

Advances in Applied Mathematics

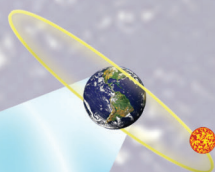
POCKET BOOK OF INTEGRALS AND MATHEMATICAL FORMULAS

5TH EDITION

RONALD J. TALLARIDA



$$L = \int_{x_1}^{x_2} \sqrt{1 + (dy/dx)^2} dx$$



$$w = \frac{(MG)^{1/2}}{(R + H)^{3/2}}$$



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POCKET BOOK OF
INTEGRALS AND
MATHEMATICAL
FORMULAS
5TH EDITION

Advances in Applied Mathematics

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Pocket Book of Integrals and Mathematical Formulas, 5th Edition
Ronald J. Tallarida

Advances in Applied Mathematics

POCKET BOOK OF INTEGRALS AND MATHEMATICAL FORMULAS 5TH EDITION

RONALD J. TALLARIDA

TEMPLE UNIVERSITY

PHILADELPHIA, PENNSYLVANIA, USA



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Preface to the Fifth Edition

Pocket Book of Integrals and Mathematical Formulas, Fifth Edition, a revision of a very successful pocket book, provides a handy reference source for students, engineers, scientists, and others seeking essential mathematical formulas, concepts, and definitions. Topics range from precalculus to vector analysis and from Fourier series to statistics. The previous editions added material on business and financial mathematics that has been well received since it provided information on progressions, especially geometric progressions, which form the basis for many formulas related to annuities, growth of funds, and interest payments. That material has been retained. The fourth edition also retained topics in statistics, nonlinear regression, and an expanded discussion in the differential equations section by adding a treatment of Runge Kutta methods and a new application to drug kinetics. This edition includes several classic calculus applications. These gems of calculus illustrate its power and practical use. Readers of the previous editions have enjoyed special topics that included the derivation leading to the geostationary satellite orbit, a timely topic, as well as an interesting set

of topics in number theory whose inclusion was motivated by the recent proof of Fermat's last theorem. An interesting Fermat offshoot, namely, "near misses," is included, thereby extending the range of interest of this popular book. The table of integrals, which contains the most useful forms, has been reformatted and has been rechecked for accuracy. Although we strive to keep the book size small, we have enlarged the type slightly without sacrificing special topics. These include Fourier series, Laplace and Z-transforms, vector analysis, complex numbers, orthogonal polynomials and infinite series. Many other handbooks go too far in their attempts, essentially trying to mimic larger comprehensive texts. The result is a reference less detailed than the full texts and too big to be conveniently portable so that users would not carry them. Through a careful selection of topics and detail, *Pocket Book of Integrals and Mathematical Formulas* truly meets the needs of students and professionals in being a convenient, compact, and usable resource that also provides worked examples where most necessary. The book is portable, comprehensive, and easy to use.

Ronald J. Tallarida

Philadelphia, Pennsylvania

Preface to the Fourth Edition

As in the previous works, this new edition preserves the content, size, and convenience of this portable reference source for students and workers who use mathematics, while introducing much new material. New in this fourth edition is an expanded chapter on series that now includes many fascinating properties of the natural numbers that follow from number theory, a field that has attracted much new interest since the recent proof of Fermat's last theorem. While the proofs of many of these theorems are deep, and in some cases still lacking, all the number theory topics included here are easy to describe and form a bridge between arithmetic and higher mathematics. The fourth edition also includes new applications such as the geostationary satellite orbit, drug kinetics (as an application of differential equations), and an expanded statistics section that now discusses the normal approximation of the binomial distribution as well as a treatment of nonlinear regression. The widespread use of computers now makes the latter topic amenable to all students, and thus all users of the *Pocket Book of Integrals* can benefit from the concise summary of this topic. The chapter on financial

mathematics, introduced in the third edition, has proved successful and is retained without change in this edition, whereas the Table of Integrals has been reformatted for easier usage. This change in format also allowed the inclusion of all the new topics without the necessity of increasing the physical size of the book, thereby keeping its wide appeal as a true, handy pocket book that students and professionals will find useful in their mathematical pursuits.

Ronald J. Tallarida

Philadelphia, Pennsylvania

Preface to the Third Edition

This new edition has been enlarged to contain all the material in the second edition, an expanded chapter on statistics that now includes sample size estimations for means and proportions, and a totally new chapter on financial mathematics. In adding this new chapter we have also included a number of tables that aid in performing the calculations on annuities, true interest, amortization schedules, compound interest, systematic withdrawals from interest accounts, etc. The treatment and style of this material reflect the rest of the book, i.e., clear explanations of concepts, relevant formulas, and worked examples. The new financial material includes analyses not readily found in other sources, such as the effect of lump sum payments on amortization schedules and a novel “in-out formula” that calculates current regular deposits to savings in order to allow the start of systematic withdrawals of a specified amount at a later date. While

many engineers, mathematicians, and scientists have found much use for this handy pocket book, this new edition extends its usage to them and to the many business persons and individuals who make financial calculations.

Ronald J. Tallarida

Philadelphia, Pennsylvania

Preface to the Second Edition

This second edition has been enlarged by the addition of several new topics while preserving its convenient pocket size. New in this edition are the following topics: z-transforms, orthogonal polynomials, Bessel functions, probability and Bayes' rule, a summary of the most common probability distributions (binomial, Poisson, normal, t, Chi square, and F), the error function, and several topics in multivariable calculus that include surface area and volume, the ideal gas laws, and a table of centroids of common plane shapes. A list of physical constants has also been added to this edition.

I am grateful for many valuable suggestions from users of the first edition, especially Lt. Col. W. E. Skeith and his colleagues at the U.S. Air Force Academy.

Ronald J. Tallarida
Philadelphia, Pennsylvania

Preface to the First Edition

The material of this book has been compiled so that it may serve the needs of students and teachers as well as professional workers who use mathematics. The contents and size make it especially convenient and portable. The widespread availability and low price of scientific calculators have greatly reduced the need for many numerical tables (e.g., logarithms, trigonometric functions, powers, etc.) that make most handbooks bulky. However, most calculators do not give integrals, derivatives, series, and other mathematical formulas and figures that are often needed. Accordingly, this book contains that information in addition to a comprehensive table of integrals. A section on statistics and the accompanying tables, also not readily provided by calculators, have also been included.

The size of the book is comparable to that of many calculators, and it is really very much a companion to the calculator and the computer as a source of information for writing one's own programs. To facilitate such use, the author and the publisher have worked together to make the format attractive and clear. Yet, an important requirement in a book of this kind is accuracy.

Toward that end we have checked each item against at least two independent sources.

Students and professionals alike will find this book a valuable supplement to standard textbooks, a source for review, and a handy reference for many years.

Ronald J. Tallarida

Philadelphia, Pennsylvania

Author

Ronald J. Tallarida holds BS and MS degrees in physics/mathematics and a PhD in pharmacology. His primary appointment is professor of pharmacology at Temple University School of Medicine, Philadelphia, Pennsylvania. For more than 30 years, he also served as an adjunct professor of Biomedical Engineering at Drexel University in Philadelphia where he received the Lindback Award for Distinguished Teaching of mathematics. As an author and researcher, he has published more than 290 works that include eight books, has been the recipient of research grants from NIH, and has served as a consultant to both industry and government agencies. His main research interests are in the areas of mathematical modeling of biological systems, feedback control, and the action of drugs and drug combinations.

Greek Letters

α	A	Alpha
β	B	Beta
γ	Γ	Gamma
δ	Δ	Delta
ε	E	Epsilon
ζ	Z	Zeta
η	H	Eta
θ	Θ	Theta
ι	I	Iota
κ	K	Kappa
λ	Λ	Lambda
μ	M	Mu
ν	N	Nu
ξ	Ξ	Xi
ο	O	Omicron
π	Π	Pi
ρ	P	Rho
σ	Σ	Sigma
τ	T	Tau
υ	Υ	Upsilon
φ	Φ	Phi
χ	X	Chi
ψ	Ψ	Psi
ω	Ω	Omega

The Numbers π and e

π	=	3.14159	26535	89793
e	=	2.71828	18284	59045
$\log_{10}e$	=	0.43429	44819	03252
$\log_e 10$	=	2.30258	50929	94046

Prime Numbers

2	3	5	7	11	13	17	19	23	29
31	37	41	43	47	53	59	61	67	71
73	79	83	89	97	101	103	107	109	113
127	131	137	139	149	151	157	163	167	173
179	181	191	193	197	199	211	223	227	229
233	239	241	251	257	263	269	271	277	281
...				...					...

Important Numbers in Science (Physical Constants)

Avogadro constant (N_A)	$6.02 \times 10^{26} \text{ kmole}^{-1}$
Boltzmann constant (k)	$1.38 \times 10^{-23} \text{ J} \cdot \text{K}^{-1}$
Electron charge (e)	$1.602 \times 10^{-19} \text{ C}$
Electron, charge/mass (e/m_e)	$1.760 \times 10^{11} \text{ C} \cdot \text{kg}^{-1}$
Electron rest mass (m_e)	$9.11 \times 10^{-31} \text{ kg}$ (0.511 MeV)
Faraday constant (F)	$9.65 \times 10^4 \text{ C} \cdot \text{mole}^{-1}$
Gas constant (R)	$8.31 \times 10^3 \text{ J} \cdot \text{K}^{-1} \text{ kmole}^{-1}$
Gas (ideal) normal volume (V_o)	$22.4 \text{ m}^3 \cdot \text{kmole}^{-1}$
Gravitational constant (G)	$6.67 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$
Hydrogen atom (rest mass) (m_H)	$1.673 \times 10^{-27} \text{ kg}$ (938.8 MeV)
Neutron (rest mass) (m_n)	$1.675 \times 10^{-27} \text{ kg}$ (939.6 MeV)
Planck constant (h)	$6.63 \times 10^{-34} \text{ J} \cdot \text{s}$
Proton (rest mass) (m_p)	$1.673 \times 10^{-27} \text{ kg}$ (938.3 MeV)
Speed of light (c)	$3.00 \times 10^8 \text{ m} \cdot \text{s}^{-1}$

1

Elementary Algebra and Geometry

1.1 Fundamental Properties (Real Numbers)

$a + b = b + a$	Commutative Law for Addition
$(a + b) + c = a + (b + c)$	Associative Law for Addition
$a + 0 = 0 + a$	Identity Law for Addition
$a + (-a) = (-a) + a = 0$	Inverse Law for Addition
$a(bc) = (ab)c$	Associative Law for Multiplication
$a\left(\frac{1}{a}\right) = \left(\frac{1}{a}\right)a = 1, \quad a \neq 0$	Inverse Law for Multiplication
$(a)(1) = (1)(a) = a$	Identity Law for Multiplication

$$ab = ba$$

Commutative Law for
Multiplication

$$a(b + c) = ab + ac$$

Distributive Law

Division by zero is not defined.

1.2 Exponents

For integers m and n ,

$$a^n a^m = a^{n+m}$$

$$a^n / a^m = a^{n-m}$$

$$(a^n)^m = a^{nm}$$

$$(ab)^m = a^m b^m$$

$$(a/b)^m = a^m / b^m$$

1.3 Fractional Exponents

$$a^{p/q} = (a^{1/q})^p$$

where $a^{1/q}$ is the positive q th root of a if $a > 0$ and the negative q th root of a if a is negative and q is odd. Accordingly, the five rules of exponents given above (for integers) are also valid if m and n are fractions, provided a and b are positive.

1.4 Irrational Exponents

If an exponent is irrational, e.g., $\sqrt{2}$, the quantity, such as $a^{\sqrt{2}}$, is the limit of the sequence $a^{1.4}$, $a^{1.41}$, $a^{1.414}$,

- *Operations with Zero*

$$0^m = 0; \quad a^0 = 1$$

1.5 Logarithms

If x , y , and b are positive and $b \neq 1$,

$$\log_b(xy) = \log_b x + \log_b y$$

$$\log_b(x/y) = \log_b x - \log_b y$$

$$\log_b x^p = p \log_b x$$

$$\log_b(1/x) = -\log_b x$$

$$\log_b b = 1$$

$$\log_b 1 = 0 \quad \text{Note: } b^{\log_b x} = x.$$

- *Change of Base ($a \neq 1$)*

$$\log_b x = \log_a x \log_b a$$

1.6 Factorials

The factorial of a positive integer n is the product of all the positive integers less than or equal to the integer n and is denoted $n!$. Thus,

$$n! = 1 \cdot 2 \cdot 3 \cdot \dots \cdot n.$$

Factorial 0 is defined $0! = 1$.

- *Stirling's Approximation*

$$\lim_{n \rightarrow \infty} (n/e)^n \sqrt{2\pi n} = n!$$

(See also Section 9.2.)

1.7 Binomial Theorem

For positive integer n ,

$$\begin{aligned}(x+y)^n &= x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 \\ &\quad + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 + \dots \\ &\quad + nxy^{n-1} + y^n.\end{aligned}$$

1.8 Factors and Expansion

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

$$(a^2 - b^2) = (a-b)(a+b)$$

$$(a^3 - b^3) = (a-b)(a^2 + ab + b^2)$$

$$(a^3 + b^3) = (a+b)(a^2 - ab + b^2)$$

1.9 Progression

An *arithmetic progression* is a sequence in which the difference between any term and the preceding term is a constant (d):

$$a, a+d, a+2d, \dots, a+(n-1)d.$$

If the last term is denoted $l [= a + (n-1)d]$, then the sum is

$$s = \frac{n}{2}(a+l).$$

A *geometric progression* is a sequence in which the ratio of any term to the preceding terms is a constant r . Thus, for n terms,

$$a, ar, ar^2, \dots, ar^{n-1}$$

The sum is

$$S = \frac{a - ar^n}{1 - r}$$

1.10 Complex Numbers

A complex number is an ordered pair of real numbers (a, b) .

Equality: $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$

Addition: $(a, b) + (c, d) = (a + c, b + d)$

Multiplication: $(a, b)(c, d) = (ac - bd, ad + bc)$

The first element of (a, b) is called the *real* part; the second, the *imaginary* part. An alternate notation for (a, b) is $a + bi$, where $i^2 = (-1, 0)$, and $i(0, 1)$ or $0 + 1i$ is written for this complex number as a convenience. With this understanding, i behaves as a number, i.e., $(2 - 3i)(4 + i) = 8 - 12i + 2i - 3i^2 = 11 - 10i$. The conjugate of $a + bi$ is $a - bi$, and the product of a complex number and its conjugate is $a^2 + b^2$.

Thus, *quotients* are computed by multiplying numerator and denominator by the conjugate of the denominator, as illustrated below:

$$\frac{2+3i}{4+2i} = \frac{(4-2i)(2+3i)}{(4-2i)(4+2i)} = \frac{14+8i}{20} = \frac{7+4i}{10}$$

1.11 Polar Form

The complex number $x + iy$ may be represented by a plane vector with components x and y :

$$x + iy = r(\cos\theta + i\sin\theta)$$

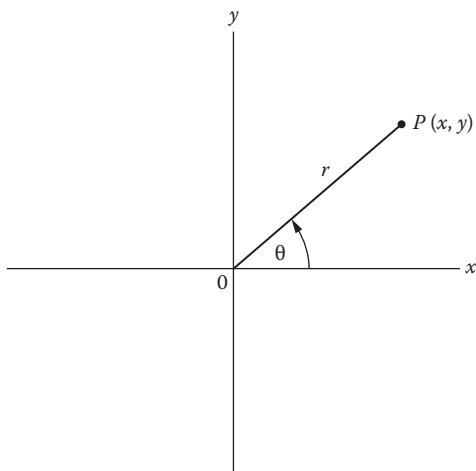
(see Figure 1.1). Then, given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, the product and quotient are:

Product: $z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)]$

Quotient: $z_1 / z_2 = (r_1 / r_2) [\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$

Powers: $z^n = [r(\cos\theta + i\sin\theta)]^n$
 $= r^n [\cos n\theta + i\sin n\theta]$

Roots: $z^{1/n} = [r(\cos\theta + i\sin\theta)]^{1/n}$
 $= r^{1/n} \left[\cos \frac{\theta + k.360}{n} + i \sin \frac{\theta + k.360}{n} \right],$
 $k = 0, 1, 2, \dots, n-1$

**FIGURE 1.1**

Polar form of complex number.

1.12 Permutations

A permutation is an ordered arrangement (sequence) of all or part of a set of objects. The number of permutations of n objects taken r at a time is

$$\begin{aligned} p(n, r) &= n(n-1)(n-2)\cdots(n-r+1) \\ &= \frac{n!}{(n-r)!} \end{aligned}$$

A permutation of positive integers is even or odd if the total number of inversions is an even