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elastic wave propagation

M.F. McCARTHY, M.A. HAYES
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ELASTIC WAVE PROPAGATION

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ELASTIC WAVE PROPAGATION

Proceedings of the Second I. U.T. A. M. - I. U. P. A. P. Symposium on Elastic Wave Propagation, Galway, Ireland, March 20–25, 1988

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PREFACE

This volume contains the texts of the contributions presented at the I.U.T.A.M. – I.U.T.A.P. Symposium on Elastic Wave Propagation held in University College, Galway, Ireland from March 20-25, 1988.

The first I.U.T.A.M. Symposium on Elastic Wave Propagation was held in 1977 at Northwestern University, Evanston, U.S.A. In constructing the programme for this second Symposium the International Scientific Committee was deeply conscious of the significant progress that has been made since 1977 and aimed at as broad a coverage as possible of the area. Because of the tremendous interest shown in the Symposium together with the excellent standard of the contributions submitted, a greater number of papers than is customary at such Symposia was accepted for presentation.

Seven distinguished scholars were invited to deliver sessional lectures which surveyed recent developments. A wide range of theoretical, experimental and numerical research was presented and discussed and an overview session in which an effort was made to identify new significant problems was held. There was an extremely good mix of topics presented and this volume contains topics as diverse as surface existence theory, spreading of nonlinear surface waves, and propagation through anisotropic microcracked rocks.

Financial support was provided by the International Union of Theoretical and Applied Mechanics, the International Union of Pure and Applied Physics, the Royal Irish Academy and University College, Galway. It would not, however, have been possible to organise the Symposium without the most generous support of the U.S. Office of Naval Research and the many Irish commerical organisations who made financial grants to the Symposium. The generosity of all our sponsors is greatly appreciated.

We are extremely grateful to the members of the International Scientific Committee for their advice and encouragement. The preparatory management of the Symposium

and related events were the responsibility of the local organising committee and we wish to express particular thanks to Dr. P.M. O'Leary and Professor J.N. Flavin who did much to ensure the success of the meeting. We also wish to express our deepest appreciation to Ms. Noelle Scully who prepared the documentation for the Symposium and who, together with Ms. Maeve McCarthy, helped to make the Symposium successful as well as enjoyable for the participants.

We hope that our efforts in organising this Symposium and editing its proceedings will motivate researchers to further study resulting in significant advances in the field of Elastic Wave Propagation.

July, 1988

Matthew F. McCarthy
Michael A. Hayes

Opening Address of Dr. Colm O'hEocha, President, University College, Galway

It is my pleasure to welcome delegates from around the globe to University College Galway to attend a prestigious Symposium on Elastic Waves. The selection of the location is a tribute to the international standing of Galway/Irish researchers in this area of science.

The meeting is, of course, held under joint auspices with the Royal Irish Academy. The first volume of the *Transactions* of the Academy (1787) contained the following:

"The researches of the mathematician are the only sure ground on which we can reason from experiments; and how far experimental science may assist the commercial interest of a state is clearly evinced by the success of those several manufacturers in the neighbouring countries of England and France, where the hand of the artificier has taken its direction from the philosopher".

It is in this spirit that I express the hope that economic, no less than intellectual and academic benefit will accrue to Ireland as a result of this Symposium being held here.

University Presidents, as they age, become repositories of out-of-date knowledge and so are increasingly more valued for what little wisdom they possess. Therefore, I asked Professor McCarthy for advice on the historical basis for holding the meeting at Galway. We searched the list of Joseph Larmor's publications and found some references to elasticity.

Joseph Larmor was Professor of Natural Philosophy in Galway during the period 1880-85.

When he died in his native Antrim in 1942, an obituary was written by D'Arcy Wentworth Thompson, whose father was Larmor's contemporary at Galway — as Professor of Greek. He noted that the young Larmor came out of St. John's College

Cambridge as Senior Wrangler in 1880 with J.J. Thompson in second place. "In the same year (1880) being twenty-three years old, he went to Queen's College, Galway, as Professor of Natural Philosophy ... Galway was then a nest of scholars, like one of the little old German Universities – as Larmor said, looking backward later on. My father was there, and Davies, the learned editor of the Eumenides; Allman was there, writing his small but famous book on the Greek Geometers; Rowney and King, chemist and geologist, were demolishing the old figment of *Eozoon canadense*; and Larmor came and wrote his first papers, including one on *Least Action as the Fundamental Formulation in Dynamics and Physics* (1884). It was a subject of which he never tired; it was a confession of his scientific faith. He spent five happy years in Galway, went back to Cambridge not without reluctance. There in 1903 he succeeded Stokes in the Lucasian Professorship – Isaac Newton's Chair – and held it for nearly thirty years.

It is topical to recall that, as D'Arcy Thompson says, Larmor, like many old Galway students and staff was unionist Irish. Towards the end of his days he referred to the contributions of such as Kelvin and Maxwell, and could not forgive the new generation for forgetting, as he believed they did, "that the Scoto-Irish school of Physics dominated the world in the middle of the last century, but has now vanished from the face of the earth".

In another letter on the relation of Time to Natural Theology, Larmor wrote that he hoped that his reflections might bear fruit "when the present phase has worn off, and people are willing to consider ideas not based on the negation of the great age of Mathematical Physics".

Sir D'Arcy Wentworth Thompson himself (1860-1948) held chairs of Natural History at Dundee University, and later, at St. Andrews. His great seminal work, in which he linked Mathematics and Biology, was on *Growth and Form* (1917). Sir Peter Medawer, Professor of Zoology, and Nobel Laureate in Medicine, wrote as follows of D'Arcy in *The Art of the Soluble*:

"D'Arcy Wentworth Thompson was an aristocrat of learning whose intellectual endowments are not likely ever again to be combined within one man. He was a classicist of sufficient distinction to have become President of the Classical Associations of England and Wales and of Scotland; a mathematician good enough to have had an entirely mathematical paper accepted for publication by the Royal Society; and a naturalist who held important chairs for sixty-four years, that is, for all but the length of time into which we must nowadays squeeze the whole of our lives from birth until professional retirement. He was a famous conversationalist and lecturer (the two are often thought to go together, but seldom do), and the author of a work which considered as literature, is the equal of anything of Pater's or Logan Pearsall Smith's in its complete mastery of the *bel canto* style. Add to all this that he was over six feet tall, with the build and carriage of a Viking and with the pride of bearing that comes from good looks known to be possessed".

And so, Galway has many contemporary and historic claims, both direct and indirect, to host this Symposium.

I wish to offer my congratulations to the organisers, notably Professor M.F. McCarthy.

It gives me pleasure to declare the Symposium on Elastic Waves officially open.

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ELASTIC SURFACE WAVES

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ELASTIC WAVE PROPAGATION
M.F. McCarthy, M.A. Hayes, (Editors)

RECENT DEVELOPMENTS IN THE THEORY OF ELASTIC SURFACE AND INTERFACIAL WAVES

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An outline of the basic theory of elastic surface waves in the state of development reached in 1977 is followed by a survey of subsequent contributions, including advances in the related theory of interfacial waves.

1. PLANE WAVES AND SURFACE WAVES

We consider a homogeneous elastic body which is anisotropic relative to a natural reference configuration N. The material forming the body is assumed to have a positive definite strain energy, so that the fourth-order linear elasticity tensor **B** in N has the properties

$$B_{jikl} = B_{klij} = B_{ijkl} \tag{1}$$

and

$$B_{pqrs}S_{pq}S_{rs} > 0 \quad \forall \text{ non-zero real symmetric tensors } \mathbf{S}. \tag{2}$$

(All vector and tensor components in this section relate to a fixed orthonormal basis.) The constitutive equations describing the mechanical response of the material to infinitesimal deformations from N are

$$\sigma_{ij} = B_{ijrs}u_{s,r}, \tag{3}$$

where **u** denotes the displacement, σ the stress and $,i$ differentiation with respect to the component x_i of the spatial position **x**. In the absence of external body forces the displacement equations of motion are

$$B_{pirs}u_{s,pr} = \rho\partial^2 u_i/\partial t^2, \tag{4}$$

t being the time and ρ the density in N.

Our basic concern in this article is with plane-wave solutions of (4) of the form

$$\mathbf{u} = \exp\{i\kappa(\mathbf{e}_1.\mathbf{x} + p\mathbf{e}_2.\mathbf{x} - vt)\}\mathbf{a}. \tag{5}$$

The positive real quantities κ and v are respectively a wave number and a speed of propagation, and the plane R spanned by the orthogonal unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ is the *reference plane* of the wave. The slowness is

$$\mathbf{s} = v^{-1}(\mathbf{e}_1 + p\mathbf{e}_2), \tag{6}$$

and the propagation condition resulting from the substitution of (5) into (4) can be expressed as

$$\mathbf{Q}(\mathbf{s})\mathbf{a} = \rho\mathbf{a},$$

or

$$[p^2\mathbf{Q}(\mathbf{e}_2) + p\{\mathbf{R}(\mathbf{e}_1,\mathbf{e}_2) + \mathbf{R}(\mathbf{e}_2,\mathbf{e}_1)\} + \mathbf{Q}(\mathbf{e}_1) - \rho v^2\mathbf{I}]\mathbf{a} = \mathbf{0}. \tag{7}$$

Here I is the identity tensor and Q(.) and R(.,.) are the acoustical and associated acoustical tensors defined by the component relations

$$Q_{ij}(v) = B_{pirj}v_p v_r, \quad R_{ij}(v,w) = B_{pirj}v_p w_r \quad \forall \text{ non-zero real vectors } v, w. \tag{8}$$

The traction produced by the wave on a surface element with inward unit normal e_2 is, from (3) and (5),

$$t = -\sigma^T e_2 = -i\kappa \exp\{i\kappa(e_1.x + pe_2.x - vt)\}\ell, \tag{9}$$

where

$$\ell = \{R(e_2,e_1) + pQ(e_2)\}a.$$

It will be convenient to refer to a and ℓ as the *displacement* and *traction amplitudes*.

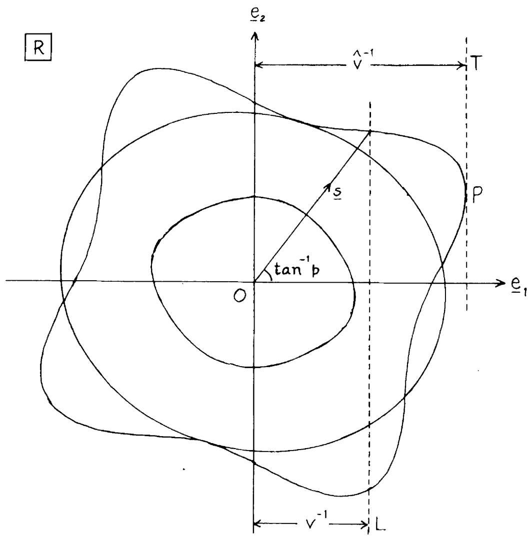

FIGURE 1
Illustration of the slowness section S∩R and the transonic state.

The eigenvalue problem (7) determines the possible values of p and, to within an arbitrary multiplier, a. The eigenvalues are the roots of the sextic equation

$$\det[p^2Q(e_2) + p\{R(e_1,e_2) + R(e_2,e_1)\} + Q(e_1) - \rho v^2 I] = 0: \tag{10}$$

in consequence of (2) they are all real when v is large enough and form three complex conjugate pairs when v is sufficiently small. This limiting behaviour of the p's can be interpreted amd amplified with reference to the slowness surface S of the material, a possible form of which is shown in Figure 1. When p is real, the representative point in slowness space of the wave (5), with position vector s given by (6) relative to the centre O, is at the intersection with S of the line L in R parallel to e_2 and at perpendicular distance 1/v from O. In Figure 1, L has four points of intersection with S∩R, indicating that (10) has four real and two conjugate complex roots for this particular value of v. The plane wave (5) is homogeneous when p is real and inhomogeneous when p is complex. When L lies to the right of the tangent T, all six eigenvalues are complex. The speed v is then in the interval $I = [0,\hat{v})$, $\hat{v}$ being the reciprocal of the perpendicular distance of O from T. T defines the *transonic state* in R relative to $\mathbf{e}_2$; the homogeneous plane wave represented by P, the point of contact of T with S, is the *limiting wave*, $\hat{v}$ the *limiting speed* and I the *subsonic interval* [1,Sect.VI]. In the transonic state illustrated in Figure 1, T touches a single sheet of S just once and the state is accordingly said to be of *type 1*. There are five other types of transonic states in which n, the number of limiting waves, is two or three [1,Sect.VI,C], but it is not necessary to particularize them here.

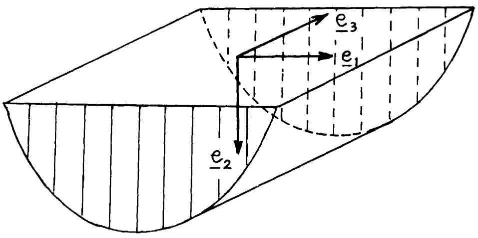

FIGURE 2
Definition of the surface-wave basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$.

A surface wave in the semi-infinite body sketched in Figure 2, propagating in the direction of $\mathbf{e}_1$, is a linear combination of inhomogeneous plane waves of the form (5) leaving the plane boundary $\mathbf{e}_2.\mathbf{x} = 0$ traction free and satisfying the decay condition

$$\mathbf{u} \to 0 \text{ as } \mathbf{e}_2.\mathbf{x} \to \infty. \tag{11}$$

In view of (11) we can only use complex roots of (10) with positive real part in the construction of a surface wave. If p_α, $\alpha = 1,2,3$, are the roots in question and $\mathbf{a}_\alpha$ and $\boldsymbol{\ell}_\alpha$ corresponding amplitudes, the displacement-traction field is

$$\{\mathbf{u},\mathbf{t}\} = \sum_{\alpha=1}^{3} \gamma_\alpha \exp\{i\kappa(\mathbf{e}_1.\mathbf{x} + p_\alpha \mathbf{e}_2.\mathbf{x} - v_S t)\}\{\mathbf{a}_\alpha, -i\kappa\boldsymbol{\ell}_\alpha\}, \tag{12}$$

and we set

$$\{\mathbf{A},\ \mathbf{L}\} = \sum_{\alpha=1}^{3} \gamma_\alpha\{\mathbf{a}_\alpha,\ \boldsymbol{\ell}_\alpha\}. \tag{13}$$

The boundary condition then takes the form

$$\mathbf{L} = 0, \tag{14}$$

equivalent to

$$[\boldsymbol{\ell}_1,\boldsymbol{\ell}_2,\boldsymbol{\ell}_3] = 0. \tag{15}$$

This is the *secular equation*, fixing the speed of propagation v_S. When v_S is known, the coefficients γ_α are determined, up to a common multiplier, by (14).

A surface wave is *subsonic* when $0 < v_S < \hat{v}$ and *supersonic* when $v_S > \hat{v}$. Since (10) has at least two real roots when $v > \hat{v}$ (see Figure 1), a supersonic surface wave consists of only two inhomogeneous plane waves. This suggests that such waves can only exist when there is a good deal of symmetry in the orientation of the crystallographic axes of the transmitting medium relative to the surface-wave basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Some examples are considered in Section 5.

2. INITIAL DEVELOPMENT OF THE THEORY OF BARNETT AND LOTHE

We now proceed to outline the main features of the theory of elastic surface waves developed by Barnett and Lothe and their coworkers between 1970 and 1976. A detailed exposition, with complete literature references, can be found in the review article [1] and a contemporary survey by the original authors [2]. Against this background subsequent advances in the basic theory of surface and interfacial waves are discussed in Sections 3 to 7.

The eigenvalue problem (7) can be recast in the standard form

$$\mathbf{N}\boldsymbol{\xi} = p\boldsymbol{\xi}, \tag{16}$$

where N is a real 6-dimensional tensor involving $Q(.)$, $Q^{-1}(.)$ and $R(.,.)$, the unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$, and the speed v. The 6-vector $\boldsymbol{\xi}$ is formed by juxtaposing the displacement and traction amplitudes $\mathbf{a}$ and $\boldsymbol{\ell}$ introduced in (5) and (9). This is the starting point of the theory of Barnett and Lothe, but (16) is referred to the rotated vectors

$$\mathbf{m}_\varphi = \sin\varphi\mathbf{e}_1 - \cos\varphi\mathbf{e}_2, \quad \mathbf{n}_\varphi = \cos\varphi\mathbf{e}_1 + \sin\varphi\mathbf{e}_2,$$

in place of $\mathbf{e}_1$ and $\mathbf{e}_2$, and then reads

$$N(\varphi,v)\boldsymbol{\xi}(\varphi,v) = p(\varphi,v)\boldsymbol{\xi}(\varphi,v). \tag{17}$$

It is found, however, that the eigenvectors of $N(\varphi,v)$ are independent of the orientation angle φ, while the eigenvalues have the property

$$(2\pi)^{-1}\int_0^{2\pi} p(\varphi,v)d\varphi = i\,\mathrm{sgn}\,\mathrm{Im}\,p(0,v).$$

The result of averaging the two sides of (17) over the interval $0 \leqslant \varphi \leqslant 2\pi$ is therefore

$$S(v)\boldsymbol{\xi}(v) = \pm i\boldsymbol{\xi}(v). \tag{18}$$

The decomposition of equation (18) into 3-vectors and 3-tensors can be set out in matrix format as

$$\begin{bmatrix} S_1(v) & S_2(v) \\ S_3(v) & S_1^T(v) \end{bmatrix}\begin{bmatrix} \mathbf{a}(v) \\ \boldsymbol{\ell}(v) \end{bmatrix} = \pm i\begin{bmatrix} \mathbf{a}(v) \\ \boldsymbol{\ell}(v) \end{bmatrix}. \tag{19}$$

The real tensors $S_1(v)$, $S_2(v)$ and $S_3(v)$ turn out to be of fundamental significance in the analysis of steady plane motions of an anisotropic elastic body. They are defined in I by

$$S_1(v) = -\pi^{-1}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} Q^{-1}(\varphi,v)R^T(\varphi,v)d\varphi,$$

$$S_2(v) = \pi^{-1}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} Q^{-1}(\varphi,v)d\varphi, \tag{20}$$

$$S_3(v) = \pi^{-1}\int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \{R(\varphi,v)Q^{-1}(\varphi,v)R^T(\varphi,v) - Q(\varphi,v)\}d\varphi, \tag{21}$$

with

$$Q(\varphi,v) = Q(\mathbf{n}_\varphi) - \rho v^2\cos^2\varphi I, \quad R(\varphi,v) = R(\mathbf{m}_\varphi,\mathbf{n}_\varphi) - \rho v^2\sin\varphi\cos\varphi I,$$

and obey the condition

$$S_1(v)S_2(v) + S_2(v)S_1^T(v) = 0, \tag{22}$$

among others [1,Sect.IV,D].

Since the p_α's in (12) have positive imaginary part, the relations between the corresponding displacement and traction amplitudes provided by (19) are

$$S_1(v)\mathbf{a}_\alpha + S_2(v)\boldsymbol{\ell}_\alpha = i\mathbf{a}_\alpha, \quad S_3(v)\mathbf{a}_\alpha + S_1^T(v)\boldsymbol{\ell}_\alpha = i\boldsymbol{\ell}_\alpha. \tag{23}$$

On multiplying equations (23) by γ_α, summing from $\alpha = 1$ to $\alpha = 3$, and evaluating at $v = v_S$, we obtain, with the use of (13) and (14),

$$S_1(v_S)A = iA, \quad S_3(v_S)A = 0. \tag{24}$$

Equation $(24)_1$ tells us that the polarization A of the surface wave is necessarily complex and we then deduce from $(24)_2$ that, since both the real and imaginary parts of A are null vectors of $S_3(v_S)$, the surface-wave speed satisfies the real form

$$\text{rank } S_3(v) = 1 \tag{25}$$

of the secular equation (15).

Equations $(1)_2$ and $(8)_1$ show that Q(.) is a symmetric tensor and it follows from (20) and (21) that $S_2(v)$ and $S_3(v)$ are also symmetric. The derivative $S_3'(v)$ of $S_3(v)$ with respect to v is positive definite in $(0,\hat{v})$ and, subject to the positive definiteness condition (2), $S_3(0)$ is negative definite [I,Sect.VI, D]. The eigenvalues of $S_3(v)$ are therefore negative at $v = 0$ and increase monotonically in I. From (25), two eigenvalues are zero simultaneously at $v = v_S$. If there were two subsonic surface waves, at least one eigenvalue would have to vanish twice in I. But this is ruled out by the monotonicity property, so a subsonic surface wave is unique whenever it exists.

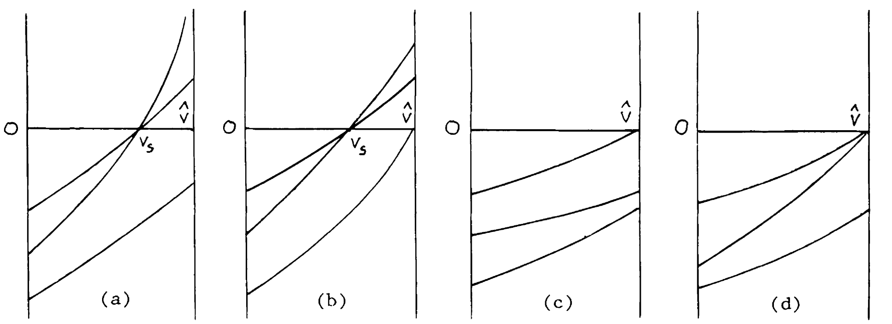

FIGURE 3
Possible patterns of behaviour of the eigenvalues of $S_3(v)$ in I.

The question of the existence of subsonic surface waves is resolved by examining the behaviour of $S_3(v)$, as given by the integral representation (21), in the limit $v \to \hat{v}$. If, for arbitrary real b, the quadratic form $\mathbf{b}.\{S_3(v)\mathbf{b}\}$ tends to infinity as $v \to \hat{v}$, at least one eigenvalue of $S_3(v)$ passes through zero and, as illustrated in Figure 3(a), a subsonic surface wave exists. This is found to be the case if the limiting wave depicted in Figure 1 produces