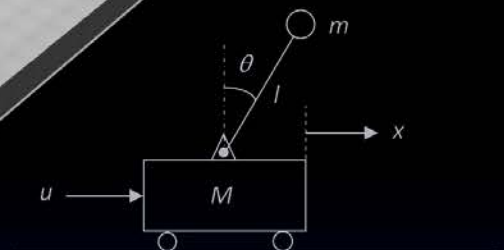
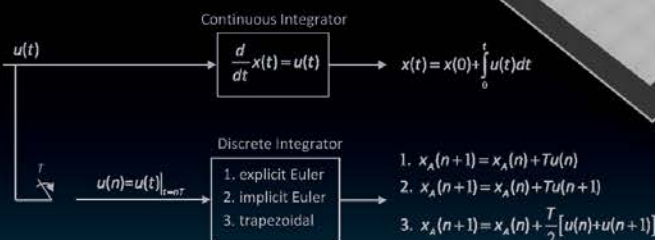
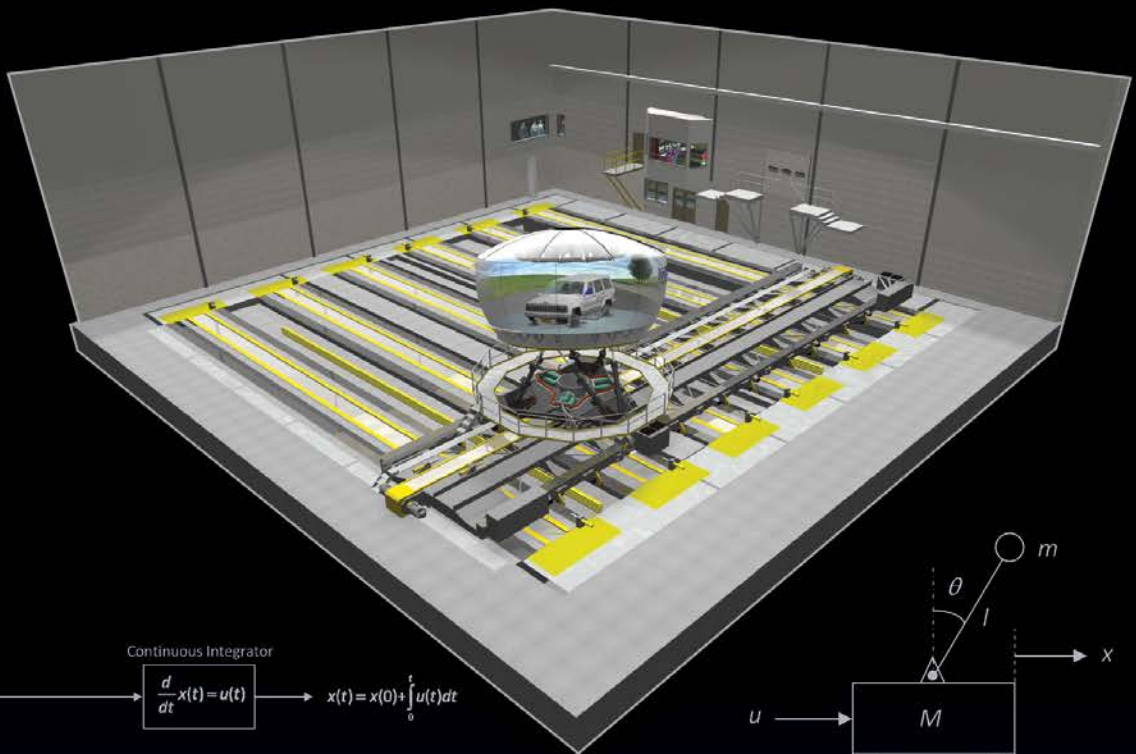


THIRD EDITION

Simulation of Dynamic Systems with MATLAB® and Simulink®

Harold Klee • Randal Allen



CRC Press
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Simulation of Dynamic Systems with MATLAB[®] and Simulink[®]

Third Edition



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*To Andrew, Cassie, and in loving memory of their
mother and my devoted wife, Laura.*

Harold Klee

*To my wife, Christine, whose inner beauty radiates with the warmth of a
sunny day at the beach. Thank you for your everlasting love and support.*

Randal Allen



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Foreword

MATLAB is used for so many applications, it defies attempts at categorization. This book demonstrates some of that interesting diversity.

As you read and use this book, you will find two kinds of knowledge. You may hope to find insight to the use of MATLAB and Simulink. That hope will be richly fulfilled, I think. But you should be mindful of another kind of knowledge; how others have solved problems. The rich collection of examples and methods go far beyond the software toolset. These span different technical disciplines and industries.

The authors show how modeling, simulation, and analysis gets done across a wide range of applications and industries, including financial markets. Work within and among various professional societies further broaden this perspective. Their university work in teaching budding scientists and engineers has honed the ability to make complexity approachable.

This book gives readers a chance to look outside their own discipline or industry, to collect ideas from afar.

I hope your imagination will be fired while your modeling and simulation skills are being honed.

Steve Roerman

Chairman & CEO of Lone Star Analysis
Dallas, TX

Simulation has come a long way since the days analog computers filled entire rooms. Yet, it is more important than ever that simulations be constructed with care, knowledge, and a little wisdom, lest the results be gibberish or, worse, reasonable but misleading. Used properly, simulations can give us extraordinary insights into the processes and states of a physical system. Constructed with care, simulations can save time and money in today's competitive marketplace.

One major application of simulation is the simulator, which provides interaction between a model and a person through some interface. The earliest simulator, Ed Link's Pilot Maker aircraft trainer, did not use any of the simulation techniques described in this book. Modern simulators, however, such as the National Advanced Driving Simulator (NADS), cannot be fully understood without them.

The mission of the NADS is a lofty one: to save lives on U.S. highways through safety research using realistic human-in-the-loop simulation. This is an example of the importance simulation has attained in our generation. The pervasiveness of simulation tools in our society will only increase over time; it will be more important than ever that future scientists and engineers be familiar with their theory and application.

The content for Simulation of Dynamic Systems with MATLAB® and Simulink® is arranged to give the student a gradual and natural progression through the important topics in simulation. Advanced concepts are added only after complete examples have been constructed using fundamental methods. The use of MATLAB and Simulink provides experience with tools that are widely adopted in industry and allow easy construction of simulation models.

May your experience with simulation be enjoyable and fruitful and extend throughout your careers.

Chris Schwarz, PhD

Iowa City, Iowa



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Preface

In the first article of *SIMULATION* magazine in the Fall of 1963, the editor John McLeod proclaimed simulation to mean “the act of representing some aspects of the real world by numbers or symbols which may be easily manipulated to facilitate their study.” Two years later, it was modified to “the development and use of models for the study of the dynamics of existing or hypothesized systems.” More than 40 years later, the simulation community has yet to converge upon a universally accepted definition. Either of the two cited definitions or others that followed convey a basic notion, namely, that simulation is intended to reinforce or supplement one’s understanding of a system. The definitions vary in their description of tools and methods to accomplish this.

The field of simulation is experiencing explosive growth in importance because of its ability to improve the way systems and people perform, in a safe and controllable environment, at a reduced cost. Understanding the behavior of complex systems with the latest technological innovations in fields such as transportation, communication, medicine, aerospace, meteorology, etc., is a daunting task. It requires an assimilation of the underlying natural laws and scientific principles that govern the individual subsystems and components. A multifaceted approach is required, one in which simulation can play a prominent role, both in validation of a system’s design and in training of personnel to become proficient in its operation.

Simulation is a subject that cuts across traditional academic disciplines. Airplane crews spend hours flying simulated missions in aircraft simulators to become proficient in the use of onboard subsystems during normal flight and possible emergency conditions. Astronauts spend years training in shuttle and orbiter simulators to prepare for future missions in space. Power plant and petrochemical process operators are exposed to simulation to obtain peak system performance. Economists resort to simulation models to predict economic conditions of municipalities and countries for policymakers. Simulations of natural disasters aid in preparation and planning to mitigate the possibility of catastrophic events.

While the mathematical models created by aircraft designers, nuclear engineers, and economists are application specific, many of the equations are analogous in form despite the markedly different phenomena described by each model. Simulation offers practitioners from each of these fields the tools to explore solutions of the models as an alternative to experimenting with the real system.

This book is meant to serve as an introduction to the fundamental concepts of continuous system simulation, a branch of simulation applied to dynamic systems whose signals change over a continuum of points in time or space. Our concern is with mathematical models of continuous-time systems (electric circuits, thermal processes, population dynamics, vehicle suspension, human physiology, etc.) and the discrete-time system models created to simulate them. The continuous system mathematical models consist of a combination of algebraic and ordinary differential equations. The discrete-time system models are a mix of algebraic and difference equations.

Systems that transition between states at randomly occurring times are called discrete-event systems. Discrete-event simulation is a complementary branch of simulation, separate from continuous system simulation, with a mathematical foundation rooted in probability theory. Examples of discrete-event systems are facilities such as a bank, a tollbooth, a supermarket, or a hospital emergency room, where customers arrive and are then serviced in some way. A manufacturing plant involving multiple production stages of uncertain duration to generate a finished product is another candidate for discrete-event simulation.

Discrete-event simulation is an important tool for optimizing the performance of systems that change internally at unpredictable times due to the influence of random events. Industrial engineering programs typically include a basic course at the undergraduate level in discrete-event simulation. Not surprisingly, a number of excellent textbooks in the area have emerged for use by the academic community and professionals.

In academia, continuous simulation has evolved differently than discrete-event simulation. Topics in continuous simulation such as dynamic system response, mathematical modeling, differential equations, difference equations, and numerical integration are dispersed over several courses from engineering, mathematics, and the natural sciences. In the past, the majority of courses in modeling and simulation of continuous systems were restricted to a specific field like mechanical, electrical, and chemical engineering or scientific areas like biology, ecology, and physics.

A transformation in simulation education is underway. More universities are beginning to offer undergraduate and beginning graduate courses in the area of continuous system simulation designed for an interdisciplinary audience. Several institutions now offer master's and PhD programs in simulation that include a number of courses in both continuous and discrete-event simulation. A critical mass of students are now enrolled in continuous simulation-related courses and there is a need for an introductory unifying text.

The essential ingredient needed to make simulation both interesting and challenging is the inclusion of real-world examples. Without models of real-world systems, a first class in simulation is little more than a sterile exposition of numerical integration applied to differential equations.

Modeling and simulation are inextricably related. While the thrust of this text is continuous simulation, mathematical models are the starting point in the evolution of simulation models. Analytical solutions of differential equation models are presented, when appropriate, as an alternative to simulation and a simple way of demonstrating the accuracy of a simulated solution. For the most part, derivations of the mathematical models are omitted and references to appropriate texts are included for those interested in learning more about the origin of the model's equations.

New and revised topics in the third edition are discussed in the later paragraphs dedicated to the content of each chapter. However, certain changes appearing in the third edition apply to the entire book, [Chapters 1](#) through [8](#). These changes consist of the following:

1. All MATLAB script and function .m files have been renamed and the references to them in the text have been changed to reflect the new file names. This eliminates the confusion present in the second edition which retained the MATLAB file names from the first edition based on the old system for naming chapter sections, figures, tables and exercise problems. Simulink model .mdl file names remain unchanged since they do not contain chapter or section references in their names.
2. With very few exceptions, nearly every graph generated in MATLAB has been redone to improve its appearance in printed form. Specifically, all line plots and markers are produced with a heavier weight, annotation and titles of most graphs have been changed to better communicate the significance of each graph. Whilst the graphs are in black and white in the text, every graph generated in MATLAB appears on screen in vivid colors to enhance their appearance. Updated MATLAB and Simulink files are accessible from CRC Press.
3. Simulink diagrams have been updated to be compatible with version R2016a of MATLAB/Simulink. Diagrams with numerous Simulink blocks have been expanded to reveal the details of each block and their interconnections.
4. Certain non-graph figures have been eliminated as a result of being unnecessary, while others have been modified to be more informative.

Simulation is best learned by doing. Accordingly, the material is presented in a way that permits the reader to begin exploring simulation, starting with a mathematical model in [Chapter 1](#). The notation used to represent discrete-time variables has been simplified in the new edition making it easier to comprehend the difference equations developed to approximate the dynamics of continuous-time systems. The latter part of Section 1.1 and all of Sections 1.2 through 1.5 have been rewritten to better explain the underlying concepts.

Chapters 2 and **4** remain basically unchanged. They present a condensed treatment of linear, continuous-time, and discrete-time dynamic systems, normally covered in an introductory linear systems course. The instructor can skip some or all of the material in these chapters if the students' background includes a course in signals and systems or linear control theory.

Numerical integration is at the very core of continuous system simulation. Instead of treating the subject in one exhaustive chapter, coverage is distributed over three chapters. Elementary numerical integration in **Chapter 3** is an informal introduction to the subject, which includes discussion of several elementary methods for approximating the solutions of first order differential equations. Presentation of the topics in **Chapter 3** has been completely revised. Much of the material in **Chapter 3** from the second edition appears in a reorganized format while some material has been deleted and new material added.

Simulink, from The MathWorks, is the featured simulation program because of its tight integration with MATLAB, the de facto standard for scientific and engineering analysis, and data visualization software. **Chapter 5** takes the reader through the basic steps of creating and running Simulink models. Monte Carlo simulation for estimation of system parameters and probability of events occurring in dynamic systems is covered. A new case study is introduced in Section 5.13 involving logically-controlled flows between two interconnected tanks.

Chapter 6 delves into intermediate-level topics of numerical integration, including a formal presentation of One-Step (Runga–Kutta) and multistep methods, adaptive techniques, truncation errors and a brief mention of stability.

Chapter 7 highlights some advanced features of Simulink useful in more in-depth simulation studies. Section 7.7 was added to demonstrate rare event modeling and portfolio risk measurement, thereby exposing potential Black Swans as they may pertain to the financial markets. Section 7.8 was added to introduce SIPmath as a means for efficiently representing uncertainty as probability distributions, enabling legacy and future simulation models to communicate with each other.

Chapter 8 is for those interested in more advanced topics on continuous simulation. Coverage includes a discussion of dynamic errors, stability, real-time compatible numerical integration and multirate integration algorithms for simulation of stiff systems.

The basic minimum requirement for anyone using this text is a first course in Ordinary Differential Equations. An outline for a one-semester, preferably senior-level course in continuous system simulation is subject to the individual requirements of the instructor as well as the prior education of the students. As a starting point, some basic recommendations by the authors for a one-semester course are:

	Chapter 1 Sections	Chapter 2 Sections	Chapter 3 Sections	Chapter 4 Sections	Chapter 5 Sections	Chapter 6 Sections
For students knowledgeable in linear systems theory	1–4	1–8	1–8	Review of 1–8	1–3, 5, 6, 8, 9, 11 or 13	1–5, 8
For students not well-versed in linear systems theory	1–4	1–8	1–8	1–5	1–3, 5, 6, 8, 11 or 13	1–4, 8

All remaining sections are appropriate for a second course in a two-semester sequence, either at the senior, or more appropriately graduate level. The material in **Chapters 7** and **8** is well suited as a reference for practicing engineers and researchers involved in more advanced simulation endeavors.

The first and second editions of this text has been field-tested for nearly a decade. Despite numerous revisions based on the scrutiny and suggestions of students and colleagues, some errors manage to go undetected. Further suggestions for improvement and revelations of inaccuracies can be brought to the attention of the authors at aerospace321@outlook.com and klee.harold@gmail.com.

Numerous individuals deserve our thanks and appreciation for making the third edition possible. Thanks to Nora Konopka at Taylor & Francis/CRC Press for committing to the third edition

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1 Mathematical Modeling

1.1 INTRODUCTION

1.1.1 IMPORTANCE OF MODELS

Models are an essential component of simulation. Before a new prototype design for an automobile braking system or a multimillion dollar aircraft is tested in the field, it is commonplace to “test drive” the separate components and the overall system in a simulated environment based on some form of model. A meteorologist predicts the expected path of a tropical storm using weather models that incorporate the relevant climatic variables and their effect on the storm’s trajectory. An economist issues a quantitative forecast of the U.S. economy predicated based on key economic variables and their interrelationships with the help of computer models. Before a nuclear power plant operator is “turned loose” at the controls, extensive training is conducted in a model-based simulator where the individual becomes familiar with the plant’s dynamics under routine and emergency conditions. Health care professionals have access to a human patient simulator to receive training in the recognition and diagnosis of disease. Public safety organizations can plan for emergency evacuations of civilians from low-lying areas using traffic models to simulate vehicle movements along major access roads.

The word “model” is a generic term referring to a conceptual or physical entity that resembles, mimics, describes, predicts, or conveys information about the behavior of some process or system. The benefit of having a model is to be able to explore the intrinsic behavior of a system in an economical and safe manner. The physical system being modeled may be inaccessible or even nonexistent as in the case of a new design for an aircraft or automotive component.

Physical models are often scaled-down versions of a larger system of interconnected components as in the case of a model airplane. Aerodynamic properties of airframe and car body designs for high-performance airplanes and automobiles are evaluated using physical models in wind tunnels. In the past, model boards with roads, terrain, miniaturized models of buildings, and landscape, along with tiny cameras secured to the frame of ground vehicles or aircraft, were prevalent for simulator visualization. Current technology relies almost exclusively on computer-generated imagery.

In principle, the behavior of dynamic systems can be explained by mathematical equations and formulae, which embody either scientific principles or empirical observations, or both, related to the system. When the system parameters and variables change continuously over time or space, the models consist of coupled algebraic and differential equations. In some cases, lookup tables containing empirical data are employed to compute the parameters. Equations may be supplemented by mathematical inequalities, which constrain the variation of one or more dependent variables. The aggregation of equations and numerical data employed to describe the dynamic behavior of a system in quantitative terms is collectively referred to as a mathematical model of the system.

Partial differential equation models appear when a dependent variable is a function of two or more independent variables. For example, electrical parameters such as resistance and capacitance are distributed along the length of conductors carrying electrical signals (currents and voltages). These signals are attenuated over long distances of cabling. The voltage at some location x measured from an arbitrary reference is written $v(x, t)$ instead of simply $v(t)$, and the circuit is modeled accordingly.

A mathematical model for the temperature in a room would necessitate equations to predict $T(x, y, z, t)$ if a temperature probe placed at various points inside the room reveals significant variations in temperature with respect to x, y, z in addition to temporal variations. Partial differential

equations describing the cable voltage $v(x, t)$ and room temperature $T(x, y, z, t)$ are referred to as “distributed parameter” models.

The mathematical models of dynamic systems where the single independent variable is “time” comprise ordinary differential equations. The same applies to systems with a single spatial independent variable; however, these are not commonly referred to as dynamic systems since variations of the dependent variables are spatial as opposed to temporal in nature. Ordinary differential equation models of dynamic systems are called “lumped parameter” models because the spatial variation of the system parameters is negligible or else it is being approximated by lumped sections with constant parameter values. In the room temperature example, if the entire contents of the room can be represented by a single or lumped thermal capacitance, then a single temperature $T(t)$ is sufficient to describe the room. We focus exclusively on dynamic systems with lumped parameter models, hereafter referred to simply as mathematical models.

A system with a lumped parameter model is illustrated in [Figure 1.1](#). The key elements are the system inputs $u_1(t), u_2(t), \dots, u_r(t)$, which make up the system input vector $\underline{u}(t)$, the system outputs $y_1(t), y_2(t), \dots, y_p(t)$, which form the output vector $\underline{y}(t)$, and the parameters p_1, p_2, \dots, p_m constituting the parameter vector \underline{p} . The parameters are shown as constants; however, they may also vary with time.

Our interest is in mathematical models of systems consisting of coupled algebraic and differential equations relating the outputs and inputs with coefficients expressed in terms of the system parameters. For steady-state analyses, transient responses are irrelevant, and the mathematical models consist of purely algebraic equations relating the system variables.

An example of a mathematical model for a system with two inputs, three outputs, and several parameters is

$$p_1 \frac{d^2}{dt^2} y_1(t) + p_2 p_3 \frac{d}{dt} y_1(t) + p_4 y_1(t) + p_5 \frac{d}{dt} y_2(t) + p_6 y_2(t) = p_7 u_1(t) \quad (1.1)$$

$$p_8 \frac{d}{dt} y_2(t) + \frac{p_9}{p_{10}} y_2(t) + p_{11} y_1(t) y_2(t) = p_{12} \frac{d}{dt} u_1(t) + p_{13} u_1(t) + p_{14} u_2(t) \quad (1.2)$$

$$p_{15} y_3(t) = \frac{p_{16} y_1^{p_{17}}(t)}{y_2(t)} \quad (1.3)$$

The order of a model is equal to the sum of the highest derivatives of each of the dependent variables, in this case $y_1(t), y_2(t), y_3(t)$, and the order is therefore $2 + 1 + 0 = 3$. Equation 1.1 is a linear differential equation. Equation 1.2 is a nonlinear differential equation because of the term involving the product of $y_1(t)$ and $y_2(t)$. The mathematical model is nonlinear due to the presence of the nonlinear differential equation and the nonlinear algebraic equation (Equation 1.3). It is to be borne in mind that it is the nature of the equations that determines whether a math model is linear

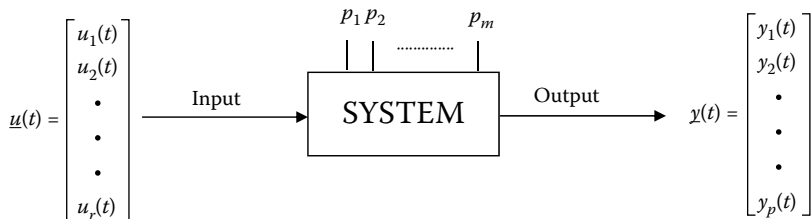


FIGURE 1.1 A system with a lumped parameter model.

or nonlinear. An adjective such as linear or nonlinear applies to the mathematical model as opposed to the actual system.

It is important to distinguish between the system being modeled and the model itself. The former is unique, even though it may exist only at the design stage, while the mathematical model may assume different forms. For example, a team of modelers may be convinced that the lead term in Equation 1.1 is likely to be insignificant under normal operating conditions. Consequently, two distinct models of the system exist, one third order and the other second order. The third-order model includes the second derivative term to accurately reflect system behavior under unusual or nontypical conditions (e.g., an aircraft exceeding its flight envelope or a ground vehicle performing an extreme maneuver). The simpler second-order model ignores what are commonly referred to as higher-order effects. Indeed, there may be a multitude of mathematical models to represent the same system under different sets of restricted operating conditions. Regardless of the detail inherent in a mathematical model, it nevertheless represents an incomplete and inexact depiction of the system.

A model's intended use will normally dictate its level of complexity. For example, models for predicting vehicle handling and responsiveness are different from those intended to predict ride comfort. In the first case, accurate equations describing lateral and longitudinal tire forces are paramount in importance, whereas passenger comfort relies more on vertical tire forces and suspension system characteristics.

Mathematical modeling is an inexact science, relying on a combination of intuition, experience, empiricism, and the application of scientific laws of nature. Trade-offs between model complexity and usefulness are routine. Highly accurate microclimatic weather models that use current atmospheric conditions to predict the following day's weather are of limited value if they require 48 h on a massively parallel or supercomputer system to produce results. At the extreme opposite, overly simplified models can be grossly inaccurate if significant effects are overlooked.

The difference between a mathematical model and a simulation model is open to interpretation. Some in the simulation community view the two as one and the same. Their belief is that a mathematical model embodies the attributes of the actual system and simulation refers to solutions of the model equations, albeit generally approximate in nature. Exact analytical solutions of mathematical model equations are nonexistent in all but the simplest cases.

Others maintain a distinction between the two and express the view that simulation model(s) originate from the mathematical model. According to this line of thinking, simulating the dynamics of a system requires a simulation model that is different in nature from a mathematical model. A reliable simulation model must be capable of producing numerical solutions in reasonably close agreement with the actual (unknown) solutions to the math model. Simulation models are commonly obtained from discrete-time approximations of continuous-time mathematical models. Much of this book is devoted to the process of obtaining simulation models in this way. More than one simulation model can be developed from a single mathematical model of a system.

Stochastic models are important when dealing with systems whose inputs and parameters are best modeled using statistical methods. Discrete event models are used to describe processes that transit from one state to another at randomly spaced points in time. Probability theory plays a significant role in the formulation of discrete event models for describing the movement of products and service times at different stages in manufacturing processes, queuing systems, and the like. In fact, the two pillars of simulation are continuous system simulation, the subject of this book, and discrete event simulation.

There is a great deal more to be said about modeling. Entire books are devoted to properly identifying model structure and parameter values for deterministic and stochastic systems. Others concentrate more on derivation of mathematical models from diverse fields and methods of obtaining solutions under different circumstances. The reader is encouraged to check the references section at the end of this book for additional sources of material related to modeling.

Modeling is essential to the field of simulation. Indeed, it is the starting point of any simulation study. The emphasis, however, in this book is on the presentation of simulation fundamentals.

Accordingly, derivation of mathematical models is not a prominent component. For the most part, the math models are taken from documented sources listed in the references section, some of which include step-by-step derivations of the model equations. The derivation is secondary to a complete understanding of the model, that is, its variables, parameters, and knowledge of conditions that may impose restrictions on its suitability for a specific application.

Simulation of complex systems requires a team effort. The modeler is a subject expert responsible for providing the math model and interpreting the simulation results. The simulationist produces the simulation model and performs the simulation study. For example, an aerodynamicist applies principles of boundary layer theory to obtain a mathematical model for the performance of a new airfoil design. Starting with the math model, simulation skills are required to produce a simulation model capable of verifying the efficacy of the design based on numerical results. Individuals with expert knowledge in a particular field are oftentimes well versed in the practice of simulation and may be responsible for formulation of alternative mathematical models of the system in addition to developing and running simulations.

A simple physical system is introduced in the next section, and the steps involved in deriving an idealized math model are presented. In addition to benefiting from seeing the process from start to finish, the ingredients for creating a simulation model are introduced. Hence, by the end of this chapter, the reader will be able to perform rudimentary simulation.

1.2 DERIVATION OF A MATHEMATICAL MODEL

We begin our discussion of mathematical modeling with a simple derivation of the mathematical model representing the dynamic behavior of an open tank containing a liquid that flows in the top and is discharged from the bottom. Referring to Figure 1.2, the primary input is the liquid flow rate $F_1(t)$, an independent variable measured in appropriate units such as cubic feet per minute (volumetric flow rate) or pounds per hour (mass flow rate). Responding to changes in the input are dependent variables $H(t)$ and $F_0(t)$ the fluid level, and flow rate from the tank.

Once the derivation is completed, we can use the model to predict the outflow and fluid level response to a specific input flow rate $F_1(t)$, $t \geq 0$. Note that we have restricted the set of possible inputs to $F_1(t)$ and in the process relegated the remaining independent variables, that is, other variables which affect $F_0(t)$ and $H(t)$, to second-order importance. Our assumption is that the eventual



FIGURE 1.2 Tank as a dynamic system with input and outputs.

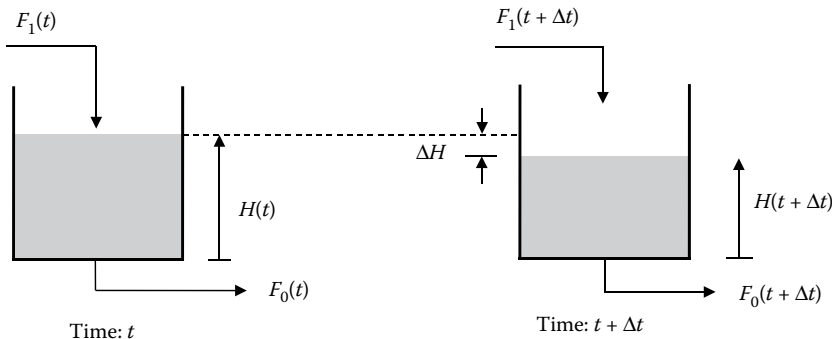


FIGURE 1.3 A liquid tank at two points in time.

model will be suitable for its intended application. It must be borne in mind that if extremely accurate predictions of the level $H(t)$ are required, it may be necessary to include second-order effects such as evaporation and hence introduce additional inputs related to ambient conditions, namely, temperature, humidity, air pressure, wind speed, and so forth.

The derivation is based on conditions of the tank at two discrete points in time, as if snapshots of the tank were available at times “ t ” and “ $t + \Delta t$,” as shown in [Figure 1.3](#).

The following notation is used with representative units given for clarity:

$F_1(t)$: Input flow at time t , ft³/min

$H(t)$: Liquid level at time t , ft

$F_0(t)$: Output flow at time t , ft³/min

A : Cross-sectional area of tank, ft²

At time $t + \Delta t$, from the physical law of conservation of volume,

$$V(t + \Delta t) = V(t) + \Delta V \quad (1.4)$$

where

$V(t)$ is the volume of liquid in the tank at time t

ΔV is the change in volume from time t to $t + \Delta t$

The volume of liquid in the tank at times t and $t + \Delta t$ is given by

$$V(t) = AH(t) \quad (1.5)$$

$$V(t + \Delta t) = AH(t + \Delta t) \quad (1.6)$$

Equations 1.5 and 1.6 assume constant cross-sectional area of the tank, that is, A is independent of H .

The change in volume from t to $t + \Delta t$ is equal to the volume of liquid flowing in during the interval t to $t + \Delta t$ minus the volume of liquid flowing out during the same period of time. The liquid volumes are the areas under the input and output volume flow rates from t to $t + \Delta t$ as shown in [Figure 1.4](#).

Expressing these areas in terms of integrals,

$$\Delta V = \int_t^{t+\Delta t} F_1(t) dt - \int_t^{t+\Delta t} F_0(t) dt \quad (1.7)$$

The integrals in Equation 1.7 can be approximated by assuming $F_1(t)$ and $F_0(t)$ are constant over the interval t to $t + \Delta t$ (see [Figure 1.4](#)). Hence,

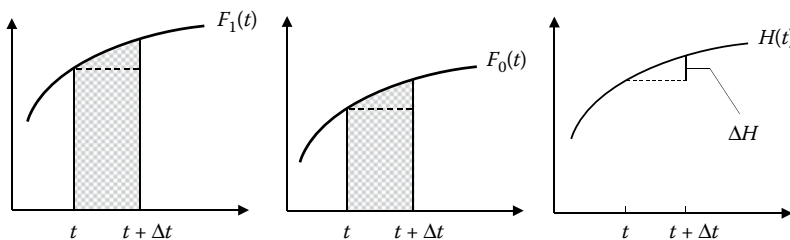


FIGURE 1.4 Volumes of liquid flowing in and out of tank from t to $t + \Delta t$.

$$\int_t^{t+\Delta t} F_1(t) dt \approx F_1(t) \Delta t \quad (1.8)$$

$$\int_t^{t+\Delta t} F_0(t) dt \approx F_0(t) \Delta t \quad (1.9)$$

Equations 1.8 and 1.9 are reasonable approximations provided Δt is small. Substituting Equations 1.8 and 1.9 into Equation 1.7 yields

$$\Delta V \approx F_1(t) \Delta t - F_0(t) \Delta t \quad (1.10)$$

Substituting Equations 1.5, 1.6, and 1.10 into Equation 1.4 gives

$$AH(t + \Delta t) \approx AH(t) + [F_1(t) - F_0(t)] \Delta t \quad (1.11)$$

$$\Rightarrow A[H(t + \Delta t) - H(t)] \approx [F_1(t) - F_0(t)] \Delta t \quad (1.12)$$

$$\Rightarrow A \left[\frac{\Delta H}{\Delta} \right] \approx F_1(t) - F_0(t) \quad (1.13)$$

where ΔH is the change in liquid level over the interval $(t, t + \Delta t)$. Note that $\Delta H/\Delta t$ is the average rate of change in the level H over the interval $(t, t + \Delta t)$. It is the slope of the secant line from pt A to pt B in Figure 1.5.

In the limit as Δt approaches zero, pt B approaches pt A, and the average rate of change in H over the interval $(t, t + \Delta t)$ becomes the instantaneous rate of change in H at time t , that is,

$$\lim_{\Delta t \rightarrow 0} \frac{\Delta H}{\Delta t} = \frac{dH}{dt} \quad (1.14)$$

where dH/dt is the first derivative of $H(t)$. From the graph, it can be seen that dH/dt is equal to the slope of the tangent line of the function $H(t)$ at t (pt A).

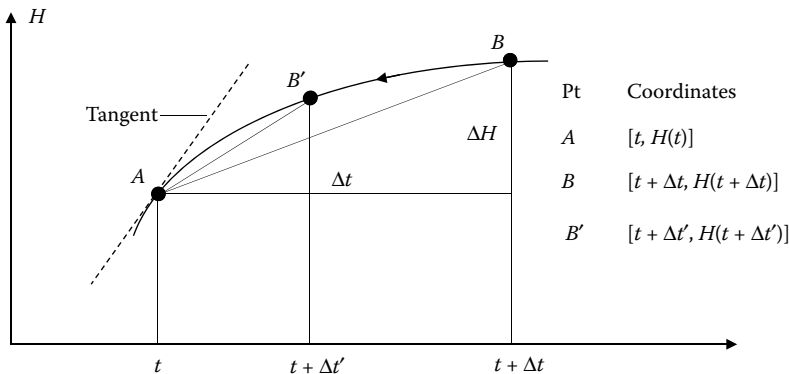


FIGURE 1.5 Average rate of change $\Delta H/\Delta t$ as Δt gets smaller.

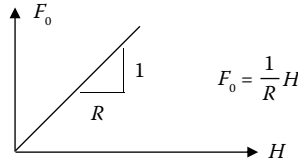


FIGURE 1.6 A tank with out-flow proportional to fluid level.

Taking the limit as Δt approaches zero in Equation 1.13 and using the definition of the derivative in Equation 1.14 give

$$\lim_{\Delta t \rightarrow 0} \left[\frac{\Delta H}{\Delta t} \right] = \lim_{\Delta t \rightarrow 0} [F_1(t) - F_0(t)] \quad (1.15)$$

$$\Rightarrow A \left[\frac{dH}{dt} \right] = F_1(t) - F_0(t) \quad (1.16)$$

Since there are two dependent variables, a second equation or constraint relating F_0 and H is required in order to solve for either one given the input function $F_1(t)$. It is convenient at this point to assume that F_0 is proportional to H , that is, $F_0 = \text{constant} \times H$ (see Figure 1.6). The constant of proportionality is expressed as $1/R$ where R is called the fluid resistance of the tank. At a later point, we will revisit this assumption.

$$F_0 = \frac{1}{R} H \quad (1.17)$$

Equations 1.16 and 1.17 constitute the mathematical model of the liquid tank, namely,

$$A \frac{dH}{dt} + F_0 = F_1 \quad \text{and} \quad F_0 = \frac{1}{R} H$$

where F_1 , F_0 , H , and dH/dt are short for $F_1(t)$, $F_0(t)$, $H(t)$, and $(d/dt)H(t)$.

In this example, the model is a coupled set of equations. One is a linear differential equation and the other is an algebraic equation, also linear. The differential equation is first order since only the first derivative appears in the equation and the tank dynamics are said to be first order.

The outflow F_0 can be eliminated from the model equations by substituting Equation 1.17 into Equation 1.16 resulting in

$$A \frac{dH}{dt} + \frac{1}{R} H = F_1 \quad (1.18)$$

Before a particular solution to Equation 1.18 for some $F_1(t)$, $t \geq 0$ can be obtained, the initial tank level $H(0)$ must be known.

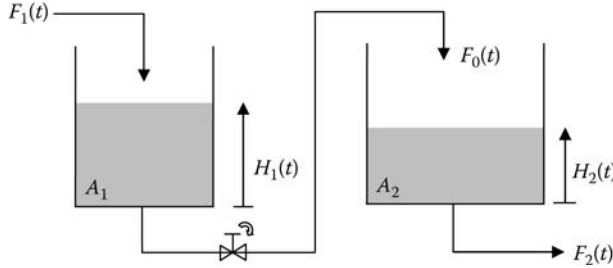
There are several reasons why an analytical approach to solving Equation 1.18 may not be the preferred method. Even when the analytical solution is readily obtainable, for example, when the system model is linear, as in the present example, the solution may be required for a number of different inputs or forcing functions. Recall from studying differential equations what happens when the right-hand side of the equation changes. A new particular solution is required that can be time-consuming, especially if the process is repeated for a number of nontrivial forcing functions.

Second, the input $F_1(t)$ may not even be available in analytical form. Suppose the input function $F_1(t)$ is unknown except as a sequence of measured values at regularly spaced points in time.

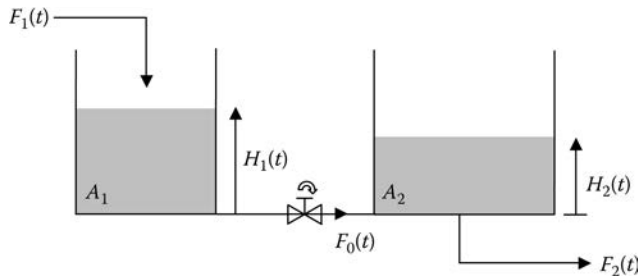
An exact solution to the differential equation model is out of the question since the input is not expressible as an analytic function of time.

EXERCISES

- 1.1 A system consists of two tanks in series in which the outflow from the first tank is the inflow to the second tank as shown in the figure below.



- Find the algebraic and differential equations comprising the mathematical model of the two tank system. Assume both tanks are linear, i.e. the outflows are proportional to the liquid levels and R_1, R_2 are the fluid resistances of the tanks.
 - Eliminate the flows $F_0(t)$ and $F_2(t)$ from the model to obtain a model in the form of two differential equations involving the system input $F_1(t)$ and the tank levels $H_1(t)$ and $H_2(t)$.
 - Obtain the model differential equations when $F_0(t)$ and $F_2(t)$ are present instead of $H_1(t)$ and $H_2(t)$.
 - The initial fluid levels in the tanks are $H_1(0)$ and $H_2(0)$. Suppose the flow in to the first tank is constant, $F_1(t) = \bar{F}_1, t \geq 0$. Obtain expressions for $H_1(\infty)$ and $H_2(\infty)$, the eventual fluid levels in Tanks 1 and 2. Do $H_1(\infty)$ and $H_2(\infty)$ depend on the initial fluid levels? Explain.
 - Find the ratio of tank resistances R_1/R_2 if $H_1(\infty) = 2H_2(\infty)$.
 - Suppose the flow between the two tanks is reduced to zero by closing the valve in the line. Show that this is equivalent to $R_1 = \infty$ and determine the values of $H_1(\infty)$ and $H_2(\infty)$ assuming the inflow to the first tank is still constant.
- 1.2 The two tanks in Exercise 1.1 are said to be non-interacting because the flow rate from the first tank only depends on the fluid level in the first tank and is independent of the fluid level in the second tank. Suppose the discharged fluid from the first tank enters the second tank at the bottom instead of the top as shown in the figure below.



The flow between the tanks is now a function of the fluid levels in both tanks. The driving force for the inter-tank flow is the difference in fluid levels and for the time being we can assume the two quantities are proportional. That is,

$$F_0(t) \propto [H_1(t) - H_2(t)] \Rightarrow F_0(t) = \frac{H_1(t) - H_2(t)}{R_{12}}$$

where R_{12} represents a fluid resistance involving both tanks. The fluid resistance of the second tank is still R_2 .

- a. The general form of the differential equation model for the system of interacting tanks is

$$\frac{dH_1}{dt} + a_{11}H_1 + a_{12}H_2 = b_1F_1$$

$$\frac{dH_2}{dt} + a_{21}H_1 + a_{22}H_2 = b_2F_1$$

Note: H_1 , H_2 and F_1 are short for $H_1(t)$, $H_2(t)$ and $F_1(t)$.

Find expressions for a_{11} , a_{12} , a_{21} , a_{22} , b_1 , b_2 in terms of the system parameters A_1 , A_2 , R_{12} , and R_2 .

- b. The tanks are initially empty, $H_1(0) = 0$ and $H_2(0) = 0$. The flow in to the first tank is constant, $F_1(t) = \bar{F}_1$, $t \geq 0$. Show that the final fluid levels in both tanks after a sufficient period of time has elapsed, $H_1(\infty)$ and $H_2(\infty)$, can be obtained from the solution of the following system of equations:

$$a_{11}H_1(\infty) + a_{12}H_2(\infty) = b_1\bar{F}_1$$

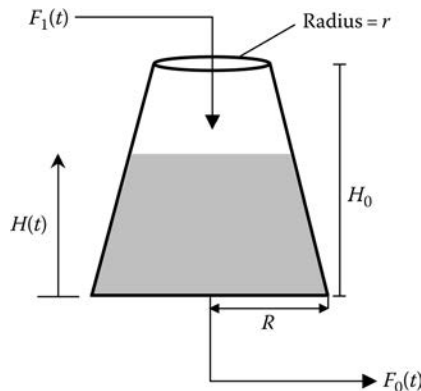
$$a_{21}H_1(\infty) + a_{22}H_2(\infty) = b_2\bar{F}_1$$

- c. Solve for $H_1(\infty)$ and $H_2(\infty)$ in terms of the system parameters A_1 , A_2 , R_{12} , R_2 and the constant inflow \bar{F}_1 . Are the results different if the tanks are not initially empty? Explain.
- d. Using the following baseline values unless otherwise stated:

$$A_1 = A_2 = 25 \text{ ft}^2, \quad R_{12} = 3 \text{ ft per ft}^3/\text{min}, \quad R_2 = 1 \text{ ft per ft}^3/\text{min}, \quad \bar{F}_1 = 5 \text{ ft}^3/\text{min}$$

Find the eventual fluid levels $H_1(\infty)$ and $H_2(\infty)$ and flows $F_0(\infty)$ and $F_2(\infty)$.

- e. Repeat Part (d) with $A_2 = 75 \text{ ft}^2$.
- f. The valve between the tanks is opened some resulting in $R_{12} = 2 \text{ ft per ft}^3/\text{min}$. The remaining baseline values remain the same. Find $H_1(\infty)$, $H_2(\infty)$ and flows $F_0(\infty)$ and $F_2(\infty)$.
- g. Suppose Tank 1 initially holds 10 ft of liquid and Tank 2 has 4 ft. Find the initial rates of change in level for both tanks.
- h. Is it possible for the fluid level in Tank 2 to exceed the level in Tank 1? Explain.
- i. How does the model change if there is a separate flow, say $F_3(t)$ directly in to the top of Tank 2?
- 1.3 Consider a cone shaped tank with circular cross sectional area like the one shown in the figure below.



- a. How does this affect the derivation of the mathematical model?
- b. Find the math model for this case.

1.3 DIFFERENCE EQUATIONS

The tank model from the previous section, given in Equation 1.18, is a first-order differential equation. First-order systems, that is, systems governed by a first-order differential equation are treated in detail in [Chapter 2](#). Simulation of a first-order system requires finding an approximate solution to the differential equation.

A first-order system is shown in [Figure 1.7](#), where $u(t)$, $t \geq 0$ is the input; $y(t)$, $t \geq 0$ is the output; $y(0)$ is the initial condition, i.e. $y(t)$ at $t = 0$; $f[y(t), u(t)]$ is the mathematical function model of the dynamic system.

Examples of $f[y(t), u(t)]$ are:

1. $f[y(t), u(t)] = ay(t)$ linear first-order system with no input
2. $f[y(t), u(t)] = ay(t) + bu(t)$ linear first-order system with input $u(t)$
3. $f[y(t), u(t)] = ay^2(t) + bu(t)$ nonlinear first-order system with input $u(t)$

Given the system model,

$$\frac{d}{dt} y(t) = f[y(t), u(t)] \quad (1.19)$$

An approximate solution for the response $y(t)$, $t \geq 0$, given the input $u(t)$, $t \geq 0$ and the initial condition $y(0)$, can be obtained at discrete points in time $t_n = nT$, $n = 0, 1, 2, \dots$. The discrete points are separated from each other by the step size T .

The approximate solution is $y_A(n)$, $n = 0, 1, 2, \dots$ where $y_A(n) \approx y(t_n) = y(nT)$, $n = 0, 1, 2, \dots$

[Figure 1.8](#) illustrates the difference between $y(t)$, $t \geq 0$ and $y_A(n)$, $n = 0, 1, 2, \dots$. It is useful for deriving an equation which can be solved to generate $y_A(n)$, $n = 0, 1, 2, \dots$

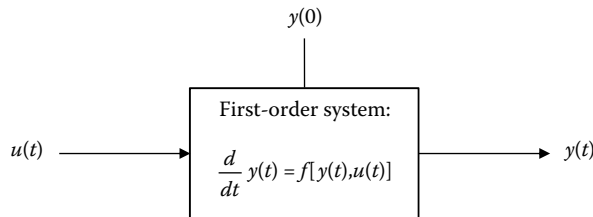


FIGURE 1.7 A first-order system with input $u(t)$ and output $y(t)$.

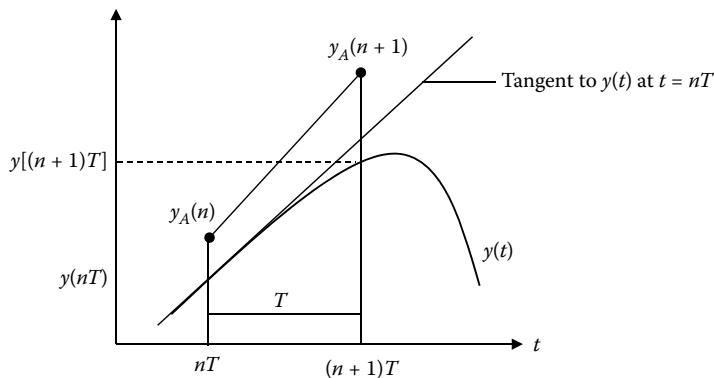


FIGURE 1.8 Illustration of the difference between $y(t)$ and $y_A(n)$.

The tangent to $y(t)$ at $t = nT$ is shown in [Figure 1.8](#). Its slope is numerically equal to the first derivative $\frac{dy}{dt}$ at $t = nT$. From Equation 1.19,

$$\text{At } t = nT, \quad \left. \frac{d}{dt} y(t) \right|_{t=nT} = f[y(nT), u(nT)] \quad (1.20)$$

Approximating $\left. \frac{d}{dt} y(t) \right|_{t=nT}$ by the slope of the line connecting $y_A(n)$ and $y_A(n+1)$,

$$f[y(nT), u(nT)] \approx \frac{y_A(n+1) - y_A(n)}{T} \quad (1.21)$$

Replacing $y(nT)$ with $y_A(n)$, and writing $u(nT)$ as $u(n)$ for short,

$$f[y_A(n), u(n)] = \frac{y_A(n+1) - y_A(n)}{T} \quad (1.22)$$

Note the use of the equality in Equation 1.22, which enables $y_A(n)$ to be solved for, giving

$$y_A(n+1) = y_A(n) + Tf[y_A(n), u(n)], \quad n = 0, 1, 2, \dots \quad (1.23)$$

Equation 1.23 is called a difference equation, and given the initial condition $y_A(0)$, is easily solved in a recursive manner. To illustrate, consider the first-order system

$$\frac{d}{dt} y(t) + 2y(t) = u(t) = 3t, \quad y(0) = 1 \quad (1.24)$$

$$\frac{dy}{dt} = f(y, u) = -2y + u \quad (1.25)$$

Using Equation 1.23, the difference equation for obtaining $y_A(n)$, $n = 0, 1, 2, \dots$ is

$$y_A(n+1) = y_A(n) + T[-2y_A(n) + u(n)], \quad n = 0, 1, 2, \dots \quad (1.26)$$

$$= (1 - 2T)y_A(n) + Tu(n), \quad n = 0, 1, 2, \dots \quad (1.27)$$

Letting $\alpha = (1 - 2T)$,

$$y_A(n+1) = \alpha y_A(n) + Tu(n), \quad n = 0, 1, 2, \dots \quad (1.28)$$

where

$$u(n) = u(nT) = u(t) \Big|_{t=nT} = 3t \Big|_{t=nT} = 3nT, \quad n = 0, 1, 2, \dots \quad (1.29)$$

Replacing $u(n)$ in Equation 1.28 with $u(n)$ in Equation 1.29 gives

$$y_A(n+1) = \alpha y_A(n) + 3nT^2, \quad n = 0, 1, 2, \dots \quad (1.30)$$

Equation 1.30 is the difference equation which is solved recursively to generate the approximate solution $y_A(n)$, $n = 0, 1, 2, \dots$

Starting with $n = 0$,

$$n = 0: \quad y_A(1) = \alpha y_A(0) + 3(0)T^2 \quad (1.31)$$

Choosing $y_A(0) = y(0)$ gives

$$n = 0: \quad y_A(1) = \alpha y(0) \quad (1.32)$$

$$n = 1: \quad y_A(2) = \alpha y_A(1) + 3(1)T^2 \quad (1.33)$$

$$= \alpha [\alpha y(0)] + 3T^2 \quad (1.34)$$

$$= \alpha^2 y(0) + 3T^2 \quad (1.35)$$

$$n = 2: \quad y_A(3) = \alpha y_A(2) + 3(2)T^2 \quad (1.36)$$

$$= \alpha [\alpha^2 y(0) + 3T^2] + 6T^2 \quad (1.37)$$

$$= \alpha^3 y(0) + 3(\alpha + 2)T^2 \quad (1.38)$$

$$n = 3: \quad y_A(4) = \alpha y_A(3) + 3(3)T^2 \quad (1.39)$$

$$= \alpha [\alpha^3 y(0) + 3(\alpha + 2)T^2] + 9T^2 \quad (1.40)$$

$$= \alpha^4 y(0) + 3T^2 [\alpha(\alpha + 2) + 3] \quad (1.41)$$

The smaller the time step T , the closer $y_A(n)$, $n = 0, 1, 2, \dots$ will be to the exact solution $y(t)$ at $t = 0, T, 2T, \dots$. Finding an approximate solution for $y(t)$ on the interval $0 \leq t \leq t_{final}$ requires $\frac{t_{final}}{T}$ iterations of the difference equation. There is a trade-off between accuracy of the approximate solution $y_A(n)$, $n = 0, 1, 2, \dots$ and the computational effort to generate it.

Equation 1.25 with input $u(t) = 3t$ and initial condition $y(0) = 1$ is easily solved by analytical methods. The exact solution is given by

$$y(t) = 1.75e^{-2t} + 1.5t - 0.75, \quad t \geq 0 \quad (1.42)$$

EXAMPLE 1.1

- Find the approximate solution to the differential equation (Equation 1.24) over the interval $0 \leq t \leq 2$ by recursive solution of the difference equation (Equation 1.30) for $T = 0.01, 0.025, 0.05$ and 0.1 . Plot the approximate solution for each value of T and the exact solution given in Equation 1.42.
- Compare approximate and exact solutions at $t = 0, 0.2, 0.5, 1, 1.5, 2$ when $T = 0.01$ and $T = 0.1$.
- Matlab program "Ch1_Ex1_1.m" solves the difference equation (Equation 1.30) recursively and also computes points along the exact solution (Equation 1.42). The results are shown in Figure 1.9.

Note, the approximate solution is known only at times $t_n = nT$, $n = 0, 1, 2, \dots$. For purposes of clarity, only a subset of the discrete points are plotted except for the case when $T = 0.1$.

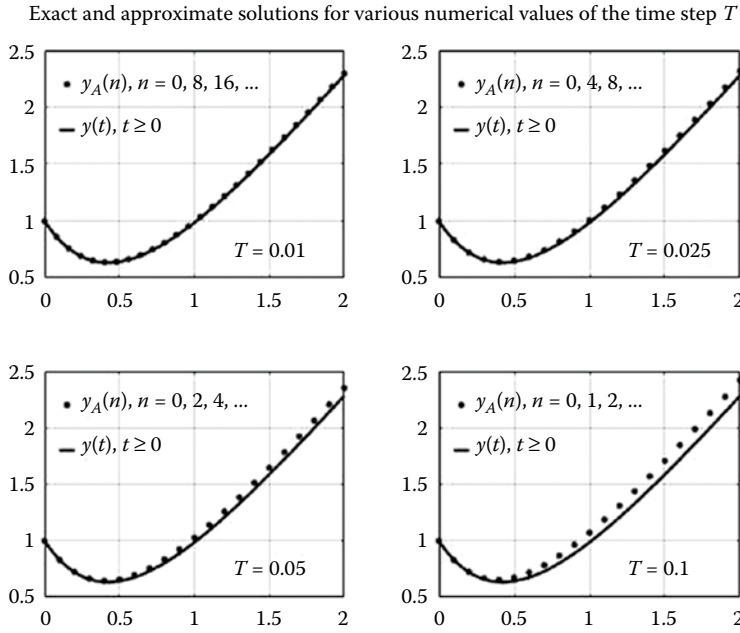


FIGURE 1.9 Comparison of exact and approximate solution of first-order system response for different values of T .

Figure 1.9 illustrates how the accuracy of the approximate solution can be improved by decreasing the time step T (at the expense of requiring additional calculations). The approximate solutions are seen to converge to the exact solution as the time step decreases. Further it appears that setting the time step below $T = 0.025$ may not be necessary unless extreme accuracy is required.

- b. Tables 1.1 and 1.2 compare the approximate and exact solution at $t = 0, 0.2, 0.5, 1, 1.5$ and 2 when $T = 0.01$ and $T = 0.1$, respectively.

Note the percent error,

$$\% \text{ Error} = 100 \times \frac{y_A(n) - y(t_n)}{y(t_n)} \quad (1.43)$$

is roughly 10 times greater when the time step $T = 0.1$ compared to when $T = 0.01$.

Let us now apply what we have learned about finding approximate solutions of first-order system response to the tank example in Section 1.2. Recall that the tank dynamics are governed by the first-order differential equation (Equation 1.18),

$$A \frac{dH}{dt} + \frac{1}{R} H = F_1 \quad (1.44)$$

Solving for the derivative function $f(F_1, H)$ gives

$$f(F_1, H) = \frac{dH}{dt} = \frac{1}{A} \left(F_1 - \frac{1}{R} H \right) \quad (1.45)$$

TABLE 1.1

Comparison of Exact and Approximate ($T = 0.01$) Solution of First-Order System Response at Different Times

n	$t_n = nT$	$y_A(n)$	$y(t_n)$	% Error
0	0	1	1	0
20	0.2	0.7233	0.7231	0.03
50	0.5	0.6468	0.6438	0.47
100	1.0	0.9951	0.9868	0.84
150	1.5	1.5988	1.5871	0.74
200	2.0	2.2955	2.2821	0.59

TABLE 1.2

Comparison of Exact and Approximate ($T = 0.1$) Solution of First-Order System Response at Different Times

n	$t_n = nT$	$y_A(n)$	$y(t_n)$	% Error
0	0	1	1	0
2	0.2	0.7240	0.7231	0.13
5	0.5	0.6743	0.6438	4.74
10	1.0	1.0718	0.9868	8.61
15	1.5	1.7063	1.5871	7.51
20	2.0	2.4148	2.2821	5.98

Based on Equation 1.23, the difference equation for approximating the tank level response is

$$H_A(n+1) = H_A(n) + Tf[F_1(n), H_A(n)] \quad (1.46)$$

$$= H_A(n) + T \cdot \frac{1}{A} \left(F_1(n) - \frac{1}{R} H_A(n) \right) \quad (1.47)$$

$$= \left(1 - \frac{T}{AR} \right) H_A(n) + \left(\frac{T}{A} \right) F_1(n) \quad (1.48)$$

Given the input flow $F_1(t)$, $t \geq 0$ and the initial tank level $H_A(0)$, Equation 1.48 is the difference equation which can be solved recursively to obtain the approximate tank level response $H_A(n)$, $n = 0, 1, 2, \dots$. Starting with $n = 0$,

$$n = 0: \quad H_A(1) = \left(1 - \frac{T}{AR} \right) H_A(0) + \left(\frac{T}{A} \right) F_1(0) \quad (1.49)$$

$$n = 1: \quad H_A(2) = \left(1 - \frac{T}{AR} \right) H_A(1) + \left(\frac{T}{A} \right) F_1(1) \quad (1.50)$$

$$= \left(1 - \frac{T}{AR} \right) \left[\left(1 - \frac{T}{AR} \right) H_A(0) + \left(\frac{T}{A} \right) F_1(0) \right] + \left(\frac{T}{A} \right) F_1(1) \quad (1.51)$$

$$= \left(1 - \frac{T}{AR} \right)^2 H_A(0) + \left(1 - \frac{T}{AR} \right) \left(\frac{T}{A} \right) F_1(0) + \left(\frac{T}{A} \right) F_1(1) \quad (1.52)$$

$$n = 2: \quad H_A(3) = \left(1 - \frac{T}{AR}\right) \left[\left(1 - \frac{T}{AR}\right)^2 H_A(0) + \left(1 - \frac{T}{AR}\right) \left(\frac{T}{A}\right) F_1(0) + \left(\frac{T}{A}\right) F_1(1) \right] + \left(\frac{T}{A}\right) F_1(2) \quad (1.53)$$

$$= \left(1 - \frac{T}{AR}\right)^3 H_A(0) + \left(1 - \frac{T}{AR}\right)^2 \left(\frac{T}{A}\right) F_1(0) + \left(1 - \frac{T}{AR}\right) \left(\frac{T}{A}\right) F_1(1) + \left(\frac{T}{A}\right) F_1(2) \quad (1.54)$$

Based on the first few iterations, a general formula for finding $H_A(n)$, for any n , is

$$H_A(n) = \left(1 - \frac{T}{AR}\right)^n H(0) + \left(\frac{T}{A}\right) \sum_{k=0}^{n-1} \left(1 - \frac{T}{AR}\right)^{n-k-1} F_1(k), \quad n = 0, 1, 2, \dots \quad (1.55)$$

where $H_A(0)$ is replaced by the initial tank level $H(0)$.

When specific values of $H_A(n)$ are required, say $H_A(100)$, Equation 1.55 eliminates the need for recursive solution of Equation 1.48 to find $H_A(n)$, $n = 0, 1, 2, \dots, 99$. The summation in Equation 1.55 requires some effort; however, the z-transform introduced in [Chapter 4](#) provides a way to avoid the sum altogether.

The quantity $\left(1 - \frac{T}{AR}\right)$ must be less than 1 for both stable and accurate results. Keeping this in mind, Equation 1.55 demonstrates the diminishing influence of the initial tank level as time progresses.

Equation 1.55 also indicates that all past inputs $F_1(k)$, $k = 0, 1, 2, \dots, n-1$ are needed to obtain $H_A(n)$, a property known as infinite memory. However, recent input flow values are weighted higher than older values (see Equation 1.54).

EXAMPLE 1.2

A tank with cross-sectional area of 10 ft² receives a constant input flow of 5 ft³/min. The fluid resistance of the tank is 2 ft per ft³/min, and the tank is initially filled to a level of 4 ft.

- Find the difference equation for obtaining an approximate solution for the level $H(t)$ using a time step of $T = 0.25$ min.
- Solve the difference equation recursively to obtain the approximate fluid level $H_A(n)$, $n = 1, 2, 3$.
- Use Equation 1.55 to find $H_A(3)$ and compare your answer to the result from part (b).

$$\text{a. } \left(\frac{T}{A}\right) = \frac{0.25}{10} = 0.025, \quad \left(1 - \frac{T}{AR}\right) = 1 - \frac{0.25}{10(2)} = 0.9875$$

$$H_A(0) = H(0) = 4, \quad F_1(n) = 5, \quad n = 0, 1, 2, \dots$$

The difference equation (Equation 1.48) is

$$H_A(n+1) = 0.9875H_A(n) + (0.025)5, \quad n = 0, 1, 2, \dots \quad (1.56)$$

- $H_A(n)$, $n = 1, 2, 3$ are computed as follows:

$$\begin{aligned} n = 0: \quad H_A(1) &= 0.9875H_A(0) + 0.125 \\ &= 4.0750 \end{aligned} \quad (1.57)$$

$$\begin{aligned} n = 1: \quad H_A(2) &= 0.9875H_A(1) + 0.125 \\ &= 0.9875(4.0750) + 0.125 \\ &= 4.1491 \end{aligned} \quad (1.58)$$

$$\begin{aligned}
n = 2: \quad H_A(3) &= 0.9875H_A(2) + 0.125 \\
&= 0.9875(4.1491) + 0.125 \\
&= 4.2222
\end{aligned} \tag{1.59}$$

c. From Equation 1.55 with $n = 3$,

$$\begin{aligned}
H_A(3) &= (0.9875)^3(4) + 0.025 \sum_{k=0}^2 (0.9875)^{3-k-1}(5) \\
&= 3.8519 + 0.025[(0.9875)^2(5) + (0.9875)(5) + 5] \\
&= 4.2222
\end{aligned}$$

Due to the simple nature of the input, that is, $F_1(t) = F$, $t \geq 0$, the analytical solution of the first-order differential equation model (Equation 1.44) is given by

$$H(t) = R\bar{F} + [H(0) - R\bar{F}]e^{-t/AR} \tag{1.60}$$

It is instructive to compare the approximate solution based on the difference equation approach with the exact solution shown in Equation 1.60. Table 1.3 includes both solutions at equally spaced intervals for the first 2 min of the response.

Graphs of the exact response $H(t)$ and the approximate response $H_A(n)$, $n = 0, 8, 16, \dots$ are shown in Figure 1.10.

By observation of Figure 1.10, it appears that the exact and approximate solutions for the tank level are in close agreement. The step size T is the determining factor in terms of how close the two solutions are at the discrete points in time where the approximate solution is defined.

Generally speaking, an assessment of whether the numerical value selected for T is reasonable cannot be made on the basis of comparing the approximate solution with the exact solution to the differential equation. Analytical solutions are rare due to the complexity of most real-world system models. A logical approach to finding an acceptable step size is to obtain approximate solutions with different step sizes (an order of magnitude apart) and comparing the results. If the approximate solutions are substantially identical, the smaller step size is eliminated from consideration. Conversely, if the approximate solutions are not close, the larger value of T is discarded. Eventually, a value for T will be found, which balances accuracy and computational requirements. This point will be revisited in greater detail after the subject of numerical integration is discussed.

TABLE 1.3
Comparison of Approximate and
Exact Tank Level Response

n	$t_n = nT$	$H_A(n)$	$H(t_n)$
0	0	4.0	4.0
1	0.25	4.0750	4.0745
2	0.5	4.1491	4.1481
3	0.75	4.2222	4.2208
4	1.0	4.2944	4.2926
5	1.25	4.3657	4.3635
6	1.5	4.4362	4.4335
7	1.75	4.5057	4.5027
8	2.0	4.5744	4.5710

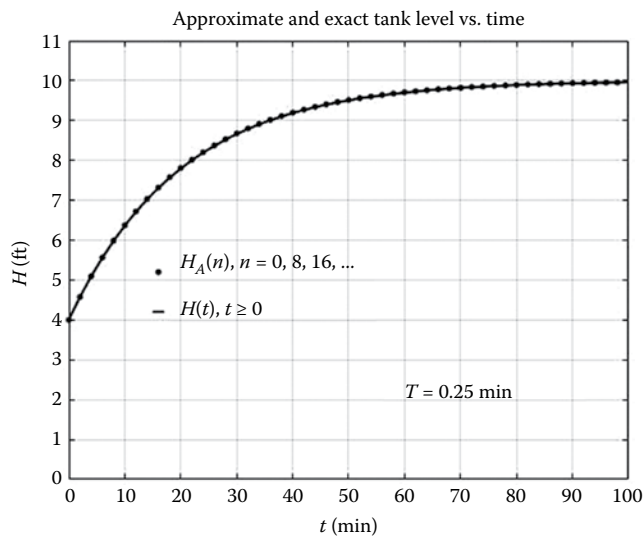


FIGURE 1.10 Approximate and exact solutions for tank level vs. time.

EXERCISES

- 1.4 Find the difference equation, similar to Equation 1.48, relating $F_0(n + 1)$ to $F_0(n)$ and $F_1(n)$.
- 1.5 A tank with cross sectional area $A = 5 \text{ ft}^2$ is initially filled to a level of 10 ft. The flow out is given by $F_0 = H/R$, $R = 1 \text{ ft per ft}^3/\text{min}$. There is no flow in to the tank.
 - a. Find $H_A(n)$, $n = 0, 1, 2, \dots, 10$ when $T = 2.5 \text{ min}$.
 - b. Find $H_A(n)$, $n = 0, 1, 2, \dots, 25$ when $T = 1 \text{ min}$.
 - c. Find $H_A(n)$, $n = 0, 1, 2, \dots, 100$ when $T = 0.25 \text{ min}$.
 - d. Plot the results and comment on the differences.
- 1.6 Repeat Exercise 1.5 for the case where the outflow is described by $F_0 = cH^{1/2}$, $c = 3 \text{ ft}^3/\text{min per ft}^{1/2}$.
- 1.7 Rework Example 2.1 using the Trial and Error method for determining a suitable value of T . Start with $T = 10 \text{ min}$ and calculate $H_A(n)$, $n = 0, 1, 2, \dots, n_f$ where $n_f T = 100 \text{ min}$. Repeat the steps with $T = 5 \text{ min}$, 2.5 min , 1.25 min , etc. until the approximations of $H(10)$, $H(20)$, $H(30)$, ..., $H(100)$ are in agreement to at least one place after the decimal point. Use Table E1.7 for comparisons. Extend the table to smaller values of T if necessary.

TABLE E1.7

$H_A(n)$		$H_A(n)$		$H_A(n)$		$H_A(n)$	
n	$T = 10$	n	$T = 5$	n	$T = 2.5$	n	$T = 1.25$
0		0		0		0	
1		2		4		8	
2		4		8		16	
3		6		12		24	
4		8		16		32	
5		10		20		40	
6		12		24		48	
7		14		28		56	
8		16		32		64	
9		18		36		72	
10		20		40		80	

- 1.8 An alternate model of the tank relates the outflow and liquid level according to

$$F_0(t) = \alpha [H(t)]^{1/2}$$

- Develop a new discrete-time model of the tank using the above relationship in conjunction with the differential equation $A(dH/dt) + F_0 = F_1$. The tank cross-sectional area is 10 ft² and the input flow is constant at 5 ft³/min. The tank is initially filled to a level of 4 ft. Assume $\alpha = 2$ ft³/min per ft^{1/2}.
- Calculate the approximate tank level for the first minute using a step size $T = 0.25$ min.
- Consider the same tank with zero flow in and an initial fluid level of 25 ft. Write a program to calculate the approximate level of the tank as it empties. Choose $T = 0.1$ min.
- The analytical solution for the level $H(t)$ when $F_1(t) = 0$, $t \geq 0$ is given by

$$H(t) = \left(H_0^{1/2} - \frac{\alpha t}{2A} \right)^2$$

where H_0 is the initial tank level. Compare the results from Part (c) to the exact solution. Present the comparison of results in tabular and graphical form.

- 1.9 A holding tank serves as an effective way of smoothing variations in the flow of a liquid. For example, suppose the liquid flow rate from an upstream process is

$$F_1(t) = \bar{F} + f \sin\left(\frac{2\pi t}{T}\right), \quad t \geq 0$$

where \bar{F} is an average flow, f is the fluctuation about the average flow, and T is the period of the fluctuations. Nominal parameter values for the input flow rate are $\bar{F} = 250$ ft³/min, $f = 50$ ft³/min and $T = 15$ min.

A holding tank is placed between the source $F_1(t)$ and a downstream process which requires a more constant input flow rate, $F_0(t)$ as shown in Figure E1.9. The downstream process requires that the sustained fluctuations in the flow $F_0(t)$ be no larger than 10 ft³/min. Assume the tank is linear and the fluid resistance $R = 0.25$ ft per ft³/min.

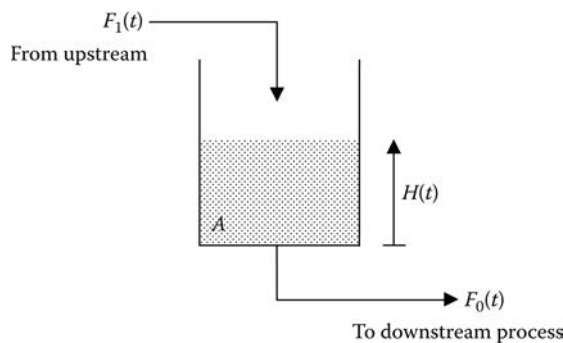


FIGURE E1.9

- Find the difference equation for $F_{0,A}(n)$, $n = 0, 1, 2, 3, \dots$. Leave the tank cross sectional area A as a parameter.
- Write a program to solve the difference equation with $\Delta t = 0.5$ min for a starting value of $A = 100$ ft². Graph both $F_{0,A}(n)$ and $H_A(n)$, $n = 0, 1, 2, \dots$ for a period of time sufficient to determine if the design criterion is satisfied. Assume the tank is initially empty.

- c. Repeat part (b) with a new value of A until the design criterion is satisfied, i.e. the sustained fluctuations in $F_0(t)$ are equal to 10 ft³/min.
- d. Graph the discrete-time signals $F_1(n)$ and $F_{0,A}(n)$, $n = 0, 1, 2, 3, \dots$ for the tank whose area is the value determined in Part (c).

1.4 FIRST LOOK AT DISCRETE-TIME SYSTEMS

The variables $F_1(t)$, $F_0(t)$, and $H(t)$ in the liquid tank shown in Figure 1.3 are referred to as continuous-time (or simply continuous) signals. The reason is because there is a continuum of values between any two points along the t -axis where the variables are defined. Equation 1.18 is a continuous-time model and the system is a continuous-time system because it involves only continuous-time variables.

In contrast to the continuous-time signals $F_1(t)$, $F_0(t)$, and $H(t)$, the sequence of sampled input flow values, $F_1(n)$, $n = 0, 1, 2, \dots$ and the sequence of approximate tank levels $H_A(n)$, $n = 0, 1, 2, \dots$ are classified as discrete-time (discrete for short) signals because the independent variable “ n ” is discrete in nature. The difference equation (Equation 1.48) is referred to as a discrete-time model, and the underlying system with purely discrete-time input and output signals is likewise a discrete-time system.

Figure 1.11 illustrates both the continuous and discrete representations of the tank dynamics.

1.4.1 INHERENTLY DISCRETE-TIME SYSTEMS

The discrete model of the tank dynamics relates $H_A(n)$, an approximation of the continuous tank level $H(t)$ and $F_1(n)$, the sampled version of the continuous flow $F_1(t)$. In contrast, inherently discrete-time systems involve discrete signals which are not related to continuous signals. For example, consider a discrete-time system governed by the difference equation

$$y(n) = \frac{1}{2} \left[y(n-1) + \frac{u(n)}{y(n-1)} \right], \quad n = 0, 1, 2, \dots \quad (1.61)$$

Equation 1.61 is simply a rule for transforming a discrete input signal $u(n)$ into an appropriate discrete output signal $y(n)$. Is this discrete-time system useful? Let us investigate its behavior for the case where $u(n)$ is a constant, for example, $u(n) = 25$, $n = 0, 1, 2, \dots$. Before we are able to compute $y(n)$, $n = 0, 1, 2, \dots$, a starting or initial value, in this case $y(-1)$ must be given. If we choose $y(-1) = 1$, the first few values of $y(n)$ are

$$n = 0: \quad y(0) = \frac{1}{2} \left[y(-1) + \frac{u(0)}{y(-1)} \right] = \frac{1}{2} \left[1 + \frac{25}{1} \right] = 13 \quad (1.62)$$

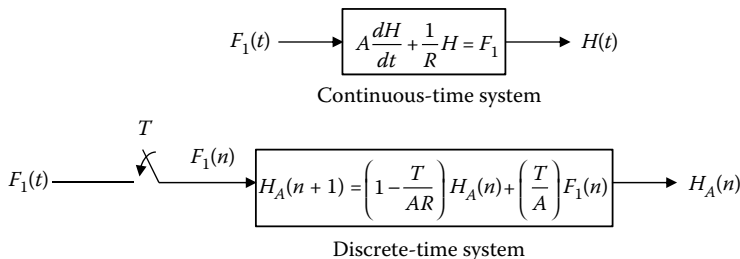


FIGURE 1.11 Continuous-time and discrete-time representations of tank dynamics.

$$n = 1: \quad y(1) = \frac{1}{2} \left[y(0) + \frac{u(1)}{y(0)} \right] = \frac{1}{2} \left[13 + \frac{25}{13} \right] = 7.4615 \quad (1.63)$$

$$n = 2: \quad y(2) = \frac{1}{2} \left[y(1) + \frac{u(2)}{y(1)} \right] = \frac{1}{2} \left[7.4615 + \frac{25}{7.4615} \right] = 5.4060 \quad (1.64)$$

$$n = 3: \quad y(3) = \frac{1}{2} \left[y(2) + \frac{u(3)}{y(2)} \right] = \frac{1}{2} \left[5.4060 + \frac{25}{5.4060} \right] = 5.0152 \quad (1.65)$$

$$n = 4: \quad y(4) = \frac{1}{2} \left[y(3) + \frac{u(4)}{y(3)} \right] = \frac{1}{2} \left[5.0152 + \frac{25}{5.0152} \right] = 5.0000 \quad (1.66)$$

Using different positive constants for $u(n)$ and other starting values for $y(-1)$ will reveal an interesting property of the system, namely

$$\lim_{n \rightarrow \infty} y(n) = \sqrt{u} \quad (1.67)$$

Hence, the primary purpose of the discrete-time system governed by Equation 1.61 is to compute the square root of its positive-valued constant input $u(n)$. The discrete signals $u(n)$ and $y(n)$ are plotted in Figure 1.12.

Another inherently discrete-time system is one we are all familiar with, namely, an interest-bearing account such as a bank account. The discrete signals of interest are $y(n)$, the account balance at the end of the n th interest period, and $u(n)$, the net deposit for the n th interest period. Figure 1.13 shows the discrete-time system.

Consider an account with an interest rate i (per interest period). The balance at the end of the n th interest period, $y(n)$ is the sum of

- the balance at the end of the $(n - 1)$ st period: $y(n - 1)$
- the interest earned for the n th interest period: $i \cdot y(n - 1)$
- the net deposit for the period: $u(n)$

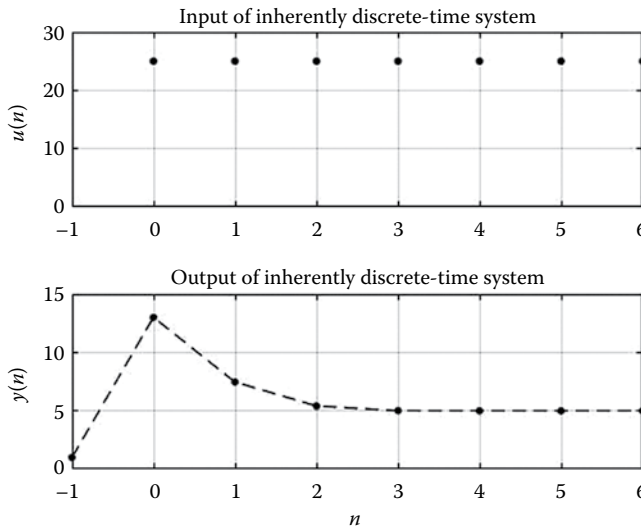


FIGURE 1.12 Illustration of using discrete-time system in Equation 1.61 to find a square root.

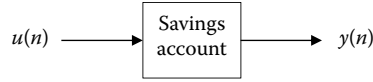


FIGURE 1.13 Example of an inherently discrete-time system.

Therefore, the model for this inherently discrete-time system is

$$y(n) = y(n-1) + iy(n-1) + u(n) \quad (1.68)$$

EXAMPLE 1.3

A college trust fund is set up with \$5000 on Jan 1, 2000. Starting on Jan 1, 2001 and every year thereafter, \$1000 is added to the fund, which earns 7.5% interest annually.

- Track the end of year fund balance for the first several years.
- Find the account balance at the end of Year 18th year.

a. The discrete-time model is

$$y(n) = y(n-1) + 0.075y(n-1) + u(n), \quad n = 1, 2, 3, \dots \quad (1.69)$$

with input

$$u(n) = 1000, \quad n = 1, 2, 3, \dots \quad (1.70)$$

and initial condition $y(0) = 5000$.

The account balance at the end of years 1, 2, and 3 are computed as follows:

$$\begin{aligned} n = 1: \quad y(1) &= y(0) + 0.075y(0) + u(1) \\ &= 5000 + 0.075(5000) + 1000 \\ &= 6375 \end{aligned} \quad (1.71)$$

$$\begin{aligned} n = 2: \quad y(2) &= y(1) + 0.075y(1) + u(2) \\ &= 6375 + 0.075(6375) + 1000 \\ &= 7853.13 \end{aligned} \quad (1.72)$$

$$\begin{aligned} n = 3: \quad y(3) &= y(2) + 0.075y(2) + u(3) \\ &= 7853.13 + 0.075(7853.13) + 1000 \\ &= 9442.11 \end{aligned} \quad (1.73)$$

- The recursive solution could be continued for $n = 4, 5, 6, \dots, 18$ resulting in the fund's balance at the end of the 18th year. However, a general solution of the discrete-time model is preferable since it can be evaluated for any value of the variable n .

Expressing the discrete-time model in Equation 1.68 as

$$y(n) = (1+i)y(n-1) + A, \quad n = 1, 2, 3, \dots \quad (1.74)$$

$$= \alpha y(n-1) + A, \quad n = 1, 2, 3, \dots \quad (1.75)$$

where $\alpha = 1 + i$ and A is the constant net deposit each interest period.

$$n = 1: \quad y(1) = \alpha y(0) + A \quad (1.76)$$

$$n = 2: \quad y(2) = \alpha y(1) + A \quad (1.77)$$

$$= \alpha[\alpha y(0) + A] + A \quad (1.78)$$

$$= \alpha^2 y(0) + \alpha A + A \quad (1.79)$$

$$n = 3: \quad y(3) = \alpha y(2) + A \quad (1.80)$$

$$= \alpha[\alpha^2 y(0) + \alpha A + A] + A \quad (1.81)$$

$$= \alpha^3 y(0) + \alpha^2 A + \alpha A + A \quad (1.82)$$

suggesting the general expression for $y(n)$ is

$$y(n) = \alpha^n y(0) + (1 + \alpha + \alpha^2 + \dots + \alpha^{n-1})A, \quad n = 0, 1, 2, \dots \quad (1.83)$$

Further simplification is possible using the closed form of the finite geometric series in Equation 1.83 resulting in

$$y(n) = \alpha^n y(0) + \left(\frac{1 - \alpha^n}{1 - \alpha} \right) A, \quad n = 0, 1, 2, \dots \quad (1.84)$$

The account balance after 18 years is computed from Equation 1.84 with $\alpha = 1.075$, $y(0) = 5000$ and $A = 1000$.

$$\begin{aligned} y(18) &= (1.075)^{18}(5000) + \left[\frac{1 - (1.075)^{18}}{1 - 1.075} \right] 1000 \\ &= 54,056.41 \end{aligned} \quad (1.85)$$

The results from part (a) can be verified using the general solution in Equation 1.84.

EXERCISES

1.10 Prove that the output of the discrete-time system in Equation 1.61 will approach the square root of the input, any positive constant “ A ”. In other words, show that

$$\lim_{n \rightarrow \infty} y(n) = \sqrt{A}$$

where $u(n) = A$, $n = 0, 1, 2, 3, \dots$

1.5 CASE STUDY: POPULATION DYNAMICS (SINGLE SPECIES)

The population of a country is under investigation. Unlike the liquid tank example, there is no scientific principle to serve as a foundation for deriving a mathematical model that can be used to predict future populations. Instead, empirical observations of historical birth and death rates, immigration and emigration patterns, and a host of other pertinent data are utilized.

One hundred years of observed population data, recorded at intervals of 10 years, is given in Table 1.4.

TABLE 1.4
Population Data for 100 Years

t (years)	$P_{\text{obs}}(t)$, millions
0	3.0000
10	3.2276
20	4.5759
30	6.9570
40	8.7618
50	9.1536
60	11.2669
70	14.5153
80	16.5059
90	17.9563
100	19.5078

Based on the available data, researchers are convinced that the population is adequately modeled by the following differential equation, referred to in the literature as logistic growth (Haberman 1977).

$$\frac{dP}{dt} = cP(P_m - P) \quad (1.86)$$

$P = P(t)$ is the population “ t ” years after the initial population was recorded. The parameters c and P_m influence the specific growth pattern behavior. The model ignores immigration and emigration and all other external inputs, which influence dP/dt , the rate at which the population changes.

The system model in Equation 1.86 is said to be autonomous, meaning there are no additional terms independent of P , as might be the case if immigration or emigration inputs as a function of time were considered. The dynamics depend solely on initial conditions and the system parameters. It is also referred to as an unforced system since there are no external inputs.

Statistical analyses of the population data has resulted in estimated values for c and P_m to be 1.25×10^{-9} and 25 million, respectively. It is now 100 years since the initial population was measured. Government planners are interested in determining what the likely population will be over the next several decades. A method is needed to obtain an approximate solution for $P(t)$, that is, a difference equation for $P_A(n) \approx P(nT)$, $n = 0, 1, 2, \dots$ is required.

Employing the method from Section 1.2 for obtaining a difference equation to approximate the dynamics of a first-order system,

$$P_A(n+1) = P_A(n) + T \cdot f[P_A(n)] \quad (1.87)$$

where $f[P_A(n)]$ is the derivative function, that is,

$$f[P_A(n)] = cP_A(n)[P_m - P_A(n)] \quad (1.88)$$

Combining Equations 1.87 and 1.88 results in

$$P_A(n+1) = P_A(n) + TcP_A(n)[P_m - P_A(n)] \quad (1.89)$$

Note, Equation 1.89 could just as easily have been found by simply replacing $\frac{dP}{dt}$ in Equation 1.86 with the divided difference term $\frac{P_A(n+1) - P_A(n)}{T}$ and replacing P with $P_A(n)$ on the right hand side of the equation.

Simplifying Equation 1.89 leads to the required difference equation, namely

$$P_A(n+1) = \{1 + Tc[P_m - P_A(n)]\}P_A(n) \quad (1.90)$$

Since our interest is in predicting populations for 100 years and beyond, we need to solve Equation 1.90 over a suitable range of values for the discrete-time variable “ n ”. The appropriate integer values depend on the size of our time step T . For simplicity, we shall choose $T = 1$ year, necessitating the calculation of $P_A(101)$, $P_A(102)$, ..., $P_A(130)$ to obtain predictions for a 30-year time span.

A recursive solution seems like our only alternative, since a general solution is not easily achievable. The Matlab file “Ch1_CaseStudy.m” includes the necessary statements to solve Equation 1.90 recursively and produce the results shown in Table 1.5. Note, the initial value $P_A(0)$ is the initial observed population of 3 million.

A comparison of the numbers for $P_{\text{obs}}(t)$ and $P_A(n)$ indicates that the modelers were justified in assuming logistic growth of the population for the 100 years corresponding to the recorded data. Naturally, this assumes that the approximate solution values $P_A(n)$, $n = 0, 1, 2, \dots, 100$ are reasonably close to the exact solution $P(t)$, $t = 0, 1, 2, \dots, 100$.

Ordinarily, dynamic population models are not amenable to exact solutions. However, the analytical solution to Equation 1.86 is known and given in Equation 1.91.

$$P(t) = \frac{P_m P(0)}{P(0) + [P_m - P(0)]e^{-cP_m t}}, \quad t \geq 0 \quad (1.91)$$

Knowing the exact solution, $P(t)$, to the continuous model differential equation, we can evaluate $P(t)$ at $t = 0, 10, 20, \dots, 100$ years for comparison with the discrete output $P_A(n)$, $n = 0, 10, 20, \dots, 100$ years obtained from solution of the difference equation. This allows us to determine if the current step size $T = 1$ year needs to be adjusted.

Comparing the last two columns in Table 1.5 should convince us that the step size T does not need to be reduced. While its possible to reduce the discrepancy between the approximate and exact solutions by lowering T , it's hardly justified in view of the fact that the logistic growth model, Equation 1.86, is itself only an approximate representation of the true population dynamics.

The data in Table 1.5 are presented in graphical form in Figure 1.14. The difference equation used to find $P_A(n)$, $n = 0, 1, 2, \dots, 100$ is used to predict the future population for the next 30 years. The projected populations for years 100, 120, and 130 are included in Table 1.5 and appear as data points in Figure 1.14.

The previous point relating to the accuracy of the approximate solution is worth reiterating. Extremely accurate discrete solutions of nonlinear differential equation models are generally not warranted unless the continuous models themselves are highly detailed with known accuracy.

Once we have an approximate solution of a continuous model response, how can we be certain if the two responses are in close agreement with each other? There is no simple answer; however, its worth remembering that

1. the difference equations in the discrete-time model converge to the differential equations of the continuous-time model in the limiting case when the step size! approaches zero.
2. the discrete-time solutions approach the exact solutions of the continuous-time model as the step size T is reduced to zero.

Systematically reducing the step size T until the changes in the discrete-time outputs are within some tolerance demonstrates this convergence and is an effective way of selecting the step size for

TABLE 1.5
Comparison of Observed, Discrete ($T = 1$ year) and Continuous Populations

t (years)	$P_{\text{obs}}(t)$, millions	n	$P_A(n)$, millions	$P(t)$, millions
0	3.0000	0	3.0000	3.0000
10	3.2276	10	3.9161	3.9276
20	4.5759	20	5.0493	5.0759
30	6.9570	30	6.4129	6.4570
40	8.7618	40	8.0003	8.0618
50	9.1536	50	9.7778	9.8536
60	11.2669	60	11.6834	11.7669
70	14.5153	70	13.6325	13.7153
80	16.5059	80	15.5321	15.6059
90	17.9563	90	17.2976	17.3563
100	19.5078	100	18.8671	18.9078
110		110	20.2076	20.2310
120		120	21.3139	21.3226
130		130	22.2012	22.1990

future runs. We touched on this in the previous section as a way of choosing an appropriate value for the step size T .

On a cautionary note, the step size may have to be readjusted as conditions change, in particular, the frequency components of the inputs. We will have more to say about this when we consider the frequency response characteristics of continuous and discrete systems in [Chapter 4](#).

Further scrutiny of the logistic growth model, Equation 1.86, reveals several important and noteworthy characteristics of the underlying population dynamics. Expressing the model in a slightly different form,

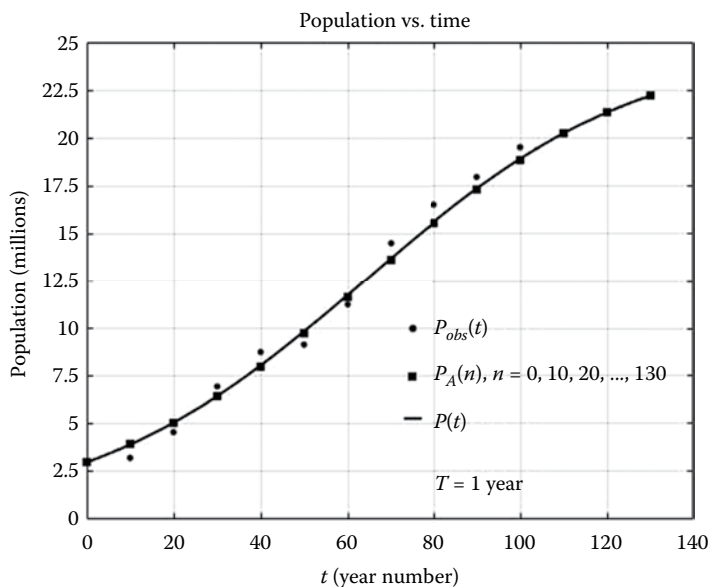


FIGURE 1.14 Observed, discrete (approximate) and continuous populations.

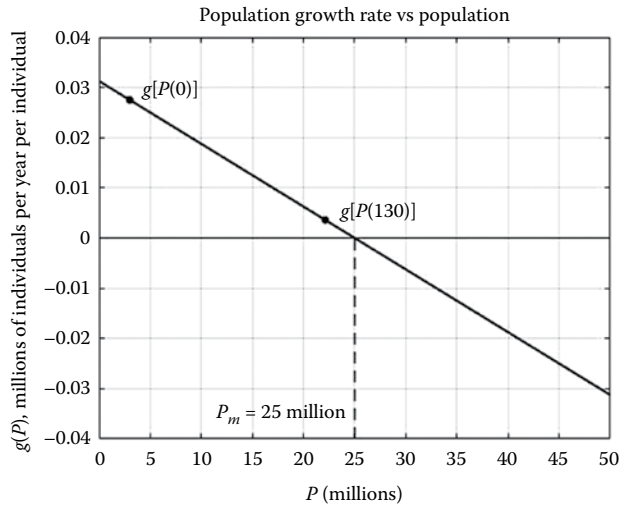


FIGURE 1.15 Plot of growth rate for a logistic growth population.

$$g(P) = \frac{1}{P} \frac{dP}{dt} = c(P_m - P) \quad (1.92)$$

where $g(P)$, the rate of change in population dP/dt divided by the population P , is called the population growth rate. Different population models are normally characterized by the terms appearing on the right hand side of Equation 1.92.

The growth rate for a logistic growth population model is shown in [Figure 1.15](#).

We expect the population to be increasing whenever the growth rate is positive, since a positive growth rate implies the instantaneous rate of change in the population, that is, the first derivative is also positive. The logistic population growth rate declines linearly with increasing population,

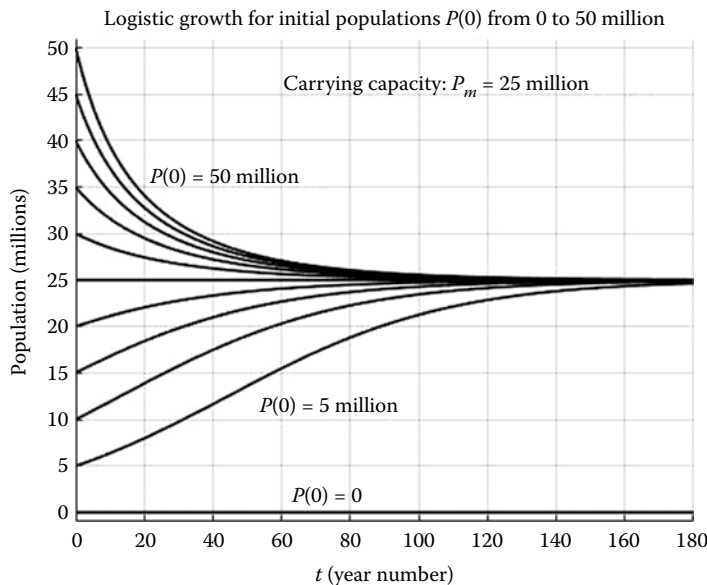


FIGURE 1.16 Logistic growth with different initial populations.

eventually reaching zero when the population reaches P_m . In logistic growth models, P_m is called the carrying capacity, which equals 25 million in this example.

Observe from Figure 1.14 that the discrete and continuous model outputs for 130 years ranged from the initial population of 3 million people to somewhere around 22 million people. Looking at the line segment in Figure 1.15, corresponding to this range of populations, we notice that the growth rate is positive. Hence the population should be monotonically increasing, as indeed it was.

Is it possible for a population $P(t)$ governed by a logistic growth model to ever assume values on both sides of its carrying capacity P_m ? For example, is the population growth shown in Figure 1.14 capable of exceeding $P_m = 25$ million if we wait long enough? Figure 1.16 shows population time histories for the logistic growth model considered previously ($c = 1.25 \times 10^{-9}$, $P_m = 25 \times 10^6$) with different starting populations.

It is clear that the population approaches its carrying capacity from below or above in asymptotic fashion. We should not be surprised if we consider what happens to the population growth rate $g(P)$ as the population approaches its carrying capacity from either direction (see Figure 1.15).

EXERCISES

- 1.11 Assume the logistic growth population model accurately predicts future populations.
- Some time in the future, the population will reach 98% of its carrying capacity. Find how many more years will it take for this to occur by using the difference equation given in Equation 1.89. Does it make a difference whether you start from $P_A(0) = 3$ million or $P_A(130) = 22.2012$ million from Table 1.5?
 - Compare the answer obtained in Part (a) with the analytical solution for $P(t)$.
 - The population growth rate $g(P)$ vs P in Figure 1.15 does not explicitly involve time. Label the points on the growth rate curve corresponding to $\{t_i, P(t_i)\}$ where $t_0 = 0$, $t_1 = 25$, $t_2 = 50$, $t_3 = 75$, $t_4 = 100$.
 - The carrying capacity P_m in a logistic growth model is an equilibrium population, meaning that if the population at some point in time were equal to P_m it would remain there forever. Investigate whether its stable or not by supposing the population were slightly less or slightly more than P_m and determine if the population returns to the carrying capacity. Obtain several approximate solutions corresponding to different initial populations reasonably close to P_m .
 - Find the other equilibrium population of the logistic growth model and determine if its stable.
- 1.12 A simpler model for population growth of a species is one in which the growth rate is assumed constant, that is independent of the population. Mathematically, this is represented by

$$\text{Growth Rate} = g(P) = \frac{1}{P} \frac{dP}{dt} = k$$

Suppose a culture of bacteria is increasing in size according to the constant growth rate model above. The initial bacteria population is P_0 .

- Develop the difference equation for the discrete system approximation of the continuous model. Denote the discrete population as $P_A(n)$.
- Find the general solution for $P_A(n)$, $n = 0, 1, 2, 3, \dots$. Leave your answer in terms of k and P_0 . The constant growth rate $k = 0.01$ bacteria/min per bacteria and the initial number of bacteria is 10,000.
- Solve the difference equation recursively using a step size $T = 1$ min for $P_A(n)$, $n = 1, 2, 3, 4, 5$. Compare the result for $P_A(5)$ to the value obtained from the general solution found in Part (b).

- d. The analytical solution to the continuous model is $P(t) = P_0 e^{kt}$, $t \geq 0$. How long does it take for the population to reach 1 million?
- e. On the same graph, plot the continuous model output $P(t)$, $0 \leq t \leq 500$ and the discrete model output $P_A(n)$, $n = 0, 50, 100, 150, \dots, 1000$ when $T = 0.5$ min.
- f. Explain what would happen to a population with constant growth rate k , if k were negative.

2 Continuous-Time Systems

2.1 INTRODUCTION

Before we start our exploration of simulation, it is important for us to have some basic knowledge of how linear time-invariant (LTI) dynamic systems behave. The analysis of linear systems and how they respond to elementary types of inputs is straightforward. Linear systems appear as building blocks in more complex systems. Our intuitive understanding of the entire system is enhanced by recognizing the fundamental behavior of its linear components. Control systems, for example, are oftentimes composed of linear continuous-time components interconnected to produce a desirable response to commanded as well as uncontrollable or disturbance inputs.

Speaking of control systems, the mathematical model of the process being controlled is often nonlinear; however, a properly designed regulatory control system will limit excursions of the process variables. In fact, the design of the controller may be based on a linearized model of the nonlinear process owing to the wealth of tools available in the field of linear control theory. Simulation can play a valuable role here by shedding light on the validity of using a linearized mathematical model to approximate a nonlinear system model.

Modern simulation software contains user interfaces employing graphical icons that serve as building blocks for representing the linear continuous- and discrete-time components within a system. In order to exploit this feature, the simulation builder must understand the meaning and differences between the assortment of linear system blocks (integrators, first-order lags, second-order systems, transfer functions, and state space models) at his or her disposal. The material on first- and second-order system response, and state variables covered in this chapter and [Chapter 4](#), is intended as an introduction (or possibly a review) to the topic of linear continuous-time systems. There are literally dozens of excellent books on the subject of linear systems theory and linear control systems. Several are included in the references and the reader is encouraged to consult one or more as necessary.

In addition to the focus on linear systems in this chapter, one section includes several examples of nonlinear systems as well. A graphical illustration of how to linearize a nonlinear system model is presented as a preview of what is to come in [Chapter 7](#) where the subject is revisited in more detail.

Simulation of continuous-time systems is not discussed in detail until [Chapter 3](#) where the subject of numerical integration is introduced. However, a simulation model based on numerical differentiation, similar to what was done in [Chapter 1](#), is presented. At the conclusion of this chapter, the reader will be capable of representing simple continuous-time systems in state variable form and generate discrete-time model approximations of them, which can be solved in a recursive fashion.

2.2 FIRST-ORDER SYSTEMS

Continuous-time dynamic systems are said to be first order if the highest derivative of the dependent variable appearing in the mathematical model is first order. Systems in which a quantity of material or energy changes at a rate dependent on the amount of material or energy present are typically first order in nature. The general representation of a scalar first-order system is

$$\frac{dy}{dt} = f(t, y, u) \quad (2.1)$$

where

t is the continuous-time variable

$u = u(t)$ is the system input

$y = y(t)$ is the system output

$f(t, y, u)$ is the derivative function, which relates the rate of change in y to all three arguments

Not all three arguments will be present in every first-order model. Furthermore, it is possible for multiple inputs $u_1(t)$, $u_2(t)$, ..., $u_r(t)$ to be present.

We begin our discussion of first-order systems with a special case, namely, where the derivative function is an explicit linear function of the input and output given by

$$f(t, y, u) = b_0 u(t) - a_0 y(t) \quad (2.2)$$

where a_0 and b_0 are constants. Combining Equations 2.1 and 2.2 gives

$$\frac{d}{dt} y(t) + a_0 y(t) = b_0 u(t) \quad (2.3)$$

Equation 2.3 is a LTI, ordinary differential equation. In the time-varying case, one or both of the linear system parameters a_0 and b_0 are functions of the independent variable t . Equation 2.3 is commonly expressed as

$$\tau \frac{d}{dt} y(t) + y(t) = K u(t) \quad (2.4)$$

where τ and K are easily related to a_0 and b_0 by

$$\tau = \frac{1}{a_0} \quad \text{and} \quad K = \frac{b_0}{a_0} \quad (2.5)$$

Many simple real-world dynamic systems are modeled by the first-order differential equation (Equation 2.4). More complex systems often behave similarly to first-order systems under certain conditions. Furthermore, higher-order system models can be reduced to a system of coupled first-order models. Familiarity with first-order system response will prove useful later on when we undertake the task of simulating higher-order linear and nonlinear systems. For this reason, we explore some basic properties of first-order systems modeled by Equation 2.4.

2.2.1 STEP RESPONSE OF FIRST-ORDER SYSTEMS

When the input $u(t)$ is constant, that is, $u(t) = A$, $t \geq 0$, the solution to Equation 2.4 for $y(t)$ is obtained using Laplace transform or classical time-domain methods. It is given below:

$$y(t) = y(0)e^{-t/\tau} + KA(1 - e^{-t/\tau}), \quad t \geq 0 \quad (2.6)$$

where $y(0)$ is the initial value of the output $y(t)$. Several graphs of $y(t)$ are shown in [Figure 2.1](#) for the cases where $y(0) = 0$, $K = 5$, $A = 2$, and $\tau = 0.5, 2, 5$, and 10 .

The graphs of $y(t)$ shown in [Figure 2.1](#) are called the step response because the input resembles a step (changing from 0 to A at $t = 0$). Note that the initial condition is zero in all the step responses.

The constant A measures the amplitude of the input and is not an inherent system parameter. The system parameters are K and τ (or a_0 and b_0 from which they are computed). The first parameter

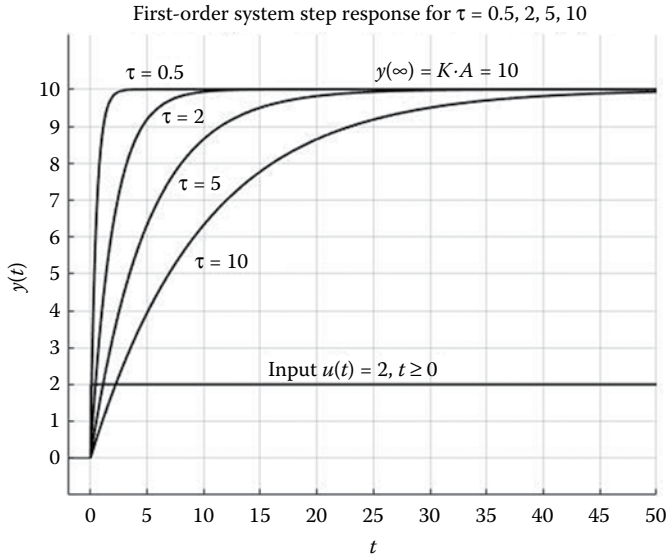


FIGURE 2.1 Step response of first-order system with different values of τ .

K is called the system DC or steady-state gain. It is so named because the final value of the output, $y(\infty)$, is easily computed from

$$y(\infty) = K \cdot u(\infty) = K \cdot A \quad (2.7)$$

which in this case is $y(\infty) = 5 \cdot 2 = 10$ (see Figure 2.1). The final value $y(\infty)$ is unaffected by the initial condition $y(0)$. However, the graph of $y(t)$ in Equation 2.6 certainly depends on $y(0)$, since that is where it starts. A first-order system like the one in Equation 2.4 is called a first-order lag because of the way the step response in Figure 2.1 lags the step input.

There are situations when the input to a first-order system is not a step; however, the input remains constant for a period of time that is largely relative to the parameter τ . Equation 2.7 enables us to readily compute the final output value prior to a change in the input. In essence, we are tracking the first-order system from one steady-state level to another, and the transient response (portion of the overall step response that decays to zero) is ignored. Even without knowledge of the transient response, it is possible to predict the amount of time necessary for the new steady state to be established.

In the first-order system modeled by Equation 2.4, the first derivative vanishes when the system is at steady state, leaving $y_{ss} = K\bar{u}$, where y_{ss} is the output at steady state in response to the constant input \bar{u} . A similar result is obtained from Equation 2.6 with A replaced by \bar{u} and t approaching ∞ .

The first-order system step responses shown in Figure 2.1 correspond to four distinct values for the parameter τ . It is apparent that while all approach the limiting value $y(\infty) = 10$, there is a noticeable difference in the amount of time required for each to get there. The individual step responses are correlated with the system parameter τ . This parameter is called the time constant of the first-order system. It is a measure of the speed of the step response as well as an indicator of the overall speed of the first-order system's dynamics. A "rule of thumb" for first-order systems is that the transient response vanishes after four or five time constants. The transient response component of the step response in Equation 2.6 with $y(0) = 0$ is

$$y_{tr}(t) = -KAe^{-t/\tau}, \quad t \geq 0 \quad (2.8)$$

when $t = 5\tau$,

$$y_{tr}(5\tau) = -KAe^{-5} = -KA(0.0067) \quad (2.9)$$

and the step response

$$y(5\tau) = KA(1 - e^{-5}) = 0.9933KA \quad (2.10)$$

is more than 99% complete. After four time constants have elapsed, the step response is slightly over 98% of its final value (see [Figure 2.1](#)).

First-order system models are commonplace in science, engineering, economics, business, etc. The liquid storage tank model in Section 1.2 and the population models considered in Section 1.5 are examples of first-order system models. Another example of a physical system described in terms of a first-order model is the simple electric circuit shown in [Figure 2.2](#) along with the tank.

The circuit components are a capacitor C , a resistor R , and a voltage source $e_0(t)$. There is also a switch that connects the source to the rest of the circuit when it is in the closed position. Like the tank that stores its energy as a column of liquid, the circuit's capacitor stores energy in the form of electric charge. The potential energy of the fluid varies as the tank level changes and the electrical energy stored in the circuit varies with the amount of electrical charge stored in the capacitor. Both systems have a mechanism for dissipating energy. The tank does so whenever the level of fluid is dropping and the circuit dissipates energy in the resistor whenever there is current flowing.

The fluid resistance of the tank tells us the amount of effort, that is, height of liquid, required to produce a unit of flow from the tank. A typical unit for fluid resistance is ft per ft³/min. The electrical counterpart is the electrical resistor that also measures the driving force, in this case, the voltage applied to the resistor, necessary to produce a unit of current flow, measured in amperes. The unit of electric resistance is volts/ampere, commonly called ohms.

Choosing the voltage across the capacitor $v_c(t)$ as the output, the circuit model is easily derived using basic principles of electrical circuits. The result is

$$RC \frac{d}{dt} v_c(t) + v_c(t) = e_0(t) \quad (2.11)$$

Comparison of Equation 2.11 with the standard form introduced in Equation 2.4 reveals the time constant of the circuit $\tau = RC$ and the steady-state gain $K = 1(\text{V/V})$. Hence, the transient response lasts for a period of time equal to approximately $5RC$. For a constant voltage applied to the circuit, that is, $e_0(t) = E_0$, $t \geq 0$, the steady-state voltage $v_c(\infty)$ is numerically equal to E_0 since $v_c(\infty) = KE_0 = 1 \cdot E_0$.

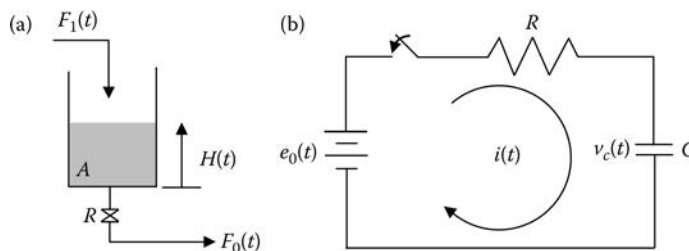


FIGURE 2.2 Examples of systems with first-order system models: (a) storage tank and (b) RC circuit.

The step response is obtained from Equation 2.6 with $y(0) = v_c(0) = 0$, $\tau = RC$, $K = 1$, and $A = E_0$. The result is

$$v_c(t) = E_0(1 - e^{-t/RC}), \quad t \geq 0 \quad (2.12)$$

The step response consists of the steady-state component

$$v_c(\infty) = E_0 \quad (2.13)$$

and the transient component

$$v_c(t) = -E_0 e^{-t/RC}, \quad t \geq 0 \quad (2.14)$$

The transient response involves the exponential $e^{-t/RC}$, which is called the natural mode of the system. To understand this, consider the circuit response with zero applied voltage ($E_0 = 0$) and a nonzero initial voltage across the capacitor $v_c(0)$. From Equation 2.6, the solution for $v_c(t)$ is

$$v_c(t) = v_c(0)e^{-t/RC}, \quad t \geq 0 \quad (2.15)$$

a constant times the natural mode. Natural modes of linear systems are exponential functions of time involving the parameters of the system, in this case, R and C . The natural modes do not depend on the system inputs. The unforced response of higher-order system models is referred to as the natural response of the system. It contains a linear combination of the natural modes (only one for the first-order system model). In general, the natural modes of linear system models appear in the transient response independent of whether the system is being forced (excited by inputs) or simply responding to initial conditions as in the case of an autonomous system.

EXAMPLE 2.1

A 12 V battery is used to charge the capacitor in the circuit shown in Figure 2.2. When the switch is closed at $t = 0$, the capacitor voltage is zero. Numerical values of the circuit parameters are $R = 5000 \, \Omega$ and $C = 0.125 \times 10^{-6} \, \text{F}$ (1 F = 1 A per V/s).

- Find the time constant τ , steady-state gain K , and natural mode of the circuit.
- Find the steady-state voltage $v_c(\infty)$ across the capacitor.
- Determine how long it takes for the capacitor to charge up to 50% of $v_c(\infty)$.
- Find and graph the transient component, steady-state component, and the complete response for the case where the capacitor is initially charged to 3 V.

a. $\tau = RC = (5000 \, \Omega) \times 0.125 \times 10^{-6} \, \text{F} = 0.000625 \, \text{s}$ ($625 \times 10^{-6} \, \text{s}$)

$K = 1 \, \text{V/V}$

Natural mode: $e^{-t/RC} = e^{-t/0.000625}$, $t \geq 0$

b. $v_c(\infty) = KE_0 = (1 \, \text{V/V}) \times 12 \, \text{V} = 12 \, \text{V}$

c. $v_c(t) = E_0(1 - e^{-t/RC}) \Rightarrow 6 = 12(1 - e^{-t/625 \times 10^{-6}})$, which can be solved using natural logarithms to give $t = 0.0004332 \, \text{s}$

d. From Equation 2.6 with initial condition $v_c(0) = 3 \, \text{V}$, the complete response is

$$\begin{aligned} v_c(t) &= v_c(0)e^{-t/RC} + KE_0(1 - e^{-t/RC}), \quad t \geq 0 \\ &= 3e^{-t/625 \times 10^{-6}} + (1)(12)(1 - e^{-t/625 \times 10^{-6}}) \end{aligned}$$

The transient component is

$$\begin{aligned} V_c(t)_{tr} &= [v_c(0) - KE_0]e^{-t/RC}, \quad t \geq 0 \\ &= [3 - (1)(12)]e^{-t/625 \times 10^{-6}} \\ &= -9e^{-t/625 \times 10^{-6}} \end{aligned}$$

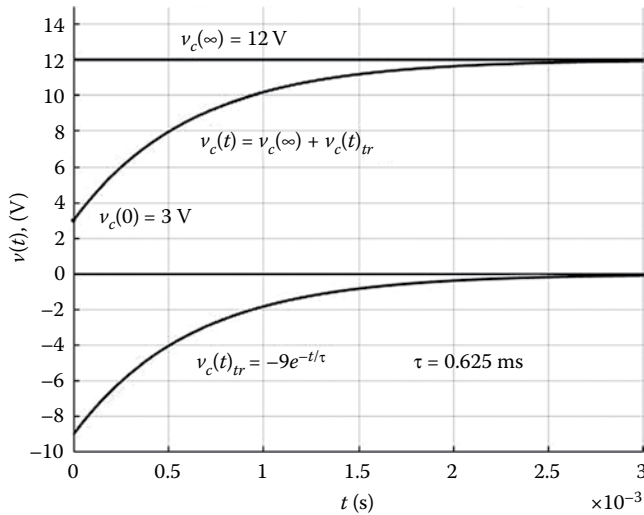


FIGURE 2.3 Steady-state, transient, and total response of an RC circuit.

and the steady-state component is

$$v_c(\infty) = KE = 1 \frac{\text{V}}{\text{V}} (12 \text{ V}) = 12 \text{ V}$$

Graphs of the steady-state, transient, and complete responses are shown in [Figure 2.3](#).

Note that the transient response has decayed to essentially zero after five time constants ($5 \times 625 \times 10^{-6} = 3.125 \times 10^{-3}$) have elapsed.

EXERCISES

- 2.1 The tank shown in [Figure 2.2](#) has a constant cross-sectional area A and fluid resistance R .
 - a. Find expressions for the time constant τ and steady-state gain K of the tank in terms of the physical parameters A and R .
 - b. The empty tank is subject to a constant flow in of \bar{F} ft³/min. Obtain an expression for the liquid level step response of the tank.
 - c. The cross-sectional area of the tank is 20 ft², and the fluid resistance is 0.5 ft per ft³/min. How high must the tank be if the inflow is constant at $\bar{F} = 15$ ft³/min for it not to overflow.
 - d. How long will it take for the tank level to reach 50% of its final height?
 - e. What size tank is needed if the time required to fill up is increased by 10%?
- 2.2 Consider the first-order system: $(d/dt)y(t) + a_0y(t) = b_0u(t)$
 - a. Under what conditions does this system reduce to a pure integrator?
 - b. For the continuous-time integrator in part (a), express the output $y(t)$ in terms of the input $u(t)$. Assume the initial condition is $y(0) = y_0$.
 - c. When is a liquid storage tank a pure integrator?
- 2.3 The amount of salt Q in a well-stirred tank shown in the [Figure E2.3](#) depends on c_1 , the concentration of salt in the brine solution entering the tank, as well as the flow rates F_1 and F_0 into and out of the tank. The continuous-time model is based on conservation of salt. It equates dQ/dt , the instantaneous rate of change in the amount of salt in the tank to the difference in the rate of salt entering the tank, c_1F_1 , and the rate of salt flowing out of the tank, cF_0 .

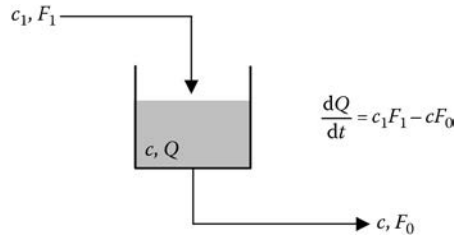


FIGURE E2.3

The tank initially contains 100 lb of salt-free water. The concentration of salt in the brine solution flowing in is 0.25 lb/ft^3 . Both the flow into and the flow out of the tank are both $1 \text{ ft}^3/\text{min}$. Note that 1 ft^3 of water weighs approximately 62.4 lb.

- Find $Q(t)$, the amount of salt in the tank as a function of time.
- Find the amount of salt in the tank at steady state.

$$\frac{dQ}{dt} = c_1 F_1 - c F_0$$

2.4 A temperature-controlled chamber is shown in Figure E2.4.

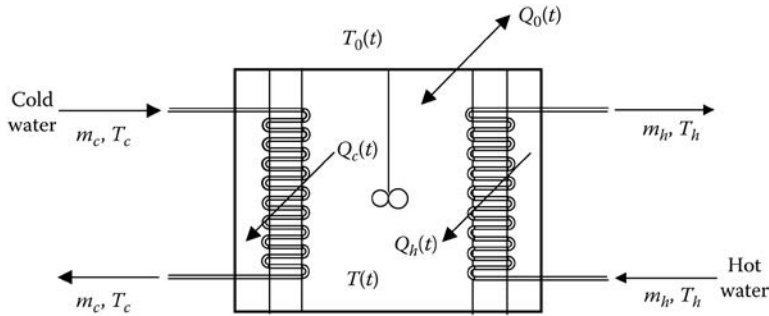


FIGURE E2.4

The air temperature inside the chamber is assumed to be the same everywhere, namely, $T(t)$. The chamber walls are insulated to reduce heat loss or gain with its surroundings. Temperature control is achieved by circulating hot or cold water through pipes located inside the chamber. Heat exchange occurs between the air inside the chamber and the circulating water in the pipes. The heat flow from the circulating hot water is $Q_h(t)$, and $Q_c(t)$ is the heat flow to the cold water. Heat exchange $Q_0(t)$ also occurs between the air inside and outside the chamber. Ambient temperature outside the chamber is denoted $T_0(t)$.

A suitable model for this thermal system is based on the conservation of energy.

$$c_A V \frac{dT}{dt} = Q_h - Q_c - Q_0$$

V is the volume (ft^3) of air in the chamber, and c_A is the thermal capacitance of air ($0.01375 \text{ Btu}/^\circ\text{F}/\text{ft}^3$). The heat flow terms on the right-hand side are given by

$$Q_h = \dot{m}_h c_p (T_h - T)$$

$$Q_c = \dot{m}_c c_p (T - T_c)$$

$$Q_0 = \frac{1}{R} (T - T_0)$$

where

\dot{m}_h and \dot{m}_c are the mass flow rates (lb/min) of the hot and cold water
 c_p is the specific heat of water (1 Btu/lb/°F)
 R is the thermal resistance (°F/Btu/min) of the chamber walls

The expressions for Q_h and Q_c assume that the flow rates of the circulating fluids are great enough that both fluids exit at the same temperature at which they entered the chamber.

- Express the mathematical model in the form of a differential equation relating the output T and its derivative to the inputs T_h , T_c , and T_0 .
- Find the time constant and the three steady-state gains of the system. Check the units to verify that the time constant is in minutes and the steady-state gains are dimensionless (°F/°F).
- Show that the air temperatures inside and outside the chamber eventually equalize after both the hot and cold circulating water flows are turned off.
- Suppose the chamber air temperature is required to be higher than the outside ambient air temperature, which remains constant, that is, $T_0(t) = \bar{T}_0$, $t \geq 0$. The hot water temperature entering the chamber is three times greater than the ambient temperature. The initial air temperature inside the chamber is the same as the outside ambient temperature. Find the analytical solution for $T(t)$, $t \geq 0$, the air temperature inside the chamber.
- Graph the solution for $T(t)$, $t \geq 0$ in part (d) using the following values:

$$V = 5000 \text{ ft}^3, R = 0.025^\circ\text{F/Btu/min}, \dot{m}_h = 50 \text{ lb/min}, \text{ and } \bar{T}_0 = 60^\circ\text{F}$$

2.3 SECOND-ORDER SYSTEMS

Input–output models of continuous-time dynamic systems where the highest derivative of the dependent variable is second order are classified as second-order systems. Second-order systems result when there are two energy storage elements present. Our interest for now is in linear second-order systems, which can be manipulated into the form shown in Equation 2.16 relating an output $y(t)$ to an input $u(t)$ involving generic system parameters, ζ , ω_n , and K .

$$\frac{d^2}{dt^2} y(t) + 2\zeta\omega_n \frac{d}{dt} y(t) + \omega_n^2 y(t) = K\omega_n^2 u(t) \quad (2.16)$$

For an actual second-order system (mechanical, electrical, biological, etc.), the generic parameters can be expressed in terms of the system's physical parameters. The importance of each will be explained shortly.

The unit step response of the second-order system is the solution for $y(t)$ in Equation 2.16 when $y(0) = 0$ and the input $u(t) = 1$, $t \geq 0$, hereafter denoted by $\hat{u}(t)$. It can be found in any text related to linear systems or controls (Palm 1983; Franklin et al. 2002; Dorf and Bishop 2005). The unit step response assumes one of three forms depending on the location of the roots of the algebraic equation

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (2.17)$$

known as the characteristic equation of the system. The characteristic roots are the solution to Equation 2.17 and are given by

$$s_1, s_2 = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n \quad (2.18)$$

The natural modes of the second-order system are $e^{s_1 t}$ and $e^{s_2 t}$. The step response depends on the value of the parameter ζ . There are three cases to consider.

Case 1: $\zeta > 1$

If we let $s_1 = -\zeta\omega_n - \sqrt{\zeta^2 - 1}\omega_n$ and $s_2 = -\zeta\omega_n + \sqrt{\zeta^2 - 1}\omega_n$, then both roots are negative (assuming $\omega_n > 0$) and $s_1 < s_2 < 0$. Introducing time constants τ_1 and τ_2 as the reciprocals of the characteristic roots s_1 and s_2 , respectively,

$$\tau_1 = -\frac{1}{s_1}, \quad \tau_2 = -\frac{1}{s_2} \quad (2.19)$$

The unit step response is

$$y(t) = K \left[1 + \frac{\tau_2 e^{-t/\tau_2} - \tau_1 e^{-t/\tau_1}}{\tau_1 - \tau_2} \right], \quad t \geq 0 \quad (2.20)$$

Case 2: $0 < \zeta < 1$

The characteristic roots are complex conjugates and can be expressed as

$$s_1, s_2 = -\zeta\omega_n \pm j\sqrt{1 - \zeta^2}\omega_n \quad (2.21)$$

It is convenient to define a new quantity ω_d in terms of ζ and ω_n according to

$$\omega_d = \sqrt{1 - \zeta^2}\omega_n \quad (2.22)$$

The unit step response is

$$y(t) = K \left[1 - e^{-\zeta\omega_n t} \left(\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin\omega_d t \right) \right], \quad t \geq 0 \quad (2.23)$$

An alternate form of Equation 2.23 is

$$y(t) = K \left[1 - \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \varphi) \right], \quad t \geq 0 \quad (2.24)$$

where the phase angle term φ is given by

$$\varphi = \tan^{-1} \frac{\omega_d}{\zeta\omega_n} = \tan^{-1} \left(\frac{\sqrt{1 - \zeta^2}}{\zeta} \right) \quad (2.25)$$

Case 3: $\zeta = 1$

From Equation 2.18, the characteristic roots are repeated, $s_1 = s_2 = -\omega_n$

The unit step response is

$$y(t) = K[1 - e^{-\omega_n t}(\omega_n t + 1)], \quad t \geq 0 \quad (2.26)$$

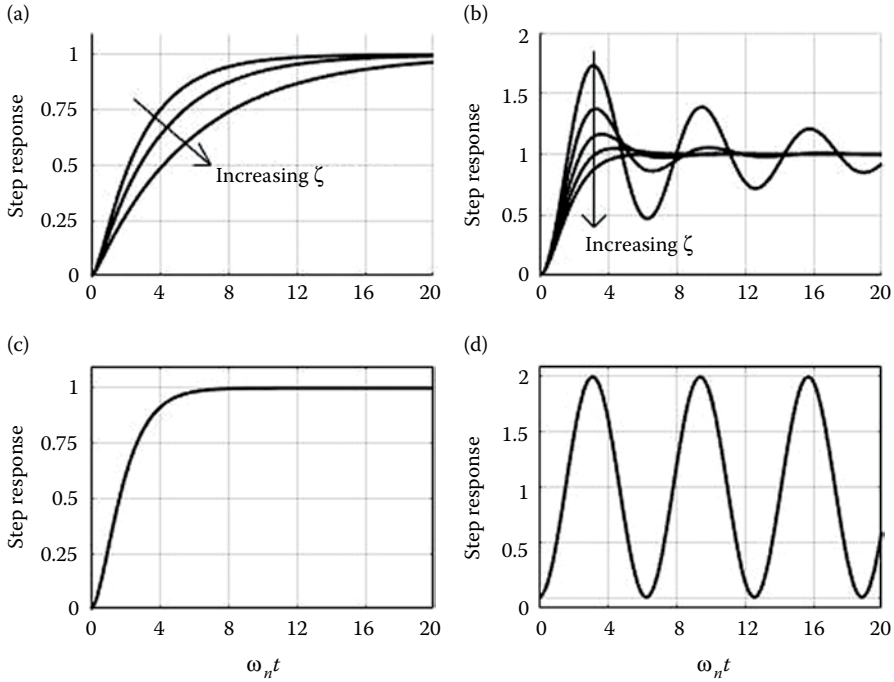


FIGURE 2.4 Unit step response of second-order system in Equation 2.16. (a) Overdamped, $\zeta = 1.5, 2, 3$. (b) Underdamped, $\zeta = 0.1, 0.3, \dots, 0.9$. (c) Critically damped, $\zeta = 1$. (d) Zero damping, $\zeta = 0$.

A graph of the unit step responses given in Equations 2.20, 2.23, and 2.26 with $K = 1$ is shown in Figure 2.4. The abscissa is $\omega_n t$, a dimensionless variable, which allows us to visualize the effect of the parameter on the step response independent of ω_n . Note that all three step responses start from zero. Furthermore, the initial slope given by $dy(0)/dt$ is also zero for all three cases (see Exercise 2.6).

There are no oscillations in Case 1 ($\zeta > 1$), that is, the response is monotonically increasing without overshooting the final value $y(\infty) = K\bar{u} = 1$ for a unit step input. The transient period increases with increasing ζ . The system is said to be overdamped.

An oscillatory step response occurs in Case 2 ($0 < \zeta < 1$), and the system is referred to as underdamped. As the value of ζ decreases, the oscillations become more pronounced, and the settling time for the transient component to die out becomes larger.

The case when $\zeta = 1$ represents the transition from Case 1 to Case 2 (or vice versa). The second-order system is called critically damped in this situation.

The graph in Figure 2.4d is the unit step response for the case when $\zeta = 0$. From Equation 2.23 with $\zeta = 0$,

$$y(t) = K(1 - \cos \omega_n t), \quad t \geq 0 \quad (2.27)$$

resulting in sustained oscillations from 0 to 2 when $K = 1$. The differential equation of the unforced system is

$$\frac{d^2}{dt^2} y(t) + \omega_n^2 y(t) = 0 \quad (2.28)$$

and the natural response resulting from the presence of initial conditions is that of harmonic motion, that is, sustained oscillations about zero at a frequency of ω_n rad/s.

Except for the case when $\zeta = 0$, the unit step response approaches the limiting or steady-state value $y(\infty) = K$, which means that K is the DC or steady-state gain of the second-order system in Equation 2.16. The parameter ζ , which determines the existence and extent of the oscillations as well as the duration of the transient response, is called the damping ratio of the system. The last two parameters ω_n and ω_d are the natural frequency and damped natural frequency of the second-order system, respectively. The first, ω_n , is the frequency of the sustained oscillations ($\zeta = 0$) in Equation 2.27, and the second, ω_d , is the frequency of the decaying oscillations ($0 < \zeta < 1$) in Equation 2.24. It follows from Equation 2.22 that $\omega_d < \omega_n$. The natural frequency ω_n is an indication of the speed of the step response (and the system in general) since the oscillatory natural modes are damped by the exponential term with time constant $1/\zeta\omega_n$ in Equation 2.23.

EXAMPLE 2.2

Figure 2.5 shows a delicate instrument placed on a table that moves as a result of a vertical force acting on it. Springs and dampers connect the table to the ground to limit the table's movement.

The combined mass of the table and instrument is m . The total stiffness of the springs is k and the total damping is c . The mechanical system is modeled by

$$m \frac{d^2}{dt^2} x(t) + c \frac{d}{dt} x(t) + kx(t) = f(t) \quad (2.29)$$

where

$x(t)$ is the displacement of the table (from its static equilibrium position)

$f(t)$ is the force acting on the platform resulting in the motion $x(t)$

- Find expressions for the steady-state gain K , the damping ratio ζ , and the natural frequency ω_n in terms of the physical parameters m , c , and k .
 - Numerical values of the physical parameters are $m = 40 \text{ lb}_m$, $k = 45 \text{ lb}_f/\text{ft}$, and $c = 4 \text{ lb}_f \cdot \text{s}/\text{ft}$. Find the response of the table when the platform is subjected to a sudden deflection due to a force of 12 lb_f .
 - Graph the solution and estimate the duration of the transient.
 - The instrument is not usable if it is moving faster than 0.04 ft/s . How long a period of time must pass after the force is applied before the instrument will function properly?
- Dividing Equation 2.29 by m for comparison with the standard form of a second-order system in Equation 2.16 gives

$$\frac{d^2}{dt^2} x(t) + \frac{c}{m} \frac{d}{dt} x(t) + \frac{k}{m} x(t) = \frac{1}{m} f(t) \quad (2.30)$$

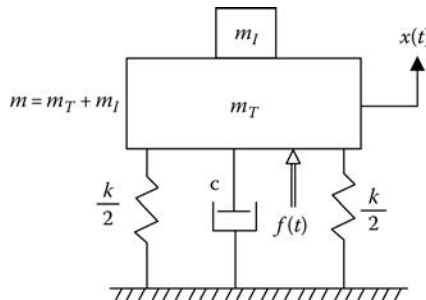


FIGURE 2.5 Mechanical system for Example 2.2.

$$\Rightarrow 2\zeta\omega_n \frac{c}{m}, \quad \omega_n^2 = \frac{k}{m}, \quad K\omega_n^2 = \frac{1}{m} \quad (2.31)$$

Solving for the parameters K , ω_n , and ζ yields

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \zeta = \frac{c}{2\sqrt{km}}, \quad K = \frac{1}{k} \quad (2.32)$$

b. Substituting the given values for m (in slugs), k , and c ,

$$\begin{aligned} \omega_n &= \sqrt{\frac{k}{m}} = \sqrt{\frac{45}{40/32.2}} = 6.0187 \text{ rad/s} \\ \zeta &= \frac{c}{2\sqrt{km}} = \frac{4}{2\sqrt{45 \cdot 40/32.2}} = 0.2675 \\ K &= \frac{1}{45} = 0.0222 \text{ in/lb}_f \end{aligned}$$

The damping ratio $\zeta = 0.2675$ indicates the system is underdamped. From Equation 2.22, the damped natural frequency is

$$\omega_d = \sqrt{1 - \zeta^2} \omega_n = \left(\sqrt{1 - 0.2675^2} \right) 6.0187 = 5.7994 \text{ rad/s}$$

and the response to a step input $f(t) = \bar{F} = 12 \text{ lb}_f$, $t \geq 0$ is

$$x(t) = K\bar{F} \left[1 - e^{-\zeta\omega_n t} \left(\cos \omega_d t + \frac{\zeta\omega_n}{\omega_d} \sin \omega_d t \right) \right], \quad t \geq 0 \quad (2.33)$$

Substituting the numerical values for K , \bar{F} , ζ , ω_n , and ω_d results in

$$x(t) = 0.2667[1 - e^{-1.6100t} (\cos 5.7994t + 0.2776 \sin 5.7994t)], \quad t \geq 0 \quad (2.34)$$

c. A graph of the step response is generated in the script file “*Chap2_Ex3_1.m*” and shown in [Figure 2.6](#).

The transient period can be approximated from the graph as roughly 3 s, or it can be computed from the time constant of the exponential envelope as

$$\text{Transient period} \approx 5 \times \frac{1}{\zeta\omega_n} = 5 \times \frac{1}{0.2675(6.0187)} = 3.1056 \text{ s}$$

d. The first derivative is obtained by differentiation of the underdamped step response in Equation 2.24. The result is

$$\frac{d}{dt} y(t) = K \frac{\omega_n}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin \omega_d t, \quad t \geq 0 \quad (2.35)$$

Substituting the numerical values for the system parameters K , ζ , ω_n , and ω_d gives

$$\frac{d}{dt} y(t) = 0.1388 e^{-1.6100t} \sin 5.7994t, \quad t \geq 0 \quad (2.36)$$

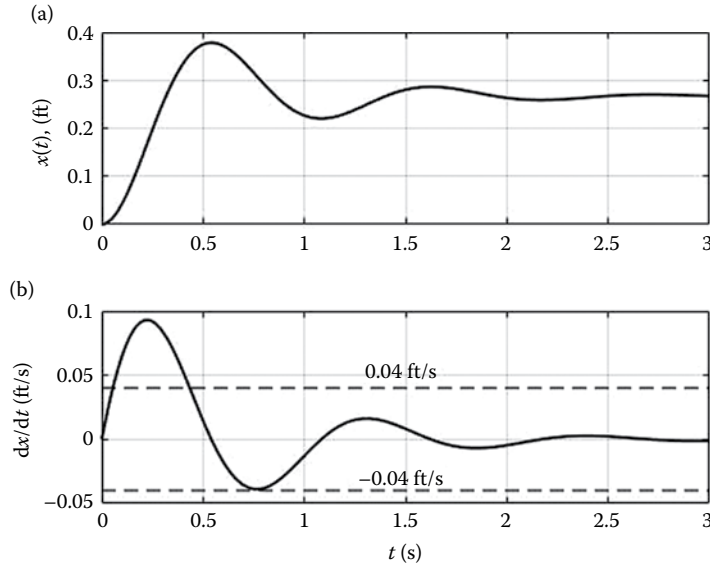


FIGURE 2.6 (a) Position and (b) velocity response of table and instrument ($\bar{F} = 12 \text{ lb}_f$).

The first derivative is graphed in the lower half of Figure 2.6. From the graph, it appears that approximately 0.77 s must elapse for the instrument to be usable, that is, the instrument is moving at less than 0.04 ft/s in either direction after that period of time. (A closeup of the response in the neighborhood of $dx/dt = -0.04 \text{ ft/s}$ reveals that the instrument's velocity actually falls a bit short of -0.04 ft/s .)

2.3.1 CONVERSION OF TWO FIRST-ORDER EQUATIONS TO A SECOND-ORDER MODEL

A linear second-order system is sometimes represented as a system of two first-order differential equations like those in Equations 2.37 and 2.38:

$$\frac{dx}{dt} = ax + by + f(t) \quad (2.37)$$

$$\frac{dy}{dt} = cx + dy + g(t) \quad (2.38)$$

Suppose a single equation relating the dependent variable $x = x(t)$ and the inputs $f = f(t)$ and $g = g(t)$ is required. The first step is to solve for $y = y(t)$ in Equation 2.37,

$$y = \frac{1}{b} \left(\frac{dx}{dt} - ax - f \right) \quad (2.39)$$

Differentiating Equation 2.39,

$$\frac{dy}{dt} = \frac{1}{b} \left(\frac{d^2x}{dt^2} - a \frac{dx}{dt} - \frac{df}{dt} \right) = cx + dy + g \quad (2.40)$$

Replacing y in Equation 2.40 with Equation 2.39 gives

$$\frac{1}{b} \left(\frac{d^2 x}{dt^2} - a \frac{dx}{dt} - \frac{df}{dt} \right) = cx + d \left[\frac{1}{b} \left(\frac{dx}{dt} - ax - f \right) \right] + g \quad (2.41)$$

and simplifying leads to the second-order differential equation,

$$\frac{d^2 x}{dt^2} - (a + d) \frac{dx}{dt} + (ad - bc)x = \frac{df}{dt} - df + bg \quad (2.42)$$

A similar procedure is used to eliminate x from Equations 2.37 and 2.38 to give a second-order differential equation in y .

EXAMPLE 2.3

The well-mixed tanks shown in Figure 2.7 contain uniform salt concentrations of $c_1 = c_1(t)$ and $c_2 = c_2(t)$, respectively. Concentration of salt in the input to the first tank is $c = c(t)$. The flow rates between the tanks are Q_1 and Q_2 , where $Q_1 > Q_2 > 0$. The liquid volumes in both tanks remain constant at V_1 and V_2 .

- Write the differential equations for the conservation of salt in each tank.
- Find the differential equation relating $c_2(t)$ and the input $c(t)$.
- Find expressions for the damping ratio, natural frequency, and steady-state gain.
- Find and plot the step response for c_2 under the following conditions:
 $Q_1 = 10$ gal/min, $Q_2 = 5$ gal/min, $V_1 = 15$ gal, and $V_2 = 15$ gal
 $c_1(0) = c_2(0) = 0$ lb of salt/gal, $c(t) = \bar{c} = 0.25$ lb salt/gal, $t \geq 0$

- Equating the accumulation of salt in each tank to the difference between the rates of salt in and out of the tanks,

$$\frac{d}{dt}(c_1 V_1) = Q_{in} c + Q_2 c_2 - Q_1 c_1 \quad (2.43)$$

$$\frac{d}{dt}(c_2 V_2) = Q_1 c_1 - Q_2 c_2 - Q_{out} c_2 \quad (2.44)$$

Since the holdup of liquid in both tanks is constant, the flows Q_{in} and Q_{out} are equal,

$$Q_{in} = Q_{out} = Q_1 - Q_2 \quad (2.45)$$

And, therefore, Equations 2.43 and 2.44 become

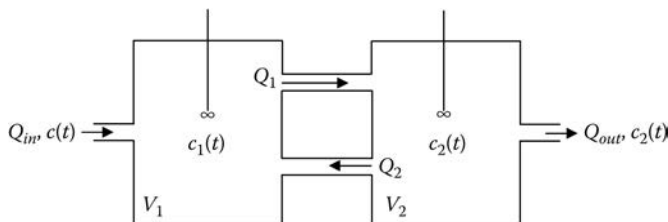


FIGURE 2.7 Two-tank mixing system.

$$V_1 \frac{dc_1}{dt} = (Q_1 - Q_2)c + Q_2c_2 - Q_1c_1 \quad (2.46)$$

$$V_2 \frac{dc_2}{dt} = Q_1c_1 - Q_2c_2 - (Q_1 - Q_2)c_2 \quad (2.47)$$

b. Rearranging Equations 2.46 and 2.47 into the form of Equations 2.37 and 2.38,

$$\frac{dc_2}{dt} = -\frac{Q_1}{V_2}c_2 + \frac{Q_1}{V_2}c_1 \quad (2.48)$$

$$\frac{dc_1}{dt} = \frac{Q_2}{V_1}c_2 - \frac{Q_1}{V_1}c_1 + \frac{(Q_1 - Q_2)}{V_1}c \quad (2.49)$$

Thinking of c_2 as x and c_1 as y , and comparing Equations 2.48 and 2.49 with Equations 2.37 and 2.38 implies

$$a = -\frac{Q_1}{V_2}, b = \frac{Q_1}{V_2}, c = \frac{Q_2}{V_1}, d = -\frac{Q_1}{V_1}, f(t) = 0, g(t) = \frac{(Q_1 - Q_2)}{V_1}c(t) \quad (2.50)$$

From Equation 2.42, the second-order differential equation relating c_2 and c is

$$\frac{d^2c_2}{dt^2} + Q_1 \left(\frac{1}{V_1} + \frac{1}{V_2} \right) \frac{dc_2}{dt} + \frac{Q_1(Q_1 - Q_2)}{V_1V_2}c_2 = \frac{Q_1(Q_1 - Q_2)}{V_1V_2}c \quad (2.51)$$

c. Comparing the left-hand side of Equation 2.51 with the standard form in Equation 2.16 gives

$$2\zeta\omega_n = Q_1 \left(\frac{1}{V_1} + \frac{1}{V_2} \right), \quad \omega_n^2 = \frac{Q_1(Q_1 - Q_2)}{V_1V_2} \quad (2.52)$$

$$\Rightarrow \omega_n = \left[\frac{Q_1(Q_1 - Q_2)}{V_1V_2} \right]^{1/2}, \quad \zeta = \frac{(V_1 + V_2)}{2} \left[\frac{Q_1}{(Q_1 - Q_2)V_1V_2} \right]^{1/2} \quad (2.53)$$

For $c(t) = \bar{c}$, the steady-state value of c_2 is obtained from Equation 2.51 by setting the derivatives equal to zero resulting in

$$\frac{Q_1(Q_1 - Q_2)}{V_1V_2}(c_2)_{ss} = \frac{(Q_1 - Q_2)}{V_1} \left(\frac{Q_1}{V_2} \bar{c} \right) \quad (2.54)$$

$$\Rightarrow (c_2)_{ss} = \bar{c} \quad (2.55)$$

Hence, the steady-state gain $K = 1$ lb salt/lb salt as expected.

d. For the given conditions, that is, $Q_2 = Q = 5$, $Q_1 = 2Q = 10$, and $V_1 = V_2 = V = 15$

$$\omega_n = \left[\frac{2Q(2Q - Q)}{VV} \right]^{1/2} = (2)^{1/2} \frac{Q}{V} = (2)^{1/2} \frac{5}{15} = 0.4714 \text{ rad/min} \quad (2.56)$$

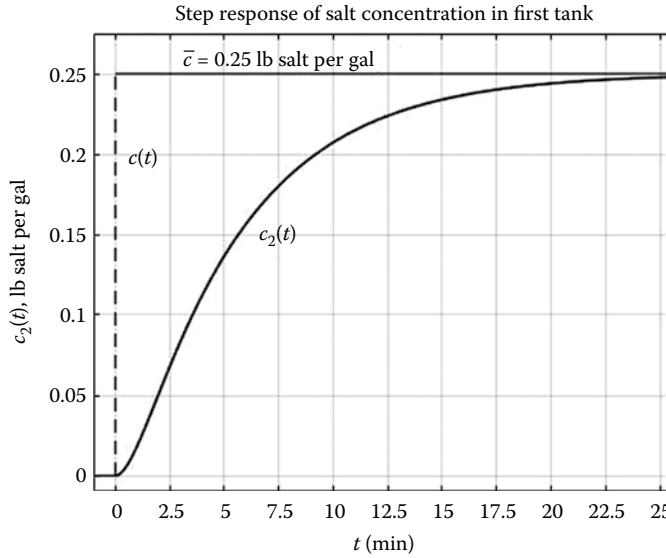


FIGURE 2.8 Response of salt concentration in second tank to step input $c(t) = 0.25, t \geq 0$.

$$\zeta = \frac{(V + V)}{2} \left[\frac{2Q}{(2Q - Q)V} \right]^{1/2} = (2)^{1/2} = 1.4142 \quad (2.57)$$

From Equation 2.18, the characteristic roots of the overdamped system are

$$s_1, s_2 = -\zeta\omega_n \pm \sqrt{\zeta^2 - 1}\omega_n = -1.1381 \text{ rad/min}, -0.1953 \text{ rad/min} \quad (2.58)$$

The time constants in Equation 2.19 are $\tau_1 = -1/s_1 = 0.8787 \text{ min}$ and $\tau_2 = -1/s_2 = 5.1213 \text{ min}$, and from the unit step response in Equation 2.20, the response to a step of magnitude \bar{c} is

$$c_2(t) = K\bar{c} \left[1 + \frac{\tau_2 e^{-t/\tau_2} - \tau_1 e^{-t/\tau_1}}{\tau_1 - \tau_2} \right] \quad (2.59)$$

$$= 0.25 \left[1 - \left(\frac{5.1213 e^{-t/5.1213} - 0.8787 e^{-t/0.8787}}{4.2426} \right) \right], \quad t \geq 0 \quad (2.60)$$

The second-order differential equation in Equation 2.51 is in standard form; however, the second-order differential equation for $c_1(t)$ contains the first derivative dc/dt on the right-hand side of the equation (see Exercise 2.6). The implication of input derivatives in the system model will be discussed in a later section. A graph of the step response is shown in Figure 2.8.

EXERCISES

- 2.5 Starting with Equations 2.37 and 2.38, obtain the second-order differential equation relating the output $y = y(t)$ and its derivatives to the inputs $f = f(t)$ and $g = g(t)$.
- 2.6 In Example 2.3,
 - a. Find the differential equation relating $c_1(t)$ and the input $c(t)$.

- b. Find the step response in $c_1(t)$ for the same initial conditions, system parameters, and input $c(t)$. Graph the step response for $c_1(t)$ and $c_2(t)$.
 - c. Show that the first derivative dc_1/dt is discontinuous at $t = 0$ while the first derivative dc_2/dt is continuous at $t = 0$.
- 2.7 The two-tank system in Exercise 1.2 is second order.
- a. Convert the model of the system from two first-order differential equations to one second-order differential equation with input $F_1(t)$ and output $H_2(t)$.
 - b. Find expressions for the damping ratio, natural frequency, and steady-state gain in terms of the physical parameters A_1 , A_2 , R_1 , and R_2 .
 - c. Use the results from part (b) to express the damping ratio in terms of the tank time constants $\tau_1 = A_1 R_1$ and $\tau_2 = A_2 R_2$.
 - d. Show that the system can never be underdamped.
For parts (e) and (f), assume the following values for the system parameters:
 $A_1 = 100 \text{ ft}^2$, $R_1 = 0.25 \text{ ft per ft}^3/\text{min}$, $A_2 = 50 \text{ ft}^2$, and $R_2 = 0.1 \text{ ft per ft}^3/\text{min}$
 - e. Find and graph the response $H_2(t)$ of the unforced system, that is, $F_1(t) = 0$, $t \geq 0$ starting from $H_1(0) = 40 \text{ ft}$ and $H_2(0) = 0 \text{ ft}$.
 - f. Find and graph the step response of $H_2(t)$ when $F_1(t) = 75 \text{ ft}^3/\text{min}$. Both tanks are initially empty. Does the first tank achieve steady state in roughly $5\tau_1$? Does the second tank achieve steady state in roughly $5\tau_2$? Explain.
- 2.8 A fundamental difference between the step response of first- and second-order linear systems in standard form is the initial rate of change, that is, the first derivative at $t = 0$.
- a. Show that the first-order system step response undergoes the maximum rate of change at $t = 0$.
 - b. Show that the initial derivative of the second-order system step response is zero regardless of whether the system is underdamped, critically damped, or overdamped.

2.4 SIMULATION DIAGRAMS

In many cases, dynamic systems are composed of individual components and subsystems. The relationship of a system's components to each other and the role they serve in the overall system design are oftentimes easier to comprehend when presented in visual form rather than by inspection of the mathematical models. Control systems for ground vehicles, aircraft, robotic devices, building environments, and so forth are typically presented in graphical form as block diagrams. The blocks are both static and dynamic depending on the component it represents. Modern continuous-time system simulation languages include extensive libraries of special purpose blocks to represent the dynamics of commonly occurring components.

It is useful to reduce the blocks in a block diagram of a continuous-time dynamic system to a level that exposes the pure integrators. The simulationist is then given the flexibility of approximating individual integrators using different numerical algorithms. This is especially useful in applications where simulation code is developed manually instead of relying on a general purpose simulation language. This point will be revisited in [Chapter 3](#) following a discussion of numerical integration.

A block diagram of a continuous-time dynamic system comprising algebraic blocks and integrators is referred to as a simulation diagram. We begin with the first-order system of Equation 2.61:

$$\frac{d}{dt} y(t) + a_0 y(t) = b_1 \frac{d}{dt} u(t) + b_0 u(t) \quad (2.61)$$

Equation 2.61 is a more general form than the first-order models introduced in Section 2.2 due to the presence of the first derivative term on the right-hand side.

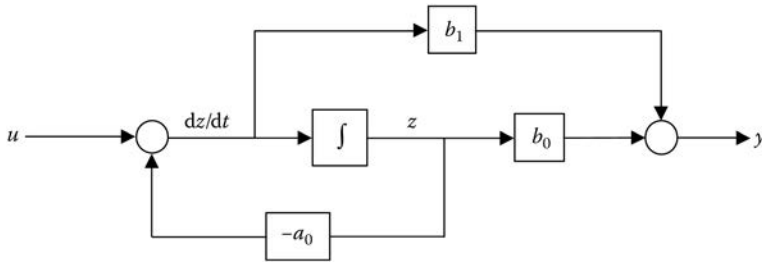


FIGURE 2.9 Simulation diagram of first-order system: $(d/dt)y(t) + a_0y(t) = b_1(d/dt)u(t) + b_0u(t)$.

If we introduce a new variable $z = z(t)$ where

$$\frac{d}{dt}z(t) + a_0z(t) = u(t) \quad (2.62)$$

the output y is related to z by

$$y(t) = b_0z(t) + b_1 \frac{d}{dt}z(t) \quad (2.63)$$

It is left as an exercise to show that Equations 2.62 and 2.63 are equivalent to Equation 2.61. In addition to the blocks required to implement Equations 2.62 and 2.63, an integrator block is needed to integrate the first derivative dz/dt to generate $z(t)$, that is,

$$z(t) = \int \frac{dz}{dt} dt \quad (2.64)$$

The simulation diagram in [Figure 2.9](#) is constructed by first drawing an integrator block and labeling the input dz/dt and output z corresponding to Equation 2.64. Next, we solve for the derivative term dz/dt in Equation 2.62 and draw a portion of the diagram to implement the result. Finally, the output y is generated from Equation 2.63 using the b_0 and b_1 gain blocks and a summing block.

The simulation diagram representation of the first-order system's dynamics involves a single dynamic block, namely, the integrator. The remaining blocks are sum blocks and gains that are algebraic in nature.

A block diagram for the same first-order system is shown in [Figure 2.10](#). The block diagram is a direct implementation of Equation 2.61 after solving for the first derivative dy/dt . An additional variable z is not required in this case. The diagram in [Figure 2.10](#) is not a simulation diagram because of the presence of the differentiator. In digital simulation, the differentiator (like the integrator) must be implemented using a numerical approximation. Numerical methods for approximating the

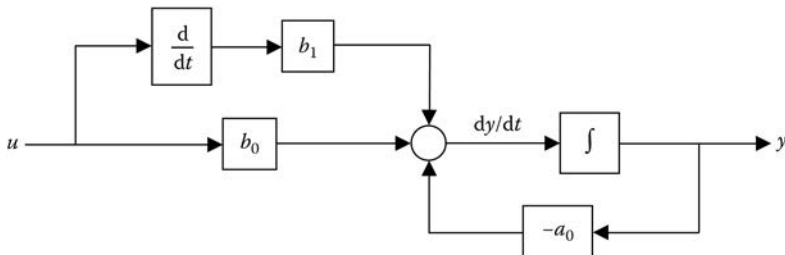


FIGURE 2.10 Block diagram of first-order system: $(d/dt)y(t) + a_0y(t) = b_1(d/dt)u(t) + b_0u(t)$.

derivative of a continuous-time function are available. However, they are rarely implemented in simulation applications due to their sensitivity to high-frequency noise components often present in continuous-time signals.

A final observation relates to the special case when b_1 in Equation 2.61 is zero. The input derivative is absent, and the first-order system assumes the simpler form of Equation 2.3 or 2.4. Recall that this form was sufficient to model the dynamics of the linear tank in [Chapter 1](#) and the simple RC circuit of Example 2.1.

EXAMPLE 2.4

Draw a simulation diagram of the linear tank modeled by

$$A \frac{d}{dt} H(t) + \frac{1}{R} H(t) = F_1(t) \quad (2.65)$$

The diagram is shown in [Figure 2.11](#).

Dividing Equation 2.65 by the parameter A and comparing the result to Equation 2.61 show

$$a_0 = \frac{1}{AR}, \quad b_0 = \frac{1}{A}, \quad b_1 = 0$$

leading to the simulation diagram shown in [Figure 2.11](#).

The simulation diagram in [Figure 2.11](#) is not unique. The “ $1/A$ ” block can be moved from the location where z is its input to the left of the summer where F_1 becomes its input. In that case, z and H are identical. The alternate simulation diagram can be obtained directly by solving the differential equation of the tank for the first derivative,

$$\frac{d}{dt} H(t) = \frac{1}{A} \left[F_1(t) - \frac{1}{R} H(t) \right] \quad (2.66)$$

and implementing Equation 2.66 directly. Integrating the derivative dH/dt to get H completes the diagram.

EXAMPLE 2.5

Suppose the current $i(t)$ in the RC circuit of [Figure 2.2](#) is considered the output. The differential equation for the circuit becomes

$$\frac{d}{dt} i(t) + \frac{1}{RC} i(t) = \frac{1}{R} \frac{d}{dt} e_0(t) \quad (2.67)$$

Draw the simulation diagram for the circuit described by Equation 2.67.

From Equation 2.61, a_0 , b_0 , and b_1 are

$$a_0 = \frac{1}{RC}, \quad b_0 = 0, \quad b_1 = \frac{1}{R}$$

and the simulation diagram is drawn in [Figure 2.12](#).

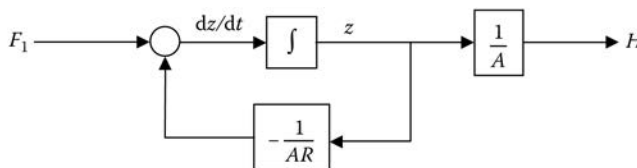


FIGURE 2.11 Simulation diagram of linear tank: $A(d/dt)H(t) + (1/R)H(t) = F_1(t)$.

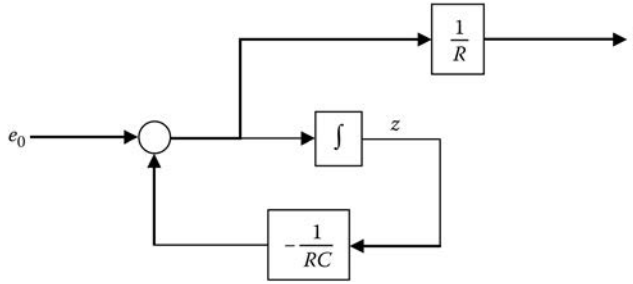


FIGURE 2.12 Simulation diagram for an RC circuit: $(d/dt)i(t) + (1/R)C i(t) = (1/R)(d/dt)e_0(t)$.

When the differential equation model of a first-order system contains a term involving the first derivative of the input, a direct link or coupling exists from the input directly to the output. In other words, when $b_1 \neq 0$ in Equation 2.61, sudden changes in the input are immediately reflected in the output. Notice the path of heavy solid lines in Figure 2.12 illustrating the direct connection of algebraic components between the voltage input $e_0(t)$ and the output current $i(t)$.

In contrast, there is no direct connection from input to output in the simulation diagram shown in Figure 2.11 for the linear tank model. This is expected since changes in the inflow $F_1(t)$ must work their way through the tank dynamics, that is, the integrator, prior to affecting the output level $H(t)$. Hence, the tank prevents abrupt changes like a step or other inputs with high-frequency components from immediately causing any significant changes in the output $H(t)$. The tank behaves like a low-pass filter (see Exercise 1.10).

Obtaining a simulation diagram for a second-order system in the standard form

$$\frac{d^2}{dt^2} y(t) + 2\zeta\omega_n \frac{d}{dt} y(t) + \omega_n^2 y(t) = K\omega_n^2 u(t) \quad (2.68)$$

is straightforward. We begin by drawing two consecutive integrators, labeling the input and output of the first with d^2y/dt^2 and dy/dt , respectively. The second integrator integrates the first derivative dy/dt producing y and is labeled accordingly. The next step is to solve for the second derivative term in Equation 2.68 resulting in

$$\frac{d^2}{dt^2} y(t) = K\omega_n^2 u(t) - 2\zeta\omega_n \frac{d}{dt} y(t) - \omega_n^2 y(t) \quad (2.69)$$

Algebraic blocks (gains and summers) are used to implement Equation 2.69 leading to the simulation diagram shown in Figure 2.13.

The simulation diagram for a second-order system with first- or second-order derivatives of the input appearing in the differential equation model is not as straightforward. Starting with Equation 2.70

$$\frac{d^2}{dt^2} y(t) + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_0 u(t) + b_1 \frac{d}{dt} u(t) + b_2 \frac{d^2}{dt^2} u(t) \quad (2.70)$$

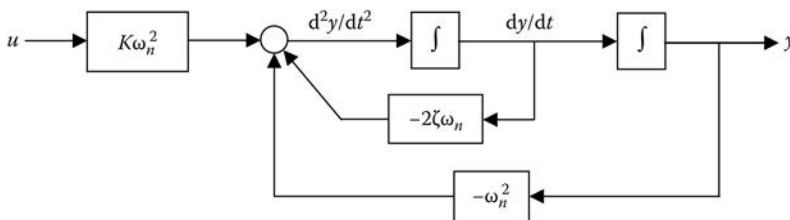


FIGURE 2.13 Simulation diagram of a second-order system in standard form.

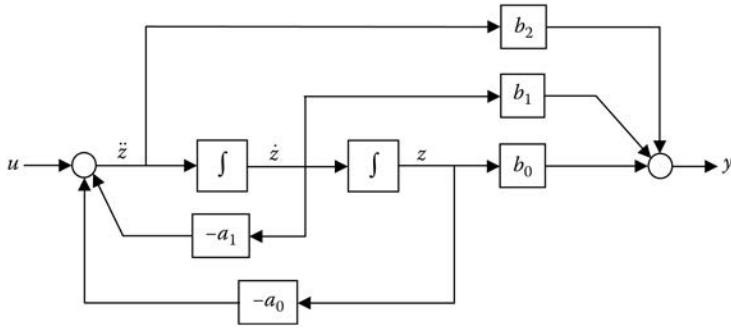


FIGURE 2.14 Simulation diagram for a second-order system with input derivatives present.

an approach similar to the method used for first-order systems with an input derivative term present is employed. An artificial variable $z(t)$ is introduced, and the output $y(t)$ is expressed as a linear combination of $z(t)$ and its two derivatives. The result is

$$\frac{d^2}{dt^2} z(t) + a_1 \frac{d}{dt} z(t) + a_0 z(t) = u(t) \quad (2.71)$$

$$y(t) = b_0 z(t) + b_1 \frac{d}{dt} z(t) + b_2 \frac{d^2}{dt^2} z(t) \quad (2.72)$$

The simulation diagram of the second-order system in Equation 2.70 is shown in [Figure 2.14](#). Note the use of the dot notation, short for differentiation with respect to time. It is clear that a direct connection from the input $u(t)$ to the output $y(t)$ exists only when b_2 , the coefficient of the input second derivative in Equation 2.70, is nonzero.

Looking at the simulation diagrams in [Figures 2.9](#) and [2.14](#) for the first- and second-order systems in Equations 2.61 and 2.70, a general pattern emerges for creating the simulation diagram of an n th-order system modeled by

$$\frac{dy^n}{dt^n} + a_{n-1} \frac{dy^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = b_0 u + b_1 \frac{du}{dt} + \cdots + b_{n-1} \frac{du^{n-1}}{dt^{n-1}} + b_n \frac{du^n}{dt^n} \quad (2.73)$$

The two equations equivalent to Equation 2.73 are

$$\frac{dz^n}{dt^n} + a_{n-1} \frac{dz^{n-1}}{dt^{n-1}} + \cdots + a_1 \frac{dz}{dt} + a_0 z = u \quad (2.74)$$

$$y = b_0 z + b_1 \frac{dz}{dt} + \cdots + b_{n-1} \frac{dz^{n-1}}{dt^{n-1}} + b_n \frac{dz^n}{dt^n} \quad (2.75)$$

The simulation diagram follows directly from Equations 2.74 and 2.75.

EXAMPLE 2.6

A unicycle is traveling over an uneven road as shown in [Figure 2.15](#).

The input is the road elevation $x_r(t)$ above some reference. The output is the vertical movement $x(t)$ of the rider and seat combination (with respect to its equilibrium position). Ignoring the compliance

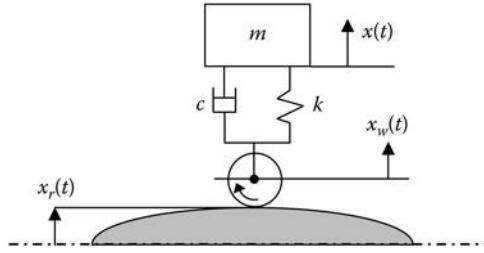


FIGURE 2.15 Unicycle traveling along an uneven road surface.

of the tire makes the wheel deflection $x_w(t) = x_r(t)$. Assume that the wheel remains in contact with the road surface. The mass of the rider and seat is m , and c and k are suspension parameters.

- Find the differential equation relating the output $x(t)$ and input $x_r(t)$.
 - Draw a simulation diagram of the system.
 - Is there a direct coupling between the input and output? Explain.
- The differential equation is obtained by equating the sum of the suspension forces acting on the rider and seat to the product of its mass and acceleration.

$$m \frac{d^2}{dt^2} x(t) = c \left[\frac{d}{dt} x_w(t) - \frac{d}{dt} x(t) \right] + k [x_w(t) - x(t)] \quad (2.76)$$

Replacing $x_w(t)$ with $x_r(t)$ gives

$$m \frac{d^2}{dt^2} x(t) = c \left[\frac{d}{dt} x_r(t) - \frac{d}{dt} x(t) \right] + k [x_r(t) - x(t)] \quad (2.77)$$

- Rearranging terms in Equation 2.77 gives

$$\frac{d^2}{dt^2} x(t) + \frac{c}{m} \frac{d}{dt} x(t) + \frac{k}{m} x(t) = \frac{k}{m} x_r(t) + \frac{c}{m} \frac{d}{dt} x_r(t) \quad (2.78)$$

Comparing Equations 2.78 and 2.70 leads to expressions for a_0 , a_1 , b_0 , b_1 , and b_2 in terms of the system parameters,

$$a_0 = \frac{k}{m}, \quad a_1 = \frac{c}{m}, \quad b_0 = \frac{k}{m}, \quad b_1 = \frac{c}{m}, \quad b_2 = 0 \quad (2.79)$$

and eventually the simulation diagram shown in [Figure 2.16](#).

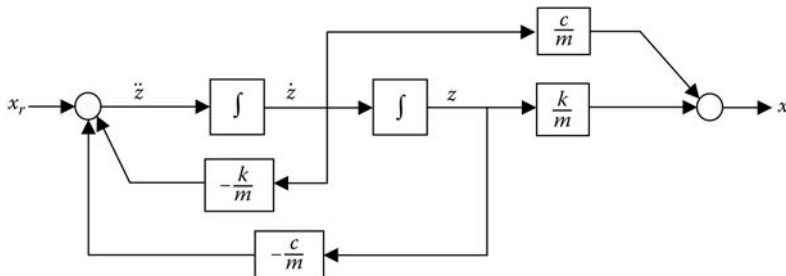


FIGURE 2.16 Simulation diagram for a unicycle suspension.

- c. Since both paths from x_r to x contain an integrator, there is no direct coupling between input and output. Consequently, an abrupt change in x_r , such as a vertical jump in the road surface height does not result in a similar type of displacement of the rider and seat combination.

2.4.1 SYSTEMS OF EQUATIONS

System models can assume the form of coupled differential and algebraic equations. The simulation diagram representation is straightforward.

EXAMPLE 2.7

A two-room building with temperatures $T_1(t)$ and $T_2(t)$ is shown in [Figure 2.17](#).

The simplified model relating the uniform room temperatures $T_1(t)$ and $T_2(t)$ to the heat supplied from the furnace $Q_f(t)$ and outside temperature $T_0(t)$ is based on conservation of energy. It consists of the following differential and algebraic equations:

$$C_1 \frac{d}{dt} T_1(t) = Q_f(t) - Q_1(t) - Q_{12}(t) \quad (2.80)$$

$$C_2 \frac{d}{dt} T_2(t) = Q_{12}(t) - Q_2(t) \quad (2.81)$$

$$Q_{12}(t) = \frac{T_1(t) - T_2(t)}{R_{12}} \quad (2.82)$$

$$Q_1(t) = \frac{T_1(t) - T_0(t)}{R_1} \quad (2.83)$$

$$Q_2(t) = \frac{T_2(t) - T_0(t)}{R_2} \quad (2.84)$$

where C_1 , C_2 , R_1 , R_2 , and R_{12} are thermal parameters of the system. The simulation diagram shown in [Figure 2.18](#) follows directly from Equations 2.80 through 2.84.

Combining Equations 2.80 through 2.84 and solving for the first derivatives give

$$\frac{d}{dt} T_1(t) = \frac{1}{C_1} \left[Q_f(t) - \frac{T_1(t) - T_0(t)}{R_1} - \frac{T_1(t) - T_2(t)}{R_{12}} \right] \quad (2.85)$$

$$= \frac{1}{C_1} \left[-\left(\frac{1}{R_1} + \frac{1}{R_{12}} \right) T_1(t) + \frac{1}{R_{12}} T_2(t) + \frac{1}{R_1} T_0(t) + Q_f(t) \right] \quad (2.86)$$

$$\frac{d}{dt} T_2(t) = \frac{1}{C_2} \left[\frac{T_1(t) - T_2(t)}{R_{12}} - \frac{T_2(t) - T_0(t)}{R_2} \right] \quad (2.87)$$

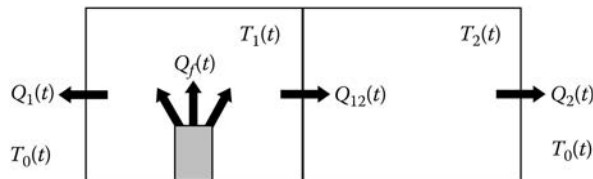


FIGURE 2.17 Heat flows and temperatures in a two-room building.

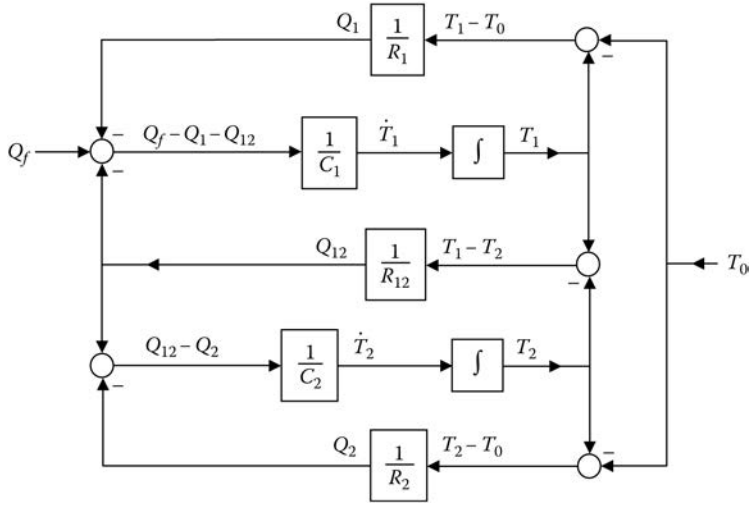


FIGURE 2.18 Simulation diagram for building room temperature model.

$$= \frac{1}{C_2} \left[\frac{1}{R_{12}} T_1(t) - \left(\frac{1}{R_2} + \frac{1}{R_{12}} \right) T_2(t) + \frac{1}{R_2} T_0(t) \right] \quad (2.88)$$

Equations 2.86 and 2.88 are of the form

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + b_{11}u_1 + b_{12}u_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + b_{21}u_1 + b_{22}u_2 \end{aligned} \quad (2.89)$$

where $x_1 = T_1$, $x_2 = T_2$, $u_1 = T_0$, and $u_2 = Q_f$ and the coefficients a_{ij} and b_{ij} ($j = 1, 2$) depend on the system parameters according to

$$a_{11} = -\frac{1}{C_1} \left(\frac{1}{R_1} + \frac{1}{R_{12}} \right), \quad a_{12} = \frac{1}{R_{12}C_1}, \quad b_{11} = \frac{1}{R_1C_1}, \quad b_{12} = \frac{1}{C_1} \quad (2.90)$$

$$a_{21} = -\frac{1}{R_{12}C_2}, \quad a_{22} = \frac{1}{C_2} \left(\frac{1}{R_2} + \frac{1}{R_{12}} \right), \quad b_{21} = \frac{1}{R_2C_2}, \quad b_{22} = 0 \quad (2.91)$$

Suppose we need to draw a simulation diagram for the system in Equation 2.89 with only x_1 or x_2 present. Using an approach similar to the one presented in Section 2.2 for converting two coupled first-order differential equations into a second-order differential equation, the second-order system in Equation 2.89 is equivalent to

$$\ddot{x}_1 + \alpha_1 \dot{x}_1 + \alpha_0 x_1 = \beta_{11} \dot{u}_1 + \beta_{10} u_1 + \beta_{21} \dot{u}_2 + \beta_{20} u_2 \quad (2.92)$$

where

$$\alpha_1 = -(a_{11} + a_{22}), \quad \alpha_0 = a_{11}a_{22} - a_{12}a_{21} \quad (2.93)$$

$$\beta_{11} = b_{11}, \quad \beta_{10} = a_{12}b_{21} - a_{22}b_{11}, \quad \beta_{21} = b_{12}, \quad \beta_{20} = a_{12}b_{22} - a_{22}b_{12} \quad (2.94)$$

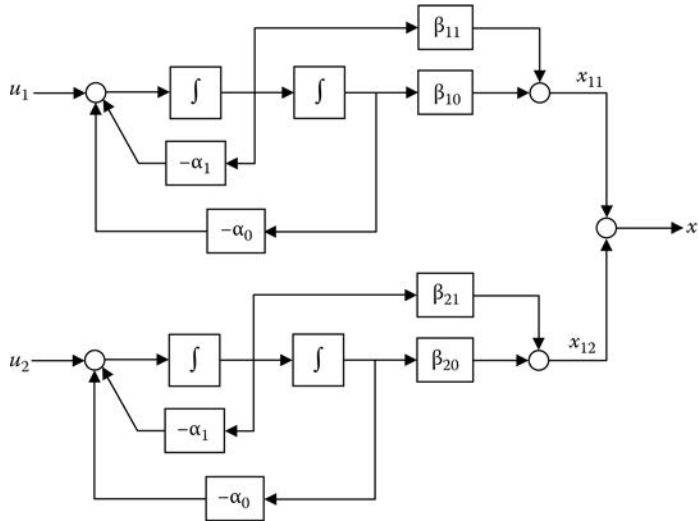


FIGURE 2.19 Simulation diagram for second-order system in Equation 2.92.

The simulation diagram for Equation 2.92 is constructed in two steps. From superposition, the output x_1 can be viewed as the sum of x_{11} and x_{12} where

$$\ddot{x}_{11} + \alpha_1 \dot{x}_{11} + \alpha_0 x_{11} = \beta_{11} \dot{u}_1 + \beta_{10} u_1 \quad (2.95)$$

$$\ddot{x}_{12} + \alpha_1 \dot{x}_{12} + \alpha_0 x_{12} = \beta_{21} \dot{u}_2 + \beta_{20} u_2 \quad (2.96)$$

Simulation diagrams for Equations 2.95 and 2.96 are drawn separately, and outputs x_{11} and x_{12} are added to yield the complete output x_1 . The result is shown in Figure 2.19.

Do not be misled into thinking that the simulation diagram shown in Figure 2.19 corresponds to a fourth-order system due to the presence of four integrators. There are two decoupled second-order systems, one with input u_1 and output x_{11} and the other with input u_2 and output x_{12} . In reality, they are the same system, that is, the second-order system governed by the second-order model in Equation 2.92.

On the other hand, if the feedback coefficients in the two systems are not identical, that is, α_0 and α_1 in both cases, the result is indeed a fourth-order system (see Exercise 2.13).

EXERCISES

2.9 Show that the system of equations

$$\frac{d}{dt} z(t) + a_0 z(t) = u(t) \quad \text{and} \quad y(t) = b_0 z(t) + b_1 \frac{d}{dt} z(t)$$

used to construct the simulation diagram for the first-order system

$$\frac{d}{dt} y(t) + a_0 y(t) = b_1 \frac{d}{dt} u(t) + b_0 u(t)$$

is equivalent to the first-order differential equation above.

Hint: The variable $z(t)$ must be eliminated from the two equations.

2.10 An alternate simulation diagram for the second-order system

$$\frac{d}{dt^2} y(t) + 2\zeta\omega_n \frac{d}{dt} y(t) + \omega_n^2 y(t) = K\omega_n^2 u(t)$$

when it is critically damped or overdamped is shown in [Figure E2.10](#) below:

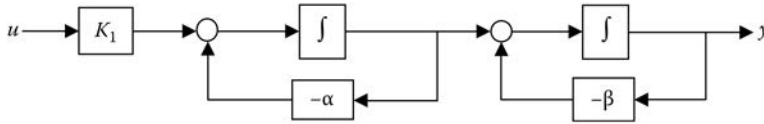


FIGURE E2.10

Find expressions for K_1 , α , and β in terms of the parameters ζ , ω_n , and K .

2.11 The circuit shown in [Figure E2.11](#) is governed by the differential equation:

$$RC \frac{d}{dt^2} v_C + \frac{d}{dt} v_C + \frac{R}{L} v_C = \frac{d}{dt} e_s$$

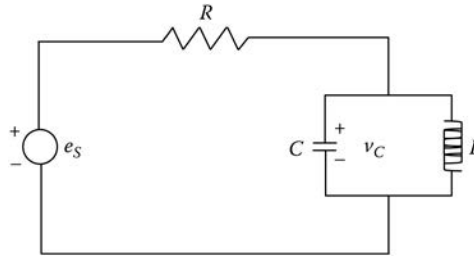


FIGURE E2.11

Draw a simulation diagram for the circuit.

2.12 Consider the building temperature example with room temperatures described by Equations 2.86 and 2.88.

- Find the second-order differential equation relating $T_2(t)$ and the system inputs $T_0(t)$ and $Q_f(t)$.
- Draw a simulation diagram like the one shown in [Figure 2.19](#).

2.13 Simulation diagrams are shown in [Figure E2.13a](#) through [E2.13c](#) below.

- Find the differential equation relating x and inputs u_1 and u_2 in [Figure E2.13a](#).
- Find the differential equation relating x and input u in [Figure E2.13b](#).
- Find the differential equation relating x and inputs u_1 and u_2 in [Figure E2.13c](#).
- Comment on the differences between the systems represented by each diagram.

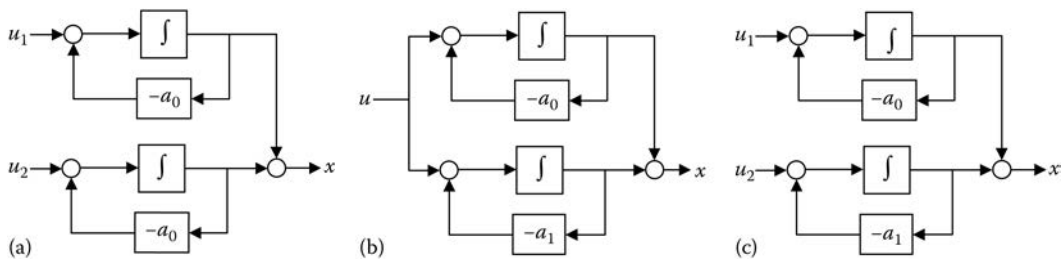


FIGURE E2.13

2.5 HIGHER-ORDER SYSTEMS

To this point, we have looked at linear continuous-time systems with first- and second-order dynamics only. Linear systems and linear controls texts include extensive coverage of lower-order

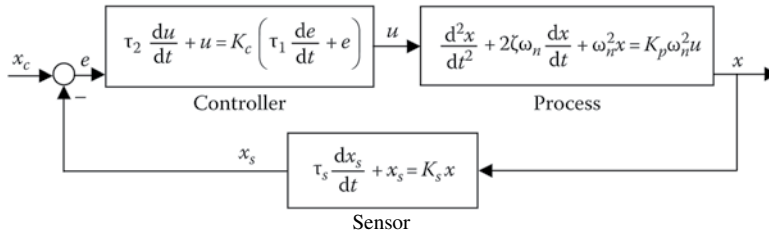


FIGURE 2.20 A control system consisting of first- and second-order components.

system response. In particular, the response of first- and second-order systems to impulse, step, and sinusoidal inputs is fully developed.

The dynamics of complex systems with linear differential equation models are invariably higher than second order. One may question why so much attention is devoted to first- and second-order systems. The explanation is simple.

High-order linear systems are oftentimes a collection of components or subsystems that are intrinsically first or second order. An electrical circuit with several capacitors and inductors is a good example. The circuit dynamics will depend on the number of these energy storage elements and their location in the circuit. In general, its order will be equal to the number of energy storage elements since each element is itself modeled as a first-order component. With n nonredundant energy storage elements, an n th-order differential equation involving an output (a voltage or current in the circuit) and an input (if an independent source is present) governs the behavior of the circuit. The same principle applies to fluid, thermal, mechanical, chemical, and so forth, systems made up of components analogous to the resistor, capacitor, and inductor of the electrical circuit.

The block diagram of a simple feedback control system is shown in [Figure 2.20](#). The controller, process, and sensor are the subsystem components, which are individually modeled as either first or second order.

The control system model comprises the three coupled differential equations

$$\tau_2 \frac{du}{dt} + u = K_c \left(\tau_1 \frac{de}{dt} + e \right) \quad (2.97)$$

$$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = K_p \omega_n^2 u \quad (2.98)$$

$$\tau_s \frac{dx_s}{dt} + x_s = K_s x \quad (2.99)$$

and the summer equation

$$e = x_c - x_s \quad (2.100)$$

The command input $x_c = x_c(t)$ is the control system input, and the output of the process $x = x(t)$ is the control system output. Dependent variables $e(t)$, the error signal, $u(t)$, the output from the controller and input to the process, and $x_s(t)$, the sensor output are internal to the control system. Eliminating these variables produces a single fourth-order $(1 + 2 + 1)$ differential equation model of the control system in the form

$$\frac{d^4x}{dt^4} + a_3 \frac{d^3x}{dt^3} + a_2 \frac{d^2x}{dt^2} + a_1 \frac{dx}{dt} + a_0 x = b_4 \frac{d^4x_c}{dt^4} + b_3 \frac{d^3x_c}{dt^3} + b_2 \frac{d^2x_c}{dt^2} + b_1 \frac{dx_c}{dt} + b_0 x_c \quad (2.101)$$

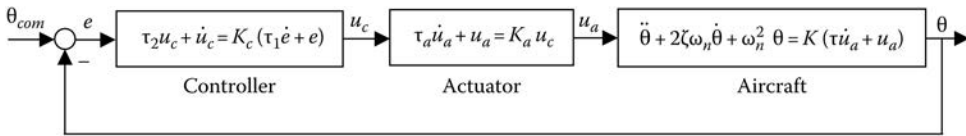


FIGURE 2.21 Control system for an aircraft pitch.

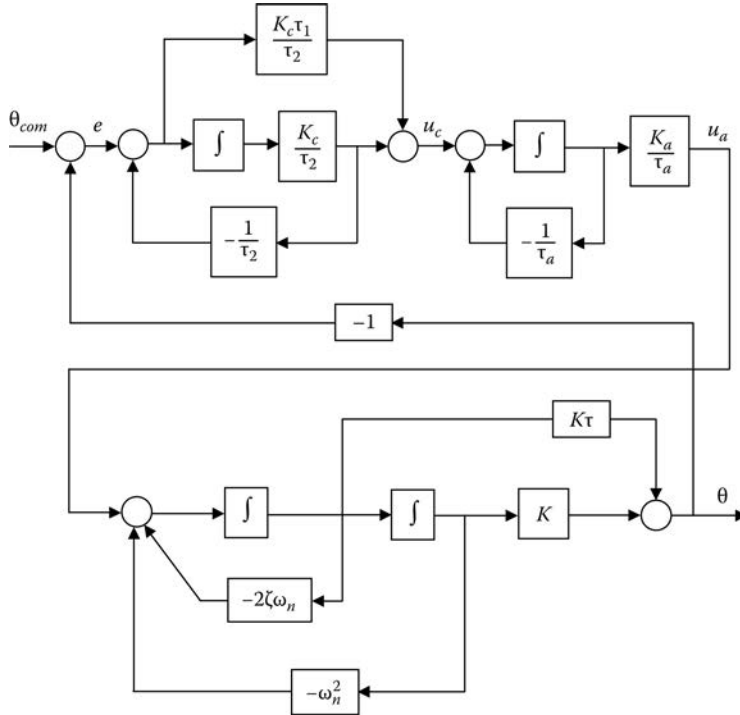


FIGURE 2.22 Simulation diagram for an aircraft pitch control system.

where several of the coefficients a_i , $i = 0, 1, 2, 3$ and b_i , $i = 0, 1, 2, 3, 4$ may be zero.

A simulation diagram of the control system can be obtained from Equation 2.101 using the procedure from the previous section. Alternatively, simulation diagrams can be developed for the individual components in Figure 2.20 and properly connected to produce a simulation diagram for the control system. Simulation of the system based on a simulation diagram using the second approach is preferable since the internal variables are readily identifiable. We can check the simulation results to verify that inputs and outputs of the controller and sensor remain within proper operating ranges.

EXAMPLE 2.8

The control system for the pitch of an aircraft is shown in Figure 2.21.

Draw a simulation diagram for the aircraft pitch control system block diagram.

Simulation diagrams of each component are connected to produce the simulation diagram of the entire control system shown in Figure 2.22.

EXERCISES

2.14 For the control system shown in Figure 2.20.

- Find the coefficients a_i , $i = 0, 1, 2, 3$ and b_i , $i = 0, 1, 2, 3, 4$ in Equation 2.101 in terms of the system parameters τ_1 , τ_2 , K_c , ω_n , K_p , τ_s and K_s .

Hint: The use of Laplace transforms (see Chapter 4) significantly reduces the amount of work necessary to eliminate the variables e , u , and x_s .

b. Draw a simulation diagram based on the fourth-order differential equation model.

- 2.15 Find the differential equation for the control system in Figure 2.21 relating the output θ and its derivatives to the input θ_{com} and its derivatives. Draw the simulation diagram based on the resulting differential equation.

Hint: The use of Laplace transforms (see Chapter 4) significantly reduces the amount of work necessary to eliminate the variables e , u_c , and u_d .

- 2.16 For the railroad cars shown in Figure E2.16,

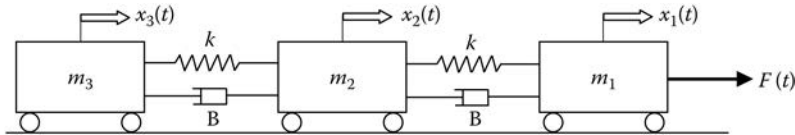


FIGURE E2.16

- Write the differential equation expressing $\sum_k F_{i,k} = m_i \ddot{x}_i$, $i = 1, 2, 3$ for each car where $F_{i,k}$ is the k th force acting on the i th car.
- Draw a simulation diagram of the system with integrators for x_i , \dot{x}_i , $i = 1, 2, 3$.
- Find the differential equation relating the input $F(t)$ and output $x_1(t)$.

Hint: The use of Laplace transforms (see Chapter 4) significantly reduces the amount of work necessary to eliminate the variables x_2 and x_3 .

2.6 STATE VARIABLES

In everyday terms, one's state of mind on a given day is determined by the history of numerous psychological factors that influence our mental well-being. The state of the national economy (weak, moderate, strong) depends on numerous factors such as energy prices, inflation, trade balances, employment, productivity, housing, tax policies, corporate earnings, transportation, agriculture, and so forth. Imagine that all the economic factors (inputs) affecting the national economy were measurable and the complex interrelationships among those variables that determine the state of the economy were fully understood. If the state of the economy were known at some point in time and the complete set of aforementioned economic factors were observed from that time forward, knowledgeable economists would (in principle) be able to predict the state of the national economy at future times.

The essential point is that if we know the state of a system at some point in time and wish to predict its future, then knowledge of the system inputs only from that time onward is required. The current state of a system reflects the effect of prior inputs that are responsible for the system's transition from some previous state to the current state.

Consider a simple spring-mass-damper system subject to an applied force acting on the mass like the one shown in Figure 2.23. The spring and mass are both capable of storing energy. At any time,

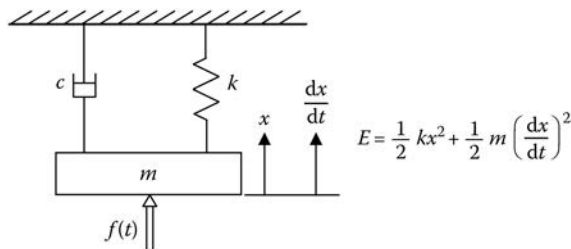


FIGURE 2.23 A spring-mass-damper system with applied force $f(t)$.

the instantaneous energy $E(t)$ stored in the system is given in Equation 2.102 where x is the position of the mass (relative to its equilibrium position) and dx/dt is the velocity of the mass.

$$E = \frac{1}{2} kx^2 + \frac{1}{2} m \left(\frac{dx}{dt} \right)^2 \quad (2.102)$$

A possible choice of state variables for the mechanical system is x and dx/dt . Given both state variables at time t_0 determines the energy $E(t_0)$. The applied force $f(t)$ for $t \geq t_0$ must be known to solve the initial value problem

$$m \frac{d^2}{dt^2} x(t) + c \frac{d}{dt} x(t) + kx(t) = f(t) \quad \text{given } x(t_0) \quad \text{and} \quad \frac{dx}{dt}(t_0) \quad (2.103)$$

and determine both state variables x and dx/dt as well as $E(t)$ for $t \geq t_0$. The same cannot be said if only the position or the velocity of the mass were known at t_0 . In that case, the initial energy in the system $E(t_0)$ would be unknown, and it would be impossible to predict future values of x and dx/dt even if the force $f(t)$ were known for $t \geq t_0$. Consequently, x or dx/dt alone is not a suitable choice for the state of the system.

The situation is illustrated for the general case of a system with two state variables $x_1(t)$ and $x_2(t)$ and single input $u(t)$ in Figure 2.24. Given $x_1(t_0)$, $x_2(t_0)$, and $u(t)$, $t \geq t_0$, both states can be determined from t_0 on.

The choice of state variables for a dynamic system model is not unique; however, the number of state variables is limited to the minimum number of variables, which satisfy the requirement of predicting future states given the current state and future inputs. This number of state variables is equal to the number of independent energy storage components present in the system. It is advantageous to choose physical (measurable) quantities as in the case of the mechanical system in Figure 2.23 whenever possible.

A simulation diagram is a valuable tool when it comes to choosing the state variables of a system. The outputs of each integrator in a simulation diagram representation of a system is a valid choice for the state variables. The choice of which integrator output is x_1 , x_2 , and so forth is arbitrary.

Consider a second-order system governed by

$$\frac{d^2}{dt^2} y(t) + a_1 \frac{d}{dt} y(t) + a_0 y(t) = b_0 u(t) \quad (2.104)$$

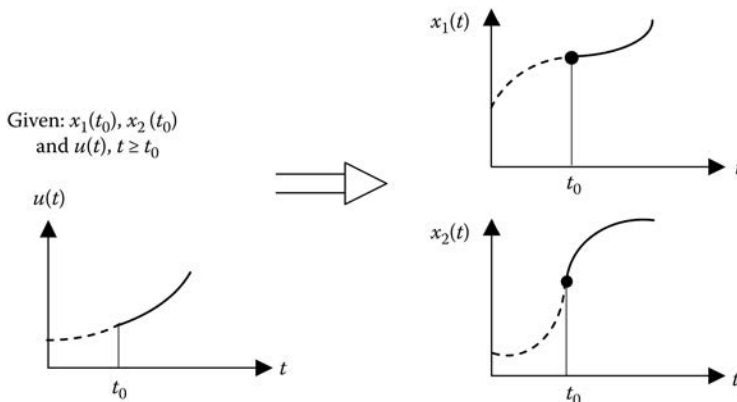


FIGURE 2.24 Dynamic system with state variables $x_1(t)$ and $x_2(t)$.

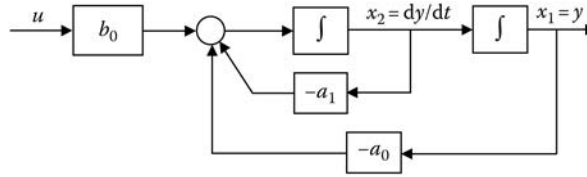


FIGURE 2.25 Simulation diagram of second-order system with state $x_1 = y$ and $x_2 = dy/dt$.

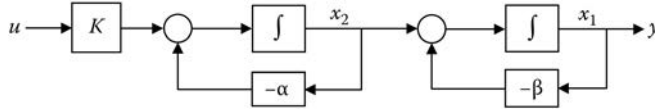


FIGURE 2.26 Simulation diagram for critically damped or overdamped second-order system using two first-order systems in a series.

A simulation diagram like the one shown in Figure 2.25 is easily constructed. State variables x_1 and x_2 are chosen as the output y and first derivative dy/dt , respectively.

The second-order system is critically damped or overdamped if $a_1^2 - 4a_0 \geq 0$. In this case, it is equivalent to two cascaded first-order systems as shown in Figure 2.26.

The parameters K , α , and β are related to a_0 , a_1 , and b_0 according to

$$K = b_0, \alpha = \frac{a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}, \beta = \frac{a_1 \mp \sqrt{a_1^2 - 4a_0}}{2} \quad (2.105)$$

State variable x_1 is again the system output y ; however, the second state variable x_2 is no longer the output derivative dy/dt .

For an n th-order linear system model with constant coefficients, the state derivatives are expressible as a linear combination of the state variables and input(s). For example, from Figure 2.25, the state derivatives are equal to

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= b_0 u - a_0 x_1 - a_1 x_2 \end{aligned} \quad (2.106)$$

whereas in Figure 2.26, the appropriate expressions are

$$\begin{aligned} \frac{dx_1}{dt} &= x_2 - \beta x_1 \\ \frac{dx_2}{dt} &= Ku - \alpha x_2 \end{aligned} \quad (2.107)$$

In the general linear case with n states x_1, x_2, \dots, x_n and r inputs,

$$\frac{dx_1}{dt} = f_1(\underline{x}, \underline{u}) = a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1r}u_r \quad (2.108)$$

$$\frac{dx_2}{dt} = f_2(\underline{x}, \underline{u}) = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2r}u_r \quad (2.109)$$

$$\frac{dx_n}{dt} = f_n(\underline{x}, \underline{u}) = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \cdots + b_{nr}u_r \quad (2.110)$$

where

$$\underline{x} \text{ is the } n \times 1 \text{ state vector } \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$\underline{u} \text{ is the } r \times 1 \text{ input vector } \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix}$$

and $f_i(\underline{x}, \underline{u})$, $i = 1, 2, 3, \dots, n$ is the state derivative function of the i th state variable.

Equations 2.108 through 2.110 can be written in the compact form

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}) = \underline{A}\underline{x} + \underline{B}\underline{u} \quad (2.111)$$

where

$$\dot{\underline{x}} = \begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix}, \quad \underline{A} = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdot & \cdot & \cdot & a_{nn} \end{bmatrix}, \quad \underline{B} = \begin{bmatrix} b_{11} & b_{12} & \cdot & \cdot & \cdot & b_{1r} \\ b_{21} & b_{22} & \cdot & \cdot & \cdot & b_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ b_{n1} & b_{n2} & \cdot & \cdot & \cdot & b_{nr} \end{bmatrix}$$

The $n \times n$ matrix \underline{A} is called the system matrix, and the $n \times r$ matrix \underline{B} is the input matrix.

Multivariable, LTI systems involve multiple inputs u_1, u_2, \dots, u_r and outputs y_1, y_2, \dots, y_p . The outputs are linearly related to the states and the inputs according to

$$\underline{y} = \underline{C}\underline{x} + \underline{D}\underline{u} \quad (2.112)$$

where

$$\underline{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}, \quad \underline{C} = \begin{bmatrix} c_{11} & c_{12} & \cdot & \cdot & \cdot & c_{1n} \\ c_{21} & c_{22} & \cdot & \cdot & \cdot & c_{2n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{p1} & c_{p2} & \cdot & \cdot & \cdot & c_{pn} \end{bmatrix}, \quad \underline{D} = \begin{bmatrix} d_{11} & d_{12} & \cdot & \cdot & \cdot & d_{1r} \\ d_{21} & d_{22} & \cdot & \cdot & \cdot & d_{2r} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ d_{p1} & d_{p2} & \cdot & \cdot & \cdot & d_{pr} \end{bmatrix}$$

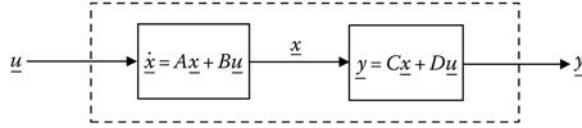


FIGURE 2.27 Dynamic system with input \underline{u} , output \underline{y} , and state \underline{x} .

The $p \times n$ constant matrix C is called the output matrix, and the $p \times r$ matrix D is the direct transmission matrix.

Equations 2.111 and 2.112 taken together are the state equations of the system. Note that the states x_1, x_2, \dots, x_n , are internal to the system as shown in Figure 2.27. Multivariable systems are easier to analyze in terms of state variables compared to the input–output model description of the system, that is, $dy/dt = f_i(y, u)$, $i = 1, 2, \dots, n$.

EXAMPLE 2.9

Interacting tanks with inflows into both tanks are shown in Figure 2.28. Choose the states to be the levels $H_1 = H_1(t)$ and $H_2 = H_2(t)$ and the single output as the volume of liquid in both tanks. Write the state equations for the system.

The continuous-time model of the linear tanks consists of the following equations:

$$A_1 \frac{dH_1}{dt} + F_{0,1} = F_1 \quad (2.113)$$

$$F_{0,1} = \frac{1}{R_{12}}(H_1 - H_2) \quad (2.114)$$

$$A_2 \frac{dH_2}{dt} + F_{0,2} = F_{0,1} + F_2 \quad (2.115)$$

$$F_{0,2} = \frac{1}{R_2} H_2 \quad (2.116)$$

Eliminating $F_{0,1}$ and $F_{0,2}$ from Equations 2.113 and 2.115 yields

$$A_1 \frac{dH_1}{dt} + \frac{1}{R_{12}}(H_1 - H_2) = F_1 \quad (2.117)$$

$$A_2 \frac{dH_2}{dt} + \frac{1}{R_2} H_2 = \frac{1}{R_{12}}(H_1 - H_2) + F_2 \quad (2.118)$$

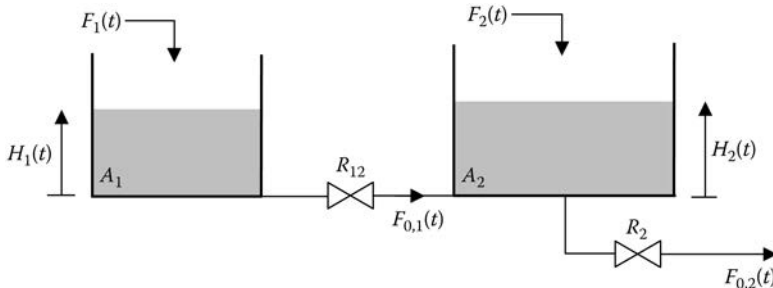


FIGURE 2.28 A system of interacting tanks.

Solving for the state derivatives in Equations 2.117 and 2.118

$$\frac{dH_1}{dt} = -\frac{1}{A_1 R_{12}} H_1 + \frac{1}{A_1 R_{12}} H_2 + \frac{1}{A_1} F_1 \quad (2.119)$$

$$\frac{dH_2}{dt} = \frac{1}{A_2 R_{12}} H_1 - \left[\frac{1}{A_2 R_2} + \frac{1}{A_2 R_{12}} \right] H_2 + \frac{1}{A_2} F_2 \quad (2.120)$$

Writing Equations 2.119 and 2.120 in matrix form gives the first part of the state equations,

$$\begin{bmatrix} \frac{dH_1}{dt} \\ \frac{dH_2}{dt} \end{bmatrix} = \begin{bmatrix} -\frac{1}{A_1 R_{12}} & \frac{1}{A_1 R_{12}} \\ \frac{1}{A_2 R_{12}} & -\frac{1}{A_2 R_2} - \frac{1}{A_2 R_{12}} \end{bmatrix} \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{A_1} & 0 \\ 0 & \frac{1}{A_2} \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \end{bmatrix} \quad (2.121)$$

The single output V_T , which represents the volume of liquid in both tanks, is

$$V_T = A_1 H_1 + A_2 H_2 = [A_1 \ A_2] \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \quad (2.122)$$

The transmission matrix D is a 1×2 matrix of zeros due to the absence of a direct coupling from either input F_1 or F_2 to the output V_T .

2.6.1 CONVERSION FROM LINEAR STATE VARIABLE FORM TO SINGLE INPUT–SINGLE OUTPUT FORM

In Section 2.3, an example was presented illustrating the conversion of a second-order state variable model into a second-order differential equation by eliminating one of the state variables (see Equations 2.37, 2.38, and 2.42). The procedure involved manipulation and substitution of terms in the time domain, an approach that quickly becomes unwieldy as the number of state variables increases. Simpler methods are described in [Chapter 4](#).

For a linear, third-order system with a single input, the starting point is the state variable model consisting of three coupled first-order differential equations expressing the state derivatives as a linear function of the states and input

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_1u \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_2u \\ \dot{x}_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + b_3u \end{aligned} \quad (2.123)$$

where the output y is x_1 , x_2 , or x_3 .

A third order, input–output differential equation model equivalent to Equation 2.123 is

$$\ddot{y} + \alpha_2 \dot{y} + \alpha_1 y = \beta_2 \ddot{u} + \beta_1 \dot{u} + \beta_0 u \quad (2.124)$$

Expressions for the system coefficients α_2 , α_1 , and α_0 and input coefficients β_2 , β_1 , and β_0 are summarized in Equations 2.125 through 2.127 and [Table 2.1](#).

$$\alpha_2 = -(a_{11} + a_{22} + a_{33}) \quad (2.125)$$

TABLE 2.1**Input Coefficients on Right-Hand Side of Equation 2.125 for $y = x_1, x_2, x_3$**

y	β_2	β_1	β_0
x_1	b_1	$-(a_{22} + a_{33})b_1 + (a_{12}b_2 + a_{13}b_3)$	$(a_{22}a_{33} - a_{23}a_{32})b_1 + (a_{13}a_{32} - a_{12}a_{33})b_2 + (a_{12}a_{23} - a_{13}a_{22})b_3$
x_2	b_2	$a_{21}b_1 - (a_{11} + a_{33})b_2 + a_{23}b_3$	$(a_{23}a_{31} - a_{21}a_{33})b_1 + (a_{11}a_{33} - a_{13}a_{31})b_2 + (a_{13}a_{21} - a_{11}a_{23})b_3$
x_3	b_3	$a_{31}b_1 + a_{32}b_2 - (a_{11} + a_{22})b_3$	$(a_{21}a_{32} - a_{22}a_{31})b_1 + (a_{12}a_{31} - a_{11}a_{32})b_2 + (a_{11}a_{22} - a_{12}a_{21})b_3$

$$\alpha_1 = a_{11}(a_{22} + a_{33}) - a_{12}a_{21} - a_{13}a_{31} + a_{22}a_{33} - a_{23}a_{32} \quad (2.126)$$

$$\alpha_0 = a_{11}(a_{23}a_{32} - a_{22}a_{33}) + a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{22}a_{31} - a_{21}a_{32}) \quad (2.127)$$

2.6.2 GENERAL SOLUTION OF THE STATE EQUATIONS

A solution to the state equation, Equation 2.111 can be found in any one of the texts on linear control theory listed in References. The solution is expressed in terms of an $n \times n$ matrix $\Phi(t)$, called the transition matrix of the system.

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t - \tau)Bu(\tau) - d\tau \quad (2.128)$$

The transition matrix depends solely on the system matrix A . One method for finding $\Phi(t)$ uses a definition based on an infinite series,

$$\Phi(t) = I + (tA) + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \cdots \quad (2.129)$$

As an illustration of how the transition matrix is used to solve the linear state equations, suppose the system matrix for an autonomous system ($u = 0$) is

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Using the infinite series expansion in Equation 2.129 or some other method (see [Chapter 4](#)) for finding $\Phi(t)$ the result is

$$\Phi(t) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \frac{1}{2} \quad (2.130)$$

and from Equation 2.128, the state $x(t)$, $t \geq 0$ is

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \quad (2.131)$$

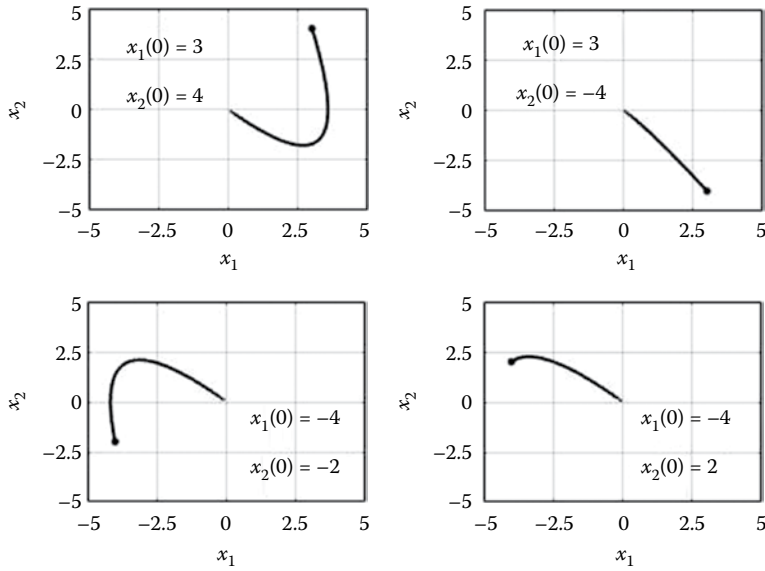


FIGURE 2.29 State trajectory in Equation 2.131 for different initial states.

The state trajectory or state portrait is a plot showing the path of the state vector in state space. In the general case, there is a separate coordinate axis for each of the state variables. The time variable “ t ” does not appear explicitly; however, each point along the state trajectory corresponds to a specific point in time. Figure 2.29 shows four different state trajectories starting from different initial states. Note that the four state trajectories all terminate at the origin, the equilibrium point of the system.

EXERCISES

2.17 For the system of interacting tanks in Example 2.9.

- Draw the simulation diagram of the system.
- Choose a new set of state variables as

$$z_1 = H_1 + H_2, z_2 = H_1 - H_2$$

and find the new system and input matrices A and B where

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = A \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

Hint: Find H_1 and H_2 in terms of z_1 and z_2 .

- Find the new output matrix C where

$$V_T = C \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

- 2.18 Write the state equations for the system of three railroad cars in Exercise 2.16. Choose the outputs to be the positions of each car.
- 2.19 An ecosystem consists of three species whose populations are denoted by F , S , and G . The growth rates of each species are given by

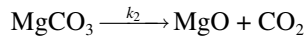
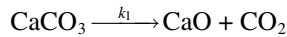
$$\text{Growth rate of } F = \frac{1}{F} \frac{dF}{dt} = a - cS - u_F$$

$$\text{Growth rate of } S = \frac{1}{S} \frac{dS}{dt} = -k + \lambda F - m_G - u_S$$

$$\text{Growth rate of } G = \frac{1}{G} \frac{dG}{dt} = -e + \sigma S + u_G$$

Write the system in state variable form $\dot{x} = f(x, u)$ $y = g(x, u)$ with the state $x = [F \ S \ G]^T$, input $u = [u_F \ u_S \ u_G]^T$, and output chosen as $y = F + S + G$.

- 2.20 Limestone is reduced to calcium oxide (CaO), magnesium oxide (MgO), and carbon dioxide (CO₂) by heating it in a reaction vessel maintained at a constant high temperature (McClamroch 1980). The limestone is made up of a fixed fraction β of calcium carbonate (CaCO₃), and the rest is magnesium carbonate (MgCO₃). The process is described by the first-order irreversible chemical reactions



where k_1 and k_2 are the rate constants for the two reactions.

Limestone is added to the reaction vessel at a rate of u mol/h. The mass (in moles) of CaCO₃, MgCO₃, CaO, and MgO in the vessel are denoted by x_1 , x_2 , x_3 , and x_4 , respectively (see Figure E2.20).

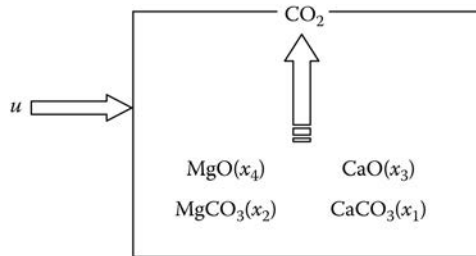


FIGURE E2.20

Since each mole of reactant that decomposes yields one mole of product (plus one mole of carbon dioxide), the state equations are

$$\dot{x}_1 = -k_1 x_1 + \beta u$$

$$\dot{x}_2 = -k_2 x_2 + (1 - \beta)u$$

$$\dot{x}_3 = k_1 x_1$$

$$\dot{x}_4 = k_2 x_2$$

- Draw a simulation diagram of the system. What is the order of the system?
- Find the matrices A , B , C , and D in the state equation model if the outputs are $y_1 = x_3$ and $y_2 = x_4$.
- Find the differential equation relating y_1 and u . Comment on the result.