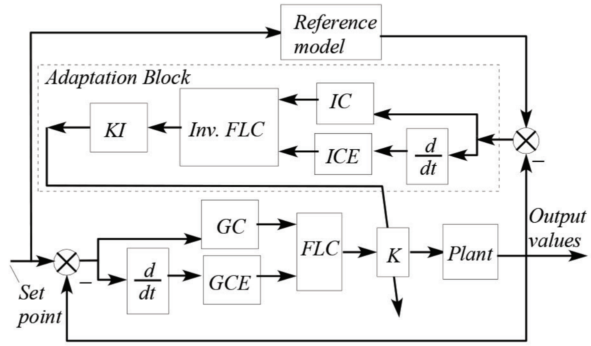

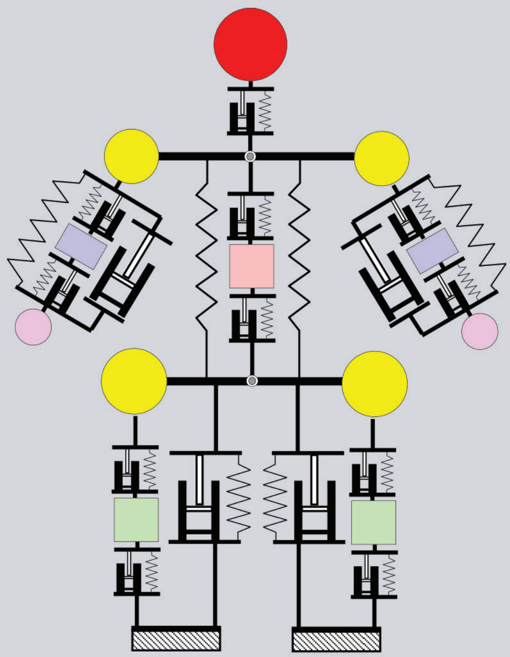

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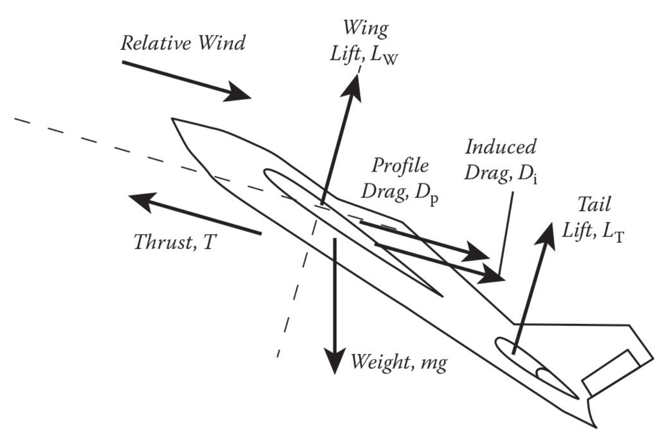
An Integrated Approach

Ranjan Vepa



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To my parents, Narasimha Row and Annapurna

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Preface

During the last two decades, much progress has been made in the application of nonlinear differential geometric control theory, first to robotic manipulators and then to autonomous vehicles. In fact, robot control is simply a metaphor for nonlinear control. The ability to transform complex nonlinear systems sequentially to simpler prototypes, which can then be controlled by the application of Lyapunov's second method, has led to the development of some novel techniques for controlling both robot manipulators and autonomous vehicles without the need for approximations. More recently, a synergy of the technique of feedback linearization with classical Lyapunov stability theory has led to the development of a systematic adaptive backstepping design of nonlinear control laws for systems with unknown constant parameters. Another offspring of the Lyapunov-based controllers is a family of controllers popularly known as sliding mode controls. Currently, sliding mode controls have evolved into second- and higher-order implementations, which are being applied extensively to robotic systems.

Some years ago, the author embarked on a comprehensive programme of research to bring together a number of techniques in an attempt to formulate the dynamics and solve the control problems associated with both robot manipulators and autonomous vehicles, such as unmanned aerial vehicles (UAVs), without making any approximations of the essentially nonlinear dynamics. A holistic approach to the two fields have resulted in new application ideas such as the morphing control of aerofoil sections and the decoupling of force (or flow) and displacement control loops in such applications. A number of results of several of these studies were also purely pedagogical in nature. Pedagogical results are best reported in the form of new learning resources, and for this reason, the author felt that the educational outcomes could be best communicated in a new book. In this book, the author focuses on control and regulation methods that rely on the techniques related to the methods of feedback linearization rather than the more commonly known methods that rely on Jacobian linearization. The simplest way to stabilize the zero dynamics of a nonlinearly controlled system is to use, when feasible, input–output feedback linearization. The need for such a book arose due to the increasing appearance of both robot manipulators and UAVs with operating regimes involving large magnitudes of state and control variables in environments that are not generally very noisy. The underpinning themes which serve as a foundation for both robot dynamics and UAVs include Lagrangian dynamics, feedback linearization and Lyapunov-based methods of both stabilization and control. In most applications, a combination of these fundamental techniques provides a powerful tool for designing controllers for a range of application tasks involving tracking, coordination and motion control. Clearly, the focus of these applications is primarily on the ability to handle the nonlinearities rather than dealing with the environmental disturbances and noise which are of secondary importance. This book is of an applied nature and is about *doing* and *designing* control laws. A number of application examples are included to facilitate the reader's learning of the art of nonlinear control system design.

The book is not meant to supplant the many excellent books on nonlinear and adaptive control but is designed to be a complementary resource. It seeks to present the methods of nonlinear controller synthesis for both robots and UAVs in a single, unified framework.

The book is organized as follows: Chapter 1 deals with the application of the Euler–Lagrange method to robot manipulators. Special consideration is given to rapidly determining the equations of motion of various classes of manipulators. Thus, the manipulators are classified as parallel and serial, as Cartesian and spherical and as planar, rotating planar and spatial, and the methods of determining the equations of motion are discussed under these categories. The definition of planar manipulators is generalized so that a wider class of manipulators can be included in this category. The methods of deriving the dynamics of the manipulators can be used as templates to derive the dynamics of any manipulator. This approach is unique to this book. Chapter 2 focuses on the application of the Lagrangian method to UAVs via the method of quasi-coordinates. It is worth remembering that the use of the Lagrangian method for deriving the equations of motion of a UAV is not the norm amongst flight dynamicists. Moreover, the chapter introduces the velocity axes, as the synthesis of the flight controller in these axes is a relatively easy task. The concept of feedback linearization is introduced in Chapter 3, while the classical methods of phase plane analysis of the stability of nonlinear systems and their features are discussed in Chapter 4 in the context of Lyapunov's first method. Chapter 5 presents an overview of the methods of robot and UAV control. Chapter 6 is dedicated to introducing the concepts of stability, and Chapter 7 is exclusively about Lyapunov stability with an enunciation of Lyapunov's second method. The methodology of computed torque control is the subject of Chapter 8, and sliding mode controls are introduced in Chapter 9. Chapter 10 discusses parameter identification, including recursive egression, while adaptive and model predictive controller designs are introduced in Chapter 11. In a sense, linear optimal control, a particular instance of the Lyapunov design of controllers, is also covered in the section on model predictive control, albeit briefly. Chapter 12 is exclusively devoted to the Lyapunov design of controllers by backstepping. Chapter 13 covers the application of feedback linearization in the task space to achieve decoupling of the position and force control loops, and Chapter 14 is devoted to the applications of nonlinear systems theory to the synthesis of flight controllers for UAVs.

It is the author's belief that the book will not be just another text on nonlinear control but serve as a unique resource to both the robotics and UAV research communities in the years to come and as a springboard for new and advanced projects across the globe.

First and foremost, I thank Jonathan Plant, without his active support, this project would not have been successful. I also thank my colleagues and present and former students at the School of Engineering and Material Science at Queen Mary University of London for their assistance in this endeavour. In particular, I thank Professor Vassili Toropov for his support and encouragement.

I thank my wife Sudha for her love, understanding and patience. Her encouragement and support provided the motivation to complete the project. I also thank our children Lullu, Satvi and Abhinav for their understanding during the course of this project.

Ranjan Vepa
London, United Kingdom

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Dr. Vepa is the author of five books: *Biomimetic Robotics* (Cambridge University Press, 2009); *Dynamics of Smart Structures* (Wiley, 2010); *Dynamic Modeling, Simulation and Control of Energy Generation* (Springer, 2013) and *Flight Dynamics, Simulation, and Control: For Rigid and Flexible Aircraft* (CRC Press, 2014). Dr. Vepa is a member of the Royal Aeronautical Society, London; the Institute of Electrical and Electronic Engineers, New York and the Royal Institute of Navigation, London. He is also a Fellow of the Higher Education Academy and a chartered engineer.

In addition, Dr. Vepa is studying techniques for automatic implementation of structural health monitoring based on observer and Kalman filters. He is involved in the design of crack detection filters applied to crack detection and isolation in aeroelastic aircraft structures such as nacelles, casings, turbine rotors and rotor blades for health monitoring and control. Elastic wave propagation in cracked structures is being used to develop distributed filters for structural health monitoring. Feedback control of crack propagation and compliance compensation in cracked vibrating structures are also being investigated. Another issue is the modeling of damage in laminated composite plates, nonlinear flutter analysis and the interaction with unsteady aerodynamics. These research studies contribute to the holistic design of vision-guided autonomous UAVs, which are expected to be used extensively in the future.

Dr. Vepa's research interests also include the design of flight control systems, aerodynamics of morphing wings and bodies with applications in smart structures, robotics, biomedical engineering and energy systems, including wind turbines. In particular, his focus is on the dynamics and robust adaptive estimation and control of linear and nonlinear aerospace, energy and biological systems with parametric and dynamic uncertainties. The research in the area of the aerodynamics of morphing wings and bodies is dedicated to the study of aerodynamics and its control, including the use of smart structures and their applications in the flight control of air vehicles, jet engines, robotics and biomedical systems. Other applications are in wind turbine and compressor control, maximum power point tracking, flow control over smart flaps and the control of biodynamic systems.

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CHAPTER ONE

Lagrangian methods and robot dynamics

Introduction

The basis of the Newtonian approach to dynamics is the Newtonian viewpoint, that motion is induced by the action of forces acting on particles. This viewpoint led Sir Isaac Newton to formulate his celebrated laws of motion. In the late 1700s and early 1800s, a different view of dynamic motion began to emerge. According to this view, particles do not follow trajectories because they are acted upon by external forces, as Newton proposed. Instead, amongst all possible trajectories between two points, they choose the one which minimizes a specific time integral of the difference between the kinetic and the potential energies called the action. Newton's laws are then obtained as a consequence of this principle, by the application of variational principles in minimizing the action integral. Also, as a consequence of the minimization of the action integral, the total potential and kinetic energies of systems are conserved in the absence of any dissipative forces or forces that cannot be derived from a potential function. The alternate view of particle motion then led to a newer approach to the formulation and analysis of the dynamics of motion. It was no longer required to isolate each and every particle or body and forces acting on them, within a system of particles or bodies. The system of particles could be treated in a holistic manner without having to identify the forces of interaction between the particles or bodies.

The variational approach seeks to derive the equations of motion for a system of particles in the presence of a potential force field as a solution to a minimization problem. The independent variable in the problem will clearly be time, and the dependent functions will be the three-dimensional (3D) positions of each particle. The aim is to find a function L such that the paths of the particles between times t_1 and t_2 extremize the integral:

$$I = \int_{t_1}^{t_2} L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}; t\right) dt. \tag{1.1}$$

The integral I will be referred to as the action of the system and the function L as the Lagrangian. In fact, we can show that when the Lagrangian L is defined as

$$L = T - V = \frac{1}{2} m v^2 - V\left(x, y, z; t\right), \tag{1.2}$$

the equations of motion are given by the Euler–Lagrange equations which are obtained by setting the variation of the Lagrangian δL to zero. Thus, we set

$$\delta L = \frac{\partial L}{\partial x}\delta x + \frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial z}\delta z + \frac{\partial L}{\partial \dot{x}}\delta \dot{x} + \frac{\partial L}{\partial \dot{y}}\delta \dot{y} + \frac{\partial L}{\partial \dot{z}}\delta \dot{z} = 0. \tag{1.3}$$

However, by expressing δL as

$$\delta L = \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)\right)\delta x + \left(\frac{\partial L}{\partial y} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{y}}\right)\right)\delta y + \left(\frac{\partial L}{\partial z} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right)\right)\delta z = 0, \tag{1.4}$$

and assuming that the variations δx, δy and δz can be varied t without placing any constraints on them, it follows that

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \tag{1.5}$$

with $q_1 = x$, $q_2 = y$ and $q_3 = z$. These are the celebrated Euler–Lagrange equations which result in Newton's second laws of motion when $L = T - V$.

Our focus in this chapter is the application of Lagrangian dynamics, not to particles in motion but to kinematic mechanisms in general, and robot manipulators in particular. To this end, a brief review of the kinematics of robot manipulators is essential.

1.1 Constraining kinematic chains: Manipulators

The primary element of a mechanical system is a link. A link is a rigid body that possesses at least two nodes that are points for attachment to other links. A joint is a connection between two or more links at specific locations known as their nodes, which allows some motion, or potential motion, between the connected links. A kinematic chain is defined as an assemblage of links and joints, interconnected in a way to provide a controlled output motion in response to a specified input motion. A mechanism is defined as a kinematic chain in which at least one link has been 'grounded', or attached, to a frame of reference which itself may be stationary or in motion. A robot manipulator is a controlled mechanism, consisting of multiple segments of kinematic chains, that performs tasks by interacting with its environment. Joints are also known as kinematic pairs and can be classified as a lower pair to describe joints with surface contact while the term *higher pair* is used to describe joints with a point or line contact. Of the six possible lower pairs, the revolute and the prismatic pairs are the only lower pairs usable in a planar mechanism. The screw, cylindrical, spherical and flat lower pairs are all combinations of the revolute and/or prismatic pairs and are used in spatial (three-dimensional) mechanisms.

Manipulator kinematics: The Denavit and Hartenberg (DH) parameters

A primary problem related to the kinematics of manipulators is the forward kinematics problem, which refers to the determination of the position and orientation of the *end effector*, given the values for the joint variables of the robot. In the robotics community, a systematic procedure for achieving this in terms of four standardized parameters of a link, namely the joint angle, the link length, the link offset and the link twist, is adopted. This convention is known as the Denavit and Hartenberg convention, and the parameters are known as the Denavit and Hartenberg (DH) parameters. The complete systematic method of defining the DH parameters will not be discussed here. The interested reader is referred to texts such as Vepa [1], where the application of the DH convention to robot manipulators is discussed in some detail.

Velocity kinematics: Jacobians

The Denavit and Hartenberg conventions are used to derive the forward and inverse position equations relating joint, link and end-effector positions and orientations. From these relations, one derives the velocity relationships, relating the linear and angular velocities of the end effector or any point on a link in the manipulator to the joint velocities. The position referenced to a frame attached to the end effector is a function of both the orientation of the frame and the position of the origin of the frame. Thus, it can be used to determine representations for both the translational and rotational velocities relating the linear and angular velocities of the end effector or any point on a link in the manipulator to the joint velocities. In particular, one could obtain the angular velocity of the end-effector frame and the linear velocity of the origin of the frame in terms of the joint velocities.

Mathematically, the Denavit and Hartenberg conventions are used to obtain the forward kinematics equations, defining functions relating the space of Cartesian positions and orientations to the space of joint positions. The velocity relationships are then determined by the Jacobian of these functions. The Jacobian is a matrix-valued function and can be thought of as the vector version of the ordinary derivative of a scalar function. The interested reader is again referred to texts such as Vepa [1], where the velocity kinematics and the derivation of the Jacobian of specific robot manipulators are discussed in some detail.

Degrees of freedom: The Gruebler criterion and Kutzbach's modification

A mechanical system's mobility (M) can be classified according to the number of degrees of freedom that it possesses. The system's degree of freedom is equal to the number of independent parameters (measurements) that are needed to uniquely define its position in space at any instant of time. The degrees of freedom of any planar assembly of links can be obtained from the Gruebler condition, $M = 3(L - G) - 2J$, where M is the *degree of freedom or mobility*, L is the *number of links*, J is the *number of joints* and G is the *number of grounded links*. In real mechanisms as there can be only one ground plane $G = 1$. Furthermore, one can distinguish between joints with one degree of freedom which are referred to as full joints and joints with two degrees of freedom which are effectively equivalent to two half joints. Thus, if the number of full joints is J_F and the number of half joints is J_H, the Gruebler condition as modified by Kutzbach is

$$M = 3\left(L-1\right)-2J_F - J_H. \tag{1.6}$$

The approach used to determine the mobility of a planar mechanism can be easily extended to three dimensions. In a three-dimensional space, a rigid body has six degrees of freedom unlike in two dimensions where a body has only three degrees of freedom. Thus, a full joint in 3D space removes five degrees of freedom. In general, if the number of joints that remove k degrees of freedom is denoted as J_{6-k}, the Kutzbach criterion is

$$M = 6\left(L-1\right)-\sum_{k=1}^{5} kJ_{6-k}. \tag{1.7}$$

Similar criteria can be established to identify the number of rotational degrees of freedom of a mechanism.

1.2 The Lagrangian formulation of dynamics

Joseph Louis Lagrange defined the so-called Lagrangian method based on sound mathematical foundations by using the concept of virtual work along with D'Alembert's principle. While Newton argued that the rate of change of momentum of a body was directly proportional to the applied force, D'Alembert proposed that the change in the momentum of a body was itself responsible for the generation of a force and that this force along with all other applied forces was responsible in maintaining the body in an equilibrium state.

Principle of virtual work

Virtual displacements are the result of infinitesimal changes to the system of coordinates that define a particular system and that are consistent with the different forces and constraints imposed on the system at a given instant of time. An element of the complete system or vector of virtual displacements is referred to as a single virtual displacement. The term *virtual* is used to distinguish these types of displacement with actual displacement occurring in a finite time interval, during which the forces could be changing.

Suppose that a system is in static equilibrium. In this case, the total force $\mathbf{F}_i$ acting on each particle that compose the system must vanish, that is, $\mathbf{F}_i = 0$. If we define the virtual work done on a particle as $\mathbf{F}_i \cdot \delta\mathbf{q}_i$ (assuming the use of Cartesian coordinates), then we have for the total virtual work done by all of the particles

$$\sum_i \mathbf{F}_i \cdot \delta\mathbf{q}_i = 0. \tag{1.8}$$

Let's now decompose the force $\mathbf{F}_i$ as the sum of the externally applied forces $\mathbf{F}_i^a$ and the forces of constraints $\mathbf{F}_i^c$ such that

$$\mathbf{F}_i = \mathbf{F}_i^a + \mathbf{F}_i^c. \tag{1.9}$$

Then the equation for the total virtual work becomes

$$\sum_i \left(\mathbf{F}_i^a \cdot \delta\mathbf{q}_i + \mathbf{F}_i^c \cdot \delta\mathbf{q}_i\right) = 0. \tag{1.10}$$

Generally, it is true that the forces of constraints satisfy

$$\sum_i \mathbf{F}_i^c \cdot \delta\mathbf{q}_i = 0. \tag{1.11}$$

Hence, it follows that

$$\sum_i \mathbf{F}_i^a \cdot \delta\mathbf{q}_i = 0. \tag{1.12}$$

Furthermore, if the applied forces are indeed equal to the rate of change of momenta, we can write

$$\mathbf{F}_i = \dot{\mathbf{p}}_i. \tag{1.13}$$

Thus, it follows that

$$\mathbf{F}_i - \dot{\mathbf{p}}_i = 0, \tag{1.14}$$

and that

$$\sum_i \left(\mathbf{F}_i - \dot{\mathbf{p}}_i\right) \cdot \delta\mathbf{q}_i = 0, \tag{1.15}$$

which reduces to

$$\sum_i \left(\mathbf{F}_i^a - \dot{\mathbf{p}}_i\right) \cdot \delta\mathbf{q}_i = 0. \tag{1.16}$$

D'Alembert argued that $-\dot{\mathbf{p}}_i$ was itself a force due to the inertia of the particle. Consequently, D'Alembert expressed

$$\mathbf{F}_i^m = -\dot{\mathbf{p}}_i. \tag{1.17}$$

Examples of the forces of inertia are the so-called centrifugal force exerted by a rotating body as well as the forces due to the Coriolis acceleration. The force due to gravity on the surface of the Earth is an example, which includes the forces due to gravitation and the centrifugal and Coriolis forces due to the Earth's rotation. Thus, by eliminating the rates of change of momenta, the total virtual work done reduces to

$$\sum_i \left(\mathbf{F}_i^a + \mathbf{F}_i^m\right) \cdot \delta\mathbf{q}_i = 0. \tag{1.18}$$

The previous relation is the principle of virtual work in its most general form. The principle naturally leads to Newton's laws of motion and to Euler's equations.

Principle of least action: Hamilton's principle

Hamilton's principle is concerned with the minimization of a quantity (i.e. the action integral) in a manner that is similar to extremum problems solved using the calculus of variations. Hamilton's principle can be stated as follows:

The motion of a system from time t_1 to time t_2 is such that the line integral (called the action or the action integral)

$$I = \int_{t_1}^{t_2} L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}; t\right) dt \tag{1.19}$$

where $L = T - V$ (with T and *V the kinetic and potential energies, respectively) has a stationary value for the actual path of the motion.*

Note that a 'stationary value' for the action integral implies an extremum for the action, not necessarily a minimum. But in almost all important applications in dynamics, a minimum does occur. Because of the dependency of the kinetic and potential energies on the coordinates x, y and z and the velocities $\dot{x}$, $\dot{y}$ and $\dot{z}$, and possibly the time t, it is found that

$$L = L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}; t\right). \tag{1.20}$$

Hamilton's principle can now be expressed mathematically by

$$\delta I = \delta \int_{t_1}^{t_2} L\left(x, y, z, \dot{x}, \dot{y}, \dot{z}; t\right) dt = 0. \tag{1.21}$$

A solution for the previous equation is obtained by setting the variation of the Lagrangian δL to zero. Thus, we get the equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0, \tag{1.22}$$

with $q_1 = x$, $q_2 = y$ and $q_3 = z$. These are the celebrated Euler–Lagrange equations in Cartesian coordinates which were shown to be equivalent to Newton's second law of motion.

Generalized coordinates and holonomic dynamic systems

If a mechanical system is made up of n interconnected particles, the positions of all particles may be specified by $3n$ coordinates. However, if there are m physical constraints resulting in an equal number of constraint equations, then the $3n$ coordinates are not all independent. Furthermore, if the m constraint equations are in the form of functional relations between the degrees of freedom, they are said to be *holonomic*. When the constraints are holonomic, there will be only $3n - m$ independent coordinates, and the system will possess only $3n - m$ degrees of freedom. Moreover, the degrees of freedom do not need to be specified as Cartesian coordinates but can be any transformation of them so long as the corresponding virtual displacements associated with the set of degrees of freedom are independent of each other. Such coordinates are known as *generalized coordinates*. Thus, one may choose to have different types of coordinate systems for different coordinates as long as they are a minimal set. Also, the degrees of freedom do not even need to share the same unit or dimensions. One could also transform a set of generalized coordinates to another set as long as the transformation is invertible over the entire domain of the generalized coordinate set. It follows naturally that Hamilton's principle can now be expressed in terms of the generalized coordinates and velocities as

$$\delta I = \delta \int_{t_1}^{t_2} L\left(q_i, \dot{q}_i; t\right) dt = 0. \tag{1.23}$$

Euler–Lagrange equations

It also follows naturally that the Euler–Lagrange can now be expressed in terms of the generalized coordinates and velocities as

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = 0. \tag{1.24}$$

We now go back to our usual coordinate transformation that relates the Cartesian and generalized coordinates. We distinguish between the two sets of coordinates by using a superscript 'C' for the Cartesian coordinates. Thus, one can express the Cartesian coordinates as functions of the generalized coordinates as

$$q_i^C = q_i^C\left(q_j, t\right). \tag{1.25}$$

Hence, it follows that

$$\dot{q}_i^C = \frac{\partial q_i^C\left(q_j, t\right)}{\partial q_j}\dot{q}_j + \frac{\partial q_i^C\left(q_j, t\right)}{\partial t}. \tag{1.26}$$

Similarly, the components δq_i^C of the virtual displacement vectors at a given instant of time t can be written as

$$\delta q_i^C = \frac{\partial q_i^C\left(q_j, t\right)}{\partial q_j}\delta q_j. \tag{1.27}$$

From the expression for the virtual work done by external forces,

$$\sum_i \mathbf{F}_i^a \cdot \delta \mathbf{q}_i^C = \sum_i \mathbf{F}_i^a \sum_j \frac{\partial \mathbf{q}_i^C\left(\mathbf{q}_j, t\right)}{\partial q_j} \cdot \delta q_j = \sum_j Q_j \cdot \delta q_j = 0. \tag{1.28}$$

Hence, one can express the generalized forces in the transformed generalized coordinates as

$$Q_j = \sum_i \mathbf{F}_i^a \frac{\partial \mathbf{q}_i^C\left(\mathbf{q}_j, t\right)}{\partial q_j}. \tag{1.29}$$

Considering the inertia forces in the principle of virtual work, we may show that

$$\sum_i \mathbf{F}_i^m \cdot \delta \mathbf{q}_i^C = \sum_j \left(\frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) \right) \delta q_j, \tag{1.30}$$

where T is the total kinetic energy expressed in terms of the generalized coordinates and generalized velocities. Hence, it follows that

$$\sum_i \left(\mathbf{F}_i^a + \mathbf{F}_i^m \right) \cdot \delta \mathbf{q}_i^C = \sum_j \left(Q_j + \frac{\partial T}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) \right) \delta q_j = 0. \tag{1.31}$$

The Euler–Lagrange equations may be expressed as

$$Q_j = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j}. \tag{1.32}$$

We can now identify two types of external forces: forces that can be derived from a potential function and other generalized forces. Forces that can be derived from a potential are then expressed in terms of a potential energy function and the other generalized forces are denoted by Q_j. The Euler–Lagrange equations now reduce to

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j} \right) - \frac{\partial L}{\partial q_j} = Q_j, \tag{1.33}$$

where $L = T - V$, with V equal to the total potential energy of all the forces that can be derived from a potential function.

1.3 Application to manipulators: Parallel and serial manipulators

Classically, a manipulator is said to be a planar manipulator if all the moving links and their motion are restricted to planes parallel to one another. A manipulator is said to be a spatial manipulator if at least one of the links of the mechanism possesses a general spatial motion in three-dimensional space. A manipulator is said to be a serial manipulator or an open-loop manipulator if all of its links form an open-loop kinematic chain. A manipulator is said to be a parallel manipulator if it is made up of one or more closed-loop kinematic chains. A manipulator is known as a hybrid manipulator if it consists of both open-loop and closed-loop kinematic chains.

Three-degree-of-freedom parallel manipulator

In this example, the motion of the platform of a parallel manipulator is along three axes which are parallel to each other. Consider a uniform homogeneous platform in the shape of an equilateral triangle with a side of length L, in the horizontal plane, supported at its three vertices by three extendable vertical legs. The moving mass of each leg is assumed to be m_l

and the three vertices of the platform are assumed at depths of z_1, z_2 and z_3. The depth of the CM of the platform is

$$z_{cm} = \frac{z_1 + z_2 + z_3}{3}. \tag{1.34}$$

The nose of the platform is represented by the vertex '*1*' and the base by the other two vertices. The longitudinal axis passes through the nose and the mid-point of the base.

The displacement of any point on the platform may be expressed in triangular area coordinates L_j, $j = 1, 2, 3$, as in Vepa [1, Section 5.2.1]

$$w = L_1 z_1 + L_2 z_2 + L_3 z_3. \tag{1.35}$$

The velocity of any point on the surface of the platform is

$$\dot{w} = L_1 \dot{z}_1 + L_2 \dot{z}_2 + L_3 \dot{z}_3. \tag{1.36}$$

The kinetic energy of the platform is given by

$$T = \frac{m_p}{2A} \int_A \left(L_1 \dot{z}_1 + L_2 \dot{z}_2 + L_3 \dot{z}_3 \right)^2 dA. \tag{1.37}$$

Employing the integration formula for polynomial functions of triangle coordinates,

$$\int_A L_1^a L_2^b L_3^c \, dA = \frac{a!b!c!}{(a+b+c+2)!} 2A. \tag{1.38}$$

The total kinetic energy of the platform is

$$T_p = \frac{1}{12} m_p \left(\dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2 + \dot{z}_1 \dot{z}_2 + \dot{z}_2 \dot{z}_3 + \dot{z}_3 \dot{z}_1 \right). \tag{1.39}$$

The kinetic energy of the moving masses of the three legs is

$$T_l = \frac{1}{2} m_l \left(\dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2 \right). \tag{1.40}$$

The total kinetic energy of the manipulator is

$$T = \frac{1}{2} \left(\left(m_l + \frac{m_p}{6} \right) \dot{z}_1^2 + \left(m_l + \frac{m_p}{6} \right) \dot{z}_2^2 + \left(m_l + \frac{m_p}{6} \right) \dot{z}_3^2 + \frac{m_p}{6} \left(\dot{z}_1 \dot{z}_2 + \dot{z}_2 \dot{z}_3 + \dot{z}_3 \dot{z}_1 \right) \right). \tag{1.41}$$

The potential energy in the legs and platform is

$$V = -\left(m_l + \frac{m_p}{3} \right) g \left(z_1 + z_2 + z_3 \right). \tag{1.42}$$

The Lagrangian may be defined as $L = T - V$.

In terms of the generalized coordinates q_j and the generalized applied forces Q_j, the Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial L}{\partial q_j} - \frac{\partial L}{\partial q_j} = Q_j \,. \tag{1.43}$$

The Euler–Lagrange equations are

$$\frac{m_p}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_2 \end{bmatrix} + m_l \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} \ddot{z}_1 \\ \ddot{z}_2 \\ \ddot{z}_2 \end{bmatrix} - \left(m_l + \frac{m_p}{3} \right) g \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \tau_1 & 0 & 0 \\ 0 & \tau_2 & 0 \\ 0 & 0 & \tau_2 \end{bmatrix}. \tag{1.44}$$

It is interesting to note that the equations are linear.

Cartesian and spherical manipulators

A manipulator is said to be a Cartesian manipulator if all its motion can be resolved to one-directional uncoupled motion along three axes which are mutually perpendicular to each other. A manipulator is said to be a spherical manipulator if all the links perform motions over the surface of a sphere referenced to a common stationary point.

In the example of the Cartesian manipulator considered in the following text, the motion of the end effector of the manipulator is resolved along three axes which are mutually perpendicular to each other. An example of a three-dimensional Cartesian manipulator is shown in Figure 1.1.

The 3D Cartesian manipulator is by far the simplest example illustrating the application of the Euler–Lagrange equations. If the mass of link and end effector moving only along the y-axis is m_y, the mass of the link moving in the x-axis alone is m_x and the block moving only along the z-axis is m_z, the total kinetic energy is given by

$$T = \frac{1}{2} m_y \dot{Y}^2 + \frac{1}{2}\left(m_x + m_y \right) \dot{X}^2 + \frac{1}{2}\left(m_z + m_x + m_y \right) \dot{Z}^2. \tag{1.45}$$

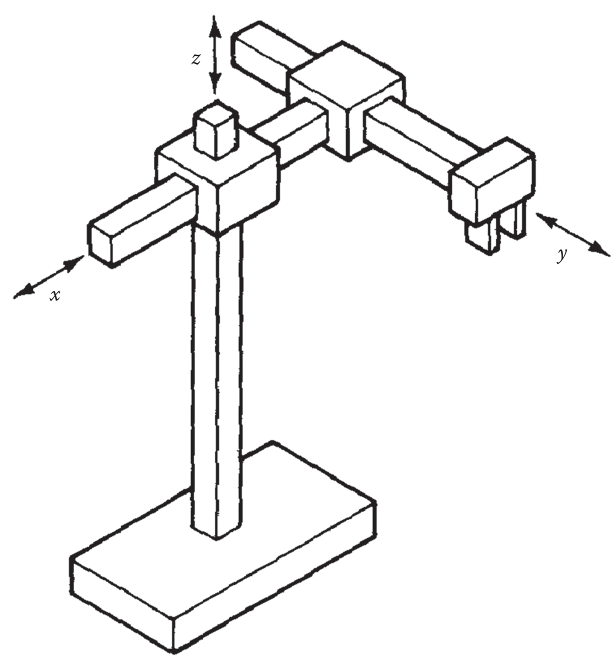

FIGURE 1.1 Example of a 3D Cartesian manipulator.

The total gravitational potential energy stored is given by

$$V = \left(m_z + m_x + m_y\right) gZ. \tag{1.46}$$

The Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{X}} + \frac{\partial V}{\partial X} = \tau_1, \tag{1.47}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{Y}} + \frac{\partial V}{\partial Y} = \tau_2, \tag{1.48}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{Z}} + \frac{\partial V}{\partial Z} = \tau_3. \tag{1.49}$$

Hence,

$$\left(m_x + m_y\right)\ddot{X} = \tau_1, \tag{1.50}$$

$$m_y\ddot{Y} = \tau_2, \tag{1.51}$$

$$\left(m_z + m_x + m_y\right)\ddot{Z} + \left(m_z + m_x + m_y\right) g = \tau_3. \tag{1.52}$$

A typical example of a spherical joint is shown in Figure 1.2.

A manipulator based on the spherical joint may be treated as a rotating planar manipulator and is discussed in a latter section. Typically, a planar manipulator is one where all motion is restricted to a single plane. Any rotational axes are orthogonal to the plane in which the motion is permitted. However, for our purposes, the definition of planar manipulators is generalized so that a wider class of manipulators could be included in this category. Thus, we define a planar manipulator as one where all motion is restricted to a plane and along one or

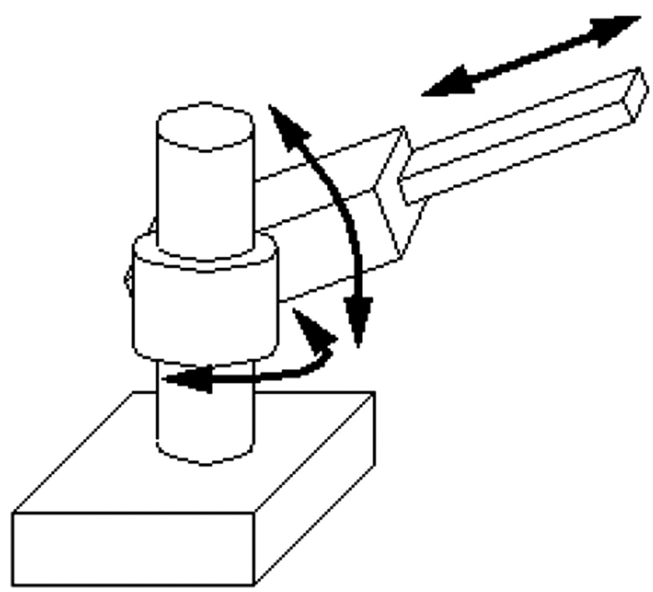

FIGURE 1.2 Example of a 3D spherical manipulator.

more axes normal to the plane; that is in a direction parallel to the axes of rotations. A typical example of such a manipulator is the selectively compliant assembly robot arm (SCARA) manipulator which is considered in a subsequent section.

1.4 Dynamics of planar manipulators: Two-link planar manipulators

Consider the two-link planar arm [1] which is a typical configuration that is a planar open-loop chain with only revolute joints as shown in Figure 1.3, where the end effector and its payload are modeled as a lumped mass, located at the tip of the outer link.

The total kinetic and potential energies will be obtained in terms of the moment of inertia and mass moment components:

$$I_{11} = m_1\left(L_{1cg}^2 + k_{1cg}^2\right) + \left(m_2 + M\right)L_1^2, \tag{1.53}$$

$$I_{21} = \left(m_2 L_{2cg} + ML_2\right)L_1 = \Gamma_{22}L_1, \quad I_{22} = m_2\left(L_{2cg}^2 + k_{2cg}^2\right) + ML_2^2, \tag{1.54}$$

$$\Gamma_{11} = \left(m_1 L_{1cg} + m_2 L_1 + ML_1\right), \quad \Gamma_{22} = \left(m_2 L_{2cg} + ML_2\right). \tag{1.55}$$

In the previous expressions, M is the tip mass and m_i, L_i, L_{icg} and k_{icg} are, respectively, the ith link mass, the ith link length, the ith link's position of the CM with reference to the ith joint and the ith link's radius of gyration about its CM.

Let $q_1 = \theta_1$, the angle of rotation of the first link with respect to the local horizontal, positive counterclockwise, and $q_2 = \theta_2$, the angle of rotation of the second link with respect to the first, positive counterclockwise.

The height of the centre of gravity (CG) of the first link from the axis of the first revolute joint is $Y_1 = L_{1cg}\sin\theta_1$. For the second link, it is $Y_2 = L_1\sin\theta_1 + L_{2cg}\sin(\theta_2 + \theta_1)$, and for the tip mass, it is $Y_{tip} = L_1\sin\theta_1 + L_2\sin(\theta_2 + \theta_1)$.

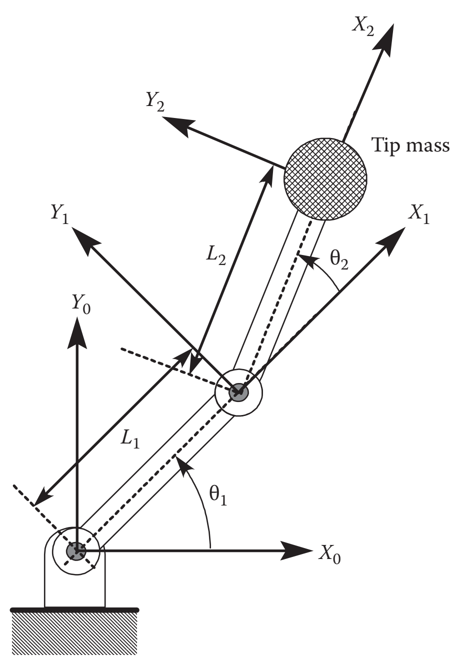

FIGURE 1.3 Two-link planar anthropomorphic manipulator (the ACROBOT); the Z axes are all aligned normal to the plane of the paper.

Increase in the potential energy of the body is

$$V = m_1 g\left[L_{1cg}\sin\theta_1\right] + m_2 g\left[L_1\sin\theta_1 + L_{2cg}\sin\left(\theta_2+\theta_1\right)\right] + Mg\left[L_1\sin\theta_1 + L_2\sin\left(\theta_2+\theta_1\right)\right].$$

Hence, $V = g(m_1L_{1cg} + m_2L_1 + ML_1)\sin\theta_1 + g(m_2L_{2cg} + ML_2)\sin(\theta_1 + \theta_2)$ which may be written as

$$V = g\Gamma_{11}\sin\theta_1 + g\Gamma_{22}\sin\left(\theta_1+\theta_2\right), \tag{1.56}$$

where

$$\Gamma_{11} = \left(m_1L_{1cg} + m_2L_1 + ML_1\right),\quad \Gamma_{22} = \left(m_2L_{2cg} + ML_2\right). \tag{1.57}$$

The horizontal positions of the CG of the first and second link and the tip mass, positive east, are $X_1 = L_{1cg}\cos\theta_1$, $X_2 = L_1\cos\theta_1 + L_{2cg}\cos(\theta_2 + \theta_1)$, $X_{tip} = L_1\cos\theta_1 + L_2\cos(\theta_2 + \theta_1)$.

The horizontal velocities of the CGs of the masses are $\dot{X}_1 = -L_{1cg}\dot{\theta}_1\sin\theta_1$, $\dot{X}_2 = -L_1\dot{\theta}_1\sin\theta_1 - L_{2cg}\left(\dot{\theta}_2+\dot{\theta}_1\right)\sin\left(\theta_2+\theta_1\right)$, $\dot{X}_{tip} = -L_1\dot{\theta}_1\sin\theta_1 - L_2\left(\dot{\theta}_2+\dot{\theta}_1\right)\sin\left(\theta_2+\theta_1\right)$.

The vertical velocities of the CGs of the masses are $\dot{Y}_1 = L_{1cg}\dot{\theta}_1\cos\theta_1$, $\dot{Y}_2 = L_1\dot{\theta}_1\cos\theta_1 + L_{2cg}\left(\dot{\theta}_2+\dot{\theta}_1\right)\cos\left(\theta_2+\theta_1\right)$, $\dot{Y}_{tip} = L_1\dot{\theta}_1\cos\theta_1 + L_2\left(\dot{\theta}_2+\dot{\theta}_1\right)\cos\left(\theta_2+\theta_1\right)$.

The translational kinetic energy for the three masses is

$$T_1 = \frac{1}{2}m_1\left(\dot{X}_1^2+\dot{Y}_1^2\right) + \frac{1}{2}m_2\left(\dot{X}_2^2+\dot{Y}_2^2\right) + \frac{1}{2}M\left(\dot{X}_{tip}^2+\dot{Y}_{tip}^2\right). \tag{1.58}$$

In Equation 1.58,

$$\dot{X}_{tip}^2+\dot{Y}_{tip}^2 = \left(L_1\dot{\theta}_1\sin\theta_1 + L_2\left(\dot{\theta}_2+\dot{\theta}_1\right)\sin\left(\theta_2+\theta_1\right)\right)^2 + \left(L_1\dot{\theta}_1\cos\theta_1 + L_2\left(\dot{\theta}_2+\dot{\theta}_1\right)\cos\left(\theta_2+\theta_1\right)\right)^2.$$

Expanding

$$\begin{aligned}\dot{X}_{tip}^2+\dot{Y}_{tip}^2 &= L_1^2\dot{\theta}_1^2\left(\sin^2\theta_1+\cos^2\theta_1\right) + L_2^2\left(\dot{\theta}_2+\dot{\theta}_1\right)^2\left(\sin^2\left(\theta_2+\theta_1\right)+\cos^2\left(\theta_2+\theta_1\right)\right)\\ &\quad + 2L_1L_2\dot{\theta}_1\left(\dot{\theta}_2+\dot{\theta}_1\right)\left(\sin\theta_1\sin\left(\theta_2+\theta_1\right)+\cos\theta_1\cos\left(\theta_2+\theta_1\right)\right).\end{aligned}$$

The expression reduces to

$$\dot{X}_{tip}^2+\dot{Y}_{tip}^2 = L_1^2\dot{\theta}_1^2 + L_2^2\left(\dot{\theta}_2+\dot{\theta}_1\right)^2 + 2L_1L_2\dot{\theta}_1\left(\dot{\theta}_2+\dot{\theta}_1\right)\cos\theta_2. \tag{1.59}$$

Furthermore,

$$\dot{X}_2^2+\dot{Y}_2^2 = L_1^2\dot{\theta}_1^2 + L_{2cg}^2\left(\dot{\theta}_2+\dot{\theta}_1\right)^2 + 2L_1L_{2cg}\dot{\theta}_1\left(\dot{\theta}_2+\dot{\theta}_1\right)\cos\theta_2; \tag{1.60}$$

$$\dot{X}_1^2+\dot{Y}_1^2 = L_{1cg}^2\dot{\theta}_1^2. \tag{1.61}$$

Substituting and simplifying,

$$T_1 = \frac{1}{2} m_1 L_{1cg}^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(L_1^2 \dot{\theta}_1^2 + L_{2cg}^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 \right) + \frac{1}{2} M \left(L_1^2 \dot{\theta}_1^2 + L_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 \right)$$
$$+ \left(m_2 L_{2cg} + M L_2 \right) L_1 \cos\theta_2 \left(\dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \right). \tag{1.62}$$

The kinetic energy of rotation of the rods is

$$T_2 = \frac{1}{2} m_1 k_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 k_2^2 \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 . \tag{1.63}$$

The total kinetic energy is

$$T = T_1 + T_2 = \frac{1}{2} \left(m_1 \left(L_{1cg}^2 + k_1^2 \right) + \left(m_2 + M \right) L_1^2 \right) \dot{\theta}_1^2 + \frac{1}{2} \left(m_2 \left(L_{2cg}^2 + k_2^2 \right) + M L_2^2 \right) \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2$$
$$+ \left(m_2 L_{2cg} + M L_2 \right) L_1 \cos\theta_2 \left(\dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \right). \tag{1.64}$$

Hence, the total kinetic energy may be expressed as

$$T = \frac{1}{2} I_{11} \dot{\theta}_1^2 + \frac{1}{2} I_{22} \left(\dot{\theta}_1 + \dot{\theta}_2 \right)^2 + I_{21} \dot{\theta}_1 \left(\dot{\theta}_1 + \dot{\theta}_2 \right) \cos\left(\theta_2 \right), \tag{1.65}$$

or as

$$T = \frac{1}{2} \left(I_{11} + I_{22} + 2 I_{21} \cos\left(\theta_2 \right) \right) \dot{\theta}_1^2 + \frac{1}{2} I_{22} \dot{\theta}_2^2 + \left(I_{22} + I_{21} \cos\left(\theta_2 \right) \right) \dot{\theta}_1 \dot{\theta}_2, \tag{1.66}$$

where

$$I_{11} = m_1 \left(L_{1cg}^2 + k_{1cg}^2 \right) + \left(m_2 + M \right) L_1^2$$

$$I_{21} = \left(m_2 L_{2cg} + M L_2 \right) L_1$$

$$I_{22} = m_2 \left(L_{2cg}^2 + k_{2cg}^2 \right) + M L_2^2$$

The total potential energy is

$$V = g \left(m_1 L_{1cg} + m_2 L_1 + M L_1 \right) \sin\theta_1 + g \left(m_2 L_{2cg} + M L_2 \right) \sin\left(\theta_1 + \theta_2 \right), \tag{1.67}$$

which may be written as

$$V = g \Gamma_{11} \sin\theta_1 + g \Gamma_{22} \sin\left(\theta_1 + \theta_2 \right), \tag{1.68}$$

where

$$\Gamma_{11} = \left(m_1 L_{1cg} + m_2 L_1 + M L_1 \right), \quad \Gamma_{22} = \left(m_2 L_{2cg} + M L_2 \right). \tag{1.69}$$

Hence, the Lagrangian may be defined as $L = T - V$.

Euler–Lagrange equations

Applying the Lagrangian energy method, it can be shown that the general equations of motion of a two-link manipulator may be expressed as

$$\dot{\theta}_1 = \omega_1, \tag{1.70}$$

$$\dot{\theta}_2 = \omega_2 - \omega_1, \tag{1.71}$$

$$\begin{bmatrix} I_{11} + I_{21}\cos(\theta_2) & I_{12} \\ I_{21}\cos(\theta_2) & I_{22} \end{bmatrix} \begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} + (m_2 L_{2cg} + M L_2) L_1 \sin(\theta_2) \begin{bmatrix} \omega_1^2 - \omega_2^2 \\ \omega_1^2 \end{bmatrix} + g \begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \tag{1.72}$$

where

$$I_{12} = m_2 \left(L_{2cg}^2 + k_{2cg}^2\right) + M L_2^2 + (m_2 L_{2cg} + M L_2) L_1 \cos(\theta_2) = I_{22} + I_{21}\cos(\theta_2), \tag{1.73}$$

$$\Gamma_1 = \Gamma_{11}\cos(\theta_1) + \Gamma_{22}\cos(\theta_1 + \theta_2), \tag{1.74}$$

$$\Gamma_2 = \Gamma_{22}\cos(\theta_1 + \theta_2). \tag{1.75}$$

The Euler–Lagrange equations are $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i$, where $q_i = \theta_i$; Q_i are the generalized forces other than those accounted for by the potential energy function and are equal to the torques applied by the joint servo motors, T_i.

The partial derivative,

$$\frac{\partial T}{\partial \dot{\theta}_1} = I_{11}\dot{\theta}_1 + I_{22}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + I_{21}\cos(\theta_2)\left(2\dot{\theta}_1 + \dot{\theta}_2\right), \tag{1.76}$$

simplifies to

$$\frac{\partial T}{\partial \dot{\theta}_1} = \left(I_{11} + I_{21}\cos(\theta_2)\right)\dot{\theta}_1 + \left(I_{22} + I_{21}\cos(\theta_2)\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right). \tag{1.77}$$

The other partial derivatives are

$$\frac{\partial T}{\partial \dot{\theta}_2} = I_{21}\cos(\theta_2)\dot{\theta}_1 + I_{22}\left(\dot{\theta}_1 + \dot{\theta}_2\right); \tag{1.78}$$

$$\frac{\partial T}{\partial \theta_1} = 0; \quad \frac{\partial T}{\partial \theta_2} = -(m_2 L_{2cg} + M L_2) L_1 \sin\theta_2 \left(\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right); \tag{1.79}$$

$$\frac{\partial V}{\partial q_1} = g\left(m_1 L_{1cg} + m_2 L_1 + M L_1\right)\cos(\theta_1) + g\left(m_2 L_{2cg} + M L_2\right)\cos(\theta_1 + \theta_2); \tag{1.80}$$

$$\frac{\partial V}{\partial q_2} = g\left(m_2 L_{2cg} + M L_2\right)\cos(\theta_1 + \theta_2). \tag{1.81}$$

Hence, the partial derivatives of the Lagrangian $L = T - V$ are

$$\frac{\partial L}{\partial \dot{q}_1} = \left(I_{11} + I_{21}\cos\left(\theta_2\right)\right)\dot{\theta}_1 + I_{12}\left(\dot{\theta}_1 + \dot{\theta}_2\right), \tag{1.82}$$

$$\frac{\partial L}{\partial \dot{q}_2} = I_{21}\cos\left(\theta_2\right)\dot{\theta}_1 + I_{22}\left(\dot{\theta}_1 + \dot{\theta}_2\right), \quad \frac{\partial L}{\partial q_1} = -g\Gamma_1, \tag{1.83}$$

and

$$\frac{\partial L}{\partial q_2} = -g\Gamma_2 - \left(m_2 L_{2cg} + ML_2\right)L_1 \sin\theta_2\left(\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right), \tag{1.84}$$

where

$$\Gamma_1 = \left(m_1 L_{1cg} + m_2 L_1 + ML_1\right)\cos\left(\theta_1\right) + \left(m_2 L_{2cg} + ML_2\right)\cos\left(\theta_1 + \theta_2\right), \tag{1.85}$$

$$\Gamma_2 = \left(m_2 L_{2cg} + ML_2\right)\cos\left(\theta_1 + \theta_2\right). \tag{1.86}$$

Hence, the two Euler–Lagrange equations of motion are

$$\begin{bmatrix} I_{11} + I_{22} + 2I_{21}\cos\left(\theta_2\right) & I_{22} + I_{21}\cos\left(\theta_2\right) \\ I_{22} + I_{21}\cos\left(\theta_2\right) & I_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} - I_{21}\dot{\theta}_2\sin\left(\theta_2\right)\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$
$$+\left(m_2 L_{2cg} + ML_2\right)L_1\sin\theta_2\left(\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + g\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}. \tag{1.87}$$

In fact, if one defines

$$\mathbf{H} = \begin{bmatrix} I_{11} + I_{22} + 2I_{21}\cos\left(\theta_2\right) & I_{22} + I_{21}\cos\left(\theta_2\right) \\ I_{22} + I_{21}\cos\left(\theta_2\right) & I_{22} \end{bmatrix}, \tag{1.88}$$

$$T = \frac{1}{2}\begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}\mathbf{H}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}, \tag{1.89}$$

and then the two Euler–Lagrange equations of motion are

$$\mathbf{H}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \left(\frac{d}{dt}\mathbf{H}\right)\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}\frac{\partial T}{\partial \theta_2} + g\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}. \tag{1.90}$$

Hence, we have

$$\mathbf{H}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \left(\dot{\theta}_2\mathbf{I} - \frac{1}{2}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 & \dot{\theta}_2 \end{bmatrix}\right)\left(\frac{\partial \mathbf{H}}{\partial \theta_2}\right)\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + g\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}; \tag{1.91}$$

that is, the equations are expressed entirely in terms of the matrix **H**, its partial derivatives and the partial derivatives of the potential energy function. There are indeed several alternate ways of expressing the two Euler–Lagrange equations of motion.

The two Euler–Lagrange equations of motion may also be expressed as

$$\begin{bmatrix} I_{11}+I_{21}\cos(\theta_2) & I_{12} \\ I_{21}\cos(\theta_2) & I_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_1+\ddot{\theta}_2 \end{bmatrix} - I_{21}\dot{\theta}_2\sin(\theta_2)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1+\dot{\theta}_2 \end{bmatrix}$$

$$+I_{21}\sin\theta_2\left(\dot{\theta}_1\left(\dot{\theta}_1+\dot{\theta}_2\right)\right)\begin{bmatrix} 0 \\ 1 \end{bmatrix} + g\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}. \tag{1.92}$$

Let

$$\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1+\dot{\theta}_2 \end{bmatrix}, \tag{1.93}$$

and it follows that

$$\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} \omega_1 \\ \omega_2-\omega_1 \end{bmatrix}. \tag{1.94}$$

Hence, the expression associated with the second term in the Euler–Lagrange equations may be expressed as

$$I_{21}\sin(\theta_2)\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_1+\dot{\theta}_2 \end{bmatrix} = I_{21}\sin(\theta_2)\begin{bmatrix} \omega_1+\omega_2 \\ \omega_1 \end{bmatrix}. \tag{1.95}$$

It follows that

$$\dot{\theta}_2 I_{21}\sin(\theta_2)\begin{bmatrix} \omega_1+\omega_2 \\ \omega_1 \end{bmatrix} = (\omega_2-\omega_1)I_{21}\sin\theta_2\begin{bmatrix} \omega_1+\omega_2 \\ \omega_1 \end{bmatrix} = I_{21}\sin\theta_2\begin{bmatrix} \omega_1^2-\omega_2^2 \\ (\omega_1-\omega_2)\omega_1 \end{bmatrix}. \tag{1.96}$$

Hence, the two Euler–Lagrange equations of motion are

$$\begin{bmatrix} I_{11}+I_{21}\cos(\theta_2) & I_{12} \\ I_{21}\cos(\theta_2) & I_{22} \end{bmatrix}\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} + I_{21}\sin\theta_2\left\{\begin{bmatrix} \omega_1^2-\omega_2^2 \\ (\omega_1-\omega_2)\omega_1 \end{bmatrix} + \omega_1\omega_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\} + g\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix} = \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}. \tag{1.97}$$

The final equations of motion may be written in state-space form as

$$\dot{\theta}_1 = \omega_1, \tag{1.98}$$

$$\dot{\theta}_2 = \omega_2-\omega_1, \tag{1.99}$$

$$\begin{bmatrix} I_{11}+I_{21}\cos(\theta_2) & I_{21}\cos(\theta_2)+I_{22} \\ I_{21}\cos(\theta_2) & I_{22} \end{bmatrix}\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix}+I_{21}\sin(\theta_2)\begin{bmatrix} \omega_1^2-\omega_2^2 \\ \omega_1^2 \end{bmatrix}+g\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}=\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}, \quad (1.100)$$

where

$$\Gamma_1=\Gamma_{11}\cos(\theta_1)+\Gamma_{22}\cos(\theta_1+\theta_2), \quad (1.101)$$

$$\Gamma_2=\Gamma_{22}\cos(\theta_1+\theta_2), \quad (1.102)$$

and

$$I_{11}=m_1\left(L_{1cg}^2+k_{1cg}^2\right)+\left(m_2+M\right)L_1^2, \quad (1.103)$$

$$I_{21}=\left(m_2L_{2cg}+ML_2\right)L_1=\Gamma_{22}L_1, \quad I_{22}=m_2\left(L_{2cg}^2+k_{2cg}^2\right)+ML_2^2, \quad (1.104)$$

$$\Gamma_{11}=\left(m_1L_{1cg}+m_2L_1+ML_1\right), \quad \Gamma_{22}=\left(m_2L_{2cg}+ML_2\right). \quad (1.105)$$

When

$$L_1=L_2=2L_{1cg}=2L_{2cg}=L, \quad m_1=m_2=m, \quad M=\mu m \quad \text{and} \quad k_{1cg}^2=k_{2cg}^2=\frac{1}{12},$$

$$I_{11}=mL^2\left(\frac{4}{3}+\mu\right), \quad I_{21}=mL^2\left(\frac{1}{2}+\mu\right), \quad I_{22}=mL^2\left(\frac{1}{3}+\mu\right), \quad (1.106)$$

$$\Gamma_{11}=mL\left(\frac{3}{2}+\mu\right), \quad \Gamma_{22}=mL\left(\frac{1}{2}+\mu\right). \quad (1.107)$$

$$\begin{bmatrix} \left(\frac{4}{3}+\mu\right)+\left(\frac{1}{2}+\mu\right)\cos(\theta_2) & \left(\frac{1}{2}+\mu\right)\cos(\theta_2)+\left(\frac{1}{3}+\mu\right) \\ \left(\frac{1}{2}+\mu\right)\cos(\theta_2) & \left(\frac{1}{3}+\mu\right) \end{bmatrix}\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix}$$
$$+\left(\frac{1}{2}+\mu\right)\sin(\theta_2)\begin{bmatrix} \omega_1^2-\omega_2^2 \\ \omega_1^2 \end{bmatrix}+\frac{g}{L}\begin{bmatrix} \left(\frac{3}{2}+\mu\right) \\ \left(\frac{1}{2}+\mu\right) \end{bmatrix}=\frac{1}{mL^2}\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}. \quad (1.108)$$

1.5 The SCARA manipulator

The dynamic model of the three-axis SCARA robot [1] is formulated using the Lagrange method. The dynamics of the first two links are identical to the two-link planar manipulator discussed in the preceding section. The dynamics of the third link moving within a prismatic joint and normal to the plane of motion of the first two links is

$$M_3\dot{v}=M_3g+F_3. \quad (1.109)$$

The rotational dynamics of the end effector is

$$I_4\left(\ddot{\theta}_1+\ddot{\theta}_2+\ddot{\theta}_4\right)=T_4. \quad (1.110)$$

1.6 A two-link manipulator on a moving base

A robotic manipulator, which was designed to clean a whiteboard, is shown in Figure 1.4. It consists of a slider constrained to move horizontally above the whiteboard. A two-link planar manipulator is attached to the slider. The two links of the manipulator are attached to each other by a revolute joint. The top end of the manipulator is attached to the slider by a revolute joint while the bottom end is attached to a duster by another revolute joint, and constrained so that the duster cannot rotate relative to the whiteboard. The slider and duster are modeled as point masses.

Our objective is to

1. Apply the Lagrangian energy method
2. Obtain the general equations of motion of a two-link manipulator

To obtain the Euler–Lagrange equations, we must obtain the total kinetic and potential energies in terms of the total mass, moment of inertia and mass moment components. The slider and duster masses are M_s and M, respectively; m_i and L_i are, respectively, the ith link mass and the ith link length; L_{icg} is the position of the CM of the ith link with reference to the ith joint and k_{icg} is the ith link's radius of gyration about its CM. All the disturbance torques and forces are ignored.

Let q_1 be the horizontal displacement of the slider d. Let $\theta_1 = q_2$, the angle of rotation of the first link w.r.t. the local horizontal, positive clockwise and $\theta_2 = q_3$, the angle of rotation of the second link w.r.t. the first, positive clockwise.

The depth of the CG of the first link from the axis of the first revolute joint is

$$X_1 = L_{1cg}\cos(\theta_1).$$

For the second link, it is

$$X_2 = L_1\cos(\theta_1) + L_{2cg}\cos(\theta_2). \tag{1.111}$$

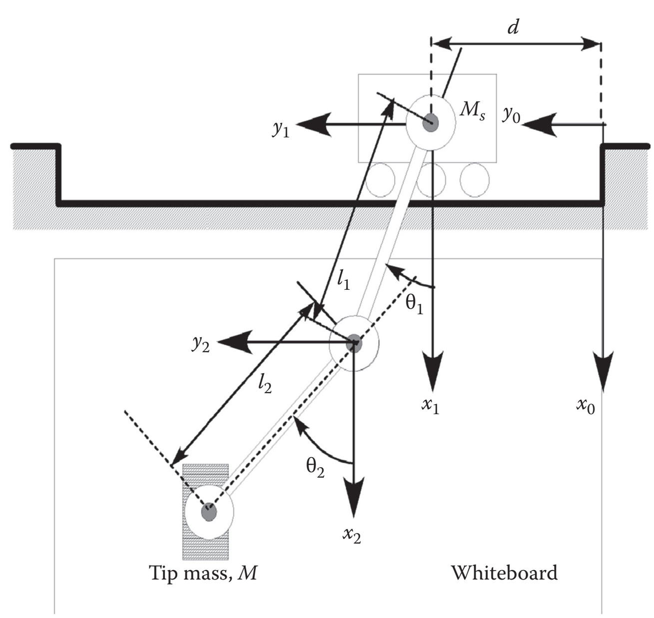

FIGURE 1.4 A robotic manipulator designed to clean a whiteboard.

For the tip mass, it is

$$X_{tip} = L_1 \cos(\theta_1) + L_2 \cos(\theta_2). \tag{1.112}$$

Increase in the potential energy of the system comprising the two links, the duster and the slider, is

$$V = -m_1 g\left[L_{1cg} \cos(\theta_1)\right] - m_2 g\left[L_1 \cos(\theta_1) + L_{2cg} \cos(\theta_2)\right] - Mg\left[L_1 \cos(\theta_1) + L_2 \cos(\theta_2)\right]. \tag{1.113}$$

It may be expressed as

$$V = -g\left(m_1 L_{1cg} + m_2 L_1 + ML_1\right)\cos(\theta_1) - g\left(m_2 L_{2cg} + ML_2\right)\cos(\theta_2), \tag{1.114}$$

or as

$$V = -g\Gamma_{11} \cos(\theta_1) - g\Gamma_{22} \cos(\theta_2), \tag{1.115}$$

where

$$\Gamma_{11} = \left(m_1 L_{1cg} + m_2 L_1 + ML_1\right) \quad \text{and} \quad \Gamma_{22} = \left(m_2 L_{2cg} + ML_2\right). \tag{1.116}$$

The horizontal positions of the CG of the first and second link and the tip mass, positive east, are $Y_1 = d + L_{1cg}\sin(\theta_1)$, $Y_2 = d + L_1\sin(\theta_1) + L_{2cg}\sin(\theta_2)$ and $Y_{tip} = d + L_1\sin(\theta_1) + L_2\sin(\theta_2)$.

The horizontal velocities of the CGs of the masses are

$$\dot{Y}_1 = \dot{d} + L_{1cg}\dot{\theta}_1 \cos(\theta_1), \tag{1.117}$$

$$\dot{Y}_2 = \dot{d} + L_1\dot{\theta}_1 \cos(\theta_1) + L_{2cg}\left(\dot{\theta}_2\right)\cos(\theta_2), \tag{1.118}$$

$$\dot{Y}_{tip} = \dot{d} + L_1\dot{\theta}_1 \cos(\theta_1) + L_2\left(\dot{\theta}_2\right)\cos(\theta_2). \tag{1.119}$$

The vertical velocities of the CGs of the masses are

$$\dot{X}_1 = -L_{1cg}\dot{\theta}_1 \sin(\theta_1), \quad \dot{X}_2 = -L_1\dot{\theta}_1 \sin(\theta_1) - L_{2cg}\left(\dot{\theta}_2\right)\sin(\theta_2), \tag{1.120}$$

$$\dot{X}_{tip} = -L_1\dot{\theta}_1 \sin(\theta_1) - L_2\left(\dot{\theta}_2\right)\sin(\theta_2). \tag{1.121}$$

The kinetic energies of translation of the slider and the manipulator masses are

$$T_s = \frac{1}{2} M_s \dot{d}^2, \tag{1.122}$$

$$T_1 = \frac{1}{2} m_1\left(\dot{X}_1^2 + \dot{Y}_1^2\right) + \frac{1}{2} m_2\left(\dot{X}_2^2 + \dot{Y}_2^2\right) + \frac{1}{2} M\left(\dot{X}_{tip}^2 + \dot{Y}_{tip}^2\right). \tag{1.123}$$

Thus,

$$T_1 = \frac{1}{2} m_1 L_{1cg}^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 \left(L_1^2 \dot{\theta}_1^2 + L_{2cg}^2 \dot{\theta}_2^2 \right) + \frac{1}{2} M \left(L_1^2 \dot{\theta}_1^2 + L_2^2 \dot{\theta}_2^2 \right) + \frac{1}{2} \left(m_1 + m_2 + M \right) \dot{d}^2$$
$$+ \left(m_2 L_{2cg} + M L_2 \right) L_1 \cos\left(\theta_2 - \theta_1\right) \left(\dot{\theta}_1 \dot{\theta}_2 \right) + \dot{d} \left(\left(m_1 L_{1cg} + m_2 L_1 + M L_1 \right) \dot{\theta}_1 \cos\left(\theta_1\right) \right)$$
$$+ \dot{d} \left(\left(m_2 L_{2cg} + M L_2 \right) \left(\dot{\theta}_2 \right) \cos\left(\theta_2\right) \right). \tag{1.124}$$

The kinetic energy of rotation of the links is

$$T_2 = \frac{1}{2} m_1 k_1^2 \dot{\theta}_1^2 + \frac{1}{2} m_2 k_2^2 \dot{\theta}_2^2. \tag{1.125}$$

If we let

$$M_{total} = \left(M_s + m_1 + m_2 + M \right), \quad I_{11} = m_1 \left(L_{1cg}^2 + k_{1cg}^2 \right) + \left(m_2 + M \right) L,$$
$$I_{21} = \left(m_2 L_{2cg} + M L_2 \right) L_1 \cos\left(\theta_2\right), \quad I_{22} = m_2 \left(L_{2cg}^2 + k_{2cg}^2 \right) + M L_2^2, \tag{1.126}$$

the total kinetic energy is

$$T = T_s + T_1 + T_2 = \frac{1}{2} \left(M_s + m_1 + m_2 + M \right) \dot{d}^2$$
$$+ \left(m_1 L_{1cg} + \left(m_2 + M \right) L_1 \right) \dot{d} \dot{\theta}_1 \cos\theta_1 + \left(m_2 L_{2cg} + M L_2 \right) \dot{d} \dot{\theta}_2 \cos\theta_2$$
$$+ \frac{1}{2} \left(m_1 \left(L_{1cg}^2 + k_1^2 \right) + \left(m_2 + M \right) L_1^2 \right) \theta_1^2 + \frac{1}{2} \left(m_2 \left(L_{2cg}^2 + k_2^2 \right) + M L_2^2 \right) \dot{\theta}_2^2$$
$$+ \left(m_2 L_{2cg} + M L_2 \right) L_1 \cos\left(\theta_2 - \theta_1\right) \left(\dot{\theta}_1 \dot{\theta}_2 \right). \tag{1.127}$$

It may be expressed as

$$T = T_s + T_1 + T_2 = \frac{1}{2} M_{total} \dot{d}^2 + \Gamma_{11} \dot{d} \dot{\theta}_1 \cos\theta_1 + \Gamma_{22} \dot{d} \dot{\theta}_2 \cos\theta_2$$
$$+ \frac{1}{2} I_{11} \dot{\theta}_1^2 + \frac{1}{2} I_{22} \dot{\theta}_2^2 + I_{21}\left(\theta\right) \dot{\theta}_1 \dot{\theta}_2. \tag{1.128}$$

since the potential energy is

$$V = -g\Gamma_{11} \cos\left(\theta_1\right) - g\Gamma_{22} \cos\left(\theta_2\right), \tag{1.129}$$

Hence, the Lagrangian may be obtained since it is defined as $L = T - V$. The Euler–Lagrange equations are

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = Q_i, \tag{1.130}$$

where Q_i are the generalized forces other than those accounted for by the potential energy function and are equal to the torques applied by the joint servo motors.

The partial derivatives in the Euler–Lagrange equations are

$$\frac{\partial L}{\partial \dot{q}_1} = M_{total}\dot{d} + \Gamma_{11}\dot{\theta}_1 \cos\theta_1 + \Gamma_{22}\dot{\theta}_2 \cos\theta_2, \tag{1.131}$$

$$\frac{\partial L}{\partial \dot{q}_2} = I_{11}\dot{\theta}_1 + I_{21}\dot{\theta}_2 + \Gamma_{11}\dot{d}\cos\theta_1, \tag{1.132}$$

$$\frac{\partial L}{\partial \dot{q}_3} = I_{21}\dot{\theta}_1 + I_{22}\dot{\theta}_2 + \Gamma_{22}\dot{d}\cos\theta_2, \tag{1.133}$$

$$\frac{\partial L}{\partial q_1} = 0, \tag{1.134}$$

$$\frac{\partial L}{\partial q_2} = -g\Gamma_1 + \left(m_2 L_{2cg} + ML_2\right) L_1\dot{\theta}_1\dot{\theta}_2 \sin\left(\theta_2 - \theta_1\right) - \Gamma_{11}\dot{d}\dot{\theta}_1 \sin\theta_1, \tag{1.135}$$

$$\frac{\partial L}{\partial q_3} = -g\Gamma_2 - \left(m_2 L_{2cg} + ML_2\right) L_1\dot{\theta}_1\dot{\theta}_2 \sin\left(\theta_2 - \theta_1\right) - \Gamma_{22}\dot{d}\dot{\theta}_2 \sin\theta_2. \tag{1.136}$$

If we let

$$\dot{d} \equiv v, \tag{1.137}$$

then

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = M_{total}\dot{v} + \Gamma_{11}\frac{d}{dt}\left(\dot{\theta}_1 \cos\theta_1\right) + \Gamma_{22}\frac{d}{dt}\left(\dot{\theta}_2 \cos\theta_2\right), \tag{1.138}$$

$$M_{total}\dot{v} = -\Gamma_{11}\frac{d}{dt}\left(\dot{\theta}_1 \cos\theta_1\right) - \Gamma_{22}\frac{d}{dt}\left(\dot{\theta}_2 \cos\theta_2\right), \tag{1.139}$$

$$\begin{aligned} &\begin{bmatrix} I_{11} & I_{21} \\ I_{21} & I_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & \dot{I}_{21} \\ \dot{I}_{21} & 0 \end{bmatrix}\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} \Gamma_{11}\ddot{d}\cos\theta_1 \\ \Gamma_{22}\ddot{d}\cos\theta_2 \end{bmatrix} \\ &\quad + \left(m_2 L_{2cg} + ML_2\right) L_1\dot{\theta}_1\dot{\theta}_2 \sin\left(\theta_2 - \theta_1\right)\begin{bmatrix} -1 \\ 1 \end{bmatrix} = -g\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \end{bmatrix}, \end{aligned} \tag{1.140}$$

with

$$\Gamma_1 = \left(m_1 L_{1cg} + m_2 L_1 + ML_1\right)\sin\left(\theta_1\right), \tag{1.141}$$

$$\Gamma_2 = \left(m_2 L_{2cg} + ML_2\right)\sin\left(\theta_2\right). \tag{1.142}$$

Differentiating the inertia expression, I_{21}, with respect to time,

$$\dot{I}_{21} = -\left(m_2 L_{2cg} + ML_2\right) L_1\left(\dot{\theta}_2 - \dot{\theta}_1\right)\sin\left(\theta_2 - \theta_1\right). \tag{1.143}$$

The final state-space equations of the manipulator are

$$\dot{d} = v, \tag{1.144}$$

$$\dot{\theta}_1 = \omega_1, \tag{1.145}$$

$$\dot{\theta}_2 = \omega_2, \tag{1.146}$$

$$M_{total}\dot{v} = -\Gamma_{11}\frac{d}{dt}\left(\dot{\theta}_1\cos\theta_1\right) - \Gamma_{22}\frac{d}{dt}\left(\dot{\theta}_2\cos\theta_2\right), \tag{1.147}$$

$$\begin{bmatrix} I_{11} & I_{21} \\ I_{21} & I_{22} \end{bmatrix}\begin{bmatrix} \dot{\omega}_1 \\ \dot{\omega}_2 \end{bmatrix} + \left(m_2 L_{2cg} + ML_2\right) L_1 \sin\left(\theta_2 - \theta_1\right)\begin{bmatrix} -\omega_2^2 \\ \omega_1^2 \end{bmatrix}$$
$$= -g\begin{bmatrix} \Gamma_{11}\sin\theta_1 \\ \Gamma_{22}\sin\theta_2 \end{bmatrix} - \ddot{d}\begin{bmatrix} \Gamma_{11}\cos\theta_1 \\ \Gamma_{22}\cos\theta_2 \end{bmatrix}. \tag{1.148}$$

1.7 A planar manipulator: The two-arm manipulator with extendable arms

Each of the two extendable arms is assumed to be made of two links, the second moving relative to the first. The CG offset between the first two links is d_1 and the second pair is d_2. The lengths of the first pair of links are assumed to be L_1 and L_2 and those of the second pair L_3 and L_4. The lower arm makes an angle θ_1 to the horizontal while the upper arm makes an angle θ_2 to the lower arm. The horizontal positions of the CG of the first, second, third and fourth links and the tip mass, positive east, are

$$X_1 = L_1\cos\theta_1, \tag{1.149}$$

$$X_2 = \left(L_1 + d_1\right)\cos\theta_1, \tag{1.150}$$

$$X_3 = \left(L_1 + d_1 + L_2\right)\cos\theta_1 + L_3\cos\left(\theta_2 + \theta_1\right), \tag{1.151}$$

$$X_4 = \left(L_1 + d_1 + L_2\right)\cos\theta_1 + \left(L_3 + d_2\right)\cos\left(\theta_2 + \theta_1\right), \tag{1.152}$$

$$X_{tip} = \left(L_1 + d_1 + L_2\right)\cos\theta_1 + \left(L_3 + d_2 + L_4\right)\cos\left(\theta_2 + \theta_1\right). \tag{1.153}$$

The height of the CG of the first link from the axis of the first revolute joint is

$$Y_1 = L_1\sin\theta_1. \tag{1.154}$$

For the second, third and fourth links, it is

$$Y_2 = \left(L_1 + d_1\right)\sin\theta_1, \tag{1.155}$$

$$Y_3 = \left(L_1 + d_1 + L_2\right)\sin\theta_1 + L_3\sin\left(\theta_2 + \theta_1\right), \tag{1.156}$$

$$Y_4 = \left(L_1 + d_1 + L_2\right)\sin\theta_1 + \left(L_3 + d_2\right)\sin\left(\theta_2 + \theta_1\right), \tag{1.157}$$

and for the tip mass, it is

$$Y_{tip} = \left(L_1 + d_1 + L_2\right)\sin\theta_1 + \left(L_3 + d_2 + L_4\right)\sin\left(\theta_2 + \theta_1\right). \tag{1.158}$$

The corresponding velocities are

$$\dot{X}_1 = -L_1\dot{\theta}_1 \sin\theta_1, \tag{1.159}$$

$$\dot{X}_2 = -\left(L_1 + d_1\right)\dot{\theta}_1 \sin\theta_1 + \dot{d}_1 \cos\theta_1, \tag{1.160}$$

$$\dot{X}_3 = -\left(L_1 + d_1 + L_2\right)\dot{\theta}_1 \sin\theta_1 + \dot{d}_1 \cos\theta_1 - L_3 \sin\left(\theta_2 + \theta_1\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right), \tag{1.161}$$

$$\begin{aligned}\dot{X}_4 = {} & -\left(L_1 + d_1 + L_2\right)\dot{\theta}_1 \sin\theta_1 + \dot{d}_1 \cos\theta_1 - \left(L_3 + d_2\right)\sin\left(\theta_2 + \theta_1\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ & + \dot{d}_2 \cos\left(\theta_2 + \theta_1\right),\end{aligned} \tag{1.162}$$

$$\begin{aligned}\dot{X}_{tip} = {} & -\left(L_1 + d_1 + L_2\right)\dot{\theta}_1 \sin\theta_1 + \dot{d}_1 \cos\theta_1 - \left(L_3 + d_2 + L_4\right)\sin\left(\theta_2 + \theta_1\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ & + \dot{d}_2 \cos\left(\theta_2 + \theta_1\right),\end{aligned} \tag{1.163}$$

$$\dot{Y}_1 = L_1\dot{\theta}_1 \cos\theta_1, \tag{1.164}$$

$$\dot{Y}_2 = \left(L_1 + d_1\right)\dot{\theta}_1 \cos\theta_1 + \dot{d}_1 \sin\theta_1, \tag{1.165}$$

$$\dot{Y}_3 = \left(L_1 + d_1 + L_2\right)\dot{\theta}_1 \cos\theta_1 + \dot{d}_1 \sin\theta_1 + L_3 \cos\left(\theta_2 + \theta_1\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right), \tag{1.166}$$

$$\dot{Y}_4 = \left(L_1 + d_1 + L_2\right)\dot{\theta}_1 \cos\theta_1 + \dot{d}_1 \sin\theta_1 + \left(L_3 + d_2\right)\cos\left(\theta_2 + \theta_1\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right) + \dot{d}_2 \sin\left(\theta_2 + \theta_1\right), \tag{1.167}$$

$$\begin{aligned}\dot{Y}_{tip} = {} & \left(L_1 + d_1 + L_2\right)\dot{\theta}_1 \cos\theta_1 + \dot{d}_1 \sin\theta_1 + \left(L_3 + d_2 + L_4\right)\cos\left(\theta_2 + \theta_1\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right) \\ & + \dot{d}_2 \sin\left(\theta_2 + \theta_1\right).\end{aligned} \tag{1.168}$$

The kinetic energy of translation is

$$\begin{aligned}T_1 = {} & \frac{1}{2}m_1 L_1^2\dot{\theta}_1^2 + \frac{1}{2}m_1\left(L_1 + d_1\right)^2\dot{\theta}_1^2 + \frac{1}{2}m_2\dot{d}_1^2 + \frac{1}{2}m_3\left(L_1 + d_1 + L_2\right)^2\dot{\theta}_1^2 + \frac{1}{2}m_3\dot{d}_1^2 + \frac{1}{2}m_3 L_3^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ & + m_3\left\{\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\left(L_1 + d_1 + L_2\right)L_3\cos\theta_2 - \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)L_3\sin\theta_2\right\} \\ & + \frac{1}{2}m_4\left(L_1 + d_1 + L_2\right)^2\dot{\theta}_1^2 + \frac{1}{2}m_4\left(\dot{d}_1^2 + \dot{d}_2^2\right) + \frac{1}{2}m_4\left(L_3 + d_2\right)^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ & + m_4\left\{\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\left(L_1 + d_1 + L_2\right)\left(L_3 + d_2\right)\cos\theta_2 + \dot{d}_1\dot{d}_2\cos\theta_2 - \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\left(L_3 + d_2\right)\sin\theta_2\right\} \\ & + \frac{1}{2}m_{tip}\left(L_1 + d_1 + L_2\right)^2\dot{\theta}_1^2 + \frac{1}{2}m_{tip}\left(\dot{d}_1^2 + \dot{d}_2^2\right) + \frac{1}{2}m_{tip}\left(L_3 + d_2 + L_4\right)^2\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ & + m_{tip}\left\{\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\left(L_1 + d_1 + L_2\right)\left(L_3 + d_2\right)\cos\theta_2 + \dot{d}_1\dot{d}_2\cos\theta_2 - \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\left(L_3 + d_2 + L_4\right)\sin\theta_2\right\} \\ & + \left(m_4 + m_{tip}\right)\dot{d}_2\dot{\theta}_1\left(L_1 + d_1 + L_2\right)\sin\theta_2.\end{aligned} \tag{1.169}$$

The kinetic energy of rotation is

$$T_2 = \frac{1}{2}\left(I_1 + I_2\right)\dot{\theta}_1^2 + \frac{1}{2}\left(I_3 + I_4\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2. \tag{1.170}$$

The total kinetic energy is

$$\begin{aligned} T = T_1 + T_2 &= \frac{1}{2} I_{11}\dot{\theta}_1^2 + \frac{1}{2} m_{22}\dot{d}_1^2 + \frac{1}{2} m_{33}\dot{d}_2^2 + \frac{1}{2} I_{22}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ &+ \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) I_{12}\cos\theta_2 - \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) I_{23}\sin\theta_2 + m_{44}\dot{d}_1\dot{d}_2\cos\theta_2 + I_{14}\dot{d}_2\dot{\theta}_1\sin\theta_2. \end{aligned} \tag{1.171}$$

where

$$\begin{aligned} I_{11} &= I_1 + I_2 + m_1 L_1^2 + m_1\left(L_1 + d_1\right)^2 + m_3\left(L_1 + d_1 + L_2\right)^2 + m_4\left(L_1 + d_1 + L_2\right)^2 \\ &+ m_{tip}\left(L_1 + d_1 + L_2\right)^2 \end{aligned} \tag{1.172}$$

$$I_{22} = I_3 + I_4 + m_3 L_3^2 + m_4\left(L_3 + d_2\right)^2 + m_{tip}\left(L_3 + d_2 + L_4\right)^2, \tag{1.173}$$

$$m_{33} = m_2 + m_3 + m_4 + m_{tip}; \quad m_{44} = m_4 + m_{tip}; \tag{1.174}$$

$$I_{12} = m_3\left(L_1 + d_1 + L_2\right)L_3 + m_4\left(L_1 + d_1 + L_2\right)\left(L_3 + d_2\right) + m_{tip}\left(L_1 + d_1 + L_2\right)\left(L_3 + d_2\right), \tag{1.175}$$

$$I_{23} = m_3 L_3 + m_4\left(L_3 + d_2\right) + m_{tip}\left(L_3 + d_2 + L_4\right), \tag{1.176}$$

$$I_{14} = \left(m_4 + m_{tip}\right)\left(L_1 + d_1 + L_2\right). \tag{1.177}$$

The total change in the potential energy is

$$\begin{aligned} V &= m_1 g L_1\sin\theta_1 + m_2 g\left(L_1 + d_1\right)\sin\theta_1 + m_3 g\left(L_1 + d_1 + L_2\right)\sin\theta_1 + m_3 g L_3\sin\left(\theta_2 + \theta_1\right) \\ &+ m_4 g\left(L_1 + d_1 + L_2\right)\sin\theta_1 + m_4 g\left(L_3 + d_2\right)\sin\left(\theta_2 + \theta_1\right) \\ &+ m_{tip} g\left(L_1 + d_1 + L_2\right)\sin\theta_1 + m_{tip} g\left(L_3 + d_2 + L_4\right)\sin\left(\theta_2 + \theta_1\right). \end{aligned} \tag{1.178}$$

The total change in the potential energy is expressed as

$$\begin{aligned} V &= \left\{m_1 L_1 + m_2\left(L_1 + d_1\right) + m_3\left(L_1 + d_1 + L_2\right) + m_4\left(L_1 + d_1 + L_2\right) + m_{tip}\left(L_1 + d_1 + L_2\right)\right\} g\sin\theta_1 \\ &+ \left\{m_3 L_3 + m_4\left(L_3 + d_2\right) + m_{tip}\left(L_3 + d_2 + L_4\right)\right\} g\sin\left(\theta_2 + \theta_1\right). \end{aligned} \tag{1.179}$$

The partial derivatives in the Euler–Lagrange equations are

$$\frac{\partial T}{\partial\dot{\theta}_1} = I_{11}\dot{\theta}_1 + I_{22}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + \left(2\dot{\theta}_1 + \dot{\theta}_2\right) I_{12}\cos\theta_2 - \dot{d}_1 I_{23}\sin\theta_2 + I_{14}\dot{d}_2\sin\theta_2, \tag{1.180}$$

$$\frac{\partial T}{\partial\dot{\theta}_2} = I_{22}\left(\dot{\theta}_1 + \dot{\theta}_2\right) + \dot{\theta}_1 I_{12}\cos\theta_2 - \dot{d}_1 I_{23}\sin\theta_2, \tag{1.181}$$

$$\frac{\partial T}{\partial \dot{d}_1} = m_{22}\dot{d}_1 - \left(\dot{\theta}_1 + \dot{\theta}_2\right) I_{23} \sin\theta_2 + m_{44}\dot{d}_2 \cos\theta_2, \tag{1.182}$$

$$\frac{\partial T}{\partial \dot{d}_2} = m_{33}\dot{d}_2 + m_{44}\dot{d}_1 \cos\theta_2 + I_{14}\dot{\theta}_1 \sin\theta_2, \tag{1.183}$$

$$\frac{\partial T}{\partial \theta_1} = 0, \tag{1.184}$$

$$\frac{\partial T}{\partial \theta_2} = -\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) I_{12} \sin\theta_2 - \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) I_{23} \cos\theta_2 - m_{44}\dot{d}_1\dot{d}_2 \sin\theta_2 + I_{14}\dot{d}_2\dot{\theta}_1 \cos\theta_2, \tag{1.185}$$

$$\frac{\partial T}{\partial d_1} = \frac{1}{2}\frac{\partial I_{11}}{\partial d_1}\dot{\theta}_1^2 + \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\frac{\partial I_{12}}{\partial d_1}\cos\theta_2 + m_{44}\dot{d}_1\dot{d}_2 \cos\theta_2 + \frac{\partial I_{14}}{\partial d_1}\dot{d}_2\dot{\theta}_1 \sin\theta_2, \tag{1.186}$$

$$\frac{\partial T}{\partial d_2} = \frac{1}{2}\frac{\partial I_{22}}{\partial d_2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 + \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\frac{\partial I_{12}}{\partial d_2}\cos\theta_2 - \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\frac{\partial I_{23}}{\partial d_2}\sin\theta_2, \tag{1.187}$$

$$\frac{\partial V}{\partial \theta_1} = g\Gamma_1 \cos\theta_1 + g\Gamma_2 \cos\left(\theta_2 + \theta_1\right), \tag{1.188}$$

$$\frac{\partial V}{\partial \theta_2} = \Gamma_2 g \cos\left(\theta_2 + \theta_1\right), \tag{1.189}$$

$$\frac{\partial V}{\partial d_1} = \left\{m_2 + m_3 + m_4 + m_{tip}\right\} g \sin\theta_1, \tag{1.190}$$

$$\frac{\partial V}{\partial d_2} = \left\{m_4 + m_{tip}\right\} g \sin\left(\theta_2 + \theta_1\right), \tag{1.191}$$

with

$$\Gamma_1 = m_1L_1 + m_2\left(L_1 + d_1\right) + m_3\left(L_1 + d_1 + L_2\right) + m_4\left(L_1 + d_1 + L_2\right) + m_{tip}\left(L_1 + d_1 + L_2\right), \tag{1.192}$$

$$\Gamma_2 = m_3L_3 + m_4\left(L_3 + d_2\right) + m_{tip}\left(L_3 + d_2 + L_4\right). \tag{1.193}$$

The Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_1} - \frac{\partial T}{\partial \theta_1} + \frac{\partial V}{\partial \theta_1} = \tau_1, \tag{1.194}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\theta}_2} - \frac{\partial T}{\partial \theta_2} + \frac{\partial V}{\partial \theta_2} = \tau_2, \tag{1.195}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{d}_1} - \frac{\partial T}{\partial d_1} + \frac{\partial V}{\partial d_1} = \tau_3, \tag{1.196}$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{d}_2} - \frac{\partial T}{\partial d_2} + \frac{\partial V}{\partial d_2} = \tau_4. \tag{1.197}$$

Hence, we have the following equations of motion:

$$
\begin{aligned}
&I_{11}\ddot{\theta}_1 + I_{22}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + \left(2\ddot{\theta}_1 + \ddot{\theta}_2\right) I_{12}\cos\theta_2 - \ddot{d}_1 I_{23}\sin\theta_2 + I_{14}\ddot{d}_2\sin\theta_2 \\
&+ \frac{\partial I_{11}}{\partial d_1}\dot{\theta}_1\dot{d}_1 + \frac{\partial I_{22}}{\partial d_2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{d}_2 + \left(2\dot{\theta}_1 + \dot{\theta}_2\right)\left(\frac{\partial I_{12}}{\partial d_1}\dot{d}_1 + \frac{\partial I_{12}}{\partial d_2}\dot{d}_2\right)\cos\theta_2 - \dot{d}_1\dot{d}_2\frac{\partial I_{23}}{\partial d_2}\sin\theta_2 \\
&+ \frac{\partial I_{14}}{\partial d_1}\dot{d}_1\dot{d}_2\sin\theta_2 + g\Gamma_1\cos\theta_1 + g\Gamma_2\cos\left(\theta_2 + \theta_1\right) = \tau_1.
\end{aligned}
\tag{1.198}
$$

$$
\begin{aligned}
&I_{22}\left(\ddot{\theta}_1 + \ddot{\theta}_2\right) + \ddot{\theta}_1 I_{12}\cos\theta_2 - \ddot{d}_1 I_{23}\sin\theta_2 + \frac{\partial I_{22}}{\partial d_2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{d}_2 \\
&\quad + \dot{\theta}_1\left(\frac{\partial I_{12}}{\partial d_1}\dot{d}_1 + \frac{\partial I_{12}}{\partial d_2}\dot{d}_2\right)\cos\theta_2 - \dot{\theta}_1\dot{\theta}_2 I_{12}\sin\theta_2 \\
&\quad - \dot{d}_1\dot{d}_2\frac{\partial I_{23}}{\partial d_2}\sin\theta_2 - \dot{d}_1\dot{\theta}_2 I_{23}\cos\theta_2 + \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) I_{12}\sin\theta_2 + \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right) I_{23}\cos\theta_2 \\
&\quad + m_{44}\dot{d}_1\dot{d}_2\sin\theta_2 - I_{14}\dot{d}_2\dot{\theta}_1\cos\theta_2 + \Gamma_2 g\cos\left(\theta_2 + \theta_1\right) = \tau_2,
\end{aligned}
\tag{1.199}
$$

$$
\begin{aligned}
&m_{22}\ddot{d}_1 - \left(\ddot{\theta}_1 + \ddot{\theta}_2\right) I_{23}\sin\theta_2 + m_{44}\ddot{d}_2\cos\theta_2 - \left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{d}_2\frac{\partial I_{23}}{\partial d_2}\sin\theta_2 \\
&\quad - \left(\dot{\theta}_1 + \dot{\theta}_2\right)\dot{\theta}_2 I_{23}\cos\theta_2 - m_{44}\dot{d}_2\dot{\theta}_2\sin\theta_2 - \frac{1}{2}\frac{\partial I_{11}}{\partial d_1}\dot{\theta}_1^2 - \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\frac{\partial I_{12}}{\partial d_1}\cos\theta_2 \\
&\quad - m_{44}\dot{d}_1\dot{d}_2\cos\theta_2 - \frac{\partial I_{14}}{\partial d_1}\dot{d}_2\dot{\theta}_1\sin\theta_2 + \left\{m_2 + m_3 + m_4 + m_{tip}\right\} g\sin\theta = \tau_3,
\end{aligned}
\tag{1.200}
$$

$$
\begin{aligned}
&m_{33}\ddot{d}_2 + m_{44}\ddot{d}_1\cos\theta_2 + I_{14}\ddot{\theta}_1\sin\theta_2 - m_{44}\dot{d}_1\dot{\theta}_2\sin\theta_2 + I_{14}\dot{\theta}_1\dot{\theta}_2\cos\theta_2 \\
&\quad + \frac{\partial I_{14}}{\partial d_1}\dot{d}_1\dot{\theta}_1\sin\theta_2 - \frac{1}{2}\frac{\partial I_{22}}{\partial d_2}\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 - \dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\frac{\partial I_{12}}{\partial d_2}\cos\theta_2 + \dot{d}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\frac{\partial I_{23}}{\partial d_2}\sin\theta_2 \\
&\quad + \left\{m_4 + m_{tip}\right\} g\sin\left(\theta_2 + \theta_1\right) = \tau_4.
\end{aligned}
\tag{1.201}
$$

1.8 The multi-link serial manipulator

The modeling of a multi-link manipulator can be done by adopting the Lagrangian formulation. With the correct choice of reference frames, the dynamics can be reduced to a standard form. The most appropriate choice of the reference frames is not the traditional frames defined by the Denavit and Hartenberg convention. A typical three-link serial manipulator is illustrated in Figure 1.5.

The positions and velocities of link CMs in planar Cartesian coordinates for the first, second, third and Nth links are, respectively, given by

$$x_1 = l_{C1}\cos\theta_1, \quad y_1 = l_{C1}\sin\theta_1, \tag{1.202}$$

$$x_2 = l_1\cos\theta_1 + l_{C2}\cos\theta_2, \tag{1.203}$$

$$y_2 = l_1\sin\theta_1 + l_{C2}\sin\theta_2, \tag{1.204}$$

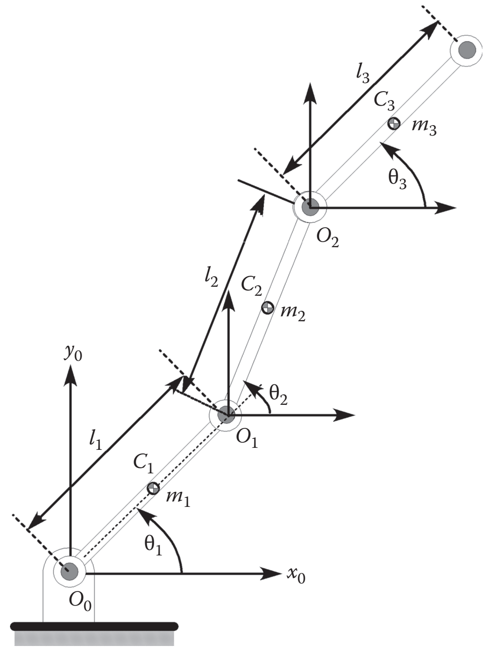

FIGURE 1.5 A typical three-link manipulator showing the definitions of the degrees of freedom.

$$x_3 = l_1\cos\theta_1 + l_2\cos\theta_2 + l_{C3}\cos\theta_3, \tag{1.205}$$

$$y_3 = l_1\sin\theta_1 + l_2\sin\theta_2 + l_{C3}\sin\theta_3, \tag{1.206}$$

$$x_i = \sum_{j=1}^{i-1} l_j\cos\theta_j + l_{Ci}\cos\theta_i, \quad y_i = \sum_{j=1}^{i-1} l_j\sin\theta_j + l_{Ci}\sin\theta_i, \tag{1.207}$$

$$v_{xi} = -\sum_{j=1}^{i-1} \dot{\theta}_j l_j\sin\theta_j - \dot{\theta}_i l_{Ci}\sin\theta_i, \quad v_{yi} = \sum_{j=1}^{i-1} \dot{\theta}_j l_j\cos\theta_j + \dot{\theta}_i l_{Ci}\cos\theta_i. \tag{1.208}$$

The kinetic energy for N links is given by the sum of the translational and rotational kinetic energy and is

$$T = \frac{1}{2}\sum_{i=1}^{N} m_i\left(v_{xi}^2 + v_{yi}^2\right) + \frac{1}{2}\sum_{i=1}^{N} I_{Ci}\dot{\theta}_i^2. \tag{1.209}$$

The potential energy for N links is given by the gravitational potential energy and is

$$V = g\sum_{i=1}^{N} m_i y_i = g\sum_{i=1}^{N} m_i\left(\sum_{j=1}^{i-1} l_j\sin\theta_j + l_{Ci}\sin\theta_i\right) = g\sum_{k=1}^{N} m_k\sum_{i=1}^{k-1} l_i\sin\theta_i + g\sum_{i=1}^{N} m_i l_{Ci}\sin\theta_i. \tag{1.210}$$

To simplify the expression for the total kinetic energy, it may be noted that

$$v_{xi}^2 + v_{yi}^2 = \left(\sum_{j=1}^{i-1} \dot{\theta}_j l_j\sin\theta_j + \dot{\theta}_i l_{Ci}\sin\theta_i\right)^2 + \left(\sum_{j=1}^{i-1} \dot{\theta}_j l_j\cos\theta_j + \dot{\theta}_i l_{Ci}\cos\theta_i\right)^2, \tag{1.211}$$

which is expressed as

$$v_{xi}^2+v_{yi}^2=\dot{\theta}_i^2 l_{Ci}^2+\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \sin\theta_j\right)^2+\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \cos\theta_j\right)^2$$
$$+2l_{Ci}\left(\dot{\theta}_i \sin\theta_i \sum_{j=1}^{i-1}\dot{\theta}_j l_j \sin\theta_j+\dot{\theta}_i \cos\theta_i \sum_{j=1}^{i-1}\dot{\theta}_j l_j \cos\theta_j\right). \tag{1.212}$$

The last term in this expression on the right-hand side of the equation is

$$2l_{Ci}\left(\dot{\theta}_i \sin\theta_i \sum_{j=1}^{i-1}\dot{\theta}_j l_j \sin\theta_j+\dot{\theta}_i \cos\theta_i \sum_{j=1}^{i-1}\dot{\theta}_j l_j \cos\theta_j\right)$$
$$=2l_{Ci}\dot{\theta}_i\sum_{j=1}^{i-1}\dot{\theta}_j l_j\left(\cos\theta_i\cos\theta_j+\sin\theta_i\sin\theta_j\right)=2l_{Ci}\dot{\theta}_i\sum_{j=1}^{i-1}\dot{\theta}_j l_j\cos\left(\theta_i-\theta_j\right). \tag{1.213}$$

The second and third terms are

$$\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \sin\theta_j\right)^2+\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \cos\theta_j\right)^2$$
$$=\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \sin\theta_j\right)\left(\sum_{k=1}^{i-1}\dot{\theta}_k l_k \sin\theta_k\right)+\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \cos\theta_j\right)\left(\sum_{k=1}^{i-1}\dot{\theta}_k l_k \cos\theta_k\right)$$
$$=\sum_{j=1}^{i-1}\dot{\theta}_j^2 l_j^2+\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \sin\theta_j\right)\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k l_k \sin\theta_k\right)+\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \cos\theta_j\right)\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k l_k \cos\theta_k\right). \tag{1.214}$$

The last term in this equation is

$$\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \sin\theta_j\right)\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k l_k \sin\theta_k\right)+\left(\sum_{j=1}^{i-1}\dot{\theta}_j l_j \cos\theta_j\right)\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k l_k \cos\theta_k\right)$$
$$=\sum_{j=1}^{i-1}\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k\dot{\theta}_j l_k l_j\left(\cos\theta_j\cos\theta_k+\sin\theta_j\sin\theta_k\right)\right)=\sum_{j=1}^{i-1}\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k\dot{\theta}_j l_k l_j\cos\left(\theta_j-\theta_k\right)\right). \tag{1.215}$$

Finally,

$$v_{xi}^2+v_{yi}^2=\dot{\theta}_i^2 l_{Ci}^2+\sum_{j=1}^{i-1}\dot{\theta}_j^2 l_j^2+2l_{Ci}\dot{\theta}_i\sum_{j=1}^{i-1}\dot{\theta}_j l_j\cos\left(\theta_i-\theta_j\right)+\sum_{j=1}^{i-1}\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k\dot{\theta}_j l_k l_j\cos\left(\theta_j-\theta_k\right)\right). \tag{1.216}$$

The complete expression for the total kinetic energy of N links is

$$T=\frac{1}{2}\sum_{i=1}^{N}m_i\left(\dot{\theta}_i^2 l_{Ci}^2+\sum_{j=1}^{i-1}\dot{\theta}_j^2 l_j^2\right)+\frac{1}{2}\sum_{i=1}^{N}I_{Ci}\dot{\theta}_i^2+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{i-1}2\dot{\theta}_j\dot{\theta}_i m_i l_{Ci} l_j\cos\left(\theta_i-\theta_j\right)$$
$$+\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{i-1}\left(\sum_{\substack{k=1\\k\neq j}}^{i-1}\dot{\theta}_k\dot{\theta}_j m_i l_k l_j\cos\left(\theta_j-\theta_k\right)\right). \tag{1.217}$$

For $N = 4$, the equations of motion may be defined by employing the Lagrangian approach (Vepa [1]), and the Euler–Lagrange equations are given by

$$\begin{bmatrix} I_{11} & I_{12}C(\theta_2-\theta_1) & I_{13}C(\theta_3-\theta_1) & I_{14}C(\theta_4-\theta_1)\\ I_{12}C(\theta_2-\theta_1) & I_{22} & I_{23}C(\theta_3-\theta_2) & I_{24}C(\theta_4-\theta_2)\\ I_{13}C(\theta_3-\theta_1) & I_{23}C(\theta_3-\theta_2) & I_{33} & I_{34}C(\theta_4-\theta_3)\\ I_{14}C(\theta_4-\theta_1) & I_{24}C(\theta_4-\theta_2) & I_{34}C(\theta_4-\theta_3) & I_{44}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\\ \ddot{\theta}_3\\ \ddot{\theta}_4\end{bmatrix}$$
$$\times\begin{bmatrix} 0 & -I_{12}S(\theta_2-\theta_1) & -I_{13}S(\theta_3-\theta_1) & -I_{14}S(\theta_4-\theta_1)\\ I_{12}S(\theta_2-\theta_1) & 0 & -I_{23}S(\theta_3-\theta_2) & -I_{24}S(\theta_4-\theta_2)\\ I_{13}S(\theta_3-\theta_1) & I_{23}S(\theta_3-\theta_2) & 0 & -I_{34}S(\theta_4-\theta_3)\\ I_{14}S(\theta_4-\theta_1) & I_{24}S(\theta_4-\theta_2) & I_{34}S(\theta_4-\theta_3) & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_2^2\\ \dot{\theta}_3^2\\ \dot{\theta}_4^2\end{bmatrix}$$
$$+g\begin{bmatrix}\Gamma_1 C\theta_1\\ \Gamma_2 C\theta_2\\ \Gamma_3 C\theta_3\\ \Gamma_4 C\theta_4\end{bmatrix}=\begin{bmatrix}M_{12}\\ M_{23}\\ M_{34}\\ M_4\end{bmatrix}. \tag{1.218}$$

In Equation 1.218, the terms in the inertia matrix are defined by

$$\begin{bmatrix} I_{11} & I_{12} & I_{13} & I_{14}\\ I_{12} & I_{22} & I_{23} & I_{24}\\ I_{13} & I_{23} & I_{33} & I_{34}\\ I_{14} & I_{24} & I_{34} & I_{44}\end{bmatrix}=\begin{bmatrix} m_1 l_{C1}^2 & m_2 l_{C2} l_1 & m_3 l_{C3} l_1 & m_4 l_{C4} l_1\\ m_2 l_{C2} l_1 & m_2 l_{C2}^2 & m_3 l_{C3} l_2 & m_4 l_{C4} l_2\\ m_3 l_{C3} l_1 & m_3 l_{C3} l_2 & m_3 l_{C3}^2 & m_4 l_{C4} l_3\\ m_4 l_{C4} l_1 & m_4 l_{C4} l_2 & m_4 l_{C4} l_3 & m_4 l_{C4}^2\end{bmatrix}$$
$$+\begin{bmatrix} I_{C_1}+l_1^2\sum_{i=2}^{N}m_i & l_1 l_2\sum_{i=3}^{N}m_i & l_1 l_3\sum_{i=4}^{N}m_i & 0\\ l_1 l_2\sum_{i=3}^{N}m_i & I_{C_2}+l_2^2\sum_{i=3}^{N}m_i & l_2 l_3\sum_{i=4}^{N}m_i & 0\\ l_1 l_3\sum_{i=4}^{N}m_i & l_2 l_3\sum_{i=4}^{N}m_i & I_{C_3}+l_3^2\sum_{i=4}^{N}m_i & 0\\ 0 & 0 & 0 & I_{C_4}\end{bmatrix}. \tag{1.219}$$

Finally, the terms Γ_i are defined as

$$\begin{bmatrix} \Gamma_1 \\ \Gamma_2 \\ \Gamma_3 \\ \Gamma_4 \end{bmatrix} = \begin{bmatrix} m_1 l_{C1} + m_2 l_1 + m_3 l_1 + m_4 l_1 \\ m_2 l_{C2} + m_3 l_2 + m_4 l_2 \\ m_3 l_{C3} + m_4 l_3 \\ m_4 l_{C4} \end{bmatrix}. \tag{1.220}$$

In Equation 1.218, the angles θ_i are defined in Figure 1.1. In Equation 1.218, m_i and l_i, i =1, 2, 3, …, are the masses and lengths of the links. In Equation 1.218, M_4 is the moment acting on the outer or tip link, while $M_{i,i+1} = M_i - M_{i+1}$ is the net moment acting on the ith link and M_i is the moment acting on the ith link at the pivot O_{i-1}. In Equation 1.219, l_{Ci} are the distances of the link CMs from the pivots while L_i are the link moments of inertia about the CMs. One could introduce a tip mass by suitably altering m_3, l_{C3} and I_{C3}. The functions $C(\cdot)$ and $S(\cdot)$ refer to the trigonometric cosine and sine functions.

1.9 The multi-link parallel manipulator: The four-bar mechanism

In the case of multi-link parallel closed-chain manipulators, one approach of modeling their dynamics is to introduce one or more virtual cuts so as to reduce them to several serial or open-chain manipulators. The closed-chain manipulator is then obtained by introducing holonomic constraints to realize the original closed-chain configuration. A typical example is a four-bar mechanism. In this case, the input crank and the coupler are treated as an independent inverted double pendulum or a two-link planar serial mechanism, while the rocker or output crank is treated as an independent single-link serial manipulator. Two constraints are then applied to realize the original four-bar mechanism.

The kinetic energy of all the bodies in the virtual serial manipulators is expressed as

$$T = T\left(\dot{q}_j, q_j\right), \quad j = 1,2,3,\ldots,J. \tag{1.221}$$

The total potential energy is expressed as

$$V = V\left(q_j\right). \tag{1.222}$$

The holonomic constraints are expressed as

$$\phi_i = \phi_i\left(q_j\right) = 0, \quad i = 1,2,3,\ldots,m. \tag{1.223}$$

The Lagrangian is defined as

$$L = T\left(\dot{q}_j, q_j\right) - V\left(q_j\right) + \sum_{i=1}^{m} \lambda_i \phi_i\left(q_j\right) \equiv T\left(\dot{q}_j, q_j\right) - \bar{V}\left(q_j\right), \tag{1.224}$$

where λ_i are Lagrange multipliers. The Euler–Lagrange equations are

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial \bar{V}}{\partial q_j} = \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial \phi_i\left(q_k\right)}{\partial q_j} = \tau_j. \tag{1.225}$$

They may be expressed as

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = \tau_j + \sum_{i=1}^{m} \lambda_i \frac{\partial \phi_i\left(q_k\right)}{\partial q_j} = \tau_j + \sum_{i=1}^{m} \lambda_i A_{i,j}\left(q_k\right), \tag{1.226}$$

where

$$\frac{\partial \phi_i\left(q_k\right)}{\partial q_j} = A_{i,j}\left(q_k\right). \tag{1.227}$$

These equations correspond to the case when the constraints are expressed in differential form as

$$\sum_{j=1}^{J} A_{i,j}\left(q_k\right)\dot{q}_j = 0, \quad i = 1,2,3,...,m. \tag{1.228}$$

The differential constraints can be either holonomic or even non-holonomic or not integrable.

To apply this method to the four-bar linkage, consider a four-bar mechanism where the fixed link is aligned with the horizontal and of length d, the length of the input crank is L_1, the length of the coupler is L_2 and that of the output crank is L_3. The corresponding masses are m_1, m_2 and m_3. The crank makes an angle θ_1 to the horizontal. The coupler is at angle θ_2 to the crank. The output crank makes an angle θ_3 to the horizontal. The subscripts '*cg*' denote the location of the CG of the corresponding link along the length of the link. k_1, k_2 and k_3 denote the radii of gyration of the input crank, coupler and output crank, respectively. The kinetic energy is expressed as

$$\begin{aligned} T = {} & \frac{1}{2}\left(m_1\left(L_{1cg}^2 + k_1^2\right) + m_2 L_1^2\right)\dot{\theta}_1^2 + \frac{1}{2} m_2\left(L_{2cg}^2 + k_2^2\right)\left(\dot{\theta}_1 + \dot{\theta}_2\right)^2 \\ & + m_2 L_{2cg} L_1 \cos\theta_2\left(\dot{\theta}_1\left(\dot{\theta}_1 + \dot{\theta}_2\right)\right) + \frac{1}{2} m_3\left(L_{3cg}^2 + k_3^2\right)\dot{\theta}_3^2. \end{aligned} \tag{1.229}$$

The potential energy is

$$V = g\left(m_1 L_{1cg} + m_2 L_1\right)\sin\theta_1 + g m_2 L_{2cg}\sin\left(\theta_1 + \theta_2\right) + g m_3 L_{3cg}\sin\theta_3. \tag{1.230}$$

The horizontal and vertical position constraints are

$$L_1\cos\theta_1 + L_2\cos\left(\theta_2 + \theta_1\right) - L_3\cos\theta_3 = d, \tag{1.231}$$

$$L_1\sin\theta_1 + L_2\sin\left(\theta_2 + \theta_1\right) - L_3\sin\theta_3 = 0. \tag{1.232}$$

From these constraint equations,

$$L_2\cos\theta_2 = L_3\cos\left(\theta_3 - \theta_1\right) + d\cos\theta_1 - L_1, \tag{1.233}$$

$$L_2\sin\theta_2 = L_3\sin\left(\theta_3 - \theta_1\right) - d\sin\theta_1. \tag{1.234}$$

Thus,

$$\begin{aligned} & \left(L_3\cos\left(\theta_3 - \theta_1\right) + d\cos\theta_1 - L_1\right)^2 + \left(L_3\sin\left(\theta_3 - \theta_1\right) - d\sin\theta_1\right)^2 \\ & \quad = L_3^2 + L_1^2 + d^2 + 2L_3 d\cos\theta_3 - 2L_1 d\cos\theta_1 - 2L_1 L_3\cos\left(\theta_3 - \theta_1\right) = L_2^2. \end{aligned} \tag{1.235}$$

The latter equation reduces to a quadratic equation for $\cos\theta_1$.

The constraints in differential form are

$$-\dot{\theta}_1\left(L_1\sin\theta_1+L_2\sin\left(\theta_2+\theta_1\right)\right)-\dot{\theta}_2L_2\sin\left(\theta_2+\theta_1\right)+\dot{\theta}_3L_3\sin\theta_3=0, \tag{1.236}$$

$$\dot{\theta}_1\left(L_1\cos\theta_1+L_2\cos\left(\theta_2+\theta_1\right)\right)+\dot{\theta}_2L_2\cos\left(\theta_2+\theta_1\right)-\dot{\theta}_3L_3\cos\theta_3=0. \tag{1.237}$$

The matrix $A_{i,j}(q_k)$ is

$$A_{i,j}\left(q_k\right)=\begin{bmatrix} L_1\sin\theta_1+L_2\sin\left(\theta_2+\theta_1\right) & L_2\sin\left(\theta_2+\theta_1\right) & -L_3\sin\theta_3 \\ L_1\cos\theta_1+L_2\cos\left(\theta_2+\theta_1\right) & L_2\cos\left(\theta_2+\theta_1\right) & -L_3\cos\theta_3 \end{bmatrix}\equiv\begin{bmatrix}\mathbf{A}_{12} & \mathbf{A}_3\end{bmatrix}. \tag{1.238}$$

Hence, the velocity constraints are

$$\mathbf{A}_{12}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}+\mathbf{A}_3\dot{\theta}_3=\begin{bmatrix}0\\0\end{bmatrix}. \tag{1.239}$$

The previous equation can be used to eliminate $\dot{\theta}_1$ and $\dot{\theta}_2$.

The acceleration constraints are obtained by differentiating the velocity constraints and are

$$\mathbf{A}_{12}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix}+\mathbf{A}_3\ddot{\theta}_3+\left\{\dot{\theta}_1\frac{\partial\mathbf{A}_{12}}{\partial\theta_1}+\dot{\theta}_2\frac{\partial\mathbf{A}_{12}}{\partial\theta_2}\right\}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}+\frac{\partial\mathbf{A}_3}{\partial\theta_3}\dot{\theta}_3^2=\begin{bmatrix}0\\0\end{bmatrix}. \tag{1.240}$$

The Euler–Lagrange equations are

$$\begin{bmatrix} I_{11}+I_{21}\cos\left(\theta_2\right) & I_{12} \\ I_{21}\cos\left(\theta_2\right) & I_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_1+\ddot{\theta}_2\end{bmatrix}-I_{21}\sin\left(\theta_2\right)\begin{bmatrix}\dot{\theta}_2 & \dot{\theta}_2\\ \dot{\theta}_2 & \dot{\theta}_1\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_1+\dot{\theta}_2\end{bmatrix}+g\begin{bmatrix}\Gamma_1\\ \Gamma_2\end{bmatrix}=\begin{bmatrix}T_1\\ T_2\end{bmatrix}+\mathbf{A}_{12}^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix} \tag{1.241}$$

$$m_3\left(L_{3cg}^2+k_3^2\right)\ddot{\theta}_3+m_3gL_{3cg}\cos\theta_3=T_3+\mathbf{A}_3^T\begin{bmatrix}\lambda_1\\ \lambda_2\end{bmatrix}, \tag{1.242}$$

where

$$I_{11}=m_1\left(L_{1cg}^2+k_{1cg}^2\right)+m_2L_1^2,\quad I_{21}=m_2L_{2cg}L_1,\quad I_{22}=m_2\left(L_{2cg}^2+k_{2cg}^2\right) \tag{1.243}$$

$$I_{12}=m_2\left(L_{2cg}^2+k_{2cg}^2\right)+m_2L_{2cg}L_1\cos\left(\theta_2\right)=I_{22}+I_{21}\cos\left(\theta_2\right) \tag{1.244}$$

$$\Gamma_1=\left(m_1L_{1cg}+m_2L_1\right)\cos\left(\theta_1\right)+m_2L_{2cg}\cos\left(\theta_1+\theta_2\right),\quad \Gamma_2=m_2L_{2cg}\cos\left(\theta_1+\theta_2\right) \tag{1.245}$$

On eliminating $\ddot{\theta}_1$, $\ddot{\theta}_2$, $\dot{\theta}_1$, $\dot{\theta}_2$, θ_1, θ_2 and the Lagrange multipliers λ_1 and λ_2, the system of equations reduces to those of a one-degree-of-freedom system.

1.10 Rotating planar manipulators: The kinetic energy of a rigid body in a moving frame of reference

The strategy adopted in evaluating the Kinetic Energy (KE) is to independently evaluate the translational KE and the rotational KE of each body at its CM. Thus, we need to find the translational velocity and rotational velocity of each body at its CM.

The velocity of a 'particle' in a body, at a fixed point x, y, z relative to a reference frame that is not fixed, with the velocity of the origin reference to inertial coordinates given as $\mathbf{v}_0$, is

$$v_x = v_{0x} + \omega_y z - \omega_z y, \tag{1.246}$$

$$v_y = v_{0y} + \omega_z x - \omega_x z, \tag{1.247}$$

$$v_z = v_{0z} + \omega_x y - \omega_y x, \tag{1.248}$$

where ω_x, ω_y and ω_z are the components of the body angular velocity in the frame.

If the frame is rotating with angular velocity components Ω_x, Ω_y and Ω_z with respect to an inertial frame, and if in addition the particle is only translating with velocities $\dot{x}$, $\dot{y}$, $\dot{z}$, relative to the frame, the velocity of a particle is

$$v_x = v_{0x} + \dot{x} + \Omega_y z - \Omega_z y, \tag{1.249}$$

$$v_y = v_{0y} + \dot{y} + \Omega_z x - \Omega_x z, \tag{1.250}$$

$$v_z = v_{0z} + \dot{z} + \Omega_x y - \Omega_y x. \tag{1.251}$$

When the point at x, y, z represents the CM of the body,

$$v_{\bar{x}} = v_{0x} + \dot{\bar{x}} + \Omega_y \bar{z} - \Omega_z \bar{y}, \tag{1.252}$$

$$v_{\bar{y}} = v_{0y} + \dot{\bar{y}} + \Omega_z \bar{x} - \Omega_x \bar{z}, \tag{1.253}$$

$$v_{\bar{z}} = v_{0z} + \dot{\bar{z}} + \Omega_x \bar{y} - \Omega_y \bar{x}. \tag{1.254}$$

The translational KE of a body idealized as a particle is

$$T_1 = \frac{1}{2} m \left(v_{\bar{x}}^2 + v_{\bar{y}}^2 + v_{\bar{z}}^2 \right). \tag{1.255}$$

The rotational KE of a body is

$$T_2 = \frac{1}{2} \begin{bmatrix} \omega_x & \omega_y & \omega_z \end{bmatrix} \mathbf{I} \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \tag{1.256}$$

where $\mathbf{I}$ is the moment of inertia matrix of the body about a set of axes passing through the body's CM and ω_x, ω_y and ω_z are the components of the body angular velocity in the body-fixed frame. When the axes are parallel to the principal axes, the rotational KE is

$$T_2 = \frac{1}{2}\left(I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2\right). \tag{1.257}$$

The total KE is the sum of the translational and rotational kinetic energies. Hence,

$$T = T_1 + T_2. \tag{1.258}$$

1.11 An extendable arm spherical manipulator

This manipulator consists of an extendable telescopic arm rotating about a horizontal revolute joint mounted on top of a capstan, along its vertical axis, as shown in Figure 1.2. The capstan can freely rotate about the vertical axis with an angular velocity $\dot{\phi}$. The length of the first link is L and the distance of the CG of the second telescoping link from the end of the first link is d. The position coordinates of the CM of the arm in a rotating frame with the capstan, in terms of the link's pointing angle θ and the distance of the link CG from the capstan axis L_{cg}, are

$$x_1 = L_{cg}\cos\theta, \quad y_1 = L_{cg}\sin\theta, \quad z_1 = 0. \tag{1.259}$$

The corresponding velocities are

$$\dot{x}_1 = -L_{cg}\dot{\theta}\sin\theta, \quad \dot{y}_1 = L_{cg}\dot{\theta}\cos\theta, \quad \dot{z}_1 = -x_1\dot{\phi} \tag{1.260}$$

The position coordinates of the extending second link in a rotating frame rotating with the capstan are

$$x_2 = (L+d)\cos\theta, \quad y_2 = (L+d)\sin\theta, \quad z_2 = 0. \tag{1.261}$$

The corresponding velocities are

$$\dot{x}_2 = -(L+d)\,\dot{\theta}\sin\theta + \dot{d}\cos\theta, \tag{1.262}$$

$$\dot{y}_2 = (L+d)\dot{\theta}\cos\theta + \dot{d}\sin\theta, \tag{1.263}$$

$$\dot{z}_2 = -x_2\dot{\phi}. \tag{1.264}$$

The translation kinetic energy is

$$T_1 = \frac{1}{2}\sum_{j=1}^{2} m_j\left\{(\dot{x}_j)^2 + (\dot{y}_j)^2 + (\dot{z}_j)^2\right\} = \frac{1}{2}m_1L_{cg}^2\left(\dot{\theta}^2 + \dot{\phi}^2\right) + \frac{1}{2}m_2(L+d)^2\left(\dot{\theta}^2 + \dot{\phi}^2\right) + \frac{1}{2}m_2\dot{d}^2. \tag{1.265}$$

The angular velocity components of the two links (links 2 and 3), corresponding to the 3–2 Euler angle sequence, are

$$\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix} = \begin{bmatrix}p_B\\ q_B\\ r_B\end{bmatrix} = \begin{bmatrix}0\\ 1\\ 0\end{bmatrix}\dot{\theta} + \begin{bmatrix}-\sin\theta\\ 0\\ \cos\theta\end{bmatrix}\dot{\phi}. \tag{1.266}$$

The moment of inertia of the capstan about its axis of rotation is I_1. Its angular velocity is $\dot{\phi}$.

Assuming that the moments of inertia of each of the two links (the second and third link) about the body transverse axes are the same and equal to I_j, $j = 2, 3$, and that about the body longitudinal axis are zero, the total kinetic energy of rotation is

$$T_2 = \frac{1}{2}\left(I_2 + I_3\right)\left(\dot{\theta}^2 + \dot{\phi}^2 \cos^2\theta\right) + \frac{1}{2} I_1 \dot{\phi}^2. \tag{1.267}$$

$$T = T_1 + T_2. \tag{1.268}$$

The potential energy is

$$V = g\left(m_1 L_{cg} + m_2\left(L + d\right)\right)\sin\theta. \tag{1.269}$$

The partial derivatives in the Euler–Lagrange equations are

$$\frac{\partial T}{\partial \dot{\phi}} = \left(m_1 L_{cg}^2 + m_2\left(L + d\right)^2 + \left(I_2 + I_3\right)\cos^2\theta + I_1\right)\dot{\phi}, \tag{1.270}$$

$$\frac{\partial T}{\partial \dot{\theta}} = \left(m_1 L_{cg}^2 + m_2\left(L + d\right)^2 + I_2 + I_3\right)\dot{\theta}, \tag{1.271}$$

$$\frac{\partial T}{\partial \dot{d}} = m_2 \dot{d}, \quad \frac{\partial T}{\partial d} = m_2\left(L + d\right)\left(\dot{\theta}^2 + \dot{\phi}^2\right), \tag{1.272}$$

$$\frac{\partial T}{\partial \theta} = -\left(I_2 + I_3\right)\dot{\phi}^2 \cos\theta \sin\theta, \quad \frac{\partial V}{\partial \theta} = g\left(m_1 L_{cg} + m_2\left(L + d\right)\right)\cos\theta, \tag{1.273}$$

$$\frac{\partial V}{\partial d} = g m_2 \sin\theta. \tag{1.274}$$

The Euler–Lagrange equations are

$$\begin{aligned}&\left(m_1 L_{cg}^2 + m_2\left(L + d\right)^2 + \left(I_2 + I_3\right)\cos^2\theta + I_1\right)\ddot{\phi} + 2m_2\left(L + d\right)\dot{d}\dot{\phi}\\ &\quad - 2\left(I_2 + I_3\right)\cos\theta \sin\theta \dot{\phi}\dot{\theta} = \tau_1,\end{aligned} \tag{1.275}$$

$$\begin{aligned}&\left(m_1 L_{cg}^2 + m_2\left(L + d\right)^2 + I_2 + I_3\right)\ddot{\theta} + 2m_2\left(L + d\right)\dot{d}\dot{\theta} + \left(I_2 + I_3\right)\dot{\varphi}^2 \sin\theta\cos\theta\\ &\quad + g\left(m_1 L_{cg} + m_2\left(L + d\right)\right)\cos\theta = \tau_2.\end{aligned} \tag{1.276}$$

$$m_2 \ddot{d} - m_2\left(L + d\right)\left(\dot{\theta}^2 + \dot{\phi}^2\right) + g m_2 \sin\theta = \tau_3. \tag{1.277}$$

Adding a point mass at the tip

If one now adds a point mass at the tip of the manipulator, the position coordinates of the tip must be obtained first. These are

$$x_{tip} = \left(L + d + L_{tip}\right)\cos\theta, \quad y_{tip} = \left(L + d + L_{tip}\right)\sin\theta, \quad z_{tip} = 0. \tag{1.278}$$

The corresponding velocities are

$$\dot{x}_{tip} = -\left(L + d + L_{tip}\right)\dot{\theta}\sin\theta + \dot{d}\cos\theta, \tag{1.279}$$

$$\dot{y}_{tip} = \left(L + d + L_{tip}\right)\dot{\theta}\cos\theta + \dot{d}\sin\theta, \tag{1.280}$$

$$\dot{z}_{tip} = -x_{tip}\dot{\phi}. \tag{1.281}$$

The additional translational kinetic energy due to the mass at the tip is

$$\Delta T_1 = \frac{1}{2}m_{tip}\left(\dot{x}_{tip}^2 + \dot{y}_{tip}^2 + \dot{z}_{tip}^2\right) = \frac{1}{2}m_{tip}\left(L + d + L_{tip}\right)^2\left(\dot{\theta}^2 + \dot{\phi}^2\right) + \frac{1}{2}m_{tip}\dot{d}^2. \tag{1.282}$$

The additional potential energy due to the mass at the tip is

$$\Delta V = gm_{tip}\left(L + d + L_{tip}\right)\sin\theta. \tag{1.283}$$

The new Euler–Lagrange equations are

$$\begin{aligned}&\left(m_1L_{cg}^2 + m_2\left(L+d\right)^2 + m_{tip}\left(L+d+L_{tip}\right)^2 + \left(I_2+I_3\right)\cos^2\theta + I_1\right)\ddot{\phi}\\ &\quad + 2\left\{m_2\left(L+d\right) + m_{tip}\left(L+d+L_{tip}\right)\right\}\dot{d}\dot{\phi} - 2\left(I_2+I_3\right)\cos\theta\sin\theta\dot{\phi}\dot{\theta} = \tau_1,\end{aligned} \tag{1.284}$$

$$\begin{aligned}&\left(m_1L_{cg}^2 + m_2\left(L+d\right)^2 + m_{tip}\left(L+d+L_{tip}\right)^2 + I_2 + I_3\right)\ddot{\theta} + 2\left\{m_2\left(L+d\right) + m_{tip}\left(L+d+L_{tip}\right)\right\}\dot{d}\dot{\theta}\\ &\quad + \left(I_2+I_3\right)\dot{\phi}^2\sin\theta\cos\theta + g\left(m_1L_{cg} + m_2\left(L+d\right) + m_{tip}\left(L+d+L_{tip}\right)\right)\cos\theta = \tau_2,\end{aligned} \tag{1.285}$$

$$\left(m_2 + m_{tip}\right)\ddot{d} - \left\{m_2\left(L+d\right) + m_{tip}\left(L+d+L_{tip}\right)\right\}\left(\dot{\theta}^2 + \dot{\phi}^2\right) + gm_2\sin\theta = \tau_3. \tag{1.286}$$

Adding a spherical 3–2–1 sequence wrist at the tip

We now wish to add a 3–2–1 sequence spherical wrist holding a body at the tip. The body components of the angular velocity vector of the tip are

$$\begin{bmatrix}\Omega_x\\ \Omega_y\\ \Omega_z\end{bmatrix} = \begin{bmatrix}p_{B,tip}\\ q_{B,tip}\\ r_{B,tip}\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}\dot{\theta} + \begin{bmatrix}-\sin\theta\\ 0\\ \cos\theta\end{bmatrix}\dot{\varphi} = \begin{bmatrix}-\dot{\phi}\sin\theta\\ \dot{\theta}\\ \dot{\phi}\cos\theta\end{bmatrix}. \tag{1.287}$$

The angular velocity components of the outer gimbal are

$$\begin{bmatrix}p_{3w}\\ q_{3w}\\ r_{3w}\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}\dot{\psi}_w + \begin{bmatrix}\cos\psi_w & -\sin\psi_w & 0\\ \sin\psi_w & \cos\psi_w & 0\\ 0 & 0 & 1\end{bmatrix}^{-1}\begin{bmatrix}p_{B,tip}\\ q_{B,tip}\\ r_{B,tip}\end{bmatrix} = \begin{bmatrix}\dot{\theta}\sin\psi_w & -\dot{\phi}\sin\theta & \cos\psi_w\\ \dot{\theta}\cos\psi_w & +\dot{\phi}\sin\theta & \sin\psi_w\\ \dot{\psi}_w & +\dot{\phi}\cos\theta & \end{bmatrix}. \tag{1.288}$$

Hence,

$$\left(p_{3w}\right)^2 = \left(\dot{\phi}\sin\theta\cos\psi_w\right)^2 + \left(\dot{\theta}\sin\psi_w\right)^2 - 2\dot{\phi}\dot{\theta}\sin\theta\cos\psi_w\sin\psi_w, \tag{1.289}$$

$$\left(q_{3w}\right)^2 = \left(\dot{\phi}\sin\theta\sin\psi_w\right)^2 + \left(\dot{\theta}\cos\psi_w\right)^2 + 2\dot{\phi}\dot{\theta}\sin\theta\cos\psi_w\sin\psi_w, \tag{1.290}$$

and

$$\left(r_{3w}\right)^2 = \dot{\psi}_w^2 + \dot{\theta}^2\cos^2\theta + 2\dot{\psi}_w\dot{\theta}\cos\theta. \tag{1.291}$$

The additional rotational kinetic energy of the outer gimbal of the wrist is

$$\Delta T_{23} = \frac{1}{2}I_{xx3w}\left(p_{3w}\right)^2 + \frac{1}{2}I_{yy3w}\left(q_{3w}\right)^2 + \frac{1}{2}I_{zz3w}\left(r_{3w}\right)^2, \tag{1.292}$$

where I_{xx3w}, I_{yy3w} and I_{zz3w} are the principal moments of inertia of the outer gimbal. (The wrist rotation axes are assumed to be coincident with the principal axes.)

The angular velocity components of the middle gimbal are

$$\begin{bmatrix} p_{2w} \\ q_{2w} \\ r_{2w} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\dot{\theta}_w + \begin{bmatrix} \cos\theta_w & 0 & \sin\theta_w \\ 0 & 1 & 0 \\ -\sin\theta_w & 0 & \cos\theta_w \end{bmatrix}^{-1} \begin{bmatrix} p_{3w} \\ q_{3w} \\ r_{3w} \end{bmatrix} = \begin{bmatrix} p_{3w}\cos\theta_w - r_{3w}\sin\theta_w \\ \dot{\theta}_w + q_{3w} \\ p_{3w}\sin\theta_w + r_{3w}\cos\theta_w \end{bmatrix}. \tag{1.293}$$

Hence,

$$\left(p_{2w}\right)^2 = \left(p_{3w}\right)^2\cos^2\theta_w + \left(r_{3w}\right)^2\sin^2\theta_w - 2p_{3w}r_{3w}\sin\theta_w\cos\theta_w, \tag{1.294}$$

$$\left(q_{2w}\right)^2 = \dot{\theta}_w^2 + q_{3w}^2 + 2\dot{\theta}_w q_{3w}, \tag{1.295}$$

$$\left(r_{2w}\right)^2 = \left(p_{3w}\right)^2\sin^2\theta_w + \left(r_{3w}\right)^2\cos^2\theta_w + 2p_{3w}r_{3w}\sin\theta_w\cos\theta_w. \tag{1.296}$$

The additional rotational kinetic energy of the middle gimbal of the wrist is

$$\Delta T_{22} = \frac{1}{2}I_{xx2w}\left(p_{2w}\right)^2 + \frac{1}{2}I_{yy2w}\left(q_{2w}\right)^2 + \frac{1}{2}I_{zz2w}\left(r_{2w}\right)^2, \tag{1.297}$$

where I_{xx2w}, I_{yy2w} and I_{zz2w} are the principal moments of inertia of the middle gimbal.

The angular velocity components of the body being held by the wrist are

$$\begin{bmatrix} p_{Bw} \\ q_{Bw} \\ r_{Bw} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\dot{\phi}_w + \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\phi_w & -\sin\phi_w \\ 0 & \sin\phi_w & \cos\phi_w \end{bmatrix}^{-1} \begin{bmatrix} p_{2w} \\ q_{2w} \\ r_{2w} \end{bmatrix}. \tag{1.298}$$

Hence,

$$\left(p_{Bw}\right)^2 = \dot{\phi}_w^2 + p_{2w}^2 + 2\dot{\phi}_w p_{2w}, \tag{1.299}$$

$$\left(q_{Bw}\right)^2 = q_{2w}^2 \cos^2\phi_w + r_{2w}^2 \sin^2\phi_w + 2q_{2w}r_{2w}\cos\phi_w \sin\phi_w, \tag{1.300}$$

$$\left(r_{Bw}\right)^2 = q_{2w}^2 \sin^2\phi_w + r_{2w}^2 \cos^2\phi_w - 2q_{2w}r_{2w}\cos\phi_w \sin\phi_w. \tag{1.301}$$

The additional rotational kinetic energy of the body being held by the inner gimbal of the wrist is

$$\Delta T_{21} \equiv \Delta T_{2B} = \frac{1}{2}I_{xxBw}\left(p_{Bw}\right)^2 + \frac{1}{2}I_{yyBw}\left(q_{Bw}\right)^2 + \frac{1}{2}I_{zzBw}\left(r_{Bw}\right)^2, \tag{1.302}$$

where I_{xxBw}, I_{yyBw} and I_{zzBw} are the principal moments of inertia of body being held by the inner gimbal of the wrist as well as that of the inner gimbal.

The total increase in the rotational kinetic energy is $\Delta T_2 = \Delta T_{21} + \Delta T_{22} + \Delta T_{23}$. The additional terms in the three equations are given by

$$F_j\left(\ddot{q}_j, \dot{q}_j, q_j\right) = \frac{d}{dt}\frac{\partial \Delta T_2}{\partial \dot{q}_j} - \frac{\partial \Delta T_2}{\partial q_j}, \quad j = 1,2,3, \tag{1.303}$$

where $q_j = \phi, \theta, d,\ j = 1,2,3$.

There are also three new equations:

$$\frac{d}{dt}\frac{\partial \Delta T_2}{\partial \dot{q}_j} - \frac{\partial \Delta T_2}{\partial q_j} = \tau_j, \quad j = 4,5,6, \tag{1.304}$$

with $q_j = \psi_w, \theta_w, \phi_w,\ j = 4,5,6$.

1.12 A rotating planar manipulator: The PUMA 560 four-link model

The programmable universal machine for assembly (PUMA) a 560 four-link manipulator is a planar manipulator rotating about a single axis in the plane in which all the translational motions of the main body of the manipulator are taking place. The main body of the manipulator is the manipulator with all the jaws of the gripper in the end effector locked in a position, as it is in this configuration that the manipulator is spatially controlled and/or regulated. The PUMA 560 four-link manipulator is shown in Figure 1.6. Also shown are the DH coordinate systems associated with each link. However, these coordinate frames do not need to be used for angular velocity determination.

The angular velocity of link 1 is $r_B^1 = \dot{\theta}_1$ about the Z_0 axis (three axis). The other two components are zero. The angular velocities of links 2 and 3 are determined from the components of the angular velocity in the Euler angle frames. Using the 32 Euler angle sequence, the body angular velocity components of link 2 are

$$\begin{bmatrix} \Omega_X^2 \\ \Omega_Y^2 \\ \Omega_Z^2 \end{bmatrix} = \begin{bmatrix} p_B^2 \\ q_B^2 \\ r_B^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\dot{\theta}_2 + \begin{bmatrix} -\sin\theta_2 \\ 0 \\ \cos\theta_2 \end{bmatrix}\dot{\theta}_1. \tag{1.305}$$