STUDIES IN LOGIC

AND

THE FOUNDATIONS OF MATHEMATICS

L. E. J. BROUWER / A. HEYTING / A. ROBINSON / P. SUPPES

EDITORS

The Theory of Models

PROCEEDINGS OF THE 1963 INTERNATIONAL SYMPOSIUM AT BERKELEY

Edited by

J. W. ADDISON / L. HENKIN / A. TARSKI

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Dedicated to

THORALF SKOLEM (1887–1963)

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PREFACE

This volume constitutes the proceedings of the International Symposium on the Theory of Models held at the University of California, Berkeley, from June 25 to July 11, 1963. The Symposium was sponsored by the Association for Symbolic Logic in cooperation with the Division of Logic, Methodology and Philosophy of Science of the International Union of History and Philosophy of Science.

The primary aim of the conference was to bring together from all countries, for review and exchange of ideas, scholars who have done significant work in the area of research in the foundations of mathematics generally known as "the theory of models." There were 160 registered participants from 17 countries; their names are recorded in a list at the end of this volume. Thirty-five scholars gave invited hour addresses and 13 additional 20-minute talks were contributed.

These proceedings contain 30 of the invited hour addresses as well as the papers of E. W. Beth (written jointly with J. J. F. Nieland), Roland Fraïssé, and Wolfram Schwabhäuser, who were invited to speak but were unable to attend the Symposium. Manuscripts for publication were not received from five invited speakers – Leon Henkin (Models of type theory), Roger C. Lyndon (Model theory and group theory), Andrzej Mostowski (Status of some theorems of model theory in the non-elementary case), Czesław Ryll-Nardzewski (Probabilities on models), and Alfred Tarski (Definability of recursive sets and undecidability of theories). In addition 4 short papers and 7 abstracts are included which cover all contributors except R. O. Gandy (β-models for analysis) and John McCarthy (A model for causality). The invited papers of C. C. Chang and E. P. Specker were written jointly with H. Jerome Keisler and Simon Kochen, respectively; the contributed paper of Walter Felscher was written jointly with Gudrun Jarfe.

These proceedings also contain a foreword on terminology and an 841-entry bibliography of the theory of models which grew out of a preliminary version prepared by Karel de Bouvère and distributed to participants in the Symposium. An essay preceding the bibliography attempts to explain what has been included under the heading "theory of models."

For convenience in finding papers in the volume the invited addresses have been arranged in alphabetical order by author, followed by the short papers and abstracts. But as an aid to students the papers and abstracts have also been organized by subject matter into eight categories, and a list of papers thus grouped precedes the foreword on terminology. The eight categories suggest the range and scope of the contemporary theory of models.

The Organizing Committee of the Symposium consisted of J. W. Addison, Alonzo Church, William Craig (Treasurer), Leon Henkin (Co-Chairman), S. C. Kleene, Roger C. Lyndon, Abraham Robinson, J. Barkley Rosser, Dana S. Scott (Secretary), Patrick Suppes, Alfred Tarski (Co-Chairman), and Robert L. Vaught.

The bulk of the support for the Symposium was granted by the U.S. National Science Foundation. Additional aid came from the U.S. National Academy of Sciences – National Research Council, from the University of California, Berkeley, and from the sponsors.

The editors are grateful for the help of a large number of logicians in the Berkeley area who read the manuscripts as they were submitted and suggested corrections and clarifications. Among these readers were E. W. Adams, R. F. Barnes, Jr., G. M. Benson, A. Cobham, A. Daigneault, T. E. Frayne, P. G. Hinman, M. Jean, D. L. Kreider, M. R. Krom, J. J. LeTourneau, J. I. Malitz, J. D. Monk, D. C. Peterson, W. N. Reinhardt, G. E. Reyes, R. M. Robinson, W. H. Rupley, A. L. Selman, J. H. Silver, P. Suppes, and B. F. Wells III. The editors also take this opportunity of acknowledging the valuable contribution made by the numerous participants in the Symposium who made helpful suggestions regarding the bibliography, and of thanking P. G. Hinman, who read the entire second printer's proof with care, independently of the editors. Finally, the editors join with all of the Symposium participants in saluting the outstanding dedication and service of Dana Scott, Secretary of the Organizing Committee, and June Lewin, Secretary of the Symposium.

University of California, Berkeley THE EDITORS

EVERT WILLEM BETH

1908 – 1964

Because of illness Professor Beth was unable to accept the invitation to speak at the Symposium, but he did agree to submit a manuscript of the paper he would have presented. Several months after the receipt of this manuscript the editors received the sad news of his death on April 12, 1964. He was an influential and prolific writer on logic; in particular he often wrote on the theory of models, contributing both to its development and to the wider understanding of its concepts, methods, results, and significance. Among his many works he is especially remembered for the beautiful theorem in the theory of definition which bears his name. His personal charm, wide ranging interests in science and education, and warm-hearted friendliness endeared him to a very wide circle of colleagues in the field. For all of these reasons his collaboration in the continued development of the theory of models will be keenly missed.

THE EDITORS

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LIST OF PAPERS
GROUPED BY SUBJECT MATTER

In a very rough way the papers presented at the Symposium can be classified into eight groups. Naturally many of the papers do not fall neatly into exactly one group — some have parts belonging to each of several groups and some, in a strict sense, do not belong to any of them. For example, Feferman's paper falls partly into Group V and partly into Group VI, while much of Julia Robinson's paper lies outside of the theory of models; the latter paper was invited because of its close interrelation with and importance for the theory of models. Actually the entire Group IV, which deals with the mathematical aspects of relational structures and categories, falls largely outside model theory. However, since the theory of models deals with the interrelationship of languages and *structures*, it was considered important to invite several papers dealing chiefly with structures themselves; and the papers on categories were solicited in view of the possibility that another general method of organizing mathematics might be revealed as closely connected with and important for the theory of models.

I. Model Theory of First-Order Languages

II. Model Theory of Richer Classical Languages

III. Model Theory of Nonclassical Languages

IV. Mathematical Structures

V. Model-Theoretic Methods in Set Theory

VI. Model-Theoretic Methods in Number Theory and Analysis

VII. Model-Theoretic Methods in Algebra and Geometry

VIII. Model-Theoretic Methods in the Empirical Sciences

FOREWORD ON TERMINOLOGY

In a relatively new field such as the theory of models it is not surprising that terminology has not yet reached a settled state. Especially for students and others beginning work in model theory this may prove a tiresome inconvenience. In an effort to make learning easier for them and to give this volume greater unity the editors decided to *suggest* to authors, on an experimental basis, some uniform terminology and notation.

By promising to summarize these terminological suggestions here in one place at the front of the volume, we have made it possible for the individual authors to omit repetitious explanations of much of their own terminology and notation. Authors were, of course, free to diverge from these suggestions and in a number of cases it proved convenient for them, for a variety of purposes, to do so. In such cases they were asked to include the necessary substitute terminological explanations in their individual papers as usual.

Following is a resume of the principal suggestions.

The basic mathematical objects considered in the theory of models have been variously called "relational systems", "relational structures", "realizations", "possible realizations", "models", "semi-models", "pseudo-models", "frames", etc. The term "*relational structure*" is suggested here.

Structures of a given similarity type, corresponding to some language L, may then be referred to as "structures *for* L" or "L-structures". This terminology preserves the word "system" for its other quite different uses in logic and preserves the word "model" for its basic use, as in "a model of a sentence" or "a model of a set of sentences". "*Universe*" is suggested for that set, associated with each structure, which has been alternatively called its "domain", "base", "set of individuals", "support", etc.

The word "language" has been used in a wide variety of ways in model theory and logic. Three chief uses have been: (i) in reference to a collection of primitive symbols and rules of formation; (ii) in reference to (i) together with a scheme of interpretation, i.e., a specification of meaning to the absolute constants and of allowable kinds of interpretation to variables (including parameters and arbitrary constants); and (iii) in reference to (ii) together with a scheme of deduction (e.g., axioms and rules of inference). It is suggested that "*language*" be reserved for entities fitting roughly into category (ii). Other words, free of semantical or deductive connotations, should be used for entities fitting roughly into category (i); one such possibility is "*grammar*". For the well-formed formal expressions of a language the words "*formula*" and "*term*" are recommended, according as the expression is of the propositional or of the denotative type. "*Sentence*" is suggested for formulas without free occurrences of variables.

For use as logical symbols in formal expressions of an object language, or as names for such symbols, the following are suggested:

$$\top \quad \bot \quad \neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \quad \forall \quad \exists \quad \triangleq$$

(where in particular $\top$ is the truth symbol, $\bot$ is the falsity symbol, and $\triangleq$ is the identity symbol). These symbols may appear either as above or in a lightface variation of them.

For the fundamental notion of semantical (or model-theoretic) consequence "$\models$" is suggested (over, e.g., "$\Vdash$" or "$\Vert\!-$"). The recommended usage is suggested by the following examples. Let Σ be a set of formulas, φ be a formula, $\mathfrak{A}$ be a structure, and $\mathscr{K}$ be a class of structures. Then "$\models \varphi$" means "φ is (logically) valid", "$\Sigma \models \varphi$" means "φ is a semantical consequence of Σ", "$\models_{\mathfrak{A}} \varphi$" means "$\varphi$ is valid in $\mathfrak{A}$", and "$\models_{\mathscr{K}} \varphi$" means "$\varphi$ is valid in all structures of $\mathscr{K}$". For deductive (or proof-theoretic) consequence "$\vdash$" is recommended.

As names for and as variables ranging over certain kinds of objects, letters from the following fonts of type are recommended:

(1) elements of the universe of a structure = *lower case italic letters*
(2) relations and operations of a structure = *capital italic letters*
(3) structures = *capital German letters*
(4) the universe of a structure = *the corresponding capital italic letter*
(5) classes of structures = *capital script letters*
(6) symbols in an object language for elements of the universe of a structure = *lower case Tamalpais (i.e. Gill lightface) letters*
(7) symbols in an object language for relations and operations of a structure = *capital Tamalpais (i.e. Gill lightface) letters*
(8) formulas of an object language = *lower case Greek letters*
(9) sets of formulas of an object language = *capital Greek letters*

It is not intended that these fonts be used exclusively for these purposes, of course. For example, lower case Greek letters might also be used for ordinal numbers. If a lower case italic letter denotes a certain object then the corresponding Tamalpais (Gill lightface) letter may sometimes be used without further explanation to denote a name for that object.

THE EDITORS

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THE METHOD OF ALTERNATING CHAINS *

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0. Introduction. A common construction occurring in contributions to the theory of models (see, for example, A. Robinson [56e], Chang [59], Keisler [60], Addison [62], Krom [63, Dissertation, Chapter V]) is the development of an *alternating chain* of structures, i.e. of a sequence of structures lying alternately in each of two given classes of structures. Although not always originally formulated in these terms, it has turned out that a substantial number of the applications of these chains can be formulated in terms of *separability* (or *interpolability*) *tests*. We begin by illustrating these ideas with a simple, concrete example.

Consider a pure first-order predicate language $\mathscr{L}$ with equality with exactly one parameter—the binary predicate parameter F. Suppose one wondered how "tight" the familiar implication $\exists a \forall b \mathsf{Fab} \models \forall b \exists a \mathsf{Fab}$ is, or equivalently, how "close together" the class $\mathscr{A}$ of models of $\exists a \forall b \mathsf{Fab}$ and the class $\mathscr{B}$ of models of $\neg \forall b \exists a \mathsf{Fab}$ are. One measure of this could be given in terms of how complicated a sentence φ of $\mathscr{L}$ must be in order to be interpolable between $\exists a \forall b \mathsf{Fab}$ and $\forall b \exists a \mathsf{Fab}$ (in the sense that $\exists a \forall b \mathsf{Fab} \models \varphi \models \forall b \exists a \mathsf{Fab}$), or how complicated a class $\mathscr{C}$ must be in order to separate $\mathscr{A}$ from $\mathscr{B}$ (in the sense that $\mathscr{A} \subseteq \mathscr{C}$ and $\mathscr{C} \cap \mathscr{B} = \emptyset$). For example, can such a $\mathscr{C}$ be both $\bigvee_2{}^0$ and $\bigwedge_2{}^0$ (in the terminology of Addison [62a])? By constructing (as is easily done) an alternating chain $\langle \mathfrak{S}_n : n \in \omega \rangle$ such that for any n in ω $\mathfrak{S}_{2n} \in \mathscr{A}$, $\mathfrak{S}_{2n+1} \in \mathscr{B}$, and $\mathfrak{S}_{n+1}$ is an extension of $\mathfrak{S}_n$, one shows that this is impossible. For assume the contrary, let $\mathfrak{T}$ be the common union of $\{\mathfrak{S}_{2n} : n \in \omega\}$ and of $\{\mathfrak{S}_{2n+1} : n \in \omega\}$, and recall that $\bigwedge_2{}^0$ classes are closed under unions of extension-chains. Then $\mathfrak{T} \in \mathscr{C}$ (since $\mathfrak{S}_{2n} \in \mathscr{A} \subseteq \mathscr{C} \in \bigwedge_2{}^0$) and $\mathfrak{T} \in \sim \mathscr{C}$ (since $\mathfrak{S}_{2n+1} \in \mathscr{B} \subseteq \sim \mathscr{C} \in \bigwedge_2{}^0$), a contradiction.

* This research was supported in part by the U.S. National Science Foundation under Grant GP–1842.

This example illustrates the fact that a useful test for the inseparability of two disjoint classes $\mathscr{A}$, $\mathscr{B}$ of structures by an $\bigvee_2^0 \cap \bigwedge_2^0$ class is the existence of an extension-chain of length ω alternating from $\mathscr{A}$ to $\mathscr{B}$.[1]

Does this *alternating chain test* constitute a *general* method for determining the $(\bigvee_2^0 \cap \bigwedge_2^0)$-inseparability of disjoint classes of structures? In the important case where the classes are elementary (or even $\bigvee_{1\Delta}^1$) it turns out that it does indeed.

The existence of an extension-chain of length ω alternating from $\mathscr{A}$ to $\mathscr{B}$ assures, of course, the existence of longer and longer extension-chains of finite length, or, equivalently (by reversing these chains), of longer and longer finite substructure-chains. For elementary $\mathscr{A}$, $\mathscr{B}$ a compactness argument can be used to show that the converse is also true. Therefore an alternate necessary and sufficient condition for the $(\bigvee_2^0 \cap \bigwedge_2^0)$-inseparability of elementary classes $\mathscr{A}$, $\mathscr{B}$ is the existence of longer and longer finite substructure-chains alternating from $\mathscr{A}$ to $\mathscr{B}$.

In the special case where the two disjoint classes are complementary, separability tests reduce to *classification tests* – in the above situation, for example, we see that a necessary and sufficient condition for an elementary class to be $\bigvee_2^0 \cap \bigwedge_2^0$ is the existence of a finite bound on the lengths of substructure chains alternating from the class to its complement. Thus, for example, a sequence of longer and longer finite substructure-chains alternating in and out of the class of models of $\exists a \forall b Fab$ assures us that this sentence is not logically equivalent to an $\bigwedge_2^0$ sentence of $\mathscr{L}$. And such simple facts as these, although often expected, are often not trivial to prove. Thus alternating chain tests are in part a contribution to one of the oldest and most central problems of logic – that of determining whether or not a given definition can be "simplified".

The alternate separability test mentioned above naturally suggests the question whether there is any simple grammatical significance to the existence or nonexistence of alternating substructure-chains of a given finite length. This question leads to the discovery of a new "hierarchy" which, rather unexpectedly, appears to share a number of properties with some of the well-known hierarchies studied in descriptive set theory, recursive function theory, and the theory of models. By the classification test this hierarchy fills up the class of $\bigvee_2^0 \cap \bigwedge_2^0$ sets and thereby sheds light on the structure of this class.

These findings have been accompanied by the discovery that as early

[1] For a still better, practical understanding of the method of alternating chains see Footnote 2.

as 1914 Hausdorff (see [14]) had studied a cousin of this hierarchy in descriptive set theory and had developed a method of residues that bears a close analogy to the method of alternating chains. Although his method of residues yields only classification tests, it can be generalized (cf. Addison [62]) to give separability tests as well.

In the present paper we initiate a general study of the class of *difference hierarchies*, which includes the hierarchy of $\bigvee_2^0 \cap \bigwedge_2^0$ sets mentioned above, Hausdorff's hierarchy, and a variety of other hierarchies in the theory of models, descriptive set theory, and recursive function theory. Included in the study is a *generalized method of alternating chains* which applies to a large number of the examples.

This method has been a powerful heuristic tool in the study of difference hierarchies. Not only has it provided the only known proofs of some results about particular hierarchies, but it has suggested general theorems about difference hierarchies which have subsequently been established by purely Boolean-algebraic methods.

For simplicity, in the present paper we restrict our study to hierarchies of length $\leqslant \omega$; however, many of the results do extend to the transfinite. We plan a separate publication on these extensions, which present special problems outside the algebraic spirit of the present paper.

In the present paper we focus attention on general methods useful for studying the structure of difference hierarchies, reserving for later the application of these methods to particular hierarchies. We begin by outlining in axiomatic form some of the important structural properties of hierarchies.

1. An axiomatic approach to hierarchies. The word "hierarchy" is usually used informally in mathematical discussions, so it no doubt suggests more or less structure to different people. We adopt here a reasonably broad definition and then introduce descriptive adjectives to cover additional structural properties that are often present.

By a *family of sets* we mean a function to a class of sets. "Hierarchy" is most often used in the literature in reference to a special kind of family of sets—namely a family of sets (or classes) of sets. The inclusion relation on the classes of sets induces a partial order on the domain (or index set) of the hierarchy, and in most cases this is a partial well order if not indeed a well order. Balancing brevity against generality we include a restriction to well order in our definition. Since the use of a family of classes in place of a class of classes is simply a matter of notational convenience, we can without any real loss of generality restrict the domains of hierarchies to be ordinal numbers (i.e. to be initial segments of ordinal numbers).

Definition 1.0. *For any* $\mathfrak{H}$

$\mathfrak{H}$ *is a* prehierarchy *if and only if* $\mathfrak{H}$ *is a function, the domain of* $\mathfrak{H}$ *is an ordinal, the range of* $\mathfrak{H}$ *is a class of classes of sets, and:*

H1. $\mathfrak{H}$ *is* nondecreasing, *i.e. for any* μ, ν *in the domain of* $\mathfrak{H}$ *if* $\mu<\nu$, *then* $\mathfrak{H}(\mu)\subseteq\mathfrak{H}(\nu)$.

Let $\mathfrak{H}$ be a prehierarchy with domain μ. We call the union of the union of the range of $\mathfrak{H}$ its *universe* and denote it by "$|\mathfrak{H}|$", and we refer to $\mathfrak{H}$ as a *μ-prehierarchy over* $|\mathfrak{H}|$. We will usually use the subscript functional notation, writing "$\mathfrak{H}_\nu$" in place of "$\mathfrak{H}(\nu)$". We call the union of the range of $\mathfrak{H}$ its *scope* and denote it by "$\mathfrak{H}_{(\mu)}$". More generally, for any ordinal ν in $\mu+1$ we write "$\mathfrak{H}_{(\nu)}$" as an abbreviation for "$\bigcup\{\mathfrak{H}_\varrho:\varrho\in\nu\}$". For any ν in μ we say that $\mathfrak{H}$ is *stationary at* ν if and only if $\mathfrak{H}_\nu=\mathfrak{H}_{(\nu)}$.

In general a prehierarchy cannot be recaptured from its range, but if it is *one-to-one* this can be done. Into the definition of hierarchy we put a slightly stronger property.

Definition 1.1. *For any* $\mathfrak{H}$

$\mathfrak{H}$ *is a* hierarchy *if and only if* $\mathfrak{H}$ *is a prehierarchy and:*

H2. $\mathfrak{H}$ *is* never stationary (*i.e. for any* ν *in the domain of* $\mathfrak{H}$: $\mathfrak{H}_\nu\neq\mathfrak{H}_{(\nu)}$).

Sometimes in the literature prehierarchies which are stationary only at limit ordinals (i.e. one-to-one prehierarchies) or only beyond a certain ordinal have been called "hierarchies", but it seems slightly advantageous to adopt the terminology chosen here.

A fundamental operation on hierarchies is *dualization* (which we denote by "$-$"). For any μ-prehierarchy $\mathfrak{H}$ by the *dual* $\mathfrak{H}^-$ of $\mathfrak{H}$ we mean $\langle\{|\mathfrak{H}|\sim A:A\in\mathfrak{H}_\nu\}:\nu\in\mu\rangle$.

Beyond the fundamental axioms H1, H2 there are a variety of special principles directly or indirectly involving dualization to which we wish to draw attention.

Definition 1.2. *For any ordinal* μ *and any* μ*-prehierarchy* $\mathfrak{H}$:

H3. $\mathfrak{H}$ *is* balanced *if and only if for any* ν *in* μ $\mathfrak{H}_{(\nu)}\cup\mathfrak{H}^-_{(\nu)}\subseteq\mathfrak{H}_\nu\cap\mathfrak{H}^-_\nu$;

H4. $\mathfrak{H}$ *is* selfdual *if and only if for any* ν *in* μ $\mathfrak{H}_\nu=\mathfrak{H}^-_\nu$;

H4′. $\mathfrak{H}$ *is* lateral *if and only if for any* ν *in* μ $\mathfrak{H}_\nu\neq\mathfrak{H}^-_\nu$;

H5. $\mathfrak{H}$ *is* separable *if and only if for any ν in μ $\mathfrak{H}_\nu$ has the first separation property with respect to $|\mathfrak{H}|$ (i.e. for any disjoint sets A, B in $\mathfrak{H}_\nu$ there exists a set C in $\mathfrak{H}_\nu \cap \mathfrak{H}_\nu^-$ such that $A \subseteq C$ and $C \cap B = \emptyset$);*

H5.′ $\mathfrak{H}$ *is* inseparable *if and only if for any ν in μ $\mathfrak{H}_\nu$ does not have the first separation property;*

H6. $\mathfrak{H}$ *is* reducible *if and only if for any ν in μ $\mathfrak{H}_\nu$ has the reduction property (i.e. for any sets A, B in $\mathfrak{H}_\nu$ there exist sets A', B' in $\mathfrak{H}_\nu$ such that $A' \cap B' = \emptyset$, $A' \subseteq A$, $B' \subseteq B$, and $A' \cup B' = A \cup B$);*

H6′. $\mathfrak{H}$ *is* irreducible *if and only if for any ν in μ $\mathfrak{H}_\nu$ does not have the reduction property;*

H7. $\mathfrak{H}$ *is* perfect *if and only if for any ν in μ*
$\mathfrak{H}_{(\nu)} \cup \mathfrak{H}_{(\nu)}^- = \mathfrak{H}_\nu \cap \mathfrak{H}_\nu^-$;

H7′. $\mathfrak{H}$ *is* coarse *if and only if for any ν in μ*
$\mathfrak{H}_{(\nu)} \cup \mathfrak{H}_{(\nu)}^- \neq \mathfrak{H}_\nu \cap \mathfrak{H}_\nu^-$.

Note that 'lateral', 'inseparable', and 'irreducible' are distinct from 'nonselfdual', 'nonseparable', and 'nonreducible', respectively.

We dualize each of these properties by use of the prefix "co". For example, we say $\mathfrak{H}$ is *coseparable* if and only if $\mathfrak{H}^-$ is separable.

We localize each of these properties by use of the preposition "at". For example, for any ν in μ we say $\mathfrak{H}$ is *separable at ν* if and only if $\mathfrak{H}_\nu$ has the first separation property with respect to $|\mathfrak{H}|$.

Certain obvious relations hold between these properties. For example, balanced = cobalanced, selfdual = coselfdual, lateral = colateral, and perfect = coperfect. Every selfdual or perfect prehierarchy is balanced and every reducible prehierarchy is coseparable.

2. Difference hierarchies. The hierarchy of $\bigvee_2^0 \cap \bigwedge_2^0$ sets, suggested by finite alternating chains and mentioned at the end of Section 0, turns out to belong to a wide class of hierarchies based on a common principle of generation. Since one of the fundamental operations involved in the generation of these hierarchies is set-theoretic difference, denoted by "$\sim$", we call such hierarchies "difference hierarchies".

Difference hierarchies are naturally defined in terms of the algebraic operation of "iterated difference", which we denote by "∇", regard as a function on the class of finite sequences of sets, and define recursively (according to the length of the sequence) as follows:

(0) $\nabla\langle A_i : i \in 0\rangle = \emptyset$;

(1) $\nabla\langle A_i : i \in n+1\rangle = A_0 \sim \nabla\langle A_{i+1} : i \in n\rangle$.

Definition 2.0. *For any class $\mathscr{R}$ of sets:*
$\mathfrak{D}(\mathscr{R}) = \langle\{VA : A : n \to \mathscr{R}\} : n \in \omega\rangle.$

Equivalently, $\mathfrak{D}(\mathscr{R})$ is the set family on ω defined recursively by: (i) $\mathfrak{D}(\mathscr{R})_0 = \{\emptyset\}$; (ii) $\mathfrak{D}(\mathscr{R})_{n+1} = \{A \sim B : A \in \mathscr{R},\ B \in \mathfrak{D}(\mathscr{R})_n\}$.

We usually write "$\mathfrak{D}_n(\mathscr{R})$" in place of "$\mathfrak{D}(\mathscr{R})_n$", and follow an analogous convention in similar situations later. For any set X and any function F we denote by "$\mathscr{P}X$" the class of all subsets of X and by "$\mathscr{D}F$" the domain of F. If $X \subseteq \mathscr{D}F$, we denote $\{F(x) : x \in X\}$ by "$F^*(X)$". For any n in ω a set sequence A on n is called *decreasing* (*nonincreasing*) if and only if for any $i+1$ in n $A_i \supset A_{i+1}$ ($A_i \supseteq A_{i+1}$).

Theorem 2.1. *For any set $\mathfrak{l}$ and any subclass $\mathscr{R}$ of $\mathscr{P}\mathfrak{l}$:*
(i) *if $\emptyset \in \mathscr{R}$, then $\mathfrak{D}(\mathscr{R})$ is an ω-prehierarchy;*
(ii) *if $\emptyset, \mathfrak{l} \in \mathscr{R}$, then $\mathfrak{D}(\mathscr{R})$ is a balanced ω-prehierarchy.*

Proof. (i) We prove $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}_n(\mathscr{R})$ by induction. Basis: $n=0$ or $n=1$. $\mathfrak{D}_{(0)}(\mathscr{R}) = \emptyset \subseteq \mathfrak{D}_0(\mathscr{R}) = \mathfrak{D}_{(1)}(\mathscr{R}) = \{\emptyset\} \subseteq \mathscr{R} = \mathfrak{D}_1(\mathscr{R})$. Induction step: $n = k+2$. Let $A \in \mathfrak{D}_{(k+2)}(\mathscr{R})$. By the hypothesis of the induction $\mathfrak{D}_{(k+2)}(\mathscr{R}) = \mathfrak{D}_{k+1}(\mathscr{R})$, so $A \in \mathfrak{D}_{k+1}(\mathscr{R})$. Hence $A = B \sim C$ for some B in $\mathscr{R}$ and C in $\mathfrak{D}_k(\mathscr{R})$. So by the hypothesis of the induction $C \in \mathfrak{D}_{k+1}(\mathscr{R})$, so $A \in \mathfrak{D}_{k+2}(\mathscr{R})$.

(ii) By (i) $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}_n(\mathscr{R})$. From this it also follows that $\mathfrak{D}^-_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}^-_n(\mathscr{R})$. From $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}^-_n(\mathscr{R})$ will follow $\mathfrak{D}^-_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}_n(\mathscr{R})$, so it suffices to prove $\mathfrak{D}_{(n)}(\mathscr{R}) \subseteq \mathfrak{D}^-_n(\mathscr{R})$. We do this by induction on n. Basis: $n=0$. $\mathfrak{D}_{(0)}(\mathscr{R}) = \emptyset \subseteq \mathfrak{D}^-_0(\mathscr{R})$. Induction step: $n = k+1$. Let $A \in \mathfrak{D}_{(k+1)}(\mathscr{R})$. By (i) $\mathfrak{D}_{(k+1)}(\mathscr{R}) = \mathfrak{D}_k(\mathscr{R})$, so $A \in \mathfrak{D}_k(\mathscr{R})$. Hence, since $\mathfrak{l} \in \mathscr{R}$, $\mathfrak{l} \sim A \in \mathfrak{D}_{k+1}(\mathscr{R})$, so $A \in \mathfrak{D}^-_{k+1}(\mathscr{R})$.

Let $\mathscr{R}$ be a class of sets containing $\emptyset$. Let ν be the greatest ordinal in $\omega+1$ such that $\mathfrak{D}(\mathscr{R})|\nu$ is a hierarchy. We call ν the *dimension* of $\mathfrak{D}_{(\omega)}(\mathscr{R})$ over $\mathscr{R}$ and we call $\mathfrak{D}(\mathscr{R})|\nu$ the *difference hierarchy generated by* $\mathscr{R}$.

To develop an understanding of difference hierarchies it is useful to study some of the algebraic properties of V. As Hausdorff noted it is convenient for this purpose to introduce a closely related algebraic operation – "alternating series" – which we denote by "$\oplus$", also regard as a function on the class of finite sequences of sets, and define recursively as follows:

(2) $\oplus\langle A_i : i \in 0\rangle = \emptyset$;

(3) $\oplus\langle A_i : i \in 2n+1\rangle = \oplus\langle A_i : i \in 2n\rangle \cup A_{2n}$;

(4) $\oplus\langle A_i : i \in 2n+2\rangle = \oplus\langle A_i : i \in 2n+1\rangle \sim A_{2n+1}$.

In the following sequence of lemmas we develop (following in part the work of Hausdorff [14]) some of the basic algebraic facts about ∇ and $\oplus$.

Lemma 2.2. *For any set X, any n in ω, and any set sequence A on n:*

(i) $\nabla\langle X \cap A_i : i \in n\rangle = X \cap \nabla\langle A_i : i \in n\rangle$;

(ii) $\oplus\langle A_i \cup X : i \in n\rangle = \begin{cases} \oplus\langle A_i : i \in n\rangle \sim X & \textit{if } n \textit{ is even,} \\ \oplus\langle A_i : i \in n\rangle \cup X & \textit{if } n \textit{ is odd.} \end{cases}$

Proof. (i) By induction on n. Basis: $n=0$. $\emptyset = X \cap \emptyset$. Induction step: $n=k+1$. $X \cap \nabla\langle A_i : i \in k+1\rangle = X \cap (A_0 \sim \nabla\langle A_{i+1} : i \in k\rangle)$ $= (X \cap A_0) \sim (X \cap \nabla\langle A_{i+1} : i \in k\rangle) =$ [by the hypothesis of the induction] $(X \cap A_0) \sim \nabla\langle X \cap A_{i+1} : i \in k\rangle = \nabla\langle X \cap A_i : i \in k+1\rangle$.

(ii) By induction on n. Basis: $n=0$. $\emptyset = \emptyset \sim X$. Induction step: $n=k+1$. Case 1: k is even. $\oplus\langle A_i \cup X : i \in k+1\rangle = \oplus\langle A_i \cup X : i \in k\rangle \cup (A_k \cup X)$ $=$ [by the hypothesis of the induction] $(\oplus\langle A_i : i \in k\rangle \sim X) \cup (A_k \cup X)$ $= \oplus\langle A_i : i \in k\rangle \cup A_k \cup X = \oplus\langle A_i : i \in k+1\rangle \cup X$. Case 2: k is odd. $\oplus\langle A_i \cup X : i \in k+1\rangle = \oplus\langle A_i \cup X : i \in k\rangle \sim (A_k \cup X) =$ [by the hypothesis of the induction] $(\oplus\langle A_i : i \in k\rangle \cup X) \sim (A_k \cup X)$ $= (\oplus\langle A_i : i \in k\rangle \sim A_k) \sim X = \oplus\langle A_i : i \in k+1\rangle \sim X$.

Lemma 2.3. *For any positive n in ω and any set sequence A on n:*

(i) $\nabla\langle A_i : i \in n\rangle = \nabla\langle A_0 \cap A_i : i \in n\rangle$;

(ii) $\oplus\langle A_i : i \in n\rangle = \oplus\langle A_i \cup A_{n-1} : i \in n\rangle$.

Proof. (i) $\nabla\langle A_0 \cap A_i : i \in n\rangle =$ [by Lemma 2.2(i)] $A_0 \cap \nabla\langle A_i : i \in n\rangle$ $= A_0 \cap (A_0 \sim \nabla\langle A_{i+1} : i \in n-1\rangle) = A_0 \sim \nabla\langle A_{i+1} : i \in n-1\rangle = \nabla\langle A_i : i \in n\rangle$.

(ii) Case 1: n is even. $\oplus\langle A_i \cup A_{n-1} : i \in n\rangle =$ [by Lemma 2.2(ii)] $\oplus\langle A_i : i \in n\rangle \sim A_{n-1} = (\oplus\langle A_i : i \in n-1\rangle \sim A_{n-1}) \sim A_{n-1} = \oplus\langle A_i : i \in n\rangle$. Case 2: n is odd. $\oplus\langle A_i \cup A_{n-1} : i \in n\rangle =$ [by Lemma 2.2(ii)] $\oplus\langle A_i : i \in n\rangle \cup A_{n-1} = (\oplus\langle A_i : i \in n-1\rangle \cup A_{n-1}) \cup A_{n-1} = \oplus\langle A_i : i \in n\rangle$.

Lemma 2.4. *For any n in ω and any nonincreasing set sequence A on $n+1$:*

$$\oplus\langle A_i : i \in n+1\rangle = A_0 \sim \oplus\langle A_{i+1} : i \in n\rangle.$$

Proof. By induction on n. Basis: $n=0$. $\emptyset \cup A_0 = A_0 = A_0 \sim \emptyset$. Induction step: $n=k+1$. Case 1: k is even. $\oplus\langle A_i : i \in k+2\rangle$ $= \oplus\langle A_i : i \in k+1\rangle \sim A_{k+1} =$ [by the hypothesis of the induction]