience with teachers suggests that engaging seriously with the structure of mathematical ideas in terms of how children come to understand them is often the way in which teachers' own understanding of the mathematics they teach is enhanced and strengthened.

MASTERY THROUGH UNDERSTANDING

In the context of the challenge to raise standards in mathematics in schools in England, the word 'mastery' has become prominent in the vocabulary of the English mathematics curriculum (see, for example, NCETM, 2014, www.ncetm.org.uk/public/files/19990433). This has developed from studying the comparative international successes of children with mathematics in Shanghai and Singapore in particular. These states redesigned their mathematics curricula around the turn of the millennium, when they realized that their previous dependence upon rote-learning, drill and frequent practice was not developing the fluent, creative mathematicians needed for competitive economic success in the 21st century. The curricula of these states have benefited particularly from the work of Bruner, who identified a progression from 'concrete' to 'pictorial' and then to 'abstract' in the development of understanding (Bruner, 1960).

The emphasis on mastery is consistent with the approach to children's learning of mathematics that we adopt in this book. Mastery involves children developing fluency in mathematics through a deep understanding of mathematical ideas and processes. Teaching approaches for mastery should 'foster deep conceptual and procedural knowledge' and 'exercises are structured with great care to build deep conceptual knowledge alongside developing procedural fluency' (NCTEM, op.cit.). This is a key principle in teaching mathematics to young children: that mastery of the subject is not achieved simply by repeated drill in various procedures. Instead, the focus is on the development of understanding of mathematical structures and on making connections. Making connections in mathematics – a recurring theme in this book – ensures that 'what is learnt is sustained over time, and cuts down the time required to assimilate and master later concepts and techniques' (NCTEM, op.cit.). We shall see frequently in our exploration of young children's understanding of mathematics that nearly all mathematical concepts and principles occur and can be applied in a wide range of contexts and situations. Because of this, the deeper understanding central to mastery in mathematics is facilitated by a wide *variation* in the experiences that embody particular mathematical ideas.

For example, mastery of the 5-times multiplication table is not just a matter of memorizing a chant that begins 'one five is five, two fives are ten …' – although that is part of it. It would also involve, for example:

- connecting each result in the table with a collection of 5p coins and the total value;
- articulating the pattern of 5s and 0s in the units position in the odd and even multiples of 5;
- explaining how to get from 4 fives to 8 fives by doubling;
- explaining how to get from 6 fives to 7 fives by adding 5;
- counting in steps of five along a counting stick;
- knowing that, say, '3 fives are fifteen' is what you use for the cost of 3 books at £5 each;
- constructing patterns with linked cubes that show 1 set of five, 2 sets of five, and so on;
- filling in the missing number in number sentences like '$6 \times \square = 30$'.

It goes without saying that to teach for this kind of mastery, even with young children, teachers themselves need a deep structural understanding of mathematics, an awareness of the range and variety of situations in which a mathematical concept or principle can be experienced, and confidence in exploring the connections that are always there to be made in understanding mathematics.

CONCRETE MATERIALS, SYMBOLS, LANGUAGE AND PICTURES

When children are engaged in mathematical activity, as in the example above, they are involved in manipulating one or more of these four key components of mathematical experience: concrete materials, symbols, language and pictures. These four components will often provide a starting point for generating the variation in experience of mathematical concepts that is essential for developing mastery.

First, they manipulate *concrete materials*. We use this term to refer to any kind of real, physical materials, structured or unstructured, that children might use to help them perform mathematical operations or to enable them to construct mathematical concepts. Examples of concrete materials would be blocks, various sets of objects and toys, rods, counters, fingers and coins.

Second, they manipulate *symbols*: selecting and arranging cards with numerals written on them; making marks representing numbers on pieces of paper and arranging them in various ways; copying exercises from a work card or a textbook; numbering the questions; breaking up numbers into tens and units; writing numerals in boxes; underlining the answer; pressing buttons on their calculator; and so on.

Third, they manipulate *language*: reading instructions from work cards or textbooks; making sentences incorporating specific mathematical words; processing the teacher's instructions; interpreting word problems; saying out loud the words that go with their recording; discussing their choices with the teacher and with other children; and so on. This language will include both formal mathematical words, such as 'subtract two, equals', and less formal language that describes particular actions or observations, such as 'take two away, how many are left?'

Finally, they manipulate *pictures*: drawing various kinds of number strips and number lines, set diagrams, arrow pictures and graphs.

AN EXAMPLE IN A NURSERY CLASS

In a nursery class some children aged 3 to 4 years are propelling themselves around the playground on tricycles. The tricycles are numbered from 1 to 9 (see photograph overleaf). At the end of the time for free play they put the tricycles away in a parking bay, where the numerals from 1 to 9 are written on the paving stones, matching their tricycle to the appropriate numbered position in the bay. There are conversations prompted by the teacher about why a particular tricycle is in the wrong place and which one should go next to which other ones. When all the tricycles are in place, the children check them by counting from 1 to 9, pointing at each tricycle in turn.

We begin to see here how an understanding of elementary mathematical ideas develops, as children begin to make connections between real objects, symbols, language and pictures. The children are making connections between the ordering of the numerical symbols and the ordering of the actual tricycles. The numerals on the paving stones form an elementary picture of part of a number line, providing a visual image to connect with the language of counting. Already the children are beginning to understand what we will call (in Chapter 2) the ordinal aspect of number, by making these simple connections between real objects lined up in order, the picture of the number line, the symbols for numbers and the associated language of counting.

UNDERSTANDING AS MAKING CONNECTIONS

A simple model that enables us to talk about understanding in mathematics is to view the growth of understanding as the building up of cognitive connections. More specifically, when we encounter some new experience there is a sense in which we understand it if we can connect it to previous experiences or, better, to a network of previously connected experiences. In this model we propose that the more strongly connected the experience is, the greater and more secure is our understanding of it. Using this model, the teacher's role in developing understanding, and thereby in promoting mastery, is to help the child to build up connections between new experiences and previous learning. Learning without making connections is what we would call learning by rote.

CONNECTIONS BETWEEN THE FOUR KEY COMPONENTS

We find it very helpful to think of understanding the concepts of number and number operations (that is, number, place value, addition, subtraction, multiplication, division, equals, number patterns and relationships, and so on) as including the building up of a network of cognitive connections between the four types of experience of mathematics that we have identified above: concrete experiences, symbols, language and pictures. Any one of the arrows in Figure 1.1 represents a possible connection between experiences that might form part of the understanding of a mathematical concept.

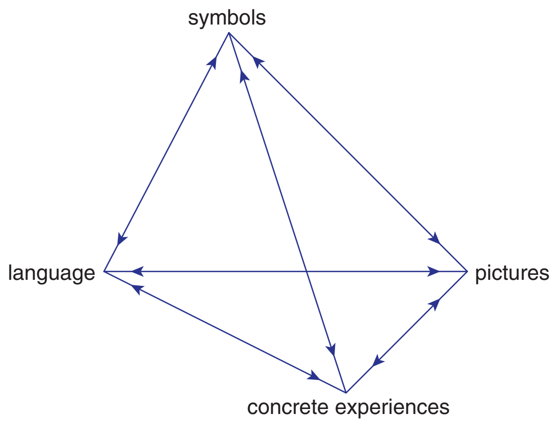

Figure 1.1 *Significant connections in understanding number and number operations*

So, for example, when a 3-year-old counts out loud as they climb the steps on the playground slide or when they stamp along a line of paving stones, they are connecting the language of number with a concrete and physical experience. Later they will be able to connect this experience and language with the picture of a number strip. When 4-year-olds play a simple board game, they are connecting a number symbol on a die with the name of the numeral and the concrete experience of moving their counter forward that number of places along the board. And so, through these connections, an understanding of number is being developed.

Consider a child just starting to use Numicon plates to explore early ideas of number. These are coloured plates carefully structured to represent numbers in sequence (see www.numicon.com). The shape for 5, as shown in Figure 1.2, is a picture that in activities with the materials is connected with the language 'five' and, eventually, the symbol 5. The shape is seen as having a position in a pattern of shapes, which places 'five' between 'four' and 'six'. The child learns to connect this shape with the physical process of filling the holes in the plate with pegs, while counting one, two, three, four, five. The materials are effective for promoting understanding of number because they enable the child to make such key connections between language, pictures and patterns, physical action and symbols.

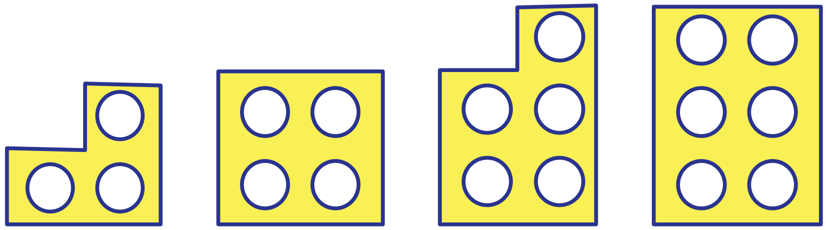

Figure 1.2 *Some Numicon shapes for promoting understanding of number*

We should make clear that the diagram in Figure 1.1 does not represent all the possible connections that might contribute to understanding mathematical concepts or processes. Particularly for young children, in their learning about number and counting, we could include physical movements, gestures and music – such as the rhythmic patterns in both the words and the music of counting songs and the actions that go with them. Then, of course, there are connections within one of the four categories of experience suggested in Figure 1.1. For example, the learner will make connections between one kind of visual image and another, such as connecting four steps of two on a number line with four rows of two dots; or between formal mathematical language, such as 'not vertical', and everyday language, such as 'tilted'. Our point is simply this: the more connections, the more secure and the more useful the understanding.

PAUSE TO REFLECT 'BY GEORGE, SHE'S GOT IT!'

These are the words of Professor Higgins in *My Fair Lady* when Eliza shows significant progress in her learning! Think about all the different expressions you might use – or hear other people use – to indicate that you (or they) understand something or do not understand something. For example, you might say things like:

- Oh, I see!
- Now it's clicked.
- Everything is falling into place.
- Sorry, I don't get it.
- I'm still in the dark.
- I can't see the sense in that!

Reflect on what insights these forms of words might give you about the nature of understanding. Do any of the expressions you identify resonate with the connections model of understanding for mathematics that we have introduced in this chapter?

AN ILLUSTRATION: 7-YEAR-OLDS AND THE CONCEPT OF DIVISION

Below is one teacher's description of some children in her class engaged in a mathematical activity designed to develop their understanding of the concept of division. The emphasis on making connections is clearly what we would recognize as developing understanding, as opposed to just the processes of handling division calculations. The children's recording is shown in Figure 1.3. This illustrates how the activity involves children in handling the four key components of mathematical experience – real objects, pictures, mathematical symbols and mathematical language – and in making connections between them.

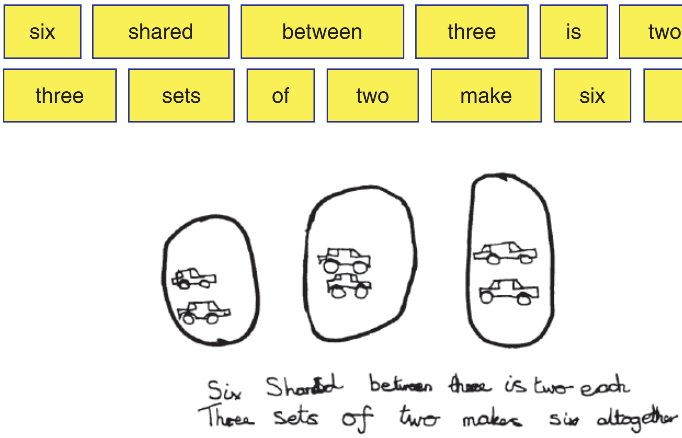

Figure 1.3 *Language patterns and a picture for a sharing experience*

Three 7-year-olds in my class were exploring the early ideas of division. On their table they had a box of toy cars, paper and pencil, a collection of cards with various words written on them (*shared*, *between*, *is*, *each*, *sets*, *of*, *make*, *altogether*, *two*, *three*, *six*, *nine*, *twelve*) and a calculator. Their first task was to share six cars between the three of them. They discussed the result. Then they selected various cards to make up sentences to describe what they had discovered. The children then drew pictures of their sharing and copied their two sentences underneath. One of the children then picked up the calculator and interpreted the first sentence by pressing these keys: 6 ÷ 3 =. She seemed delighted to see appear in the display a symbol representing the 2 cars that they each had. She then carried on and interpreted their second sentence by pressing these keys: x 3 =. As she expected, she got back to the 6 she started with. She demonstrated this to the other children who then insisted on doing it themselves. When they next recorded their calculations as 6 ÷ 3 = 2 and 2 × 3 = 6, the symbols

(Continued)

(Continued)

were a record of the keys pressed on the calculator and the resulting display. Later on I will get them to include with their drawings, their sentences, their recording in symbols, and a number line showing how you can count back from 6 to 0 in jumps of 2 (see Figure 1.4).

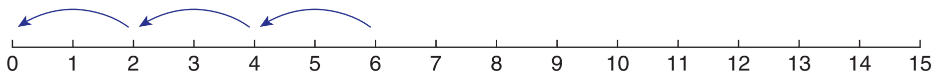

Figure 1.4 *Division connected with a number line*

We can identify some of the connections being made by these children in this activity. They make connections between concrete experience and language when they relate their manipulation of the toy cars to the language patterns of '… shared between … is … each', and '… sets of … make … altogether'. They connect their concrete experience with a picture of three sets of two things. The language of their sentences is connected with the symbols on the keys and display of the calculator. And then, later, they will be learning to connect these symbols with a picture of three steps of two on a number line. It is because of these opportunities to make so many connections between language, concrete experience, pictures and symbols that we would recognize this as an activity promoting mathematical understanding.

We should comment here on the role of calculators in this example, since they are not normally used with children in this age range. However, here they helped children to connect the mathematical symbols on the keys with the concrete experiences and the language of division, showing how calculator experiences, even with young children, can be used selectively to promote understanding. A more detailed analysis of the understanding of multiplication and division is provided in Chapter 5.

We can see, therefore, that much of what is involved in understanding can be identified as the building up of a complex network of connections between language, symbols, concrete materials and pictures. We find it helpful to regard many mathematical concepts as networks of such connections.

THE FUNCTION OF A MATHEMATICAL SYMBOL

Other than just marks we learn to write on paper and manipulate according to certain rules, what are mathematical symbols? What is the function of a symbol in mathematics? What is the relationship of a mathematical symbol to our experiences of doing mathematics and handling mathematical ideas? These are some comments from some of our Foundation Stage and Key Stage 1 teachers, in response to these questions:

- I think of mathematical symbols as abbreviations. They're a sort of shorthand.
- They have very ambiguous meanings for me. They have different meanings depending on the situation you're using them in.
- They sometimes mean you have to do something. Perform an operation. Move some blocks around.

MATHEMATICAL SYMBOLS ARE NOT JUST ABBREVIATIONS

Are mathematical symbols just abbreviations? Of course, there is a sense in which mathematical symbols (such as 4, 28, ÷, =) are abbreviations for mathematical ideas or concepts. But it is important to note that this does not mean that a symbol in mathematics is just an abbreviation for a specific word or phrase. It is tempting to think of, say, the division sign as being essentially an abbreviation for the words 'shared between'. Children often appear to view mathematical symbols in this way. One 9-year-old was using a calculator to do '28 ÷ 4', saying to himself as he pressed the keys, 'Twenty-eight shared …'. At this point he turned to the teacher and asked, 'Which button's *between*?' It was as though each word had to have a button or a symbol to represent it. When we see children writing 41 for fourteen it is clear that they often say 'four' and write 4, then say 'teen' and write 1, again using the symbols as abbreviations for the sounds they are uttering. And so the same child will happily write 41 for forty-one a few lines later! It is a similar error when children record a number like three hundred and seventy-five as 30075 or 3075. The zeros are written down as abbreviations for the word 'hundred'.

But once we think of understanding, particularly understanding of number and number operations, as the building up of connections between concrete experiences, symbols, words and pictures, we begin to see that a mathematical symbol is not simply an abbreviation for just one category of concrete experiences, or just one word or phrase, or just one picture. The child has to learn to connect one symbol with what, at times, can seem to be a confusing variety of concrete situations, pictures and language.

A SYMBOL REPRESENTS A NETWORK OF CONNECTIONS

Hence we suggest that a symbol in mathematics is a way of representing a concept, by which we mean a network of connections. The symbol then becomes a means whereby we can manipulate that concept according to various rules. Without the symbols it would be virtually impossible for us to manipulate the concepts. The symbols of mathematics allow us to both discover and express relationships between various concepts. For example, when we write down a statement in symbols like $4 + 2 = 6$ we are expressing a relationship between the concepts of four, two and six, addition and equality, each of which, as we shall see, is itself a complex network of connections represented by a specific symbol.

The teacher's suggestion above that mathematical symbols have different meanings depending on the situation in which they are being used is a very perceptive observation. One symbol can indeed represent a complex network of connections. It can therefore be applied to a variety of situations and pictures. And it can be associated with a variety of language. This is one of the major themes in our discussion of understanding number and number operations in this and subsequent chapters. We explore in considerable detail how a statement in symbols, such as $4 + 2 = 6$, can be connected to an extensive range of different pictures, language and concrete situations.

For example, these symbols could be connected with such different experiences as: throwing a 4 on one die and a 2 on another and combining these to get a score of 6; starting on square 4 and moving on 2 squares to get to square 6 in a board game; or the price of an item costing £4 being increased by £2 to give a new price of £6. So, the symbols for the numbers, the symbol for the operation of addition and the equals sign itself each has a variety of meanings depending on the situation and the manner in which they are being used. And there is a wide variety of language required in these differing contexts: altogether, counting on, increasing, and so on. Put all this together and the simple-looking statement $4 + 2 = 6$, represents a surprisingly complex network of connections. This is at one and the same time the reason why mathematics is so powerful and the reason why it is for many such a difficult subject to understand.

THE SYMBOL FOR ZERO

Consider, for example, understanding the concept of zero. The formal mathematical language is the word 'zero' and the symbol is 0. Sometimes we use the symbol 0 to represent 'nothing' or 'none'. If a child playing a game has six counters and then loses six, the child has no counters left: 0 counters, nothing. If children in a class are sorted into sets according to their ages and there are no 7-year-olds, then the set of 7-year-olds is an 'empty set'. There is nothing in it. The number in the set is zero. Similarly, if Derek's bank balance is zero, then he has nothing; the zero indicates a complete absence of money in his account. In discussing zero as a place holder in Chapter 3 we shall see that in a number like 206 the zero represents an absence of tens. So when we say the number 'two hundred and six' the zero does not even get a mention!

But we shall also see in Chapter 2 and elsewhere in this book that there are other situations where the same symbol (0) and the word 'zero' are connected with something that is not 'nothing'. For example, the point on a number line connected with 0 is not nothing; it is a very important point on the line. The point labelled 0 degrees on a thermometer does not indicate 'no temperature'. It is not surprising then that understanding the concept of zero is such a challenge for young learners. It involves building up a varied and complex network of pictures, language and real-life situations all to be connected with the symbol. And it is the symbol 0 that represents for us this network of connections.

PAUSE TO REFLECT AN ABSENCE OF SOMETHING

In this chapter we have introduced the idea that although zero sometimes represents 'nothing' there are also situations in which it represents something (for example, a temperature or a point on a number line). We should also make the point that in real-life contexts, even when it represents *nothing*, a zero usually indicates *nothing of something* – in other words an absence of something. The 'something' that is absent is a significant component of the meaning of the zero.

Having spent years thinking of zero as 'nothing' rather than 'an absence of something', we may find it quite hard to think of real-life examples to make this point. Thoughts of zero teeth or zero money might be pretty shocking for us, while the thought of zero tasks to be done may be highly appealing! (See Figure 1.5.)

Children may be less likely to be taken by these particular examples. So how might you portray for them this idea of zero being 'an absence of something'? Reflect on the day-to-day experiences that young children might have of a total absence of something. Come up with some examples where zero of something is desirable for the children as well as some examples where it is undesirable.

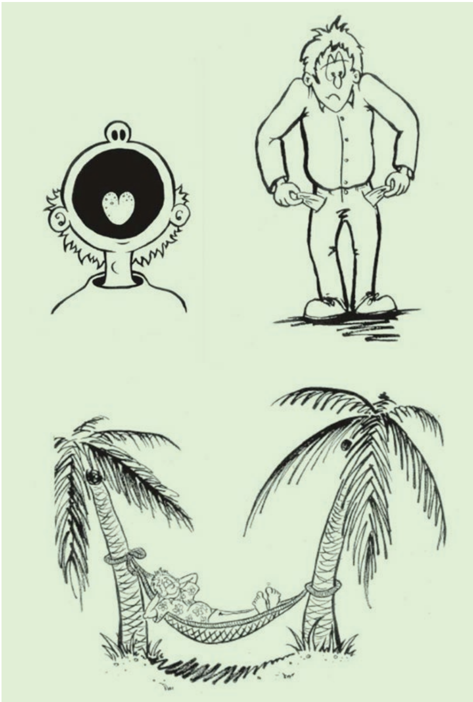

Figure 1.5 *Zero teeth, zero money, zero tasks*

TRANSFORMATION AND EQUIVALENCE

The concept of 'equals' is another example of a complex network of connections, which we represent by the equals sign. To analyse this concept we first discuss two fundamental ideas that run right through mathematics. These are the notions of *transformation* and *equivalence*.

WHAT IS DIFFERENT? WHAT IS THE SAME?

We introduce these principles with an example from our everyday experience, namely, paper sizes. Figure 1.6 represents two pieces of paper, one A4 size and the other A5 size. We should explain that you get A1 paper by folding A0 paper in half, A2 by folding A1 in half, and so on, and that the dimensions of the paper are cunningly chosen so that each rectangle has exactly the same proportions as the original. So the A5 rectangle is the shape produced by folding in half the A4 rectangle. Now when we begin to make mathematical statements about the relationships between the two rectangles in Figure 1.6, we find they fall into two categories. On the one hand, we may look at the rectangles and make observations about the ways in which they differ from each other:

- One rectangle is on the left and the other is on the right.
- One is half the area of the other.
- The length of one is about 1.4 times the length of the other.

Statements like these are essentially using the notion of transformation. We are concerned with the changes that are observed when we move our attention from one rectangle to the other. We are hinting at what would have to be done to one rectangle to transform it into the other. But, on the other hand, we may look at these rectangles and make statements about the ways in which they are the same. Statements like these are essentially using the notion of equivalence. We are concerned with what stays the same when we move our attention from one rectangle to the other:

- They are both rectangles.
- They are the same shape.
- Their sides are in the same proportion.

More generally, then, when we make statements about what has changed in a situation, what is different about two things, what something has become, and so on, we are using the idea of transformation. When we concern ourselves with what is the same, with similarities rather than differences, what remains unchanged in spite of the transformation, then we are talking about equivalence. The key questions in everyday language for teachers to prompt children to recognize transformations and equivalences are simply 'What is different?' and 'What is the same?'

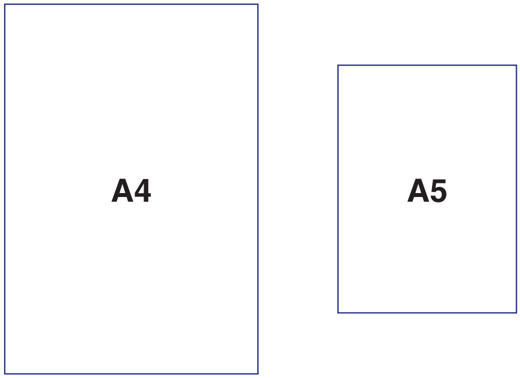

Figure 1.6 *The same but different*

So when my 4-year-olds sort the shapes in the shape box into sets and put all the squares together, for example, they are seeing an *equivalence*. And when they look at two of the squares and say something like 'this one is bigger than that one' they are seeing a *transformation*.

WHAT STAYS THE SAME WHEN THINGS CHANGE?

Much of mathematics, not just geometrical experiences like the example above, is concerned with recognizing and applying equivalences and transformations. Often a crucial mathematical principle involves the recognition of which equivalences are preserved under which transformations. An example of this, which illustrates the point nicely, is that of equivalent fractions. The reader may recall that you can transform the fraction $^4/_6$ by dividing top and bottom by 2 to produce the equivalent fraction $^2/_3$ (and record this as $^4/_6 = {}^2/_3$). This is a transformation that preserves the equivalence. But it is apparently not in order to transform $^4/_6$ by, say, adding one to top and bottom, because $^4/_6$ does not equal $^5/_7$. This transformation does not preserve the equivalence.

But there are other situations where adding 1 to each of two numbers does preserve an equivalence, such as when calculating the difference between two numbers. For example, 77 – 49 can be correctly and usefully transformed into 78 – 50. At times one feels sorry for the poor child trying to make sense of this subject, particularly if the processes are taught as recipes and routines without understanding. If the appropriate connections are not made, it must seem entirely arbitrary as to whether a particular transformation is acceptable and warrants a tick or is unacceptable and generates a cross.

THE EQUALS SIGN

Finally, to bring together the major ideas about understanding introduced in this chapter, we return to the equals sign (=) and the difficulties illustrated by Gemma at the start. We can now see that the essence of the problem with this symbol is that the concept of equals is such a complex network of ideas and experiences. We find that there is not just one form of words that goes with the symbol (=) but that there is a range of language and situations to which the symbol may become attached, including both the ideas of transformation and equivalence. Some of the teachers with whom we worked articulated their anxieties about the meaning of this symbol:

- My 6-year-olds had problems with some questions in their maths books where they had to put in the missing numbers, like this: $6 = 2 + \square$. Most of them put in 8, of course. When I tried to explain to them how to do these sums I realized I didn't actually know what the equals sign meant myself. We would say 'two add something makes six' if it were written the other way round, but 'six makes two add something' doesn't make sense.
- Is it wrong to say 'four add two makes six'? Should I insist that the children say 'equals six'?
- The word 'equals' doesn't mean anything to them. It's just a symbol, just some marks on paper that you make when you're doing sums.
- Doesn't it confuse children to say 'makes' when you're adding and then to say 'leaves' when you're taking away?
- And sometimes we just read it as 'is', like 'three add four is seven'.

THE EQUALS SIGN REPRESENTING AN EQUIVALENCE

Strictly speaking, the equals sign represents the idea of equivalence. When we write down $2 + 4 = 6$ we are expressing an equivalence between '$2 + 4$' and '6'. We are making a statement that there is something the same about 'two added to four' and 'six'. Probably the most straightforward language to go with this statement is 'two add four is the same as six'. To emphasize the underlying equivalence in statements in arithmetic that use the equals sign, the phrase 'is the same as' is particularly significant. It connects very clearly with the concrete experience of doing addition and subtraction with some structured materials, as shown in Figure 1.7.

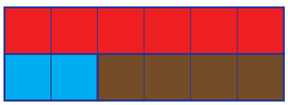

Figure 1.7 *Two add four is the same as six*

When the child makes a train with a 2-rod and a 4-rod, the problem is to find another rod to match this train. Recording this experience as 2 + 4 = 6 is an expression of the equivalence: the 2-rod added to the 4-rod is in one sense the same as the 6-rod. Of course, they are only the same in that they are the same length. A train made up of a blue rod and a brown rod is very different from a red rod. But lying side-by-side they represent an equivalence, and this is expressed by the symbols 2 + 4 = 6. It is worth noting that this interpretation of the equals sign makes sense of the problem that Gemma started with: '6 = 2 + $\square$' can be read as 'six is the same as two add something'.

THE EQUALS SIGN REPRESENTING A TRANSFORMATION

However, when the child puts out sets of two counters and four counters, forms their union and counts the new set to discover that there is now a set of six counters, it is a bit obscure to suggest that this is an experience of 'two add four is the same as six'. The child has actually *transformed* the two sets of two and four counters into a set of six (see Figure 1.8). The child's attention therefore is focused on the transformation that has taken place. This being so, it seems perfectly natural, and surely appropriate, to use the language 'two and four makes six' to describe the transformation the child has applied. One of the teachers quoted above said that she regarded the symbols as instructions to do something. In other words, the equals sign tells you to apply some sort of transformation. There is evidence that this is how children most frequently interpret the equals sign.

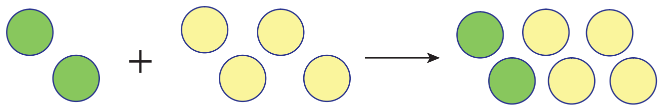

Figure 1.8 *Two add four makes six*

So, in practice, the equals sign represents both the equivalence and the transformation aspects of the relationship between 2 + 4 and 6. Thus we would not want to suggest that it is wrong or in some way mathematically incorrect to associate 'makes', 'leaves', 'is', and so on, with the equals sign, and insist on using only one particular form of words, such as 'is the same as' or even 'equals'. Rather, we would advocate a combination of experiences emphasizing the notions of both equivalence and transformation. As we have already argued, mathematical symbols are not just abbreviations for particular words or phrases. We have to recognize that the statement 2 + 4 = 6 is actually, at one and the same time, a representation in symbols of the transformation that has been applied to 2 and 4, and the equivalence that has emerged between 2 + 4 and 6.