Matrix Methods Applied to Engineering Rigid Body Mechanics

T Crouch

Coventry (Lanchester) Polytechnic, England

PERGAMON INTERNATIONAL LIBRARY
of Science, Technology, Engineering and Social Studies
The 1000-volume original paperback library in aid of education, industrial training and the enjoyment of leisure
Publisher: Robert Maxwell, M.C.

MATRIX METHODS APPLIED TO ENGINEERING RIGID BODY MECHANICS

Other Titles of Interest

BENSON & WHITEHOUSE
Internal Combustion Engines (in 2 volumes)

DIXON
Fluid Mechanics, Thermodynamics of Turbomachinery, 3rd Edition

DUNN & REAY
Heat Pipes, 2nd Edition

HAYWOOD
Analysis of Engineering Cycles, 3rd Edition

HEARN
Mechanics of Materials (in 2 volumes)

HOPKINS & SEWELL
Mechanics of Solids

LAI et al
Introduction to Continuum Mechanics, SI Edition

LIVESLEY
Matrix Methods of Structural Analysis, 2nd Edition

NEMAT-NASSER
Mechanics Today, Volumes 1-5

NEMAT-NASSER
Variational Methods in the Mechanics of Solids

REAY & MACMICHAEL
Heat Pumps

Related Pergamon Journals

(Free Specimen Copies Available on Request)

COMPUTERS AND STRUCTURES
INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES
INTERNATIONAL JOURNAL OF MECHANICAL SCIENCES
MECHANISM AND MACHINE THEORY

MATRIX METHODS APPLIED TO ENGINEERING RIGID BODY MECHANICS

T. CROUCH

B.Sc.Mech.Eng., M.I.Mech.E., C.Eng.

Lecturer in the Department of Mechanical Engineering
Coventry (Lanchester) Polytechnic

PERGAMON PRESS

OXFORD · NEW YORK · TORONTO · SYDNEY · PARIS · FRANKFURT

U.K.	Pergamon Press Ltd., Headington Hill Hall, Oxford OX3 0BW, England
U.S.A.	Pergamon Press Inc., Maxwell House, Fairview Park, Elmsford, New York 10523, U.S.A.
CANADA	Pergamon of Canada, Suite 104, 150 Consumers Road, Willowdale, Ontario M2J 1P9, Canada
AUSTRALIA	Pergamon Press (Aust.) Pty. Ltd., P.O. Box 544, Potts Point, N.S.W. 2011, Australia
FRANCE	Pergamon Press SARL, 24 rue des Ecoles, 75240 Paris, Cedex 05, France
FEDERAL REPUBLIC OF GERMANY	Pergamon Press GmbH, 6242 Kronberg-Taunus, Hammerweg 6, Federal Republic of Germany

First edition 1981

British Library Cataloguing in Publication Data

Crouch, T
Matrix methods applied to engineering rigid body mechanics. - (Pergamon international library).
1. Mechanics, Applied 2. Vector analysis
3. Matrices
I. Title
531'.01'51563 TA350 80-41186

ISBN 0 08 024245 6 (Hardcover)
ISBN 0 08 024246 4 (Flexicover)

In order to make this volume available as economically and as rapidly as possible the authors' typescripts have been reproduced in their original forms. This method has its typographical limitations but it is hoped that they in no way distract the reader.

Printed in Great Britain by A. Wheaton & Co. Ltd., Exeter

Preface

The purpose of this book is to present the solution of a range of rigid body mechanics problems using a matrix formulation of vector algebra. The treatment has other notable features. It employs a coherent letter and number suffix notation and also exploits the relationship between the orthogonal transformation matrix and angular velocity. Particular emphasis is placed upon the positioning of appropriate frames of reference and specifying their relative position.

In writing this text it has been assumed that the reader will have a knowledge of mathematics and mechanics normally associated with the first year of an Engineering Degree course.

The plan of the book is simple. There are four chapters, Chapter 1 Kinematics, Chapter 2 Dynamics, Chapter 3 Solution of Kinematics Problems and Chapter 4 Solution of Dynamics Problems. Chapters 1 and 2 give a succinct statement of the essential theory formulated in terms of matrix algebra, while Chapters 3 and 4 give a selection of solved problems and problems for solution. The reader is therefore advised to study the problems to which reference is made at various points in the text as they occur. A proper approach to the solution of dynamics problems demands that kinematic considerations have priority. It is suggested, therefore, that the reader studies Chapters 1 and 3 before proceeding the Chapters 2 and 4. Answers to the problems for solution are provided, with some indication of the salient features of their solution in most cases.

Coventry 1980 T. Crouch

Contents

Principal Symbols and Notation xiii

Chapter 1 Kinematics 1

1.1 The Position Vector 1
1.2 The Relative Position Vector 2
1.3 Transformation of Vectors 3
1.4 The Rotation Matrix for Simple Rotations 8
1.5 Consecutive Rotations 10
1.6 Successive Transformation of Vectors 12
1.7 The Velocity and Acceleration of a Point 14
1.8 Small Rotations 18
1.9 Angular Velocity and the Derivative of the Rotation Matrix 20
1.10 The Relative Velocity of Points Fixed in a Rigid Body 23
1.11 The Central and Instantaneous Axes 26
1.12 The Relative Acceleration of Points Fixed in a Rigid Body 29
1.13 The Relative Acceleration of Coincident Points Which Have Relative Motion 33
1.14 Differentiation of the Angular Velocity Vector, a Special Case 34
1.15 Relative Angular Velocity 35
1.16 Relative Angular Acceleration 37

Chapter 2 Dynamics 38

2.1 Newton's Laws of Motion 38
2.2 The Measurement of Force 39
2.3 Work, Potential and Kinetic Energy 40
2.4 The Activity of a Force and its Relationship to the Rate of Change of Kinetic Energy 43
2.5 Impulse and Momentum 43

2.6 Centre of Mass 44
2.7 Force Moment, Moment of Momentum and Moment of Rate of Change of Momentum 45
2.8 The Linear Momentum of a Rigid Body 47
2.9 The Moment of Momentum of a Rigid Body About its Centre of Mass 47
2.10 The Relationship Between Moments of Inertia Measured in Different Frames 50
2.11 The Rate of Change of Angular Momentum of a Rigid Body About its Centre of Mass 52
2.12 The Moment of Momentum of a Rigid Body About Any Point Q and the Rate of Change of Moment of Momentum About That Point 53
2.13 The Relationship Between the Moment of the External Forces and Couples on a Rigid Body About Any Point Q and the Rate of Change of Moment of Momentum About That Point 54
2.14 The Kinetic Energy of a Rigid Body 55
2.15 The Rate of Change of Kinetic Energy of a Rigid Body 56
2.16 The Special Case of the Motion of a Solid of Revolution 57
2.17 Rotation About a Fixed Axis 58
2.18 Principal Axes and Principal Moments of Inertia of a Rigid Body With a Plane of Symmetry 58
2.19 Principal Axes and Principal Moments of Inertia For Any Rigid Body 61

Chapter 3 Solution of Kinematics Problems 65

3.1 Solved Problems 65

Problem

3.1 Velocity and Acceleration of a Point Moving in a Circular Path 65
3.2 Summation of Finite Angles of Rotation 69
3.3 Velocity and Acceleration of a Point Moving on a Straight Rotating Path 71
3.4 Angular Velocity Determination From Linear Velocity Data 74
3.5 Velocity Determination For a Mechanism 77
3.6 Plane Motion of a Disc 80
3.7 Velocity and Acceleration of a Point on a Disc in Conic Rolling 85
3.8 Velocity and Acceleration Determination For a Mechanism 92
3.9 Velocity and Acceleration Determination For a Linkage 97
3.10 Velocity and Accleration of a Point Moving in Earth Fixed Axes 101
3.11 Velocity and Acceleration of a Point in a 'Plane' Epicyclic Gear Train 103
3.12 Angular Velocity and Acceleration of a Precessing Rotor 106
3.13 Angular Velocity and Angular Acceleration of Ball in a Ball Thrust Race 109

3.14 Angular Velocity and Angular Acceleration of a Roller in a Taper-Roller Thrust Race 113
3.15 Angular Acceleration of a Nutating and Precessing Rotor 115
3.16 Rubbing Velocities Associated With a Rolling Rotor 118
3.17 Discs in Rolling Contact 120
3.18 Rolling Disc With Bevel Gear Drive 123
3.19 Bevel Wheel Epicyclic Gear Train 126
3.20 Hooke's Joint 128
3.21 Rolling Disc With Constant Velocity Joint 131
3.22 Euler's Theorem on the Motion of Rigid Bodies 133

3.2 Problems For Solution 141

Problem

3.23 Velocity and Acceleration of a Point Moving on a Rotating Circular Path 141
3.24 A Vector Which is Unchanged by a Transformation
3.25 Differentiation of the Rotation Matrix For Simple Rotations 142
3.26 Relative Motion of Aircraft 1 143
3.27 Relative Motion of Aircraft 2
3.28 Derivation of a Transformation Matrix 1 144
3.29 Derivation of a Transformation Matrix 2 144
3.30 Aircraft Tracking 145
3.31 Derivation of Angular Velocity From Linear Velocity Data 146
3.32 Location of Central Axis 1 147
3.33 Location of Central Axis 2
3.34 Euler Angles 149
3.35 Bryant Angles 150
3.36 Acceleration of a Point on a Rotating Disc 1
3.37 Acceleration of a Point on a Rotating Disc 2 151
3.38 Acceleration of a Point on a Rotating Disc 3
3.39 Acceleration of a Point on a Rotating Disc 4
3.40 Rolling Wheel on Moving Surface 152
3.41 Wheels and Axle 153
3.42 Rotating and Extending Antenna
3.43 Articulated Trailer 154
3.44 Precession of Rotating Cylinder 156
3.45 Thrust Bearing
3.46 Automotive Differential 157
3.47 A Bevel Wheel Gear Train 158
3.48 Rotor on Rotating Pivoted Axle 159
3.49 Rotating Epicyclic 160
3.50 Velocity and Acceleration Determination For a Mechanism
3.51 Velocities and Acceleration of a Rolling Disc 161

Chapter 4 Solution of Dynamics Problems 164

4.1 Solved Problems 164

Problem

4.1 The Conditions Which a Moving Reference Frame

	Must Satisfy to Allow it to be Treated as an Inertial Frame	164
4.2	Gravitational Potential	165
4.3	Strain Potential 1	166
4.4	Strain Potential 2	168
4.5	Work Done by a Non-conservative Force 1	171
4.6	Work Done by a Non-conservative Force 2	174
4.7	Gravitational and Elastic Potential	179
4.8	Del V Referred to a Rotating Frame	181
4.9	Work Done and Change of Kinetic Energy When Force is a Given Function of Time	183
4.10	Potential and Kinetic Energy of a Spring and Mass System	186
4.11	The Motion of a Particle on a Helical Path	188
4.12	The Motion of a Particle Falling Freely Near the Surface of the Earth	191
4.13	The Motion of a Rotating Spring and Mass System	192
4.14	Forced Pendulum Motion	194
4.15	The Foucault Pendulum	198
4.16	Location of Centre of Mass 1	201
4.17	Location of Centre of Mass 2	204
4.18	Force Moment	205
4.19	Static Equilibrium 1	206
4.20	Static Equilibrium 2	208
4.21	Static Equilibrium 3	212
4.22	The Wrench	214
4.23	The Perpendicular and Parallel Axis Theorems	217
4.24	Inertia Matrix For a Thin Disc	218
4.25	Inertia Matrix For a Three Bladed Airscrew	219
4.26	Determination of Inertia Matrix, Angular Momentum and Rate of Change of Angular Momentum 1	221
4.27	Determination of Inertia Matrix, Angular Momentum and Rate of Change of Angular Momentum 2	225
4.28	Motion of a Rigid Body About its Centre of Mass	229
4.29	Stability of the Free Motion of a Rigid Body	232
4.30	Free Motion of a Rigid Body Having Axial Symmetry	234
4.31	Motion of a Constrained Body 1	240
4.32	Motion of a Constrained Body 2	243
4.33	Forces on a Ball in a Ball Thrust Race	245
4.34	Motion of a High Speed Rotor Mounted in Gimbals	248
4.35	Relationships Between Kinetic Energy and Momentum For a Constrained Rotor	251
4.36	Forces and Moments Due to Constraints on a Rotor	255
4.37	Motion of a Constrained Rod 1	257
4.38	Motion of a Constrained Rod 2	260
4.39	Motion of a Constrained Rod 3	263
4.40	Forces in a Mechanism	266
4.41	Inertia Determination For a Hemisphere	270
4.42	Inertia Determination For a Thin Rod	275
4.43	Inertia Determination For a Rod System	277

4.2 Problems For Solution 282

Problem

4.44 Activity of a Force 282
4.45 Work Done by a Conservative Force
4.46 Work and Potential in a Conservative System
4.47 Del V Referred to a Rotating Frame 283
4.48 A Property of the Couple
4.49 A Component of Force Moment
4.50 Resultants of a Given Force System 285
4.51 Equilibrium of a Simple Structure 286
4.52 Equilibrium of a Mechanism
4.53 The Wrench
4.54 Inertia Determination For a Rectangular Parallelepiped 288
4.55 Inertia Determination For a Right Circular Cone
4.56 Couple Due to a Two Bladed Airscrew
4.57 Couple Due to Aircraft Control Surfaces
4.58 Couple Due to Wobble of Circular Saw Blade 289
4.59 Measurement of Angular Velocity Relative to an Inertial Reference 290
4.60 Bearing Forces Due to Rotor Precession
4.61 Bearing Forces Due to Rotor Misalignment 292
4.62 Attitude of Pivoted Rectangular Parallelepiped
4.63 Frequency of Vibrations of Thin Pivoted Rod
4.64 Frequency of Vibrations of Constrained Rod 294
4.65 Contact Force Due to the Precession of a Rotor 1
4.66 Contact Force Due to the Precession of a Rotor 2 295
4.67 Forces on a Rolling Cone 296
4.68 Contact Force Due to the Precession of a Rotor 3 297
4.69 Steady Motion of a Top
4.70 Motion of a Precessing Rotor 1 298
4.71 Motion of a Precessing Rotor 2 299
4.72 Motion of a Pendulously Mounted Rotor 1 300
4.73 Motion of a Spring Controlled Gimbal Mounted Rotor 301
4.74 Motion of a Pendulously Mounted Rotor 2
4.75 Bearing Forces Due to Rotor Motion 1 302
4.76 Bearing Forces Due to Rotor Motion 2 303
4.77 A Torsional Vibration Absorber
4.78 A Rudimentary Gyro-compass 305
4.79 Steady Motion of a Disc 306
4.80 Initial Motion of a Thin Rod
4.81 Vibration of a Spinning Projectile 308
4.82 Motion of a Constrained Disc
4.83 Conservation of Energy and Momentum in Free Motion 309
4.84 Inertia Determination For a Body Having Plane Symmetry
4.85 Inertia Determination For a Body Without Plane Symmetry 1 310
4.86 Inertia Determination For a Body Without Plane Symmetry 2 311

Answers to Problems For Solution Chapter 3 313

Answers to Problems For Solution Chapter 4 321

Bibliography 339

Principal Symbols and Notation

The following lists give only the principal use of the symbols for scalar quantities. A given symbol might be used to denote a variety of physical quantities. The interpretation to be given to a symbol will be clear from the context in which it is employed.

Kinematics

1 Scalars

a, b, c, d, u, v, w, r, s, t	Length, components of vectors
α, β, γ, θ, ϕ, ψ	Angles
ω, Ω	Components of angular velocity
$\dot{\omega}$, $\dot{\Omega}$	Components of angular acceleration
ℓ	Direction cosine

2 Vectors

With the exception of the lower case Greek letter omega, upper case letters written inside braces are used to designate vector quantities as follows:

$\{R\}$	Position and relative position
$\{V\}$	Linear velocity
$\{A\}$	Linear acceleration
$\{B\}$	Any vector
$\{\omega\}$	Angular velocity
$\{\dot{\omega}\}$	Angular acceleration

These general symbols for vector quantities are qualified in two ways by appropriate suffixes. Thus

$$\{R_{\mathbf{A}}\}_1 \quad \text{or} \quad \{R_{\mathbf{AO}_1}\}_1$$

specifies the position of the point A measured in frame 1, where O_1, often omitted, is the origin of frame 1, while

$$\{R_{BA}\}_4$$

specifies the position of the point B relative to the point A measured in frame 4. It so happens the the relative position vector is independent of the frame in which its is measured, but the number suffix is retained for reasons explained in the text.

Similarly,

$$\{V_A\}_1 \quad \text{or} \quad \{V_{AO_1}\}_1 \quad \text{and} \quad \{V_{PQ}\}_3$$

specify the velocity of the point A relative to O_1 measured in frame 1 and the velocity of the point P relative to the point Q measured in frame 3 respectively.

Also

$$\{A_A\}_1 \quad \text{or} \quad \{A_{AO_1}\} \quad \text{and} \quad \{A_{DC}\}_1$$

specifiy the acceleration of the point A relative to the point O_1 measured in frame 1 and the acceleration of the point D relative to the point C measured in frame 1 respectively.

Numbers are also used as suffixes inside the braces to qualify position, velocity and acceleration. Thus

$$\{R_4\}_1 \quad , \quad \{V_4\}_1 \quad \text{and} \quad \{A_4\}_1$$

specify the position, velocity and acceleration respectively of the centre of mass of body 4 measured in frame 1.

The angular velocity vector is qualified by number suffixes. Thus

$$\{\omega_3\}_2 \quad \text{or} \quad \{\omega_{32}\}$$

specify the angular velocity of body 3 measured with respect to body 2 or the angular velocity of body 3 relative to body 2. A similar notation is used for angular acceleration. The angular velocity and acceleration vectors can be further qualified by lower case superscript letters inside the braces. Thus

$$\{\omega_2^n\}_1 \quad \text{and} \quad \{\omega_2^p\}_1$$

specify, respectively, the components of the angular velocity vector normal to and parallel to to some line joining points(specified in a particular context) fixed in body 2. Similarly,

$$\{A_{BA}^n\}_1 \quad \text{and} \quad \{A_{BA}^p\}_1$$

specify, respectively, the components of the linear acceleration of B relative to A normal to and parallel to the line joining B and A.

The usual modulus notation is employed to indicate the magnitude of a vector. Thus

$$|R_A|_1 \ , \ |V_{BA}|_1 \text{ and } |\omega_2|_1 \text{ or } |\omega_{21}|$$

are magnitudes of the corresponding vectors. In the case of the relative position vector, which is independent of the frame used for its measurement the number suffix is omitted. Thus the magnitude of

$$\{R_{BA}\}_1 = \{R_{BA}\}_4 = \{R_{BA}\}_n$$

is written

$$|R_{BA}| .$$

A vector can be described by resolving it along the axes of a particular reference frame, when it is said to be referred to that frame. The frame to which a vector is referred is written outside the braces after the first number suffix and separated from it by a solidus or oblique stroke. Thus

$$\{V_{BA}\}_{1/3} = \begin{bmatrix} u_3 \\ v_3 \\ w_3 \end{bmatrix}$$

is the column matrix which describes the velocity of B relative to A, measured in frame 1, in frame 3.

3 The transformation or rotation matrix

The transformation matrix is a 3×3 orthogonal matrix of direction cosines written

$$[\,\ell\,] .$$

It is used to change the frame to which a vector is *referred*. If, for example, a vector $\{B\}_n$ is referred to frame 1, then the transformation matrix which changes the reference frame to frame 2 is

$$[\,\ell_1\,]_2 .$$

Thus

$$\{B\}_{n/2} = [\ell_1\,]_2\{B\}_{n/1} .$$

The transformation matrix can be regarded as the matrix which specifies a rotation, or sequence of rotations, which a frame undergoes to align it with another. If, for example, frame 1 is to be aligned with frame 2, then the rotation matrix would be written

$$[\,\ell_2\,]_1 .$$

If this alignment is achieved by a sequence of simple rotations about a single axis of appropriately positioned intermediate frames 3 and 4, then this operation would be specified by the product of rotation matrices

$$[\,\ell_2\,]_1 = [\,\ell_3\,]_1[\,\ell_4\,]_3[\,\ell_2\,]_4 .$$

Dynamics

1 Scalars

A, B, C, D, E, F, I, J	Terms in the inertia matrix
k	Spring rate, constant
m	Mass
T	Kinetic energy
V	Potential energy
g	Magnitude of gravitational acceleration
W	Work

The general symbols can be qualified by appropriate suffixes.

I can take the suffixes xx, xy, xz etc. to denote the axes involved. A, B, C etc. can take number suffixes to denote the reference frame.

m can take a suffix P to indicate that it refers to a particle, or a number suffix to indicate the body to which it refers.

T can take a suffix P to indicate that it refers to a particle, or a number suffix to indicate the body to which it refers. It can be further qualified to indicate that the energy is evaluated at some particular position. Thus, for example

$$T_{4rot}\Big|_{\alpha}$$

is the rotational kinetic energy in body 4 when in some position defined by the angle α. V can be qualified in a similar manner.

W can take suffix statements such as $A \rightarrow B \rightarrow C$ to specify the path traced out by the point of application of the force involved.

2 Vectors

Upper case letters written inside braces are used to designate vector quantities as follows:

$\{F\}$	Force
$\{G\}$	Linear momentum
$\{H\}$	Angular momentum (Angular momentum)
$\{L\}$	Couple moment
$\{M\}$	Force moment
$\{W\}$	Weight
$\{\nabla\}$	Vector operator del

The general symbols for vector quantities are qualified in two ways by appropriate suffixes and also by superscripts.

In the case of the force vector, number suffixes inside the braces are used to specify a contact force between two bodies. As, for example,

$\{F_{23}\}$

which is the force *on* body 2 *due to* body 3. Similarly,

$\{L_{23}\}$

is the couple on body 2 due to body 3. Also

$\{F_2\}$

is the external force on body 2. It might be, for example,

$$\{F_2\} = \{F_{23}\} + \{F_{24}\} + \{F_{25}\} + \ldots\ldots$$

where 3, 4 and 5 are bodies which exert a force on body 2. A similar notation can be used in respect of couples. Components of $\{F\}$ and $\{L\}$ can be singled out by writing an appropriate superscript inside the braces, as for example,

$\{F^x\}$ and $\{L^y\}$

or

$\{F^n\}$ and $\{L^p\}$

where the superscripts n and p refer to components parallel to some reference direction.

A number suffix is used outside the braces to specify the frame to which the vector is referred. Thus

$\{F_{34}\}_3$

is the column matrix which describes the force on body 3 due to body 4 which is referred to frame 3. The $\{L\}$ can be similarly subscripted.

In the case of linear momentum a number suffix inside the braces specifies the body concerned and the first number suffix outside the braces specifies the frame in which the momentum is measured. This frame will invariably be an inertial reference frame which, in this text, is always designated 1. It is always included by way of emphasis. The second number suffix outside the braces, written after a solidus, specifies the frame to which the vector is referred. Thus

$\{G_3\}_{1/4}$

is the column matrix which describes the linear momentum of body 3, measured with respect to frame 1, the vector being referred to frame 4.

In the case of angular momentum of a body about its centre of mass, a number suffix inside the braces specifies the body concerned and the number suffixes outside the braces have the same significance as in the case of linear momentum. Thus

$\{H_3\}_{1/4}$

is the column matrix which describes the angular momentum of body 3 about its centre of mass, measured with respect to frame 1, the vector being referred to frame 4. If the angular momentum about a point other than the centre of mass is to be specified, point Q say, then this is written

$$\{H_{3Q}\}_{1/4} \ .$$

In the case of the moment vector, a single letter suffix is used to specify the point about which moments are taken and a single number suffix outside the braces specifies the frame to which both force and position vectors are referred. Thus

$$\{M_A\}_3$$

is the column matrix which describes the moment of a force, or system of forces and couples about A, the vector being referred to frame 3.

In the case of the weight vector, a number suffix specifies the body to which it refers and a single number suffix outside the braces specifies the frame to which the vector is referred. Hence

$$\{W_4\}_2$$

is the column matrix which describes the weight of body 4, the vector being referred to frame 2.

3 The inertia matrix

The inertia matrix is a 3×3 symmetric matrix written

$$[I]$$

Number suffixes are used in the same way as for vectors. Thus

$$[I_3]_{3/3}$$

describes the inertia of body 3, measured with respect to frame 3 and referred to frame 3. Unless expressly stated otherwise, the centre of of mass of body 3 will be at the origin of frame 3. Similarly,

$$[I_3]_{4/5}$$

describes the inertia of body 3, measured with respect to frame 4 and referred to frame 5.