Ordinary Differential Equations

1971 NRL–MRC Conference

Edited by

Leonard Weiss

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Leonard Weiss

University of Maryland
College Park, Maryland
and
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Washington, D. C.

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CONTENTS

I. INVITED PAPERS

II. SEMINAR PAPERS

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PREFACE

This volume represents the Proceedings of a Conference on Ordinary Differential Equations held in Washington, D. C., June 14-23, 1971, and sponsored by the Mathematics Research Center of the Naval Research Laboratory.

The aim of this meeting was to stimulate research in ordinary differential equations by bringing together persons who were actively pursuing research in this field so they could exchange information and ideas.

Approximately 90 mathematicians representing 8 nations attended the conference, whose program consisted of 30 formal lectures and 27 seminar presentations. In addition to the regular program, a number of informal talks were given. The invited formal lectures covered geometric and qualitative theory, analytic theory, functional differential equations, dynamical systems, and algebraic theory, with applications to control theory, celestial mechanics, and biomedicine. The seminar presentations were scheduled under 6 headings: functional differential equations, oscillations and dynamical systems, analytic theory, boundary-value problems, stability and control, and differential equations on Banach spaces.

It is always difficult to capture the spirit of a meeting merely by offering a collection of technical papers. In the present case, it is impossible. The combination of concentrated mathematical talent and more time than usual to think about and discuss mathematics at a meeting with many different colleagues proved to be a potent formula for success.

My thanks go to all the authors whose papers appear here for the needed cooperation to produce this volume so relatively soon after the conference. A special word of gratitude goes to Dr. Paul B. Richards, Superintendent of the Mathematics and Information Sciences Division of NRL, who conceived of this conference and who gave unwavering support at every stage of the planning. Finally, no list of credits would be complete without noting, with appreciation, the contributions of Dr. William Gordon and Professor Philip Hsieh, who served with me on the Organizing Committee, the Office of Naval Research, for providing some financial assistance for the conference, and the staff at the Mathematics Research Center of NRL, who exhibited unfailing competence in a variety of tasks.

Leonard Weiss

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I

INVITED PAPERS

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A GENERAL APPROACH TO LINEAR PROBLEMS FOR NONLINEAR ORDINARY DIFFERENTIAL EQUATIONS*

H. A. Antosiewicz

This is a brief summary of a few recent results on the existence of periodic solutions of a differential equation of the form

$$\dot{x} = f(t,x) .$$

My aim is to present these results as illustrations of a broad approach to various linear problems in which the desired solutions are required to satisfy a given set of much more general linear constraints.

Throughout, I will stress basic ideas rather than utmost generality and omit all details, which may be found in the references listed at the end.

1. Let me begin with the following simple result for a differential equation

$$\dot{x} = Ax + g(t,x) , \tag{1.1}$$

where A is a linear mapping in $\mathbb{R}^n$ and g is continuous and sufficiently smooth in $\mathbb{R} \times \mathbb{R}^n$ so that the solutions of (1.1) are uniquely determined by (and hence depend continuously upon) the initial conditions.

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*This work was done with partial support from the U.S. Army Research Office (Durham).

(1.2) Suppose A is stable. There exist positive constants μ, ρ such that, if $t \mapsto g(t,x)$ has period 1 for each $x \in \mathbb{R}^n$ and $\|g(t,x)\| < \mu$ for each $t \in [0,1]$ and $\|x\| \leq \rho$, then (1.1) has at least one solution in $\mathbb{R}$ with period 1.

Actually, (1.2) is a special case of much more general results which, in their original form, are due to Poincaré (cf. e.g., [1, 8, 10]).

(1.2) has been proved in many different ways. One method of proof applies Brouwer's fixed point theorem to the classical Poincaré mapping associated with (1.1) [1]. It hinges on the fact that there exists a real valued function V of class C^1 in $\mathbb{R}^n$ such that

$$\lim_{\|x\| \to +\infty} V(x) = +\infty \tag{1.3}$$

and, for every $t \in [0,1]$ and every $x \in \mathbb{R}^n$ with $\|x\| \geq \rho$

$$(DV(x),\, Ax + g(t,x)) < 0 \,. \tag{1.4}$$

Indeed, since A is stable by assumption, there is a positive definite quadratic form $V(x) = (x, Px)$ in $\mathbb{R}^n$ for which $(DV(x), Ax) = -(x,x)$ for every $x \in \mathbb{R}^n$.

Another method of proof depends upon Schauder's fixed point theorem and the admissibility, with respect to A, of the function space pair (P,P), where P is the Banach space of continuous mappings of $\mathbb{R}$ into $\mathbb{R}^n$ which are periodic with period 1 [10]. For A being stable implies that the linear differential equation $\dot{x} = Ax + b(t)$, for each $b \in P$, has precisely one solution $v(b)$ that belongs to P; in fact, v is a continuous linear mapping of P into itself. Thus, (1.1) has a solution $\phi \in P$ if and only if

$$\phi = v \circ \omega(\phi) \tag{1.5}$$

where $\omega\colon P \to P$ is the substitution mapping induced by g.

Both of these methods admit various extensions to far more general settings.

2. One such extension is the basis of Krasnoselskii's method of guiding functions [12, 13] for a differential equation

$$\dot{x} = f(t,x) \ , \tag{2.1}$$

where f is continuous and sufficiently smooth in $\mathbb{R} \times \mathbb{R}^n$ so that the solutions are uniquely determined by the initial conditions, and $t \mapsto f(t,x)$ has period 1 for each $x \in \mathbb{R}^n$.

(2.2) If there exist a real valued function V of class C^1 in $\mathbb{R}^n$ and a constant $\rho > 0$ such that (1.3) holds and

$$(DV(x),\ f(t,x)) < 0 \tag{2.3}$$

for every $t \in [0,1]$ and every $x \in \mathbb{R}^n$ with $\|x\| \geqslant \rho$, then (2.1) has at least one solution in $\mathbb{R}$ with period 1.

Observe that the set $\{x \in \mathbf{R}^n\colon V(x) \leqslant \text{const.}\}$ need not be convex so that Brouwer's fixed point theorem cannot be applied directly, as in the case (1.2). Instead, the proof of (2.2) depends upon the notion of the degree of a mapping and the equivalent assertion to Brouwer's theorem that the identity mapping of the sphere in $\mathbb{R}^n$ is not null-homotopic.

A similar argument has been used by Hartman [11] to prove the existence of a solution of a general functional equation

$$F(x) = 0 \tag{2.4}$$

where F is a continuous mapping, into $\mathbb{R}^n$, of a compact convex subset K of $\mathbb{R}^n$ which contains the origin in its interior.

(2.5) Let V be a real valued positive definite function of class C^1 in K such that $DV(x) = 0$ if and only if $x = 0$. If, for every $x \in \operatorname{bd} K$,

$$(DV(x),\ F(x)) \leqslant 0 \ , \tag{2.6}$$

then there exists at least one point $\hat{x} \in K$ for which $F(\hat{x}) = 0$.

Hartman himself extended this result to equations in locally convex Hausdorff topological linear spaces and, in turn, deduced from this extension a general existence theorem on the solution of initial value problems for nonlinear ordinary differential equations in Hilbert space [11].

3. The solution of an equation, such as (2.4), in Hilbert space can often be accomplished by the use of projection methods which yield solutions to corresponding finite-dimensional equations (cf. [15]). Analogous techniques can be employed for the construction of fixed points.

Let H be a (real) Hilbert space with orthonormal basis (e_K), $K \geq 1$, let H_n be the (closed) linear subspace of H spanned by $e_1, e_2, \dots, e_n$, and denote by P_n the usual projection of H onto H_n.

(3.1) Suppose f is a mapping of a closed bounded convex set $K \subset H$ into H with these properties:

(i) if (x_n) is a sequence of points of K converging weakly to a point $x_0 \in K$, then $(f(x_n))$ converges weakly to $f(x_0)$;

(ii) for each integer $n \geq 1$

$$P_n \circ f \circ P_n(K) \subset P_n(K) .$$

Then there exists at least one point $\hat{x} \in K$ such that $\hat{x} = f(\hat{x})$.

Indeed, the fixed point $\hat{x} \in K$ is the (strong) limit of a sequence (x_n) of projectional fixed points for which $(x_n, e_K) = (f(x_n), e_K)$ for $K = 1, 2, \dots, n$.

If $K \subset H$ is simply a closed ball centered at the origin and f is defined everywhere in H, (3.2) may be satisfied by requiring that, for each $x \in \cup H_n$,

$$|(f(x), e_K)| \leq \alpha_K \|x\| + \beta_K \qquad K = 1, 2, \dots, n ,$$

where (α_K), (β_K) are sequences with

$$\Sigma \alpha_K^2 < 1 , \qquad \Sigma \beta_K^2 < \infty .$$

This last remark yields the following generalization of a classical result of Hammerstein [9] for the scalar differential equation

$$(3.5) \qquad x'' = f(t, x, x')$$

where f is defined and continuous in $[0,\pi] \times \mathbb{R} \times \mathbb{R}$ [6].

(3.6) Suppose there are positive constants $a < 1$, b such that

$$|f(t,x,y)| \leq a\,|x| + b \tag{3.7}$$

holds for every $(t,x,y) \in [0,\pi] \times \mathbb{R} \times \mathbb{R}$. Then (3.5) has at least one solution ϕ in $[0,\pi]$ for which $\phi(0) = \phi(\pi) = 0$.

Originally, the condition (3.7) was required to hold with $a < \sqrt{3/\pi}$.

4. A general framework for the concept of admissibility is simple to formulate [4].

Let E, G be Banach spaces and let F be a Fréchet space which contains G algebraically and topologically (in the sense that the topology of G is stronger than the topology induced by F on G). Suppose $u: \mathbb{R}^n \to F$ is an injective homomorphism, $v: E \to F$ a continuous linear mapping, and $\omega: G \to E$ an arbitrary continuous mapping.

The problem of determining points $x \in \mathbb{R}^n$ and $z \in G$ such that

$$z = u(x) + v \circ \omega(z) , \tag{4.1}$$

is central to nearly all questions in the qualitative thoory of differential equations. Evidently, it has a solution only if there is a point $x \in \mathbb{R}^n$ and a point $y \in E$ such that $u(x) + v(y) \in G$. Thus, it is natural to include among sufficient conditions for the solution of (4.1) the requirement that, for each $y \in E$, there exist at least one point $x \in \mathbb{R}^n$ for which $u(x) + v(y) \in G$. This is the general notion of admissibility, of the pair of Banach spaces (E,G), relative to the pair of mappings (u,v) [4].

If (E,G) is admissible, there exists a constant $\mu > 0$ such that for each $y \in E$ there is a point $x \in \mathbb{R}^n$ for which $u(x) + v(y) \in G$ and

$$\mu \|u(x) + v(y)\| \leq \|y\| . \tag{4.2}$$

Moreover, if $X_0 = \{x \in \mathbb{R}^n : u(x) \in G\}$ and $\mathbb{R}^n = X_0 \oplus X_1$, there is a linear mapping $s: E \to X_1$ such that $v_0 = u \circ s + v$ is a linear mapping of E into G which is continuous.

Thus, if (E,G) is admissible, it is sufficient to find, for a given point $x_0 \in X_0$, a point $z \in G$ such that

$$z = u_0(x_0) + v_0 \circ \omega(z) \tag{4.3}$$

where u_0 is the restriction of u to X_0. This can be done, by use of Banach's fixed point theorem when ω is lipschitzian, and by use of Schauder's principle when v has additional properties (cf. e.g., [4, 5, 10, 14]).

In the latter case, v_0 can often be represented as the product of two suitable (continuous linear) mappings $v_0 = v_1 \circ v_2$, where v_1 maps an auxiliary space H into G and v_2 maps E into H. This device is particularly effective when x_0 is taken to be $0 \in X_0$ in (4.3) or when v_0, in fact, is identical with v (and hence (4.1) reduces to (1.5)). For, in that case, instead of determining a solution ϕ belonging to G for which

$$\phi = v_1 \circ v_2 \circ \omega(\phi) \tag{4.4}$$

one first finds a point Ψ belonging to H such that

$$\Psi = v_2 \circ \omega \circ v_1(\Psi) \tag{4.5}$$

and then obtains the solution of (4.4) as $\phi = v_1(\Psi) \in G$.

This method may be used to obtain the following improvement on (3.6) (cf. e.g., [12]).

(4.6) Suppose there are positive constants $a < 1$, b such that f in (3.5) satisfies

$$x \cdot f(t,x,y) < ax^2 + b \tag{4.7}$$

at every $(t,x,y) \in [0,\pi] \times \mathbb{R} \times \mathbb{R}$. Then (3.5) has at least one solution ϕ in $[0,\pi]$ for which $\phi(0) = \phi(\pi) = 0$.

Other illustrations of the use of (4.3) are given in [5] (cf. e.g., [2, 3, 4, 7]).

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DIFFERENTIAL RELATIONS

D. Bushaw

1. Introduction

In the early 1930's geometry, and more recently control theory, have drawn attention to conditions of the form

$$y' \in f(x,y) \,, \tag{1}$$

where f is defined on $\mathbb{R} \times \mathbb{R}^n$ and has as values subsets of $\mathbf{R}^n$. A <u>solution</u> of (1) is usually defined as an absolutely continuous function $\phi\colon I \to \mathbb{R}^n$ (where I is some real interval) such that

$$\phi'(x) \in f(x,\phi(x)) \tag{2}$$

almost everywhere on I. A <u>classical solution</u> of (1) is required to be continuously differentiable (the derivative being interpreted as the appropriate one-sided derivative at endpoints, if any, of I) and to satisfy (2) for all $x \in I$. Much is known about the existence and other qualitative properties of solutions and classical solutions of (1) under certain assumptions - typically involving continuity, compactness, and convexity - on the function f (see references).

The point of departure for this paper is a simple reinterpretation of (1)-(2). The condition (2) is equivalent to

$$(x,\phi(x),\phi'(x)) \in F \,, \tag{3}$$

where

$$F = \{(x,y,z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n : z \in f(x,y)\} ,$$

and we may thus define solutions, or classical solutions, of the "differential relation"

$$(4) \qquad (x,y,y') \in F$$

just as before, with (3) in place of (2). In the absence of any special assumptions on f, the set F in (3) or (4) may be a perfectly arbitrary subset of $\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ (or, as we shall say henceforth, of $\mathbb{R}^{2n+1}$).

For simplicity, we shall consider only classical solutions of (4).

Let $\mathcal{D}$ denote the set of all continuously differentiable functions $\phi: I \to \mathbb{R}^n$, where I is some interval (connected subset of $\mathbb{R}$ having at least two points) which may vary with ϕ and which, accordingly, will usually be denoted by I_ϕ. We define a map W from $\mathcal{D}$ into the collection of all subsets of $\mathbb{R}^{2n+1}$ by

$$(5) \qquad W(\phi) = \{(x,\phi(x),\phi'(x)) : x \in I_\phi\} .$$

The range of this map, $W(\mathcal{D})$, will be denoted by $\mathcal{W}$.

With these notations, a classical solution of (4) may be defined simply as a $\phi \in \mathcal{D}$ such that $W(\phi) \subset F$.

In fact, the map from $\mathcal{W}$ into the collection of subsets of $\mathbb{R} \times \mathbb{R}^n$ induced by the projection $(x,y,z) \to (x,y)$ is an inverse for W, which accordingly is one-one. Thus without real loss we may think wholly in terms of subsets of $\mathbb{R}^{2n+1}$ and regard as solutions of (4) - the word "classical" will be omitted henceforth - those $w \in \mathcal{W}$ which are contained in F.

The situation may now be described as follows. We have a collection $\mathcal{W}$ of subsets of $\mathbb{R}^{2n+1}$ which is universal in the sense that it depends on no differential equation or differential relation, but in fact only on n; we have a subset F of $\mathbb{R}^{2n+1}$ whose definition involves no differentiation; and the problem is to learn something about those members of $\mathcal{W}$ that lie in F. This problem subsumes the study of all "generalized differential equations" (1), and in particular of all real ordinary differential equations. Moreover, the pattern could be extended to problems where $\mathbb{R}^n$ is replaced by some more general space.

The rest of this paper will consist of some rudimentary and fragmentary observations based on this viewpoint.

2. The Collection $\mathcal{W}$

It is useful to have a geometrical-relational characterization of the collection $\mathcal{W}$. Such a characterization may be based on the concept of a wedge. Let $P_0 = (x_0,y_0,z_0) \in \mathbb{R}^{2n+1}$ (specifically, $x_0 \in \mathbb{R}$ and $y_0,z_0 \in \mathbb{R}^n$), and let $\varepsilon > 0$. The corresponding forward and backward wedges are:

$$V_\varepsilon^+(P_0) = \{(x,y,z)\colon x_0 < x < x_0+\varepsilon,\ |y-y_0-(x-x_0)z_0| < \varepsilon|x-x_0|, \text{ and } |z-z_0| < \varepsilon\} \cup \{P_0\},$$

$$V_\varepsilon^-(P_0) = \{(x,y,z)\colon x_0-\varepsilon < x < x_0,\ |y-y_0-(x-x_0)z_0| < \varepsilon|x-x_0| \text{ and } |z-z_0| < \varepsilon\} \cup \{P_0\}.$$

PROPOSITION 1. A subset w of $\mathbb{R}^{2n+1}$ belongs to $\mathcal{W}$ if and only if it is the graph of a function $\Phi\colon I_w \to \mathbb{R}^n \times \mathbb{R}^n$, where I_w is a real interval, which satisfies: for every $x_0 \in I_w$ and $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$(x,\Phi(x)) \in V_\varepsilon^-(x_0,\Phi(x_0)) \cup V_\varepsilon^+(x_0,\Phi(x_0))$$

for all $x \in I_w \cap [x_0-\delta,\ x_0+\delta]$.

The proof is elementary, and is based directly on the definition of W. The concluding condition in Proposition 1 is a continuity condition; the sets $V_\varepsilon(P_0) = V_\varepsilon^-(P_0) \cup V_\varepsilon^+(P_0)$ form a basic system of neighborhoods at each $P_0 \in \mathbb{R}^{2n+1}$, and thus define a topology $\mathcal{T}_V$ on $\mathbb{R}^{2n+1}$. The above condition is that the map $x \to (x,\Phi(x))$ be continuous relative to the usual topology on $\mathbb{R}$ and the topology $\mathcal{T}_V$ on $\mathbb{R}^{2n+1}$. Note that this is not equivalent to the continuity, in some sense, of Φ; the topology $\mathcal{T}_V$ is not a product topology. In fact, this topology is rather unpleasant: it is not locally compact, for example, and a translation of $\mathbb{R}^{2n+1}$ is continuous relative to this topology if and only if its z-component vanishes.

Topologies $\mathcal{T}_{V^+}$ and $\mathcal{T}_{V^-}$ may be defined analogously in terms of the sets $V_\varepsilon^+(P_0)$ and $V_\varepsilon^-(P_0)$.

For a given n, $\mathcal{W}$ is an extremely rich cover of $\mathbb{R}^{2n+1}$, as one would expect. For example, if $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are any two points of $\mathbb{R}^{2n+1}$ with $x_1 \neq x_2$, there exists a $w \in \mathcal{W}$ such that $\{P_1, P_2\} \subset w$; it may be found by choosing carefully the coefficients (in $\mathbb{R}^n$) in $\phi(x) = x^3a + x^2b + xc + d$ and taking $w = W(\phi)$.

Furthermore, it follows directly from the definitions that the restriction of every $w \in W$ to any subinterval of I_w again belongs to $\mathcal{W}$; and that if $w_1, w_2 \in \mathcal{W}$ and agree on $I_{w_1} \cap I_{w_2} \neq \emptyset$, then $w_1 \cup w_2 \in \mathcal{W}$.

3. Initial Value Problems

For given $F \subset \mathbb{R}^{2n+1}$ and $P_0 = (x_0, y_0, z_0) \in F$, two questions naturally arise:

The strong initial value problem. Does there exist a $w \in \mathcal{W}$ such that $P_0 \in w \subset F$?

The weak initial value problem. Do there exist $z \in \mathbb{R}^n$ and $w \in \mathcal{W}$ such that $(x_0, y_0, z_0) \in w \subset F$?

Plainly, when F is functional (i.e., the sets $f(x,y)$ in (1) are singletons) there is no difference between the problems; so the distinction is rarely made. The following discussion will be limited mainly to the strong problem.

From Proposition 1, it is clear that the strong problem has a solution at P_0 only if P_0 is a cluster point of F relative to the topology $\mathcal{T}_V$. (Thus an F at every point of which the strong problem has a solution is perfect relative to this topology.) Similarly, if a $w \in \mathcal{W}$ starts at $P_0 \in F$, then P_0 is a cluster point relative to $\mathcal{T}_{V^+}$. The converse, as crude examples show, is false.

Clearly, the strong problem will have a solution at a point $P_0 = (x_0, y_0, z_0)$ if there exists a continuous function ψ from some neighborhood of (x_0,y_0) into $\mathbb{R}^n$ whose graph lies in F and which satisfies $z_0 = \psi(x_0,y_0)$: for such a ψ, Peano's classical existence theorem establishes the existence of a $\phi \in \mathcal{D}$ such that $\phi'(x) = \psi(x,\phi(x))$ and $\phi(x_0) = y_0$; and then $W(\phi)$ is a solution of the problem. One of Filippov's existence theorems for classical solutions works by giving sufficient conditions for the existence of such a ψ, and another adapts Peano's proof to a different set of conditions. Both are very general, but somewhat unsatisfactory because they derive conclusions that are essentially local in character (in $\mathbb{R}^{2n+1}$) from assumptions that are not.

A sufficient condition for existence which is vastly more primitive, but more natural as far as it goes, is the following.

<u>PROPOSITION 2</u>. If $P_0 \in F$, while $r \in \mathbb{R}^n$ and $\varepsilon > 0$ are such that

$$(6) \qquad \{(x,y,z)\colon 0 < x-x_0 < \varepsilon,\ |y-y_0-(x-x_0)z_0| < \varepsilon|x-x_0|, \\ |z-z_0-(x-x_0)r| < \varepsilon|x-x_0|\} \subset F ,$$

then there exists a solution of the strong problem that starts at P_0.

Such a solution is $W(\phi)$, where ϕ is defined on a suitable interval by

$$\phi(x) = y_0 + (x-x_0)z_0 + \frac{1}{2}(x-x_0)^2 r .$$

Not every member of $\mathcal{W}$ starting at a point P_0 begins in such a pyramid (6), however. This is shown by the example defined by $n = 1$, $\phi(0) = 0$, $\phi(x) = x^{5/2}\sin(1/x)$ for $x > 0$.

It does follow from Proposition 2 (and also from several preceding remarks) that at any interior point P_0 of F, there are a great many solutions of the strong initial value problem. Thus the problem can be interesting only on the border $F \cap \partial F$ of F - interesting in the sense that there is any danger of nonexistence of solutions, or any possibility of their uniqueness. Both Proposition 2 and the Peano approach may be applicable at such points.

4. An Example

Let us choose $n = 1$ and take F to be the set of all (x,y,z) satisfying the conditions

$$y \sin x - z \cos x = 0 ,$$
$$y^2 + z^2 \leqslant 1 .$$

Geometrically, F may be described as a twisted ribbon along the x-axis. This set has no interior, but at points where $|y| < |\cos x|$ the Peano theorem may be applied and the strong problem has a (unique) solution both ways. Points where $|y| = |\cos x|$ and $0 < x < \pi/2 \pmod{\pi}$ are isolated points of F relative to the topology $\mathcal{T}_{V^+}$, so no solutions of the strong problem start at such points. (It is easy to show, although it does not follow immediately from anything in this paper, that certain solutions <u>end</u> at such points.) Similarly, solutions start but do not end at points where $|y| = |\cos x|$ and $-\pi/2 < x < 0 \pmod{\pi}$. It may be seen in various ways that there are no solutions at points $(k\pi, \pm 1, 0)$. This leaves the points where $x = \pi/2 \pmod{\pi}$. The section of F perpendicular to the x-axis for such x are segments where $y = 0$, $|z| \leqslant 1$. Simple geometrical arguments show that all such points are isolated points of F relative to $\mathcal{T}_V$ except those where $z = 0$. Thus any solution $W(\phi)$ such that (say) $\pi/2 \in I_\phi$ must satisfy not only $\phi(\pi/2) = 0$ but $\phi'(\pi/2) = 0$; and this is a conclusion that may not be obvious from the statement of the problem. (Of course there exists such a solution, namely the x-axis.)

5. Conclusion

The brief and elementary discussion offered here is intended merely to suggest a way of looking at ordinary differential equations and generalized differential equations that seems to have been used little, if at all, although in some ways it is natural enough. In at least one way it is decidedly unnatural: for problems usually represented geometrically in n or $n+1$ dimensions, it uses a representation in $2n+1$ dimensions. This alone would be ample to account for its being a viewpoint toward which workers in the field would not have gravitated.

Nevertheless, it has some promise as an angle of attack in dealing with the fundamental theory of ordinary differential equations and their generalizations, such problems as approximation, stability and other kinds of limiting behavior, control problems, and perhaps more subtle matters like the existence and structure of periodic solutions. If nothing else, it provides another way of feeding and guiding the intuition about these matters.

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