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Nonlinear Reaction-Diffusion-Convection Equations

Lie and Conditional Symmetry, Exact Solutions, and Their Applications

Roman Cherniha
Mykola Serov
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Preface

Second-order partial differential equations (PDEs) have played a crucial role in mathematical modeling a wide range of processes in natural and life sciences since the 18th century. Typically linear PDEs are used in order to describe various processes, while *nonlinear partial differential equations* have been widely involved for such purposes only since the beginning of the 20th century. Nowadays it is generally accepted that a huge number of real world processes arising in physics, biology, chemistry, material sciences, engineering, ecology, economics etc. can be adequately described only by nonlinear PDEs.

At the present time, there is no existing general theory for integration of nonlinear PDEs, hence construction of particular exact solutions for these equations remains an important mathematical problem. Finding exact solutions that have a clear interpretation for the given process is of fundamental importance. In contrast to linear PDEs, the well-known principle of linear superposition cannot be applied to generate new exact solutions for nonlinear PDEs. Thus, the classical methods for solving linear PDEs are not applicable to nonlinear PDEs. Of course, a change of variables can sometimes be found that transforms a nonlinear PDE into a linear equation (the classical example is the Cole–Hopf substitution for the Burgers equation). It was stated by W.F. Ames in 1965 that “transformations are perhaps the most powerful general analytic tool currently available in this area”. However, finding exact solutions of a large majority of nonlinear PDEs requires new methods. Nowadays, 50 years later, the most powerful methods for construction of exact solutions to nonlinear PDEs are the symmetry-based methods, in particular the Lie method and the method of nonclassical (i.e., non-Lie) symmetries.

The Lie method (the terminology “the Lie symmetry analysis” and “the group analysis” are also used) is based on finding Lie’s symmetries of a given PDE and using the symmetries obtained for the construction of exact solutions. The method was created by the prominent Norwegian mathematician Sophus Lie in the 1880s. It should be pointed out that Lie’s works on application Lie groups for solving PDEs were almost forgotten during the first half of the 20th century. In the end of the 1950s, L.V. Ovsiannikov inspired by Birkhoff’s works devoted to application of Lie groups in hydrodynamics, rewrote Lie’s theory using modern mathematical language and published a monograph in 1962, which was the first book (after Lie’s works) devoted fully to this subject. The Lie method was essentially developed by L.V. Ovsiannikov, W.F. Ames, G. Bluman, W.I. Fushchych, N. Ibragimov, P. Olver, and other researchers in the 1960s–1980s. Several excellent textbooks devoted to the Lie method were published during the last 30 years, therefore one may claim that it is the well-established theory at the present time. Notwithstanding the method still attracts the attention of many researchers and new results are published on a regular basis. In particular, solving the so-called problem of group classification (Lie symmetry classification) still remains a highly non-trivial task and such problems are not solved for several classes of PDEs arising in real world applications.

On the other hand, it is well-known that some nonlinear RDC equations arising in applications have a "poor" Lie symmetry. For example, the Fisher and Fitzhugh–Nagumo equations, which are widely used in mathematical biology, are invariant only under the time and space translations. The Lie method is not efficient for such equations since it enables one to construct only those exact solutions, which can be obtained without using this cumbersome algorithm. Taking into account this fact, one needs to apply other approaches for solving such equations. The best known among them is the method of nonclassical symmetries proposed by G. Bluman and J. Cole in 1969. Although this approach was suggested almost 50 years ago its successful applications for solving nonlinear equations were accomplished only in the 1990s owing to D.J. Arrigo, P. Broadbridge, P. Clarkson, J.M. Hill, E.L. Mansfield, M.C. Nucci, P. Olver, E. Pucci, G. Saccomandi, E.M. Vorob'ev, P. Winternitz and others. A prominent role in applications and further development of the nonclassical symmetry method belongs to the Ukrainian school of symmetry analysis, which was created in the early 1980s and led by W.I. Fushchych (V.I. Fyshchich) until 1997 when he passed away. In particular, a concept of conditional symmetry was worked out and its applications to a wide range of nonlinear PDEs were realized by M. Serov, I. Tsyfra, R. Zhdanov, R. Popovych, R. Cherniha and others. Notably, following Fushchysh's proposal dating back to 1988, we continuously use the terminology "Q-conditional symmetry" instead of "nonclassical symmetry".

We also note that several other approaches for solving nonlinear PDEs (in particular, evolution equations) were independently suggested in the 1990s–2000s. Not pretending to completeness and precise statement, the following of them should be mentioned: the method of linear invariant subspaces, the method of generalized conditional symmetries, the method of heir-equations, the method of linear determining equations, the method of additional generating conditions etc. Notwithstanding some of these methods formally do not use any symmetries, a deep analysis shows that they are related to symmetry-based methods.

The main mathematical object of this book is the class of nonlinear reaction-diffusion-convection equations (RDC). In our opinion, nonlinear RDC equations possess the most important role among other nonlinear equations. One cannot imagine a correct mathematical model describing heat and mass transfer, filtration of liquid, solute transport in tissue, diffusion in chemical reactions, tumor growth and many other processes without RDC equations. The importance of RDC equations in real world applications follows from the fact that they model three main transport mechanisms: diffusion (heat transfer), reaction (source/sink), and convection (advection). Thus, they have been extensively studied by means of different mathematical methods and techniques, including symmetry-based methods.

This book is devoted to (i) search Lie and Q-conditional (non-classical) symmetries of nonlinear RDC equations; (ii) constructing exact solutions using the symmetries obtained and using the method of additional generating conditions; (iii) applications of the solutions derived for solving some biologically and physically motivated problems.

The monograph summarizes in a unique way the results derived by the authors during the last 20 years. Notably, the first joint paper was written by R. Cherniha and M. Serov about 20 years ago during our stay at the Mathematisches Forschungsinstitut Oberwolfach (Germany), while the last joint paper of M. Serov and O. Pliukhin was published in 2015. It should be pointed out that a number of misprints, inexactness and even mistakes arising in our papers were corrected during the book preparation, and new unpublished results were included in Chapters 3, 4 and 5. Moreover, our results are supplemented by those obtained by other authors. As a result, the reader will realize a huge progress, which has been done in study of nonlinear RDC equations by means of symmetry-based methods since the 1990s.

The book presents a most complete (at the present time) description of Lie and conditional symmetries for nonlinear RDC equations, which are very common in real world applications. The most interesting subclasses from this class (like equations with power-law and exponential nonlinearities) are extensively studied. In particular, an essential stress is made on finding symmetries and exact solutions for the widely used equations in bio-medical applications, including Fisher, Murray, Fitzhugh-Nagumo and Kolmogorov-Petrovskii-Piskunov type equations. Concerning the equations listed above and their generalization, a number of examples are presented, in which the relevant real world models are analytically solved, and a biological/physical interpretation of the solutions obtained is provided.

In Introduction (Chapter 1), some mathematical models based on nonlinear RDC equations are discussed, methods for constructing exact solutions of nonlinear PDEs are briefly presented together with a short historical review. The remaining part of this chapter is devoted to the main notions, definitions, and theorems, which form theoretical background of the Lie method and other symmetry-based methods.

Chapter 2 is partly devoted to the linear and nonlinear diffusion (heat) equations, including the multi-dimensional case. Here we present the well-known results of the Lie symmetry classification (the group classification) together with some applications for constructing exact solutions. The main part of Chapter 2 is devoted to the complete Lie symmetry classifications (LSC) of the general class of RDC equations

$$u_t = [A(u)u_x]_x + B(u)u_x + C(u),$$

where $u = u(t,x)$ is an unknown function, $A(u), B(u), C(u)$ are arbitrary smooth functions and the subscripts t and x denote differentiation with respect to these variables. First, LSC is derived using the well-known Lie-Ovsiannikov approach based on the equivalence transformations. Afterwards the second LSC is obtained via so-called form-preserving transformations. An extensive discussion including nontrivial examples is presented to show advantages of application of the form-preserving transformations in order to solve the LSC problem.

Chapter 3 is devoted non-Lie symmetries of nonlinear PDEs. First, we present a historical review concerning conditional symmetry of PDEs, introduce notion of Q-conditional symmetry (nonclassical symmetry) and repeat some well-known results about non-Lie symmetry of nonlinear diffusion equations. The main part of Chapter 3 is devoted to the Q-conditional symmetry classifications of the general class of RDC equations. In contrast to the LSC problem, the result is incomplete, however, a complete classification is derived for several important (from applicability point of view) subclasses of the general class. Probably the most important among them is the Burgers type equations of the form

$$u_t = u_{xx} + \lambda u u_x + C(u), \quad \lambda \in \mathbb{R}.$$

In fact, the above class of equations contains as particular cases the Fisher, Murray, and Fitzhugh–Nagumo equations and their natural generalizations used widely in modeling of biomedical and ecological processes.

Chapter 4 is fully devoted to construction of exact solutions. A wide range of RDC equations are examined in order to search for both Lie and non-Lie solutions. Several examples are presented, which show how nontrivial exact solutions can be constructed for some well-known nonlinear equations. In particular, our attention is addressed to the nonlinear RDC equations with constant and power-law diffusivities, arising in bio-medical and ecological applications. For such equations, we construct exact solutions, examine their properties and (in some cases) provide their biological interpretation. The RDC equations with exponential nonlinearities are also under study in this section.

Chapter 5 is devoted to the method of additional generating conditions and its application for solving some nonlinear RDC equations. This method can be treated as a particular case of the method of differential constraints. Basic ideas of the method of differential constraints have roots in Darboux's works. In the 1960s, N.N. Yanenko formalized the method using the modern mathematical language. However, he has not provided any constructive algorithm for finding the compatible differential constraints for a PDE in question. The method of additional generating conditions solves this problem for PDEs, which can be reduced to those with quadratic nonlinearities. The chapter contains a detailed description of the method, examples demonstrating its efficiency in the case of the nonlinear RDC equations with power-law and exponential nonlinearities. An extensive comparison of the solutions obtained with those derived via other techniques is also presented.

Chapters 2, 3 and 4, which form the main part of this monograph and are essentially connected each with another, were written by all the authors and they contributed equally to these chapters.

Chapters 1 and 5 were written by R. Cherniha.

The book is a monograph. Its academic level suits graduate students and higher. Some parts of the book may be used in "Mathematical Biology" and "Nonlinear Partial Differential Equations" courses for master students and in the final year of undergraduate studies. Nowadays such courses are common in all leading universities over the world.

The book was typeset in LaTeX using the CRC Press templates, the figures were drawn using the computer algebra package `Maple` and some calculations were done using `Mathematica`.

Last but not the least, we are grateful to our colleagues and our teacher Wilhelm Fushchych (1936–1997), who was the supervisor for R. Cherniha and M. Serov in the 1980s. This book could not have been written without his innovative scientific ideas and many years of his support. The authors thank their Ukrainian colleagues for fruitful discussions, valuable critique, and helpful suggestions, which helped us to write this modest work. Especially, we are grateful to Vasyl' Davydovych, Sergii Kovalenko, Inna Rassokha, and Valentyn Tychynin.

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Kyiv, Ukraine *Roman Cherniha*
Poltava, Ukraine *Mykola Serov, Oleksii Pliukhin*

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Acronyms

DE(s) – determining equation(s)

Eq(s). – equation(s)

ET(s) – equivalence transformation(s)

FN – Fitzhugh–Nagumo

FPT(s) – form-preserving transformation(s)

KPP – Kolmogorov–Petrovskii–Piskunov

LSC – Lie symmetry classification

MAGC – method of additional generating conditions

MAI(s) – maximal algebra(s) of invariance

MGI – maximal group of invariance

ODE – ordinary differential equation

QSC – Q-conditional symmetry classification

PDE – partial differential equation

RD – reaction-diffusion

RDC – reaction-diffusion-convection

w.r.t. – with respect to

Chapter 1

Introduction

1.1 Nonlinear reaction-diffusion-convection equations in mathematical modeling

Since the 17th century when G. Leibnitz and I. Newton discovered differential and integral calculus, differential equations are the most powerful tools for mathematical modeling various processes in physics, chemistry, biology, medicine, ecology, economics etc. Of course, pioneering models were created in order to express some classical laws (like the second Newton law) in physics and astronomy, later differential equations came to be used for describing a wide range of processes not only in physics but also in other natural and life sciences. It can be noted that almost all mathematical models created before the end of the 19th century were based on ordinary differential equations (ODEs) and linear partial differential equations (PDEs). *Nonlinear PDEs* are widely used in mathematical modeling real world processes since the beginning of the 20th century only. Probably, one of the first attempts in applying and solving a nonlinear PDE of the parabolic type was made by J. Boussinesq who studied the porous diffusion equation describing the water filtration in soil [32]. One of the first applications of nonlinear reaction-diffusion (RD) equations in biology was proposed by R.A. Fisher [98, 99].

Nowadays, i.e., 100 years later, it is generally accepted that a huge number of real processes arising in physics, biology, chemistry, material sciences, engineering, ecology, economics etc. can be adequately described only by *nonlinear PDEs* (or systems of such equations). The most widely used type of equations for modeling such processes are the nonlinear reaction-diffusion-convection (RDC) (advection) equations. Since 1952 when A.C. Turing published the remarkable paper [240], in which he proposed a revolutionary idea about mechanism of morphogenesis (the development of structures in an organism during the life), nonlinear RD equations (including those with convective terms) play a crucial role in real world applications and have been extensively studied by means of different mathematical methods/techniques. As a result, in the 1970s several monographs were published, which are devoted to study and application of the nonlinear reaction-diffusion-convection (RDC) equations in physics [5, 6, 158], biology [97, 179] and chemistry [10, 11]. In our opinion, these books had a great impact attracting many scholars to study the nonlinear RDC equations and use them for modeling real world processes. Since that time many other excellent monographs and textbooks appeared, especially for models related to life sciences (see, e.g., [35, 96, 160, 181, 182, 194, 231, 245]). We concentrate ourselves mostly on biologically motivated models in what follows.

Typically the RDC equation describing a process in the 1D space approximation has the form

$$u_t = [A(u)u_x]_x + B(u)u_x + C(u),$$

where A, B and C are some given functions, while $u(t, x)$ is an unknown function (hereafter the lower indices t and x denote differentiation with respect to (w.r.t.) these variables). In the models related to biomedical applications, the function $u(t, x)$ means the concentration of cells (population, drugs, molecules). The functions A, B and C are related to the three most common types of transport mechanisms occurring in real world processes. The diffusivity $A > 0$ (typically it is a constant) is the main characteristic of the diffusion process, the term $B(u)u_x$ (B typically means velocity, which can be positive and/or negative) describes the convective transport (in contrast to diffusion, one is not random) and the reaction term $C(u)$ describes the process kinetics (for example, this function presents interaction of the population u with the environment). A natural multidimensional analog of the above equation reads as

$$u_t = \nabla \cdot (A(u)\nabla u) + V(u) \cdot \nabla u + C(u). \tag{1.1}$$

Here u is the function of t and $x_1, \dots, x_n$, $\nabla = \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)$, the velocity $B(u)$ is replaced by the velocity vector $V(u)$ and $\cdot$ means the scalar product.

In (1.1) the variable diffusivity $D(u)$ (typically it is a power-law function but can be the function with a more complicated structure [40, 160]) arises in more and more modeling situations of biomedical importance from diffusion of genetically engineered organisms in heterogeneous environments to the effect of white and grey matter in the growth and spread of brain tumors [160, 182]. For example, the power-law diffusivity occurs as an extension of the classical diffusion model, when there is an increase in diffusion due to population pressure (see Section 11.3 in [182] and references therein).

The velocity vector V can be a vector function depending on the concentration u (if $V = const$ then the term $V \cdot \nabla u$ is removable from (1.1) by the Galilei transformations) and the simplest case when V is linear w.r.t. u was firstly studied in [179]. Notably, the velocity vector typically has the structure

$V = (B(u), 0 \dots, 0)$ in real world models (for example, the evolution of fish population in a river can be adequately described if one takes into account the stream velocity only in the direction x_1 and neglects in other directions).

Although the reaction term C can possess a great variety of forms depending on the model in question (see examples in the books cited above), the most typical form of the reaction term is $C(u) = \lambda_1 u^p - \lambda_2 u^q$ with the positive exponents p and q (see Sections 11.3 and 13.4 in [182]). Depending on values of p and q several well-known equations arising in biomedical applications can be identified. For example, setting $p = 1$, $q = 2$, $A(u) = d = const$ and $V = 0$, the famous Fisher equation

$$u_t = d\Delta u + u(1-u),$$

(here Δ is the Laplace operator) is obtained describing the spread in space of a favored gene in a population. Setting $p = 2$ and $q = 3$, the Huxley equation [43, 88]

$$u_t = d\Delta u + u^2(1-u)$$

is derived, which can be thought as a limiting case of the famous Fitzhugh–Nagumo (FN) equation [100, 185]

$$u_t = d\Delta u + u(u-\delta)(1-u),\ 0 < \delta < 1.$$

The latter is a simplification of the celebrated Hodgkin–Huxley model [133] describing the ionic current flows for axonal membranes. The FN equation reduces the Hodgkin–Huxley model, which has a very complex structure, and describes the nerve impulse propagation. The function $u(t, x)$ means the electric potential across the cell membrane.

In conclusion, we note that there are real world processes, which are described by the RDC equations involving coefficients A, B and C depending on derivatives of the function u (see, e.g., the recent paper [69] and works cited therein). Examination of such equations lies beyond the scope of this monograph.

1.2 Main methods for exact solving nonlinear reaction-diffusion-convection equations

As it was already pointed out, it is a generally accepted fact at the present time that a huge number of real processes arising in physics, biology, chemistry etc. can be adequately described by *nonlinear* partial differential equations (PDEs) only. On the other hand, the well-known principle of linear superposition cannot be applied to generate new exact solutions to nonlinear PDEs. Thus, the classical methods (the Fourier method, the Green function method, the method of the Laplace transformations, and so forth) are not applicable for solving nonlinear equations.

At the present time, there are many methods/techniques, which allow us to construct particular solutions of some nonlinear PDEs, however those are applicable to correctly-specified classes of PDEs only and any general integration theory is unknown. While there is no existing general theory for integrating nonlinear PDEs, construction of particular exact solutions for these equations is *a nontrivial and important problem*. Finding exact solutions that have a physical, chemical or biological interpretation is of *fundamental importance*.

One may say that the oldest technique for solving nonlinear differential equation is finding an appropriate transformation for a given PDE. In fact, a change of variables can sometimes be found that transforms the given nonlinear PDE into a linear equation. Transformation of the Burgers equation into the linear heat equations via the Cole–Hopf substitution is the classical example in this direction. However, finding exact solutions of most nonlinear PDEs generally requires other methods/thechniques than those for linear equations.

Nowadays the most powerful methods for construction of exact solutions for a wide range of classes of nonlinear PDEs are symmetry-based methods. All these methods have the common idea stating that exact solutions (at least particular ones) can be found for a given PDE provided its symmetry (a set of symmetries) is known. These methods originated from the Lie method, which was created by the prominent Norwegian mathematician Sophus Lie in the end of 19th century [166, 167] (see also the reprints in [168, 169]). The method was essentially developed using modern mathematical language by L.V. Ovsiannikov, G. Bluman, N. Ibragimov, W.F. Ames and some other researchers in the 1960s–1970s. Although the technique of the Lie method is well-known, the method still attracts attention of researchers and new results are published on a regular basis. In the next section a short historical review, basic notions, examples and theorems of the Lie method are presented.

In 1969, G. Bluman and J. Cole introduced an essential generalization of the Lie symmetry notion [26], which later was called nonclassical symmetry (in order to distinguish the new kind of symmetry from the classical Lie symmetry). Although nonclassical symmetries were not used for examination of nonlinear PDEs almost for 20 years (until the late 1980s), nowadays it is a powerful tool for constructing exact solutions of nonlinear equations. In particular, many important results for evolution PDEs were obtained during the last two decades. In the first section of Chapter 3, a short historical review, basic notions and theorems of the nonclassical method are presented. Notably, the terminology "nonclassical symmetry" is not generally accepted because notions "Q-conditional symmetry" and "reduction operator" are widely used too (see discussion on this matter in Section 3.1).

In the 1980s–1990s, a few new types of symmetries were introduced, which also allow us to construct exact solutions of nonlinear PDEs. The notion of conditional symmetry was suggested by Fushchych and his collaborators [112], [114, Section 5.7]. Note that the notion of nonclassical symmetry can be derived as a particular case from conditional symmetry but not vice versa (see highly nontrivial examples in Section 3.1 and in paper [63, 64]). Weak symmetry was suggested in [198, 199], potential symmetry was introduced in [29, 30, 159], while generalized conditional symmetry was independently formulated in [101, 173] and [251] (the terminology "conditional Lie–Bäcklund symmetry" was used in the latter).

The crucial idea used for introducing new types of symmetries can be formulated as follows. Let us consider an arbitrary PDE. For simplicity we restrict ourselves to the second-order two-dimensional equation

$$L\left(t, x, u, u_t, u_x, u_{tt}, u_{tx}, u_{xx}\right) = 0. \tag{1.2}$$

Hereafter $u = u(t, x)$ is an unknown smooth function[1], while L is a given smooth function. Almost all known in literature symmetries of Eq. (1.2) can be written as the differential operator

$$X = \xi^0 \partial_t + \xi^1 \partial_x + \eta \partial_u, \tag{1.3}$$

where the coefficients ξ^0, ξ^1 and η should be found according to the definition (criteria) of symmetry in question. In the case of the Lie, nonclassical and conditional symmetries, the coefficients depend on dependent and independent variables at maximum (those can be simply constants) and the corresponding criteria for their finding are presented in Sections 1.3 and 3.1.

However, the operator X has a more complicated structure if one is looking for other types of symmetries. For example, the coefficient η depends on derivatives of the function u in the case of generalized conditional symmetries [101, 173, 220].

In the case of potential symmetries, the coefficients depend on integrals of u, so that nonlocal operators are obtained. Thus, the terminology "nonlocal symmetry" is also used (see, e.g., [241] and references therein). A substantial number of examples involving potential symmetries for examination of nonlinear PDEs is presented in [25] (see also references therein).

A vast literature (see, e.g., [196] and citations therein) is devoted to higher-order symmetries (the terminology "generalized" and "Lie–Bäcklund" symmetry is also used) of the form

$$Z = X + \varsigma^0 \partial_{u_t} + \varsigma^1 \partial_{u_x} + \varsigma^{11} \partial_{u_{xx}} + \varsigma^{01} \partial_{u_{tx}} + \ldots, \tag{1.4}$$

where coefficients depend on derivatives of the function u. Such symmetries were introduced by Noether in her remarkable work [190]. Here we want only to stress that integrability of nonlinear PDEs via the method of inverse scattering problem [1, 93] is related with higher-order symmetries and conservation laws [177].

[1]Throughout the book the notion "smooth function" means that one is differentiable with respect to (w.r.t.) its variables up to the equation order, i.e., in the case of Eq. (1.2), u is the twice differentiable function w.r.t. t and x (at least in an open domain)

At the present time, it is a widely accepted hypothesis that each known exact solution of a given nonlinear PDE can be derived using an appropriate symmetry. On the other hand, there are some efficient methods/techniques allowing to construct exact solutions without knowledge of any symmetry. Here we are not going to present all of them because nowadays there are too many techniques proposed by a huge number of authors and it is very difficult to classify them (notably some new methods are particular cases of those proposed in the 1990s and earlier). In our opinion, the most general approach called the method of differential constraints was formulated in the 1960s [249] and was further developed in monograph [232]. Actually, basic ideas of the method of differential constraints have roots in Darbouxs works [103](see also excellent historical reviews on this matter in [214] and [119]). The main idea of the method of differential constraints is very simple: to define suitable constraint(s) for a given PDE in such a way that the overdetermined system obtained will be compatible and can be (partly) solved using the existing methods. Methods for solving overdetermined systems of differential equations were known since the first half of the 20th century [42, 139] and can be successfully applied in many cases (for instance, see examples in [232]). However, the main problem of the method is how to define the suitable constraint(s). At the present time, the corresponding algorithm does not exist in general case and one may claim that the method of differential constraints is rather a fruitful idea without a constructive algorithm. Interestingly, the symmetry-based methods implicitly use this idea in a constructive way. In fact, in order to find exact solutions, one solves the given nonlinear PDE (system of PDEs) together with the differential constraint(s) generated by a symmetry operator.

There are several techniques, which propose to use the correctly-specified differential constraints in order to find exact solutions for some correctly-specified classes of PDEs. In particular, the method of additional generating conditions [45, 46, 48] and the method of determining equations [143, 144] were independently worked out in the 1990s. Both methods are very similar and Chapter 5 is devoted to the first of them.

Several approaches based on substitutions of the special form, which are often called *ansatz*[2], should be mentioned. Such substitution reduces the given PDE to a simpler equation (e.g., ODE) or a system of simpler equations, which can be integrated (at least partly). In order to construct the relevant ansatz, either some physical (biological, chemical etc.) motivation or an ad hoc approach are used. The most typical is the plane wave ansatz

$$u = \phi(\omega), \ \omega = x - vt, \ v \in \mathbb{R}, \tag{1.5}$$

which reduces Eq. (1.2) to an ODE provided the function L does not depend on t and x. Although solving the nonlinear ODE obtained can be also a nontrivial problem, the solution (at least partial) can be usually found by using the classical methods or handbooks like [142, 212]. Notably, there exist some

[2] Ansatz (pl. ansätze) is a German word