Numerical Solution of Systems of Nonlinear Algebraic Equations

Edited by
George D. Byrne
AND
Charles A. Hall

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PREFACE

This volume contains invited lectures of the NSF-CBMS Regional Conference on the Numerical Solution of Nonlinear Algebraic Systems with Applications to Problems in Physics, Engineering and Economics, which was held July 10-14, 1972. The host for the conference was the Department of Mathematics, University of Pittsburgh and the principal lecturer was Professor Werner C. Rheinboldt of the University of Maryland.

Professor Rheinboldt's lectures will appear in a companion volume, which will be published in the SIAM Regional Conference Series in Applied Mathematics, as required by CBMS. Since his lectures did serve as the main theme of the conference, those contained in this volume serve fairly specific purposes. These purposes include motivating methods for solving nonlinear systems by examining their origins in methods for linear systems, discussing where nonlinear systems arise, reviewing methods for the nonlinear least squares problem, presenting and reviewing specific methods for solving nonlinear problems, and reviewing the contractor theory for nonlinear systems.

The conference committee which was responsible for the arrangements consisted of Professors George D. Andria and Martin J. Marsden, along with the editors.

We gratefully acknowledge the support of the National Science Foundation (Grant GJ-33612); the Conference Board of the Mathematical Sciences for their choice of the University of Pittsburgh as host for the conference; the understanding advice of the Academic Press staff; and the patience and fortitude of our typist, Miss Nancy Brown. We also acknowledge Professor Werner C. Rheinboldt for his thorough preparation of the principal lectures, his personal interest in and advice on the execution of the conference. Finally, we thank the participants in the conference for their enthusiasm and exhiliarating discussions.

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NONLINEAR ALGEBRAIC EQUATIONS IN CONTINUUM MECHANICS

W. F. Ames

1. Introduction.

Nonlinear algebraic equations are not ubiquitous in continuum mechanics. However, they do occur regularly and in a variety of forms which are often difficult to analyze. The diversity of forms, ranging from complex polynomials to simultaneous transcendental forms, is discussed here by means of five examples. These correspond to mathematical models of problems in mechanics whose solution depends substantially upon the solution of nonlinear algebraic equations.

The first example arises during the stability analysis of a density stratified gas flow over a liquid. The possible occurrence of periodic free vibrations of a coupled nonlinear system generates the second set of nonlinear equations. The third example is that of diffusion in distinct regions, separated by a moving boundary or interface. Problem four concerns the approximate development, by collocation, of an invariant solution for boundary layer flow of a viscous fluid. The last set of nonlinear equations is obtained when an implicit numerical method is employed to study equations of the form $u_{xx} = \psi(x,t,u,u_x,u_t)$.

2. Polynomials with Complex Coefficients (Sontowski et al., [14]).

Consider two inviscid fluids in a steady state of horizontal streaming (x direction) and superimpose a disturbance upon this state. Assuming the disturbance to be small and neglecting terms of order higher than the first, we obtain six linear perturbation equations in six unknowns as a result of the physical requirements of flow continuity, incompressibility, momentum and interface kinematics. The method of normal modes, discussed by Chandrasekhar [6], is now employed. Solutions are sought whose dependence upon x (horizontal direction), y (normal direction), z (vertical direction) and time t is given by

$$(1)\qquad w(z)\ \exp\left[i\ (k_x x + k_y y + nt)\right].$$

Upon substitution of (1) into the perturbation equations six equations are obtained which, by a process of elimination, reduce to one equation in one unknown. This equation is

$$(2)\qquad \left\{(n+k_x U)\left\{\frac{d^2}{dz^2}-k^2\right\}w - k_x\frac{d^2U}{dz^2}w - \frac{gk^2}{\rho}\frac{d\rho}{dz}\frac{w}{n+k_x U} + \frac{1}{\rho}\frac{d\rho}{dz}\left\{(n+k_x U)\frac{dw}{dz} - k_x\frac{dU}{dz}w\right\}\right\} = 0$$

where w depends only on z and the product w times $\exp[i(k_x x + k_y y + nt)]$ is the perturbation of the z-component of velocity. The wave number k is defined to equal $(k_x^2 + k_y^2)^{1/2}$. In addition it is required that the quantity $w/(n+k_x U)$ be continuous across the interface and also that

$$\Delta_s\left\{\rho\,(n+k_xU)\,\frac{dw}{dz} - \rho k_x\,\frac{dU}{dz}\,w\right\}$$

(3)

$$= gk^2\left\{\Delta_s(\rho) - \frac{k^2}{g}T_s\right\}\left\{\frac{w}{n+k_xU}\right\}_s$$

where $\Delta_s(f) = \lim_{\varepsilon\to 0}[f_{z=z_s+\varepsilon} - f_{z=z_s-\varepsilon}]$ and z_s represents the undisturbed interface position.

Application of the above mathematics to the case described in Fig. 1 generates the equations

$$\frac{d^2w}{dz^2} - \beta\,\frac{dw}{dz} - k^2\left\{1 - \frac{\beta g}{(n+k_xU_a)^2}\right\}w = 0 \qquad z > 0$$

$$d^2w/dz^2 - k^2w = 0 \qquad z < 0,$$

whose general solutions are

$$w = A_ae^{m_+z} + B_ae^{m_-z} \qquad z > 0,$$

$$w = A_be^{kz} + B_be^{-kz} \qquad z < 0,$$

where A_a, B_a, A_b and B_b are arbitrary constants and

$$m_{\pm} = \frac{\beta}{2} \pm \left[\left(\frac{\beta}{2}\right)^2 + k^2\left(1 - \frac{g\beta}{(n+k_xU_a)^2}\right)\right]^{1/2}.$$

Boundary conditions disallow disturbances which

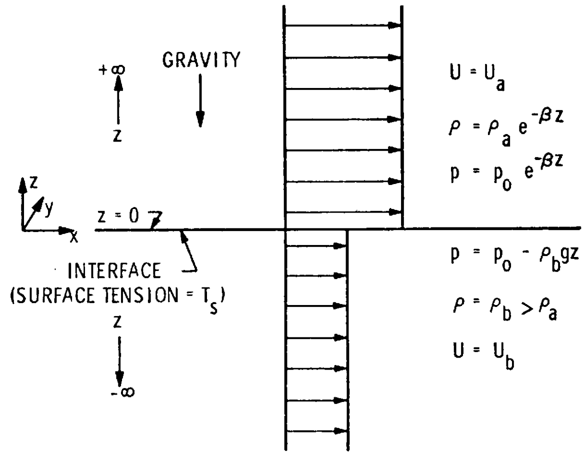

Fig. 1 The assumed stationary state

increase exponentially as the outer bounds of the fluids are approached. Thus

$$w = B_a e^{m-z} \qquad z > 0,$$

$$w = A_b e^{kz} \qquad z < 0,$$

with the requirement that

$$\mathrm{Re}\left[\left(\frac{\beta}{2}\right)^2 + k^2\left(1 - \frac{g\beta}{(n+k_x U_a)^2}\right)\right]^{1/2} \geq \frac{\beta}{2} .$$

Continuity of $w/(n+k_x U)$ across the interface leads to the following solution in terms of one arbitrary constant A:

$$w = A(n+k_x U_a)e^{m-z} \qquad z > 0,$$

$$w = A(n+k_x U_b)e^{kz} \qquad z < 0.$$

Substitution into the second interface condition, (3), yields the eigenvalue equation whose dimensionless form is

(4) $$\rho_* \beta_k (\nu+k_x \overline{U}_a)^2 - K(\nu+k_* \overline{U}_b)^2 + [(1-\rho_*) + \sigma_k]$$

$$= \rho_* (\nu + k_* \bar{U}_a)^2 \left(1 - \frac{2\beta_k}{(\nu + k_* \bar{U}_a)^2}\right)^{1/2}$$

in terms of the eigenvalue ν. The additional requirement

$$\text{Re}\left(1 - \frac{2\beta_k}{(\nu + k_* \bar{U}_a)^2}\right)^{1/2} \geq \beta_k \tag{5}$$

completes the specification of the eigenvalue problem. According to the definition,

$$\nu = \frac{\left[\left(\frac{\beta}{2}\right)^2 + k^2\right]^{1/4}}{kg^{1/2}} n, k_* = \frac{k_x}{k}, \rho_* = \frac{\rho_a}{\rho_b},$$

$$\beta_k = \frac{\left(\frac{\beta}{2}\right)}{\left[\left(\frac{\beta}{2}\right)^2 + k^2\right]^{1/2}}, \; K = \frac{k}{\left[\left(\frac{\beta}{2}\right)^2 + k^2\right]^{1/2}}, \rho_k = \frac{T_s}{g\rho_b} k^2,$$

$$\bar{U}_a = \frac{\left[\left(\frac{\beta}{2}\right)^2 + k^2\right]^{1/4}}{g^{1/2}} U_a, \quad \bar{U}_b = \frac{\left[\left(\frac{\beta}{2}\right)^2 + k\right]^{1/4}}{g^{1/2}} U_b$$

From the form of the disturbance in (1), it follows that the flow is unstable if and only if any one or more of the eigenvalues ν has a negative imaginary part. For a complete stability analysis the characteristic values of ν must be examined for all values of the vector $k = [k_x, k_y]$.

The nature of the eigenvalues must now be determined. With the substitutions

$$(6) \qquad \xi = \nu + k_* \overline{U}_a, \quad \eta = \nu + k_* \overline{U}_b - \frac{\rho_* \beta_k}{K - \rho_* \beta_k} [k_* (\overline{U}_a - \overline{U}_b)],$$

$$(7) \qquad \eta_0^2 = \frac{K \rho_* \beta_k}{(K - \rho_* \beta_k)^2} [k_* (\overline{U}_a - \overline{U}_b)]^2 + \frac{(1-\rho_*) + \sigma_k}{K - \rho_* \beta_k}$$

in (4) and (5) we find

$$(8) \qquad \eta^2 + \frac{\rho_*}{K - \rho_* \beta_k} \xi^2 \left(1 - \frac{2\beta_k}{\xi^2}\right)^{1/2} = \eta_0^2$$

and

$$(9) \qquad \mathrm{Re}(1 - 2\beta_k / \xi^2)^{1/2} \geq \beta_k,$$

which, in conjunction with the auxiliary relationship

$$(10) \qquad \xi - \eta = (K/(K - \rho_* \beta_k))[k_* (\overline{U}_a - \overline{U}_b)],$$

is equivalent to the eigenvalue problem. Because of (9) this problem is nonalgebraic. Therefore, it is advantageous to construct a parent algebraic system possessing the eigenvalue problem as a subsystem. If all restrictions on $\mathrm{Re}(1-(2\beta_k/\xi^2))^{1/2}$ are removed then (8), together with (10), is equivalent to a fourth degree polynomial in ν and this is taken as the parent system. Let us distinguish two branches of the parent system, calling our eigenvalue problem the principal or P-branch and the remainder of the system, where $\mathrm{Re}(1-(2\beta_k/\xi^2))^{1/2} < \beta_k$, the subsidiary or S-branch. In mathematical form

$$\text{P branch}\begin{cases} \eta^2 + \dfrac{\rho_*}{K-\rho_*\beta_k}\,\xi^2\left(1-\dfrac{2\beta_k}{\xi^2}\right)^{1/2} = \eta_0^2, \\[2ex] \mathrm{Re}\left(1-\dfrac{2\beta_k}{\xi^2}\right)^{1/2} \geq \beta_k, \end{cases}$$

$$\begin{array}{c}\text{S-branch}\\ \text{Part 1}\\ S_1\end{array}\begin{cases} \eta^2 + \dfrac{\rho_*}{K-\rho_*\beta_k}\,\xi^2\left(1-\dfrac{2\beta_k}{\xi^2}\right)^{1/2} = \eta_0^2 \\[2ex] 0 < \mathrm{Re}\left(1-\dfrac{2\beta_k}{\xi^2}\right)^{1/2} < \beta_k, \end{cases}$$