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Multilinear Algebra

Russell Merris

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Preface

It is not uncommon to find a special richness and vitality at the boundary between mathematical disciplines. With roots in linear algebra, group representation theory, and combinatorics, multilinear algebra is an important example. Serious expeditions into any of these fertile areas require substantial preparation, and multilinear algebra is no exception. The first four chapters of this book consist of self-contained introductions to a variety of prerequisite notions. Multilinear algebra, proper, begins in Chapter 5 with the development of the tensor product. Ironically, it is there, within sight of the goal, that one encounters what is perhaps the most formidable obstacle. In order to prevail over what Cartan has described as *une débauche d'indices*, one must slog through an obscuring foliage of superscripts and subscripts before reaching the heart, in Chapters 6 and 7, of this elegant and beautiful subject.

Many of the topics developed throughout the book are unified in the final chapter by means of the rational representations of the general linear group. Emerging as characters afforded by these representations, the classical Schur polynomials are one of the keys to the overall unification.

Throughout the book, some of the easier proofs are left to the exercises and some of the more difficult ones to the references. Apart from facilitating the flow of material, it is hoped this approach will encourage the reader to become a more active participant in exploring the subject.

Applications of multilinear algebra can be found in many areas of mathematics and physical science, some of them well beyond the author's interest or comprehension. Among those selected for inclusion in the book, graph theoretic applications are dominant. This does not reflect any particularly close connection between graph theory and multilinear algebra. However, applications to graphs suffice to give the flavor of more general combinatorial applications and, by keeping the focus on a single topic, one is able to probe a little deeper than might otherwise be possible.

Despite the book's broad scope, remarkably little prior experience is expected from the reader. It suffices to be familiar with the contents of the standard third year undergraduate courses in abstract and linear algebra. Ideally suited for a fourth year 'capstone' course, *Multilinear Algebra* is also an attractive choice for a beginning graduate course.

The book began as a series of handwritten lecture notes for an MPhil course at the Quaid-I-Azam University of Islamabad in 1973. A revised typescript was prepared later that same year for a seminar at the Instituto de Fisica e Matemática in Lisbon. These early versions were designed to supplement a series of lectures given to students whose native language was something other than English. Nevertheless, the lecture notes were circulated widely by the Institute for the Interdisciplinary Applications of Algebra and Combinatorics at the University of California, Santa Barbara. The present text is dedicated to the hearty folks who struggled through that primitive manuscript *without* the benefit of the author's lectures.

That multilinear algebra has flourished in the years since 1973 can be seen by browsing through the references. Much of this activity was stimulated by the appearance in that year of the first part of Marvin Marcus's monumental *Finite Dimensional Multilinear Algebra*. With the appearance of part II in 1975, *FDMA* became the standard reference, eclipsing the earlier classics of Bourbaki (1948) and Greub (1967), and overshadowing the compact treatises of Amir-Moez (1970s) and Oliveira (1973).

Among the individuals who have contributed to the author's scholarly research are José Dias da Silva, Amélia Fonseca, Bob Grone, Tom Pate, Steve Pierce, and Bill Watkins. He is also grateful for the professional competence of editors Donald Degenhardt, Katie Emblen, Matt Giarratano, Rebecca Stubbs and Brian Wyreweden.

CHAPTER 1

Partitions

The integer 6 is said to be "perfect" because it is the sum of its proper divisors: $6 = 1 + 2 + 3$. In this context, $1 + 2 + 3$ is the same as $2 + 3 + 1$ but different from $4 + 2$. In expressing the perfection of 6 what interests us is the *unordered* collection of its proper divisors, the "partition" of 6 whose "parts" are 3, 2, and 1.

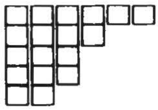
DEFINITION 1.1 A **partition** of n of **length** m is an unordered collection of m positive integers that sum to n. The m summands are the **parts** of the partition.

NOTATION 1.2 A partition of n is typically represented by a sequence $\pi = [\pi_1, \pi_2, \ldots, \pi_m]$, in which the parts of the partition are arranged so that $\pi_1 \geq \pi_2 \geq \cdots \geq \pi_m > 0$. This convention is expressed by the shorthand notation $\pi \vdash n$. The length of π is denoted $L(\pi)$. In the present instance, $L(\pi) = m$.

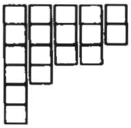
In ordinary English usage, arranging the parts of a partition from largest to smallest would typically be called "ordering" the parts. This semantic difficulty can be the source of some confusion. It is precisely because a partition is unordered that we are free to arrange its parts any way we like.

EXAMPLES 1.3 The partitions of 5 are [5], [4,1], [3,2], [3,1,1], [2,2,1], [2,1,1,1], and [1,1,1,1,1]. The partitions of 6 having 3 parts are [4,1,1], [3,2,1], and [2,2,2]. □

Already, it seems convenient to introduce another shorthand notation. Rather than [3,1,1], [2,2,1], [2,1,1,1], and so on, we will write $[3,1^2]$, $[2^2,1]$ and $[2,1^3]$, respectively. The partition [5,5,5,3,3,3,3,2,1,1] is abbreviated $[5^3,3^4,2,1^2]$. In this notation superscripts are used, not as exponents, but to denote multiplicities. In particular, $[5^3,3^4,2,1^2]$ is a 10-part partition of 31.



$F([6,4,3^2,2])$



$F([5^2,4,2,1^2])$

FIGURE 1.1

Partitions are frequently illustrated by means of so-called **Ferrers diagrams.**[1] If π is a partition of n having m parts, the corresponding Ferrers diagram, $F(\pi)$, consists of m rows of "boxes". The number of boxes in row i of $F(\pi)$ is π_i. The Ferrers diagrams for $[6,4,3^2,2]$ and $[5^2,4,2,1^2]$ are illustrated in Figure 1.1.

DEFINITION 1.4. Suppose $\pi \vdash n$. The **conjugate** of π is the partition π^* whose j-th part is the number of boxes in *column* j of $F(\pi)$. (So, $F(\pi^*)$ is the transpose of $F(\pi)$.)

The conjugate of $[6,4,3^2,2]$ is $[5^2,4,2,1^2]$ as can easily be seen by glancing at Figure 1.1. The length of π^* is the largest part of π, that is, $L(\pi^*) = \pi_1$. Finally, the number of boxes in the j-th column of $F(\pi)$ is equal to the number of rows of $F(\pi)$ that contain at least j boxes, that is, to the number of parts of π that are bigger than or equal to j. In other words, the j-th part of π^* is

$$\pi_j^* = o(\{i : \pi_i \geq j\}), \tag{1.1}$$

where $o(S)$ denotes the **cardinality** of the set S.

EXAMPLE 1.5 The partition π is said to be **self conjugate** if $\pi = \pi^*$, that is, if $F(\pi)$ is symmetric. There is just one self conjugate partition of 6, namely, [3,2,1]. The self conjugate partitions of 9 are illustrated in Figure 1.2. □

FIGURE 1.2 The self conjugate partitions of 9.

[1] After Norman Macleod Ferrers (1829–1903).

Because $\pi_i \geq \pi_{i+1}$ for any $\pi \vdash n$,

$$\begin{aligned} \pi_i - i &\geq \pi_{i+1} - i \\ &> \pi_{i+1} - (i+1). \end{aligned}$$

Thus, the integers $\pi_1 - \pi_i + i$, $1 \leq i \leq L(\pi)$, are all different, that is, $o(\{\pi_1 - \pi_i + i : 1 \leq i \leq L(\pi)\}) = L(\pi)$. Similarly, the cardinality of $\{\pi_1 + \pi_i^* - i + 1 : 1 \leq i \leq \pi_1\}$ is π_1. What may not be so obvious is that these two sets are disjoint.

LEMMA 1.6 *Suppose* $\pi \vdash n$. *Let* $N = \pi_1 + L(\pi)$. *Then* $\{1, 2, \ldots, N\}$ *is the disjoint union of* S *and* T, *where* $S = \{\pi_1 - \pi_i + i : 1 \leq i \leq L(\pi)\}$ *and* $T = \{\pi_1 + \pi_i^* - i + 1 : 1 \leq i \leq \pi_1\}$.

Proof It suffices to show that $S \cap T = \phi$. Observe that

$$\pi_1 - \pi_i + i = \pi_1 + \pi_j^* - j + 1,$$

if and only if $i + j - 1 = \pi_i + \pi_j^*$. To see that this is impossible, suppose first that $\pi_i \geq j$. Then, from Equation (1.1), $\pi_j^* = o(\{k : \pi_k \geq j\}) \geq i$, and $\pi_i + \pi_j^* \geq j + i > i + j - 1$. Therefore, we may assume $\pi_i \leq j - 1$, in which case, $\pi_j^* = o(\{k : \pi_k \geq j\}) < i$. But, $\pi_i \leq j - 1$ and $\pi_j^* \leq i - 1$ imply $\pi_i + \pi_j^* \leq i + j - 2 < i + j - 1$. □

We now discuss "ordering" the different partitions of n.

DEFINITION 1.7 Let $(a) = (a_1, a_2, \ldots, a_r)$ and $(b) = (b_1, b_2, \ldots, b_s)$ be two sequences of real numbers satisfying $a_1 \geq a_2 \geq \cdots \geq a_r \geq 0$ and $b_1 \geq b_2 \geq \cdots \geq b_s \geq 0$. Then (a) **majorizes** (b), written $(a) \succ (b)$, if

$$\sum_{i=1}^{t} a_i \geq \sum_{i=1}^{t} b_i, \quad 1 \leq t \leq r, \tag{1.2}$$

and

$$\sum_{i=1}^{r} a_i = \sum_{i=1}^{s} b_i. \tag{1.3}$$

EXAMPLE 1.8 Suppose n is a fixed positive integer. If $\pi \vdash n$, then $\pi = [\pi_1, \pi_2, \ldots, \pi_m]$ is a nonincreasing sequence of positive real numbers. If $\rho = [\rho_1, \rho_2, \ldots, \rho_k]$ is another partition of n, then $\pi_1 + \pi_2 + \cdots + \pi_m = n = \rho_1 + \rho_2 + \cdots + \rho_k$, and Equation (1.3) is satisfied automatically.

Suppose $n = 8$. If $\pi = [5,2,1]$, and $\rho = [3^2,1^2]$, then $\pi \succ \rho$ because $5 \geq 3$, $5+2 \geq 3+3$, $5+2+1 \geq 3+3+1$, and $5+2+1 = 3+3+1+1$. If $\pi = [5,2,1]$ and $\rho = [4^2]$, then neither partition majorizes the other. Thus, majorization is a partial order. Figure 1.3 exhibits the "Hasse Diagram" for the partitions of 6 partially ordered by majorization. □

Of the many conditions equivalent to majorization, one of the most useful involves doubly stochastic matrices.

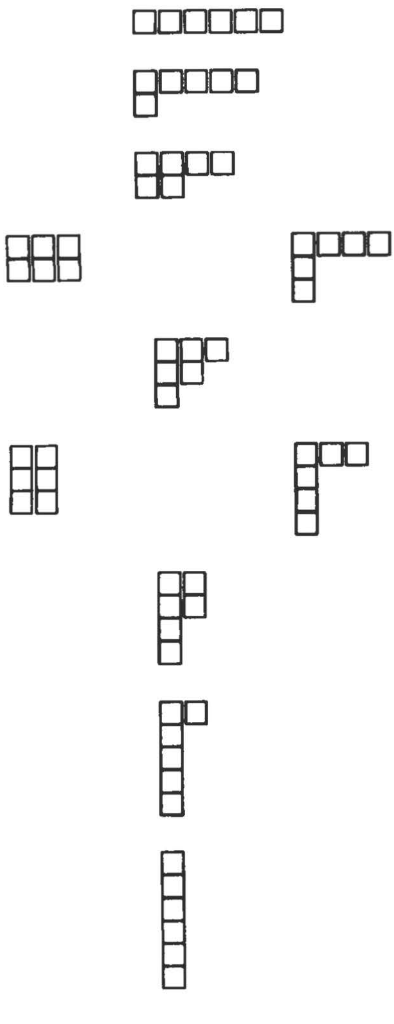


FIGURE 1.3 The partitions of 6 partially ordered by majorization.

DEFINITION 1.9 The n-by-n (entrywise) nonnegative matrix $S = (s_{ij})$ is **doubly stochastic** if its rows and columns all sum to 1, that is, if

$$\sum_{j=1}^{n} s_{ij} = 1, \quad 1 \le i \le n, \quad \text{and} \quad \sum_{i=1}^{n} s_{ij} = 1, \quad 1 \le j \le n.$$

THEOREM 1.10 *Let $(a) = (a_1, a_2, \ldots, a_n)$ and $(b) = (b_1, b_2, \ldots, b_n)$ be two sequences of real numbers satisfying $a_1 \ge a_2 \ge \cdots \ge a_n \ge 0$ and $b_1 \ge b_2 \ge \cdots \ge b_n \ge 0$. Then (a) majorizes (b) if and only if there is a doubly stochastic matrix S such that $(b) = (a)S$.*

Theorem 1.10 is stated for the case in which both sequences have the same length. Because adding zeros to the end of the shorter sequence does not affect majorization, this hypothesis does not impose any real restriction. A proof can be found in [Hardy, Littlewood & Pólya (1967), pp. 47–49] or [Marshall & Olkin (1979), p. 22].

EXAMPLE 1.11 We saw in Example 1.8 that $[5,2,1] \succ [3^2,1^2]$. As an illustration of Theorem 1.10, observe that $(3,3,1,1) = (5,2,1,0)S$, where

$$S = \frac{1}{6}\begin{pmatrix} 2 & 3 & 1 & 0 \\ 4 & 0 & 0 & 2 \\ 0 & 3 & 1 & 2 \\ 0 & 0 & 4 & 2 \end{pmatrix}.$$

If S is an n-by-n doubly stochastic matrix then [Birkhoff (1946)] there exist permutation matrices $P_1, P_2, \ldots, P_k$ and positive real numbers $\theta_1, \theta_2, \ldots, \theta_k$ such that $\theta_1 + \theta_2 + \cdots + \theta_k = 1$ and

$$S = \theta_1 P_1 + \theta_2 P_2 + \cdots + \theta_k P_k.$$

In other words, S is a **convex combination** (or "weighted average") of permutation matrices. Using these terms, Theorem 1.10 can be restated as follows: (a) majorizes (b) if and only if (b) is a convex combination of rearrangements of (a). In particular,

$$(3,3,1,1) = \frac{1}{3}(2,5,0,1) + \frac{1}{3}(5,1,0,2) + \frac{1}{6}(2,5,1,0) + \frac{1}{6}(2,1,5,0).$$

□

Apart from their intrinsic interest, the partitions of n have a variety of uses, one of which involves symmetric polynomials.

DEFINITION 1.12 A polynomial $f(x_1, x_2, \ldots, x_k)$ is **symmetric** in $x_1, x_2, \ldots, x_k$ if its value is unchanged by any permutation of the k variables, that is, if $f(x_1, x_2, \ldots, x_k) = f(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(k)})$, for every permutation σ of $\{1, 2, \ldots, k\}$.[2]

Perhaps the most natural way to begin a discussion of symmetric polynomials is with the notorious "multinomial theorem".

THE MULTINOMIAL THEOREM 1.13 *If n is a positive integer, then*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum \binom{n}{r_1, r_2, \ldots, r_k} x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}, \tag{1.4}$$

where the sum is over all nonnegative integer sequences, $(r_1, r_2, \ldots, r_k)$, satisfying $r_1 + r_2 + \cdots + r_k = n$, and

$$\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \ldots r_k!}$$

is the corresponding ***multinomial coefficient.***

Proofs can be found in any of the standard books on combinatorics.[3]

EXAMPLE 1.14 The coefficient of b^4c^2 in $(a + b + c)^6$ is

$$\binom{6}{0, 4, 2} = \frac{6!}{0!4!2!} = \frac{6!}{4!2!} = 15.$$

Because $(a + b + c)^6$ is symmetric in a, b, and c, the coefficients of a^4b^2 and a^2c^4 in $(a+b+c)^6$ must be 15 as well. One "piece" of the multinomial expansion of $(a + b + c)^6$ is $15p(x)$, where

$$p(x) = a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4. \tag{1.5}$$

□

DEFINITION 1.15 Let k and n be positive integers, and π be a partition of n of length $m \leq k$. **The monomial symmetric function**

$$M_\pi(x_1, x_2, \ldots, x_k) = \sum x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}, \tag{1.6}$$

where the sum is over all *different* rearrangements, $(r_1, r_2, \ldots, r_k)$, of the k-tuple $(\pi_1, \pi_2, \ldots, \pi_m, 0, 0, \ldots, 0)$, obtained by appending $k - m$ zeros to the end of π. If $m > k$, then $M_\pi(x_1, x_2, \ldots, x_k) = 0$.

---

[2] This is why the group of all permutations of {1,2,...,k} is called the "symmetric" group.

[3] See, for example, [Merris (1996)].

If $m = 2, k = 3$, and $\pi = [\pi_1, \pi_2] = [2, 2]$, then the "different rearrangements" of (2,2,0) are

$$(2, 2, 0), \quad (2, 0, 2), \quad \text{and} \quad (0, 2, 2),$$

not the six rearrangements of the different looking symbols π_1, π_2, and 0. Thus,

$$M_{[2,2]}(x, y, z) = x^2y^2 + x^2z^2 + y^2z^2.$$

The "piece" of the multinomial expansion of $(a + b + c)^6$ exhibited in Equation (1.5) is

$$M_{[4,2]}(a, b, c) = a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4.$$

Any symmetric polynomial is a linear combination of minimally symmetric pieces, namely, the monomial symmetric functions. We shall have more to say about this presently.

EXAMPLE 1.16 There are exactly seven partitions of 6 having three or fewer parts. So, there are seven (nonzero) monomial symmetric functions of degree 6 in the three variables a, b, and c. They are

$$M_{[6]}(a, b, c) = a^6 + b^6 + c^6,$$

$$M_{[5,1]}(a, b, c) = a^5b + a^5c + ab^5 + ac^5 + b^5c + bc^5,$$

$$M_{[4,2]}(a, b, c) = a^4b^2 + a^4c^2 + a^2b^4 + a^2c^4 + b^4c^2 + b^2c^4,$$

$$M_{[3^2]}(a, b, c) = a^3b^3 + a^3c^3 + b^3c^3,$$

$$M_{[4,1^2]}(a, b, c) = a^4bc + ab^4c + abc^4,$$

$$M_{[3,2,1]}(a, b, c) = a^3b^2c + a^3bc^2 + a^2b^3c + a^2bc^3 + ab^3c^2 + ab^2c^3,$$

and

$$M_{[2^3]}(a, b, c) = a^2b^2c^2.$$

Setting $M_\pi = M_\pi(a, b, c)$ we obtain, from the multinomial theorem, that

$$\begin{aligned}(a + b + c)^6 = {} & M_{[6]} + 6M_{[5,1]} + 15M_{[4,2]} + 20M_{[3^2]} \\ & + 30M_{[4,1^2]} + 60M_{[3,2,1]} + 90M_{[2^3]}.\end{aligned}$$

□

If $\pi = [\pi_1, \pi_2, \ldots, \pi_m]$ is some fixed but arbitrary partition of n, define

$$\binom{n}{\pi} = \frac{n!}{\pi_1!\pi_2!\ldots\pi_m!}.$$

Using this notation, the multinomial theorem can be restated as follows:

THEOREM 1.17 *If n is a positive integer, then*

$$(x_1 + x_2 + \cdots + x_k)^n = \sum_{\pi \vdash n} \binom{n}{\pi} M_\pi(x_1, x_2, \ldots, x_k). \tag{1.7}$$

We now give special names to the two "extreme" monomial symmetric functions, the ones corresponding to the partitions $[n]$ and $[1^n]$.

NOTATION 1.18 Let $P_n(x_1, x_2, \ldots, x_k) = M_{[n]}(x_1, x_2, \ldots, x_k)$ and $E_n(x_1, x_2, \ldots, x_k) = M_{[1^n]}(x_1, x_2, \ldots, x_k)$.

It is easy to recognize P_n; it is the n-th **power sum**,

$$P_n(x_1, x_2, \ldots, x_k) = x_1^n + x_2^n + \cdots + x_k^n.$$

What about E_n?

EXAMPLE 1.19 Let's choose $k = 4$. Then

$$E_1(a, b, c, d) = M_{[1]}(a, b, c, d) = a + b + c + d;$$

$$E_2(a, b, c, d) = M_{[1^2]}(a, b, c, d) = ab + ac + ad + bc + bd + cd;$$

$$E_3(a, b, c, d) = M_{[1^3]}(a, b, c, d) = abc + abd + acd + bcd; \quad \text{and}$$

$$E_4(a, b, c, d) = M_{[1^4]}(a, b, c, d) = abcd.$$

Evidently, $E_n(a, b, c, d)$ is the sum of all $C(4, n)$ (**binomial coefficient** $\binom{4}{n}$) products of the x's taken n at a time. □

If $(r_1, r_2, \ldots, r_k)$ is some rearrangement of the sequence $(1, 1, \ldots, 1, 0, 0, \ldots, 0)$ consisting of n ones followed by $k - n$ zeros, then

$$x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k} = x_{i_1} x_{i_2} \ldots x_{i_n},$$

where $i_1 < i_2 < \cdots < i_n$. Summing over the different rearrangements gives

$$E_n(x_1, x_2, \ldots, x_k) = \sum x_{i_1} x_{i_2} \ldots x_{i_n}, \tag{1.8}$$

where the summation is over all $C(k, n)$ sequences $(i_1, i_2, \ldots, i_n)$ satisfying $1 \le i_1 < i_2 < \cdots < i_n \le k$.

DEFINITION 1.20 Denote by $\Gamma_{n,k}$ the set of all functions from $\{1, 2, \ldots, n\}$ into $\{1, 2, \ldots, k\}$. Let $Q_{n,k}$ be the subset of $\Gamma_{n,k}$ consisting of the $C(k, n)$ strictly increasing functions.

There is a natural one-to-one correspondence between the functions $\beta \in \Gamma_{n,k}$ and the integer sequences $(i_1, i_2, \ldots, i_n)$ satisfying $1 \le i_t \le k$, $1 \le t \le n$, namely, $\beta \leftrightarrow (\beta(1), \beta(2), \ldots, \beta(n))$. We will feel free to abuse the language by identifying $\Gamma_{n,k}$ with a set of sequences. Thus,

$$\Gamma_{2,3} = \{(1,1), (1,2), (1,3), (2,1), (2,2), (2,3), (3,1), (3,2), (3,3)\},$$

and $Q_{2,3} = \{(1,2), (1,3), (2,3)\}$.

Using Definition 1.20, we may rewrite Equation (1.8) as

$$E_n(x_1, x_2, \ldots, x_k) = \sum_{\beta \in Q_{n,k}} x_{\beta(1)} x_{\beta(2)} \ldots x_{\beta(n)}. \tag{1.9}$$

DEFINITION 1.21 The "extreme" monomial symmetric function, $E_n(x_1, x_2, \ldots, x_k)$ is the n-th **elementary symmetric function** of $x_1, x_2, \ldots, x_k$. It is useful to define $E_0(x_1, x_2, \ldots, x_k) = 1$.

Elementary symmetric functions are familiar objects. They express the coefficients of a monic polynomial in terms of its roots. If, for example, a, b, c, and d are complex numbers, then (Example 1.19)

$$(x-a)(x-b)(x-c)(x-d) = x^4 - E_1x^3 + E_2x^2 - E_3x + E_4, \tag{1.10}$$

where $E_n = E_n(a, b, c, d)$, $1 \le n \le 4$.

FUNDAMENTAL THEOREM OF SYMMETRIC FUNCTIONS 1.22 *Any polynomial, symmetric in the variables $x_1, x_2, \ldots, x_k$, is a polynomial in the elementary symmetric functions $E_n(x_1, x_2, \ldots, x_k)$, $1 \le n \le k$.*

Proof Let $f = f(x_1, x_2, \ldots, x_k)$ be a symmetric polynomial of (total) degree p. Write $f = f_0 + f_1 + \cdots + f_p$, where $f_i = f_i(x_1, x_2, \ldots, x_k)$ is the (homogeneous) part of f consisting of all terms of degree i. It will suffice to show that f_i is a polynomial in the elementary symmetric functions for a fixed but arbitrary i.

Suppose

$$cx_1^{r_1}x_2^{r_2}\ldots x_k^{r_k} \tag{1.11}$$

is one of the monomial terms that occur in f_i. Then $r_1 + r_2 + \cdots + r_k = i$. By symmetry, we may assume that

$$r_1 \ge r_2 \ge \cdots \ge r_t > 0 = r_{t+1} = \cdots = r_k.$$

Among all partitions of i occurring as the sequence of exponents in the monomials of f_i, assume $[r_1, r_2, \ldots, r_t]$ is last in **lexicographic** (dictionary) order. That is, without loss of generality, we may assume r_1 is the largest single exponent that occurs in any monomial in f_i; r_2 is the maximum second largest exponent among all the monomials that occur in f_i and have r_1 as their largest exponent; r_3 is the maximum third largest exponent among all the monomials that occur in f_i and have r_1 and r_2 as their two largest exponents; and so on.

Consider

$$E_1^{s_1} E_2^{s_2} \ldots E_k^{s_k}, \tag{1.12}$$

where $E_n = E_n(x_1, x_2, \ldots, x_k)$, $1 \le n \le k$. In lexicographic order of its *exponents* the last monomial that occurs in (1.12) is

$$x_1^{s_1}(x_1x_2)^{s_2}(x_1x_2x_3)^{s_3} \ldots (x_1x_2 \ldots x_k)^{s_k}. \tag{1.13}$$

We would like to choose s_1, s_2, s_3, and so on, so that

$$x_1^{s_1}(x_1x_2)^{s_2}(x_1x_2x_3)^{s_3} \ldots (x_1x_2 \ldots x_k)^{s_k} = x_1^{r_1} x_2^{r_2} \ldots x_k^{r_k}.$$

This requires that

$$\begin{aligned}
r_1 &= s_1 + s_2 + s_3 + \cdots + s_k, \\
r_2 &= \phantom{s_1 +{}} s_2 + s_3 + \cdots + s_k, \\
r_3 &= \phantom{s_1 + s_2 +{}} s_3 + \cdots + s_k, \\
&\phantom{= s_1 + s_2 + s_3 + \cdots{}} \ldots \\
r_k &= \phantom{s_1 + s_2 + s_3 + \cdots + {}} s_k.
\end{aligned}$$

These equations are satisfied when $s_k = r_k$, $s_{k-1} = r_{k-1} - r_k, \ldots, s_2 = r_2 - r_3$, and $s_1 = r_1 - r_2$. If we make these choices, then either

$$f_i - cE_1^{s_1} E_2^{s_2} \ldots E_k^{s_k} = 0,$$

or it is a symmetric homogeneous polynomial of degree i, each of whose monomial terms comes before (1.11) in lexicographic order. Because dictionary ordering is a total order, the result follows by induction. □

Suppose $f = f(x_1, x_2, \ldots, x_k)$ is a symmetric homogeneous polynomial of degree n. Then f is, simultaneously, a polynomial in the elementary symmetric functions $E_n(x_1, x_2, \ldots, x_k)$, $1 \le n \le k$, and a linear combination of the monomial symmetric functions $M_\pi(x_1, x_2, \ldots, x_k)$, $\pi \vdash n$. Conversely, if c_π, $\pi \vdash n$, are constants, then

$$g(x_1, x_2, \ldots, x_k) = \sum_{\pi \vdash n} c_\pi M_\pi(x_1, x_2, \ldots, x_k)$$

defines a symmetric homogeneous polynomial of degree n. If

$$c_\pi = \binom{n}{\pi}$$

for all π, then $g(x_1, x_2, \ldots, x_k) = (x_1 + x_2 + \cdots + x_k)^n$. What about some other choices? An important and interesting example arises when $c_\pi = 1$ for all π.

DEFINITION 1.23 Let $x_1, x_2, \ldots, x_k$ be independent variables. Their n-th **homogeneous symmetric function** is defined by

$$H_n(x_1, x_2, \ldots, x_k) = \sum_{\pi \vdash n} M_\pi(x_1, x_2, \ldots, x_k). \tag{1.14}$$

It is convenient to define $H_0(x_1, x_2, \ldots, x_k) = 1$.

EXAMPLES 1.24

$$\begin{aligned} H_2(a, b, c) &= M_{[2]}(a, b, c) + M_{[1^2]}(a, b, c) \\ &= P_2(a, b, c) + E_2(a, b, c) \\ &= (a^2 + b^2 + c^2) + (ab + ac + bc), \\ H_3(a, b, c) &= M_{[3]}(a, b, c) + M_{[2,1]}(a, b, c) + M_{[1^3]}(a, b, c) \\ &= (a^3 + b^3 + c^3) + (a^2b + a^2c + ab^2 + ac^2 + b^2c + bc^2) + abc, \end{aligned} \tag{1.15}$$

and

$$\begin{aligned} H_4(a, b, c) &= M_{[4]}(a, b, c) + M_{[3,1]}(a, b, c) + M_{[2^2]}(a, b, c) + M_{[2,1^2]}(a, b, c) \\ &= (a^4 + b^4 + c^4) + (a^3b + a^3c + ab^3 + ac^3 + b^3c + bc^3) \\ &\quad + (a^2b^2 + a^2c^2 + b^2c^2) + (a^2bc + ab^2c + abc^2). \end{aligned}$$

□

From the definition, each monomial of (total) degree n in the variables $x_1, x_2, \ldots, x_k$ occurs in $H_n(x_1, x_2, \ldots, x_k)$ exactly once. This leads to a formula for $H_n(x_1, x_2, \ldots, x_k)$ analogous to Equation (1.9).

DEFINITION 1.25 Denote by $G_{n,k}$ the subset of $\Gamma_{n,k}$ consisting of all $C(n+k-1, n)$ nondecreasing functions from $\{1, 2, \ldots, n\}$ into $\{1, 2, \ldots, k\}$.

For all n and k, $Q_{n,k} \subset G_{n,k}$. As (lexicographically ordered) sequence sets, $G_{2,3} = \{(1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 3)\}$, and

$$\begin{aligned} G_{3,3} = \{&(1, 1, 1), (1, 1, 2), (1, 1, 3), (1, 2, 2), (1, 2, 3), (1, 3, 3), \\ &(2, 2, 2), (2, 2, 3), (2, 3, 3), (3, 3, 3)\}. \end{aligned} \tag{1.16}$$

Using Definition 1.25, we can rewrite Equation (1.14) as

$$H_n(x_1, x_2, \ldots, x_k) = \sum_{\beta \in G_{n,k}} x_{\beta(1)} x_{\beta(2)} \cdots x_{\beta(n)}. \tag{1.17}$$

We now return to the observation that any symmetric polynomial is a linear combination of "minimally symmetric pieces".

DEFINITION 1.26 Suppose $x_1, x_2, \ldots, x_k$ are independent indeterminates (variables) over the field $\mathbb{C}$ of complex numbers. Denote by $\mathbb{C}[x_1, x_2, \ldots, x_k]$ the set of polynomials in $x_1, x_2, \ldots, x_k$ with complex coefficients. Let $S\mathbb{C}_n[x_1, x_2, \ldots, x_k]$ be the subset of $\mathbb{C}[x_1, x_2, \ldots, x_k]$ consisting of the zero polynomial together with all symmetric homogeneous polynomials of degree n.

THEOREM 1.27 *The set* $\{M_\pi(x_1, x_2, \ldots, x_k) : \pi \vdash n,\ L(\pi) \le k\}$ *is a basis of the vector space* $S\mathbb{C}_n[x_1, x_2, \ldots, x_k]$.

Proof Let $M_\pi = M_\pi(x_1, x_2, \ldots, x_k)$, $\pi \vdash n$. The only thing remaining to be proved is the linear independence of $\{M_\pi : \pi \vdash n,\ L(\pi) \le k\}$.

Suppose

$$\sum_{\substack{\pi \vdash n \\ L(\pi) \le k}} c_\pi M_\pi = 0, \tag{1.18}$$

the zero polynomial. Let $\rho = [\rho_1, \rho_2, \ldots, \rho_r]$, $r \le k$, be a partition of n. Consider the term

$$c_\rho x_1^{\rho_1} x_2^{\rho_2} \ldots x_r^{\rho_r}$$

occurring in Equation (1.18). Taking partial derivatives of (1.18) with respect to x_1, ρ_1-times, with respect to x_2, ρ_2-times, ..., and with respect to x_r, ρ_r-times, we deduce that

$$\rho_1! \rho_2! \ldots \rho_r! c_\rho = 0.$$

$\square$

Application to Graphs

Let V be a set. Denote the family of its 2-element subsets by $V^{(2)}$. Then, for example,

$$\{a, b, c\}^{(2)} = \{\{a, b\}, \{a, c\}, \{b, c\}\};$$

$$\{1, 2, 3, 4\}^{(2)} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\};$$

and $\{x, y\}^{(2)} = \{\{x, y\}\}$. If $o(V) = n$, then $o(V^{(2)}) = C(n, 2)$.

DEFINITION 1.28 A **graph** consists of two things, a nonempty finite set V, and a (possibly empty) subset E of $V^{(2)}$. If $G = (V, E)$ is a graph, the elements of V are its **vertices** and the elements of E its **edges**. When more than one graph is under consideration, it may be useful to write $V(G)$ and $E(G)$, respectively, for the sets of vertices and edges. If $e = \{u, v\} \in E(G)$, then u and v are **adjacent** vertices, **incident** with e. Two edges are **adjacent** if their set-theoretic intersection consists of a single vertex.

EXAMPLE 1.29 If $V = \{1, 2, 3, 4, 5\}$, then $V^{(2)}$ has 10 elements and 2^{10} subsets. Hence, there are 1024 different graphs with vertex set $\{1, 2, 3, 4, 5\}$.

It is common to draw pictures of graphs in which vertices are represented by points and points representing adjacent vertices are joined by line segments (or arcs). If $E = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$, then each of the pictures in Figure 1.4 illustrates $H = (V, E)$. Note that H is not connected; vertex 5 is an "**isolated**" vertex. □

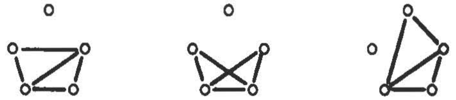


FIGURE 1.4 Pictures of graph H.

EXAMPLE 1.30 Not only can one graph be illustrated by different pictures, but one picture can represent different graphs! If $W = \{p, q, r, s, t\}$ and $F = \{\{q, s\}, \{q, t\}, \{r, s\}, \{r, t\}, \{s, t\}\}$, then the four pictures in Figure 1.4 also illustrate $K = (W, F)$. □

We are not so much interested in different graphs as in nonisomorphic graphs.

Definition 1.31 Let $G_1 = (V, E)$ and $G_2 = (W, F)$ be graphs. Then G_1 is **isomorphic** to G_2 if there is a one-to-one function $f: V \to W$ such that vertices u and v are adjacent in G_1 if and only if $f(u)$ and $f(v)$ are adjacent in G_2, that is, such that $\{u, v\} \in E$ if and only if $\{f(u), f(v)\} \in F$. The function f is an **isomorphism** from G_1 onto G_2.

If G_1 and G_2 *can* be illustrated by the same picture, then they are isomorphic. To each point of the picture there corresponds a unique vertex v_1 of G_1 and a unique vertex v_2 of G_2. The function that sends v_1 to v_2 (for every point of the picture) is an isomorphism. It is more challenging to tell when graphs illustrated by different pictures are isomorphic.

Example 1.32 The so-called "Petersen" graph, G_1, is illustrated in Figure 1.5. It is isomorphic to the graph G_2, pictured in the same figure. The proof that G_1 and G_2 are isomorphic is "by the numbers". If $V(G_1) = \{0, 1, 2, \ldots, 9\} = V(G_2)$, then $f(i) = i$, $0 \leq i \leq 9$, is an isomorphism. (Check it out: Confirm that i and j are adjacent in G_1 if and only if they are adjacent in G_2.) Such a pair of labeled figures may be considered a proof of isomorphism (provided, of course, that it "checks out"). □

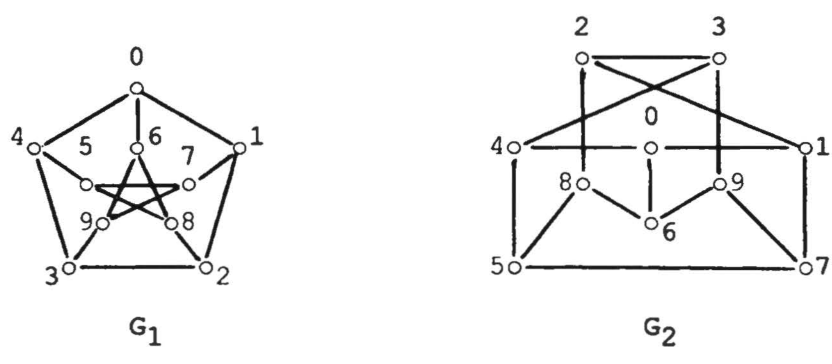


FIGURE 1.5 The Petersen graph.

It is an immediate consequence of the definition that isomorphic graphs have the same numbers of vertices and edges. Consequently, if G_1 and G_2 do not share these properties, they cannot be isomorphic. Properties that isomorphic graphs must share, are called **graph invariants**. We now introduce another graph invariant.

Definition 1.33 Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. The **degree** of $v \in V$, denoted $d(v)$, is the number of edges of G that are incident with v (which is equal to the number of vertices of G that are adjacent to v). When more than one graph is under consideration, it may be useful to write $d(v) = d_G(v)$.

The **degree sequence** is $d(G) = (d_1, d_2, \ldots, d_n)$, where $d_1 \geq d_2 \geq \cdots \geq d_n \geq 0$ are the degrees of the vertices of G, arranged in nonincreasing order. (We are not necessarily assuming that $d_i = d(v_i)$.)

THEOREM 1.34 *The degree sequence is a graph invariant.*

We can determine from $d(G)$ both n, the number of vertices of G, and m, the number of its edges: n is just the length of the sequence $d(G)$, and m is given by what has come to be known as the "first theorem" of graph theory.

THEOREM 1.35 *Let $G = (V, E)$ be a graph with vertex set $V = \{v_1, v_2, \ldots, v_n\}$. If $o(E) = m$, then*

$$\sum_{i=1}^{n} d(v_i) = 2m.$$

Proof By definition, $d(v)$ is the number of edges incident with vertex v. Thus, in summing the vertex degrees, each edge is counted twice, once at each of its vertices. □

FIGURE 1.6 Nonisomorphic graphs with the same degree sequence.

EXAMPLE 1.36 The nonisomorphic graphs G_1 and G_2 in Figure 1.6 share the degree sequence (2,2,2,1,1). □

If G is a graph with n vertices and m edges, it follows from Theorem 1.35 that, were it not for isolated vertices (of degree 0), $d(G) = (d_1, d_2, \ldots, d_n)$ would be a partition of $2m$. When speaking of the Ferrers diagram of $d(G)$ it will be understood that vertices of degree 0 go unrepresented. Similarly, let $d_j^* = o(\{i : d_i \geq j\})$. Then the **conjugate degree sequence**, $d^*(G) = (d_1^*, d_2^*, \ldots, d_k^*)$, is the conjugate of the partition of $2m$ whose parts are the nonzero vertex degrees of G.

THEOREM 1.37 *Let G be a graph with n vertices, m edges, and degree sequence $d(G)$. Then $d^*(G)$ majorizes $d(G)$.*

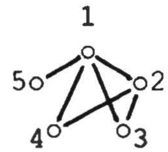


FIGURE 1.7

Proof Consider the graph G, illustrated in Figure 1.7, in which the vertices are numbered in such a way that $d(v_i) = d_i$. Figure 1.8(a) exhibits a variation on the Ferrers diagram for $d(G) = (4, 3, 2, 2, 1)$ in which the boxes have been replaced by numbers. Because vertex 1 has degree 4, there are four 1's in the first row of the diagram. The three 2's in the second row correspond to the degree of vertex 2, and so on. Now, rearrange the numbers, but not the shape, of this "Young Tableau" so that row i contains, in increasing order, the numbers of the vertices of G adjacent to vertex i. Figure 1.8(b) is the result.

$$\begin{array}{llll} 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & \\ 3 & 3 & & \\ 4 & 4 & & \\ 5 & & & \end{array} \qquad\qquad \begin{array}{llll} 2 & 3 & 4 & 5 \\ 1 & 3 & 4 & \\ 1 & 2 & & \\ 1 & 2 & & \\ 1 & & & \end{array}$$

(a) (b)

FIGURE 1.8

Note that the first column of variation (b) contains all the 1's. All the 2's are contained in the first two columns, all the 3's in the first three columns, and so on. In general, for any graph, the first r columns of the analog of variation (b) contain all the 1's, all the 2's, ..., and all the r's. In particular, the sum of the lengths of the first r rows of the analog of variation (a) is at most the sum of the lengths of the first r columns of the analog of variation (b). Because the two variations have the same shape, the proof is complete. □

Theorems 1.35 and 1.37 give necessary conditions for a nonincreasing sequence of nonnegative integers to be the degree sequence of a graph.

DEFINITION 1.38 Let m be a positive integer. A partition $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ of $2m$ is **graphic** if there is a graph G such that $d(G) = \pi$.

DEFINITION 1.39 The **trace** of partition π is $f(\pi) = o(\{i : \pi_i \geq i\})$.

If $F(\pi)$ is the Ferrers diagram corresponding to π, then $f(\pi)$ is the length of its main diagonal.

THEOREM 1.40[4] *Suppose $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ is a partition of the positive integer $2m$. Let $\pi^* = [\pi_1^*, \pi_2^*, \ldots, \pi_k^*]$ be its conjugate partition. Then π is graphic if and only if*

$$\sum_{i=1}^{r} \pi_i^* \geq \sum_{i=1}^{r} (\pi_i + 1), \quad 1 \leq r \leq f(\pi). \tag{1.19}$$

Proof The proof uses the same variations, (a) and (b), of the Ferrers diagram of $d(G)$ that were useful in the proof of Theorem 1.37. (See Figure 1.8.) Because no vertex is adjacent to itself, no row in variation (b) contains its own number. In particular, the (1,1)-entry is not less than 2. Therefore, in addition to all the 1's, the first column of variation (b) contains a number larger than 1, so $d_1^* \geq d_1 + 1$.

Since the (1,1)-entry of variation (b) is at least 2, and since the numbers in the first row are strictly increasing, the (1,2)-entry must be at least 3. If $d_2 \geq 2$ then, because the second vertex is not adjacent to itself, the (2,2)-entry can be no less than 3 as well. Therefore, all the 1's, all the 2's, and at least two numbers no smaller than 3 occur in the first two columns of variation (b). This means $d_1^* + d_2^* \geq d_1 + d_2 + 2 = (d_1 + 1) + (d_2 + 1)$. As long as $d_r \geq r$, we can use the same argument to prove that

$$\sum_{i=1}^{r} d_i^* \geq \left(\sum_{i=1}^{r} d_i \right) + r,$$

thus establishing the necessity of Condition (1.19).

To prove sufficiency, suppose $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ is a partition of $2m$ that satisfies Inequalities (1.19). Consider the extreme case in which $\pi_i^* = \pi_i + 1$, $1 \leq i \leq f(\pi)$. To produce a graph with degree sequence π, begin with the vertex set $V = \{1, 2, \ldots, n\}$. "Construct" edges from vertex 1 to each of $2, 3, \ldots, n$. Because $n = \pi_1^* = \pi_1 + 1$, we have produced a graph in which $d_1 = \pi_1$, and $d_2 = d_3 = \cdots = d_n = 1$. If $f(\pi) = 1$, then $\pi_2 = 1$, and we are finished. Otherwise, construct edges from vertex 2 to each of $3, 4, \ldots, \pi_2 + 1$. (This is possible because we have reserved "room" for $\pi_2^* = \pi_2 + 1$ vertices of degree 2 or more.) So far, we have produced a graph in which $d_1 = \pi_1$ and $d_2 = \pi_2$. If $f(\pi) = 2$, we are finished because $F(\pi)$ is completely determined by its first $f(\pi)$ rows and columns. If $\pi_3 \geq 3$, draw edges from vertex 3 to each of $4, 5, \ldots, \pi_3 + 1$ (which is possible because $\pi_3^* = \pi_3 + 1$). After three steps, we have $d_1 = \pi_1$,

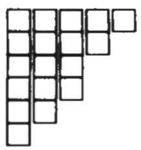
---

[4] While this result has been attributed to Hasselbarth [Sierksma & Hoogeveen (1991)], it seems to have been published first in [Ruch & Gutman (1979)].

$d_2 = \pi_2$, and $d_3 = \pi_3$. At the end of $f(\pi)$ steps, we will have produced a graph satisfying $d_i = \pi_i$, $1 \le i \le n$.

To complete the proof of sufficiency, two additional facts are required: (1) if ρ is majorized by a graphic partition π, then ρ is graphic; and (2) every partition satisfying the inequalities in (1.19) is majorized by one which is extreme in the sense that equality holds in each of the inequalities. The details are omitted. □

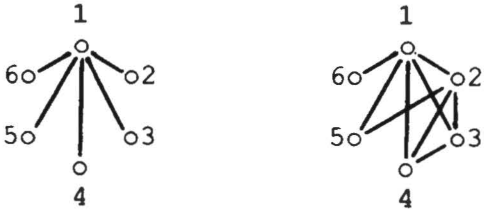
EXAMPLE 1.41 Consider the partition $\pi = [5, 4, 3, 3, 2, 1]$, whose Ferrers diagram, $F(\pi)$ appears in Figure 1.9. Because π is a partition of 18, the first condition of Theorem 1.40 is satisfied: $m = 9$. In this case, the length of the main diagonal of $F(\pi)$ is $3 = f(\pi)$. Glancing at Figure 1.9, we can write down $\pi^* = [6, 5, 4, 2, 1]$. Observe that $\pi_i^* = \pi_i + 1$, for $i = 1, 2, 3$.



$F([5, 4, 3^2, 2, 1])$

FIGURE 1.9

Draw six points in the plane and label them $1, 2, \ldots, 6$. Construct (draw) edges from vertex 1 to vertices 2, 3, 4, 5, and 6, as shown in Figure 1.10(a). This gives a vertex of degree 5 and five vertices of degree 1. Now, draw edges connecting vertex 2 to vertices 3, 4, and 5. Finally, drawing an edge from vertex 3 to vertex 4, one obtains the graph G, illustrated in Figure 1.10(b), having degree sequence $d(G) = \pi$. □



(a) (b)

FIGURE 1.10

Example 1.41 illustrates the "greedy" algorithm used in the proof of Theorem 1.40 to construct a graph whose degree sequence is extreme in the sense that equality holds in each of the inequalities in (1.19). We now give a formal name to the graphic partitions that are extreme in this sense.

DEFINITION 1.42 Let $\pi = [\pi_1, \pi_2, \ldots, \pi_n]$ be a partition of $2m$. Then π is a **maximal** (graphic) partition if $\pi_i^* = \pi_i + 1$, $1 \le i \le f(\pi)$. A graph whose degree sequence is maximal is a **threshold graph**.

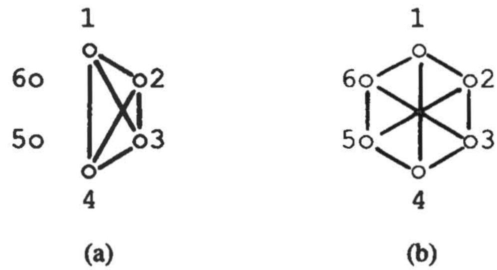


FIGURE 1.11

EXAMPLE 1.43 Let $\pi = [3^6]$. Then $\pi^* = [6^3]$ and, while π is graphic, it is not maximal. Let's see what happens if we try to use the greedy algorithm illustrated in Example 1.41 to construct a graph with degree sequence $[3^6]$. Begin by drawing six points in the plane and labeling them $1, 2, \ldots, 6$. Draw edges from vertex 1 to vertices 2, 3, and 4. Now draw edges from vertex 2 to vertices 3 and 4, producing two vertices of degree 3, two of degree 2, and two of degree 0. When an edge is drawn between vertices 3 and 4, we find ourselves in the position illustrated in Figure 1.11(a). Pretty clearly, a graph with degree sequence $[3^6]$ cannot be obtained from this figure by adding more edges. On the other hand, the existence of a graph with degree sequence $[3^6]$ is established by Figure 1.11(b). □

EXAMPLE 1.44 The connected threshold graphs having $2 \le m \le 6$ edges are illustrated in Figure 1.12.

DEFINITION 1.45 Let V be an n-element set. The **complete graph** $K_n = (V, V^{(2)})$ is the graph in which every pair of vertices is adjacent.

Strictly speaking, Definition 1.45 defines the complete graph with vertex set V. However, because any two complete graphs on n vertices are isomorphic, we will abuse the language and speak about *the* complete graph on n vertices. The complete graphs K_3 and K_4 are illustrated in Figure 1.12(b) and (h), respectively.

DEFINITION 1.46 Let $G = (V, E)$ be a graph. The **complement** of G is the graph $G^c = (V, V^{(2)} \backslash E)$.

If G is a graph, then $e = \{u, v\}$ is an edge of G if and only if e is not an edge of G^c. In particular, the complement of K_n is the graph consisting of n isolated vertices, that is, K_n^c has no edges at all.

DEFINITION 1.47 Let $G = (V, E)$ be a graph. A **cycle** in G is a sequence of distinct vertices $v_1, v_2, \ldots, v_n$, $n > 2$, such that $\{v_i, v_{i+1}\} \in E$, $1 \leq i < n$, and $\{v_1, v_n\} \in E$. A connected graph without cycle is a **tree**.

Graphs (a), (b), (e), (h), and (i) in Figure 1.12 are trees.

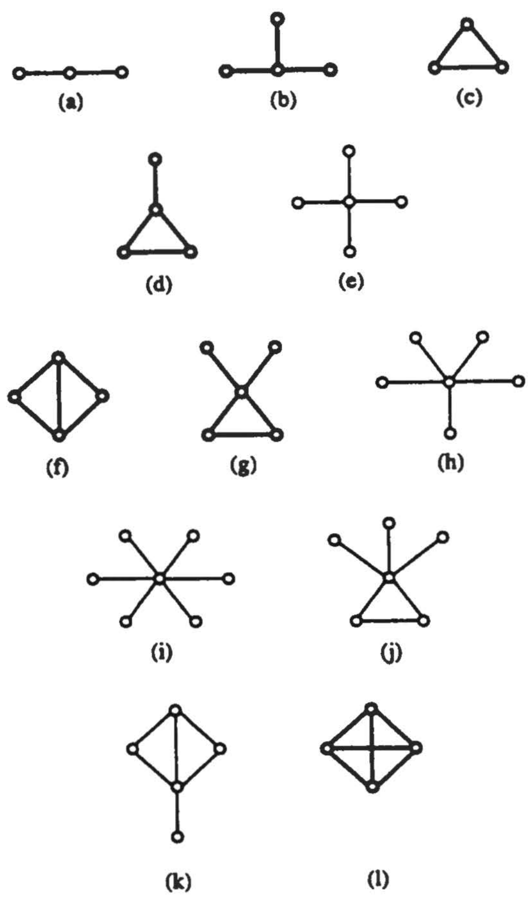


FIGURE 1.12 The Threshold Graphs with $2 \leq m \leq 6$ edges.

Exercises

1. Denote by $p_m(n)$ the number of partitions of n having m parts. Show that
   a. $p_{n-2}(n) = 2,\ n \geq 4.$
   b. $p_{n-3}(n) = 3,\ n \geq 6.$
   c. $p_2(n) = [n/2]$, the greatest integer not exceeding $n/2$.
   d. $p_m(n) = p_{m-1}(n-1) + p_m(n-m),\ 1 < m < n.$
   e. Construct a table exhibiting $p_m(n),\ 1 \leq m \leq n,\ 1 \leq n \leq 7.$
   f. The number of partitions of n is the **partition number**

$$p(n) = \sum_{m=1}^{n} p_m(n).$$

Compute $p(n),\ 1 \leq n \leq 7.$

2. Explicitly write down
   a. all 11 partitions of 6.
   b. all 8 partitions of 7 having 3 or fewer parts.
   c. all 8 partitions of 7 whose largest part is at most 3.
3. Let $\pi = [6, 4, 2^3]$. Find π^*
   a. using Ferrers diagrams.
   b. using Equation (1.1).
4. Which of the following is a self conjugate partition?
   a. $[5,4,3,2,1]$
   b. $[5,3^2,1^2]$
   c. $[4,3^2,1]$
   d. $[5,3^2,2,1^2]$
   e. $[5,4^2,3,1^2]$
   f. $[6,4,3,1^2]$
5. Find π^* and use it confirm Lemma 1.6 when $\pi =$
   a. $[5,4,3,2,1]$
   b. $[5,3^2,1^2]$
   c. $[4,3^2,1]$
   d. $[5,3^2,2,1^2]$
   e. $[5,4^2,3,1^2]$
   f. $[6,4,3,1^2]$
6. Find the smallest integer n having three different self conjugate partitions.

7. Suppose $\pi \vdash n$. Show that $\pi^*_{j+1} = \pi^*_j - o(\{i : \pi_i = j\})$.
8. Let $\pi = [\pi_1, \pi_2, \ldots, \pi_m]$ and $\rho = [\rho_1, \rho_2, \ldots, \rho_k]$ be partitions of n. Show that $\pi \succ \rho$ only if $m \leq k$.
9. Find all the partitions of 7 that
   a. majorize [5,2].
   b. are majorized by $[2^2, 1^3]$.
10. Prove that $\pi \succ \rho$ if and only if $\rho^* \succ \pi^*$.
11. Show that the doubly stochastic matrix S given in Example 1.11 is not unique by finding another one that satisfies (3,3,1,1) = (5,2,1,0)S.
12. Show that there are $C(k+n-1, n)$ nonnegative integer solutions to the equation $r_1 + r_2 + \cdots + r_k = n$.
13. When $(a+b+c+d)^{10}$ is expressed as a linear combination of monomial symmetric functions, compute the coefficient of
   a. $M_{[8,1^2]}(a,b,c,d)$.
   b. $M_{[7,2,1]}(a,b,c,d)$.
   c. $M_{[4^2,2]}(a,b,c,d)$.
   d. $M_{[3^2,2,1^2]}(a,b,c,d)$.
14. Write out in full
   a. $M_{[4,1]}(x,y,z)$.
   b. $M_{[3,2]}(x,y,z)$.
   c. $M_{[1^2]}(x,y,z)$.
15. Confirm Equation (1.10) for $a=1, b=2, c=3$, and $d=4$ by
   a. using Example 1.19 to compute $E_n(1,2,3,4)$, $1 \leq n \leq 4$.
   b. computing the product $(x-1)(x-2)(x-3)(x-4)$.
16. Denote the roots of $p(x) = x^4 - x^2 + 2x + 2$ by a, b, c, and d. Compute the elementary symmetric functions $E_r(a,b,c,d)$, $1 \leq r \leq 4$,
   a. from the coefficients of $p(x)$. (Hint: Equation (1.10).)
   b. from the definition of E_r. (Hint: $(x+1)^2$ divides $p(x)$.)
17. Suppose k is a fixed but arbitrary positive integer. Let $P_n = P_n(x_1, x_2, \ldots, x_k)$ and $E_n = E_n(x_1, x_2, \ldots, x_k)$, $n \geq 1$, be the n-th power sum and the n-th elementary symmetric function, respectively. It was shown by Isaac Newton (1642–1727) that, for any $n \geq 1$,

$$P_n - P_{n-1}E_1 + P_{n-2}E_2 - \cdots + (-1)^{n-1}P_1E_{n-1} + (-1)^n nE_n = 0.$$

Thus, $P_1 - E_1 = 0$, $P_2 - P_1E_1 + 2E_2 = 0$, $P_3 - P_2E_1 + P_1E_2 - 3E_3 = 0$, and so on.

a. Use Newton's identities to prove that

$$P_4 = E_1^4 - 4E_1^2E_2 + 4E_1E_3 + 2E_2^2 - 4E_4.$$

b. Use Newton's identities to prove that

$$E_4 = \frac{1}{24}(P_1^4 - 6P_1^2P_2 + 8P_1P_3 + 3P_2^2 - 6P_4).$$

c. Show that the general formula for E_r as a polynomial in the power sums is $r!E_r = \det(L_r)$, where

$$L_r = \begin{pmatrix} P_1 & 1 & 0 & 0 & \dots & 0 & 0 \\ P_2 & P_1 & 2 & 0 & \dots & 0 & 0 \\ P_3 & P_2 & P_1 & 3 & \dots & 0 & 0 \\ & & & & \dots & & \\ P_r & P_{r-1} & P_{r-2} & P_{r-3} & \dots & P_2 & P_1 \end{pmatrix}.$$

(Hint: Use Cramer's rule on the following matrix version of Newton's identities:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & \dots \\ P_1 & -2 & 0 & 0 & \dots \\ P_2 & -P_1 & 3 & 0 & \dots \\ P_3 & -P_2 & P_1 & -4 & \dots \\ & & \dots & & \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \\ E_4 \\ \vdots \end{pmatrix} = \begin{pmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ \vdots \end{pmatrix}.)$$

d. Prove that any polynomial, symmetric in $x_1, x_2, \dots, x_k$, is a polynomial in the power sum functions $P_n(x_1, x_2, \dots, x_k)$, $1 \le n \le k$.

18. If $2 \le r \le k$, prove that $E_r(x_1, x_2, \dots, x_k) = E_r(x_1, x_2, \dots, x_{k-1}) + x_kE_{r-1}(x_1, x_2, \dots, x_{k-1})$.

19. Use Equation (1.16) to confirm that Equation (1.17) yields Equation (1.15) when $n = k = 3$, $x_1 = a$, $x_2 = b$, and $x_3 = c$.

20. If $r \ge 2$, prove that $H_r(x_1, x_2, \dots, x_k) = H_r(x_1, x_2, \dots, x_{k-1}) + x_kH_{r-1}(x_1, x_2, \dots, x_k)$.

21. Use Exercise 18 and mathematical induction to prove that

$$\prod_{i=1}^{n}(x - a_i) = \sum_{r=0}^{n}(-1)^r E_r(a_1, a_2, \dots, a_n)x^{n-r}.$$

22. Suppose $\alpha \in \Gamma_{m,n}$. Prove that $\alpha \in G_{m,n}$ if and only if $\alpha\sigma \succ \alpha$ for all permutations $\sigma \in S_m$.

23. Denote by $m_i(\pi)$ the multiplicity of i in the partition π, that is, the number of times i occurs as a part of π. Prove that $m_i(\pi) = \pi_i^* - \pi_{i+1}^*$.

24. Suppose $\pi, \rho \vdash n$. Let $\pi + \rho$ be the partition of $2n$, the i-th part of which is $\pi_i + \rho_i$ (with the convention that $\pi_i = 0$ if $i > L(\pi)$). Denote by $\pi \cup \rho$ the partition of $2n$ the parts of which are the parts of π together with the parts of ρ.

    a. Prove that $(\pi \cup \rho)^* = \pi^* + \rho^*$.

    b. Is $(\pi + \rho)^* = \pi^* \cup \rho^*$?

25. Suppose $\pi \vdash n$. Let $\mu_i = \pi_i - i$ and $\nu_i = \pi_i^* - i$, $1 \le i \le f(\pi)$. Frobenius used $(\mu|\nu)$ to denote the partition π. Show that the Frobenius notation for π^* is $(\nu|\mu)$.

26. Among the many results known about elementary symmetric functions is that they are **Schur concave**, that is, $E_r(a) \le E_r(b)$ whenever (a) majorizes (b).

    a. Show that majorization imposes a linear order on the five partitions of 8 having 3 parts.

    b. Confirm the Schur concavity of E_r by computing $E_r(\pi)$, $1 \le r \le 3$, for each three-part partition of 8.

27. Among the many results known about homogeneous symmetric functions is that they are **Schur convex**, that is, $H_r(a) \ge H_r(b)$ whenever (a) majorizes (b).

    a. Confirm the Schur convexity of H_r by computing $H_r(\pi)$, $1 \le r \le 3$, for each three-part partition of 8.

    b. If you were to compute $H_4(\pi)$ for each partition π of 24 having 3 parts, which partition would produce the maximum? Which would produce the minimum?

28. Let $E_r = E_r(a_1, a_2, \ldots, a_n)$, $1 \le r \le n$. Show that $(1 - a_1x)(1 - a_2x)\ldots(1 - a_nx) = 1 - E_1x + E_2x^2 - \cdots + (-1)^n E_nx^n$.

29. Show that the dimension of $SC_7[x, y, z]$ is 8. (Hint: Exercise 2b.)

30. Compute

    a. $\dim(SC_7[x_1, x_2, \ldots, x_7])$. (Hint: Exercise 1f.)

    b. $\dim(SC_7[x_1, x_2, \ldots, x_8])$.

31. If A is an m-by-n matrix, denote its i-th row and j-th column sums, respectively, by $r_i(A)$ and $c_j(A)$. Suppose

$$R = (r_1, r_2, \ldots, r_m) \quad \text{and} \quad C = (c_1, c_2, \ldots, c_n)$$

are integer vectors satisfying $r_1 \ge r_2 \ge \cdots \ge r_m \ge 0$ and $c_1 \ge c_2 \ge \cdots \ge c_n \ge 0$. Then ([Gale (1957)] and [Ryser (1957)]) there exists an m-by-n, (0,1)-matrix A such that $r_i(A) = r_i$, $1 \le i \le m$, and $c_j(A) = c_j$, $1 \le j \le n$, if and only if $R^* \succ S$.

a. Use the Gale-Ryser theorem to prove the existence of a 5-by-4, (0,1)-matrix A having row sum vector $R = (3,2,1,1,1)$ and column sum vector $(3,3,1,1)$.

b. Write down such a matrix.

Application Exercises

32. Draw pictures of the 11 nonisomorphic graphs on four vertices.
33. Prove Theorem 1.34.
34. Draw Ferrers diagrams for all the maximal graphic partitions of 6. (Hint: Figure 1.3.)
35. Let $\pi = [4, 2^3, 1]$.
    a. Show that π satisfies Criteria (1.19).
    b. Explain why π is not graphic.
36. Prove that

$$\sum_{i=1}^{L(\pi)} i\pi_i = \sum_{i=1}^{\pi_1} C(\pi_i^* + 1, 2).$$

(Hint: Figure 1.8(a)).

37. Confirm that the graphs in Figure 1.12 are threshold graphs.
38. Prove that K_n is a threshold graph, $n \ge 2$.
39. Prove that, apart from isolated vertices, the complement of a threshold graph is a threshold graph.
40. If $T = (V, E)$ is a tree, prove that it has one fewer edges than vertices.