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An Introductory Course in the Mathematics of Sports Science and Sports Analytics

Roland B. Minton

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Roland B. Minton
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Preface

This is a textbook for a course that does not exist. Like the myths that are busted in Chapter 3, that statement has some truth to it. As Obi Wan Kenobi would say, it is true "from a certain point of view." There are a number of existing courses with Sports Science and Sports Analytics in the titles, created by intrepid professors venturing into the unknown. The topics and emphases of such courses vary dramatically, so that there is no consensus on what a course in Sports Science and/or Sports Analytics should be.

There are conferences on sports analytics. The MIT/ESPN Sloan Sports Analytics Conferences are graced with outstanding speakers, and the demand for tickets grows exponentially. The topics at a conference can range from *Moneyball* to marketing strategies, management strategies or technological breakthroughs. Regional conferences such as the Carolina Sports Analytics Meeting provide support for the increasingly large number of faculty and students doing research in sports-related areas.

To use a golfing analogy, writing a book like this is like hitting a drive at a driving range; there are many directions you can go without going out of bounds. At the driving range, I pick out a small target to focus on, and that is what I have done here. I have chosen a sample of topics that I know something about and that I find very interesting. Ideally, users of this book will have enough to choose from to suit whichever version of a sports course is being run.

The course that I have taught at Roanoke College since 1988 is a mix of physics, physiology, mathematics, and statistics. The order (and level of emphasis) of the topics has changed over the years; this book reflects the current status of my course. It is, admittedly, an eclectic mix of topics (at the driving range, I may aim at one target, but I do tend to spray balls all over the range). I hope to provide ideas and resources to help students launch projects. An important part of my course is the term project, and I have almost always been pleasantly surprised at the quality of work done in a short period of time.

I suspect that the high quality of work is due to the students' high level of motivation; not from any talents of mine, but because many students (of both genders) find it exciting to think about sports and to complete a research agenda. Sports problems are easy to create and state, even for students who do not live sports 24/7. Sports are part of their culture and knowledge base, and the opportunity to be an expert on some area of sports is invigorating.

This should be the primary reason for the growth of sports courses: the topic provides intrinsic motivation for students to do their best work.

This, as I said, is a textbook. That fact alters the literary qualities of the writing. My intention is for it to be easy and enjoyable to read, but examples and exercises necessarily interrupt the normal flow of text. As well, the exercises guide students to some very interesting results, so that some of the best discoveries about sports may be hiding in the exercises. I encourage you to look for fun facts in the exercises.

The choice of mathematical level is problematic for a book like this. Some of my favorite results require calculus or even differential equations for a full explanation, but I do not want to narrow the audience to the mathematically advanced. I have split the difference on calculus. I am not assuming that you know calculus, but I will show you some of the things that calculus can do for you. Those of you who have taken calculus can read the "calculus box" sections in the text and work the exercises labeled as calculus exercises. If you have not taken calculus, simply navigate around those well-marked areas of the book.

The extent to which a background in probability and statistics is required is more difficult to say. Sports analytics relies heavily on sound statistical reasoning. Statistical "common sense" is assumed throughout, but the details of tests and calculations are all provided. Similarly, a familiarity with the ideas of computing is assumed, but no programming is required. The reader's experience will be greatly enhanced by frequent use of the internet, spreadsheets, and calculations.

I should admit that I like to read books; I enjoy holding physical books. On the other hand, I now buy most of my books and music in digital format. And I am slowly allowing myself to stream a movie or music online and let it slip away without claiming possession. The point of this ramble is that while I recognize that the future of sports research is digital with remote access, this book has a fairly standard format. There will be a website at www.roanoke.edu/mcsp/minton/SportsMath.html (I know, I'm showing my age by posting a url that will change. A search for "Minton Sports Math" should do it, but you don't need me to tell you that). I'll post links, references, notes, and anything else that comes to mind that could be useful and does not fit the classic book mold. Ideally, part of the site will even be wiki-like.

In the last thirty years, data collection has progressed from repeated viewings of grainy videos to nearly continuous data streaming from sensors attached to every part of an athlete, from Bill James painstakingly copying box score numbers from *The Sporting News* to a one-minute online search that lists the top fifty hitting streaks in MLB history. My hope is that this book opens up some of the astounding possibilities of sports research, while helping you learn more about the games you enjoy.

This book would not exist without the encouragement of my editor, Bob Ross, who over the years has furthered my career in multiple ways. Thanks, Bob! My Roanoke College family has provided support in several ways. Dave

Taylor has listened to countless musings and rants on all aspects of the book, and provided good counsel at all times. His assistance with the joys of TeX is invaluable. Adam Childers provided much-needed statistical backing, plus hours of enjoyable sports talk. Thanks to Karin Saoub and Chris Lee for their assistance. The athletic department, especially Ryan Pflugrad, Matt McGuire, Page Moir, Scott Allison, and Chris Kilcoyne are great to work with. An important chunk of the time to do this enjoyable work was provided by the M. Paul Capp and Constance Whitehead Endowed Chair, for which I am very grateful. Paul is a great supporter of education, especially in mathematics and physics. Thanks to Dean Richard Smith for his support; it is very cool to get to cite my Dean's publication in this book! Finally, to my wife Jan and children Kelly and Greg, who deal with me in writing mode, which is even grumpier than usual: thanks for being who you are, and for your love.

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Chapter 1

Projectile Motion

Introduction

Basketball star Stephen Curry launches a 3-point shot. As the ball traces its high arc toward the basket, fans rise to their feet in anticipation. Will it go in? Is it a little short? Similar tension accompanies a Jordan Spieth tee shot, an Andy Murray passing shot, a long football pass by Peyton Manning or Lionel Messi, or a long fly ball by Mike Trout. We will analyze the flights of balls in this chapter as we explore the area of physics known as mechanics.

Along the way, we will answer such questions as: How does Blake Griffin hang in the air when dunking? What is the optimal angle to shoot a free throw? Why do golf balls have dimples? Does a knuckleball really dance? The answers are to be found in the fundamentals of physics.

Figuring with Newton

Sir Isaac Newton (1643-1727) constructed a framework for the analysis of objects in motion. The second of his three Laws of Motion is the launching point for most of our investigations in this chapter. The shorthand version of Newton's Second Law is

$$F = ma$$

where F is the sum of all forces acting on an object, m is the object's mass, and a is the acceleration of the object. One of the most remarkable aspects of

Newton's Second Law is that it can also be written as $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ and $\mathbf{a}$ appear in bold to indicate that they are multidimensional vector quantities. We will return to this form of the equation when we look at motion in two and three dimensions. The mass m is a **scalar** (real number) that is related to weight: for earthbound sports, weight is approximately equal to mass times the gravitational constant g.

To keep it simple, let's start with one-dimensional motion; vertical motion, to be precise. In this case, the object's position can be tracked by its height h above some reference point (e.g., the ground). We define **velocity** as the rate of change of position with respect to time. At a constant speed, this means that velocity equals change in height divided by change in time: $v = \frac{\Delta h}{\Delta t}$. This gets complicated when velocity is not constant. In general,

$$\text{Average velocity} = \frac{\Delta h}{\Delta t}$$

and, for small time intervals, (instantaneous) velocity is approximately equal to average velocity: $v \approx \frac{\Delta h}{\Delta t}$. With calculus, we can simply say that velocity is the derivative of height. Either way, note that v can be negative (if height is decreasing) or positive (if height is increasing). The **acceleration** a of the object is, in turn, the rate of change of velocity. Then $a \approx \frac{\Delta v}{\Delta t}$ and acceleration is the derivative of velocity.

Example 1.1 Suppose a ball falls from a height of 50 meters. If gravity is the only force on the ball, find the velocity of the ball after $t = 1$ second and $t = 1.5$ seconds.
Solution. For most sports situations, we can assume that the acceleration due to gravity is a constant $-g$ with $g \approx 9.8$ m/s^2 or $g \approx 32$ ft/s^2. An acceleration of 9.8 m/s^2 in the negative direction means that in every second the velocity decreases by 9.8 m/s. Assuming that the ball starts with velocity 0, then at $t = 1$ second the velocity has decreased to -9.8 m/s. In the next half-second, the velocity decreases by 0.5(9.8) m/s = 4.9 m/s. At time $t = 1.5$ s the velocity has decreased to $(-9.8 - 4.9)$ m/s = -14.7 m/s. The ideas from this basic example will be used again for the more complicated situation of Figure 1.9.

Speed is defined as the absolute value of velocity. In Example 1.1 above, at time $t = 1$ the ball's velocity is -9.8 m/s but its speed is 9.8 m/s (downward).

Notice that Example 1.1 did not ask for heights. Because the ball's velocity is changing, the calculation of position from velocity requires more than multiplying velocity by time. Fortunately, calculus gives us some simple formulas to use, shown below in Table 1.1.

In Example 1.1, we have $c = -g$, $v_0 = 0$, and $p_0 = 50$, so the height at time t is $-4.9t^2 + 50$ m. At $t = 1$, the ball is at height $-4.9 + 50$ m = 45.1 m, while at $t = 1.5$ the ball is at height $-4.9(1.5)^2 + 50$ m = 38.975 m.

TABLE 1.1: Formulas for Constant Acceleration

acceleration	$a = c$
velocity	$v = ct + v_0$
position	$p = \frac{1}{2}ct^2 + v_0 t + p_0$

Hangin' with MJ: 1-D Motion

Using the equations in Table 1.1, we can discover an interesting fact about vertical motion. We start with a straightforward calculation.

Example 1.2 A man jumps from the ground with an initial velocity of 16 ft/s, under the force of gravity. (a) How long does he stay in the air? (b) How high does he go?

Solution. We use Table 1.1 with $c = -32$, $v_0 = 16$, and $p_0 = 0$. (Note that gravity pulls in the negative direction, while the jump is in the positive direction.) Then velocity is $v = -32t + 16$ ft/s and position is $h = -16t^2 + 16t$ ft. Now, let's decipher the questions being asked. (a) What does "in the air" mean? He is in the air from launch time (height 0) to landing time (height 0). Both times occur at height 0, when $h = -16t^2 + 16t = 0$. So, solve this equation! If $-16t(t-1) = 0$, then $t = 0$ or $t = 1$. He launches at $t = 0$ and lands at $t = 1$, hence is in the air for 1 second. (b) At the top of a jump, velocity is 0: no longer going up, not yet coming down. This occurs when $v = -32t + 16 = 0$ or $t = \frac{1}{2}$. Now that we know *when* he reaches his peak, we can determine his height using the position function. The height at time $t = \frac{1}{2}$ is $h = -16\left(\frac{1}{2}\right)^2 + 16\left(\frac{1}{2}\right) = -4 + 8 = 4$ feet.

The solution of Example 1.2 follows a pattern that you should use in most such problems. First, get the equations of motion by filling in the constants in Table 1.1. Then, solve one of the equations for time t based on the situation (e.g., how long the object is in the air, or when it reaches its peak). Finally, substitute this time value into another equation to find the quantity of interest.

The 48-inch jump of Example 1.2 is in legendary leaper status, up there with Michael Jordan and Blake Griffin. But, why do these prodigious leapers seem to hang in the air? One reason is that *all* objects hang in the air. The graph of height versus time in Figure 1.1 and Table 1.2 below show the height for the jumper in Example 1.2 at equal quarter marks in time.

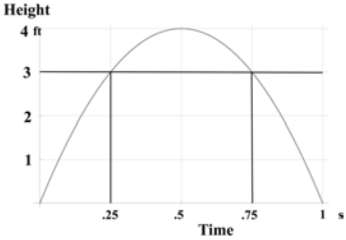

FIGURE 1.1: Jump

Notice that from time $t = 1/4$ to $t = 3/4$ (which is half of the time of the jump) the height is 3 feet or above (with a peak height of 4 feet). That is,

TABLE 1.2: Heights and Times for Jump

Time (s)	Height (ft)
0	0
1/4	3
1/2	4
3/4	3
1	0

half the time is spent in the top one-quarter of the jump! The speed is smallest at the top of the flight, so the object "hangs" at the top.

A second reason that great athletes can appear to defy gravity has to do with **center of mass**. The center of mass is where the sums of mass-times-distance quantities balance. For a standing human being, it is not far from the geometric center of the body. Newton's equations track the center of mass of the object in flight. Figure 1.1 does not show a body in flight, but the path of a single point. That point is the center of mass of the person. (Which means that a "height" of 0 does not actually mark the location of the ground; it marks the location of the center of mass of the object at launch time.) While the dunker's center of mass is tracking the nice parabola shown, he is free to pull up his legs, bob his head, and extend an arm in entertaining ways that may cause an individual body part such as the head to remain at the same height for a noticeable amount of time.

Raining 3's with Steph: 2-D Motion

Let's return to Stephen Curry's 3-point shot. We can analyze its flight with Newton's Second Law, but the fact that the ball now moves both horizontally and vertically complicates the calculations.

From nba.com/Stats, we can get an idea of the location of Curry's shot. In 2014-15, only 79 of Curry's 618 3-pointers were from the corners. (Remarkably, he made well over 40% of his shots from every 3-point zone and 62% from the left corner, plus an outrageous 91% from the left corner during the playoffs.) Most of his shots were from beyond the arc that is 23.75 feet from the basket. Let's say his shot is from 25 feet away. Align the x-axis horizontally from Curry to the basket, and the y-axis vertically.

We will assume that Curry's impeccable form keeps the ball from curving left or right. Newton's Second Law is the vector equation $\mathbf{F} = m\mathbf{a}$ where the vectors $\mathbf{F}$ and $\mathbf{a}$ have two components. That is, the acceleration has a horizontal component a_x and a vertical component a_y. Assuming that gravity

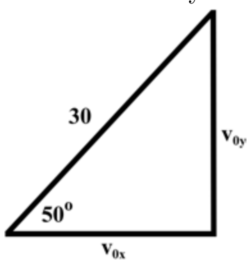

FIGURE 1.2: Velocity

is the only force, then $a_y = -g$ as before, and $a_x = 0$ (no forces acting horizontally). This allows us to separate the x- and y-equations. To use Table 1.1, we need the initial velocities and initial positions. We assume that $p_{0x} = 0$ ft for convenience and $p_{0y} = 7$ ft (assuming the ball is released from a height of 7 feet). If the ball is launched with speed 30 ft/s at an angle of 50 degrees, then v_{0x} and v_{0y} are obtained from the triangle in Figure 1.2.

Using basic trigonometry, we get initial velocities $v_{0x} = 30\cos(50°)$ ft/s and $v_{0y} = 30\sin(50°)$ ft/s, or $v_{0x} \approx 19.28$ ft/s and $v_{0y} \approx 22.98$ ft/s. Pulling this all together, we have $x \approx 19.28t$ and $y \approx -16t^2 + 22.98t + 7$.

Example 1.3 Is this shot good or not?
Solution. In this case, a perfect shot would pass through $x = 25$ and $y = 10$ (the height of the basket). We will solve for t in one equation and plug into the other equation, but that can be done in two ways. For reasons you will see, it is more convenient to start with the y-equation. We want $y = 10$ and so solve $-16t^2 + 22.98t + 7 = 10$ for t. There are two solutions, one representing the ball rising up through the height $y = 10$ and the other representing the ball dropping through the height $y = 10$; the second solution is clearly the one of interest. We get $t \approx 1.29$ s. If the shot is perfect, then at this t-value we get $x = 25$ (be sure this makes sense to you!). Instead, our equation gives $x \approx 24.90$ feet. Not perfect, but is this close enough? The center of the basket is at $x = 25$, so $x = 24.90$ represents 0.1 foot or 1.2 inches from the center. The basket has diameter 18 inches and the ball has diameter 9.5 inches, so the ball can move a little over 4 inches from the center and still be inside the basket. (This assumes that the shot is exactly on line.) Count the three!

The work in Example 1.3 does not fully prove that the shot is good. Can you think of what is missing?

Even if the center of the ball (theoretically) passes inside the basket, in real life if the trajectory of the ball is too flat some portion of the ball will hit the rim. You will show in exercise 1.41 that the ball in Example 1.3 enters the basket at an angle of about 43 degrees, more than steep enough to safely pass through the basket.

We can now develop a method to determine the best angle at which to shoot a free throw. An important part of our interpretation of the numbers in Example 1.3 is the margin of error inherent in playing with a ball that is smaller than the basket. You could imagine decreasing the initial speed from 30 ft/s until the shot is no longer good; call this speed s_1. Then find the maximum speed s_2 for which the shot is good. For the angle 50 degrees, $s_2 - s_1$ is the margin of error in speed. The bigger the margin of error, the better, since the shooter does not have to be as precise with the launch speed. Peter Brancazio has done this study and found that a free throw angle of about 49 degrees gives the largest margin of error. We will explore an interesting aspect of this angle in exercise 1.9.