REAL-VARIABLE METHODS IN HARMONIC ANALYSIS

ALBERTO TORCHINSKY

Real-Variable Methods in Harmonic Analysis

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Preface

This book is based on a set of notes from a course I gave at Indiana University during the academic year 1984–1985. My purpose in those lectures was to present some recent topics in harmonic analysis to graduate students with varied backgrounds and interests, ranging from operator theory to partial differential equations. The book is an exploration of the unity of several areas in harmonic analysis, emphasizing real-variable methods, and leading to the study of active areas of research including the Calderón–Zygmund theory of singular integral operators, the Muckenhoupt theory of A_p weights, the Fefferman-Stein theory of H^p spaces, the Burkholder–Gundy theory of good λ inequalities, and the Calderón theory of commutators.

Because I wanted this book to be essentially self-contained for those students with an elementary knowledge of the Lebesgue integral and since ideas rather than generality are stressed, the point of departure is the classical question of convergence of Fourier series of functions and distributions. Chapter I deals with pointwise convergence, Chapter II with Cesàro $(C, 1)$ convergence, Chapters III and V with norm convergence and Chapter VII with Abel convergence. Chapter IV contains the basic working principles of harmonic analysis, centered around the Calderón–Zygmund decomposition of locally integrable functions. Chapter VI discusses fractional integration, and Chapter VIII the John–Nirenberg class of *BMO* functions. A one semester course in Fourier series can easily be extracted from these first eight chapters.

From this point on our setting becomes R^n. In Chapter IX the Muckenhoupt theory of A_p weights is developed, and in Chapter X, in addition to briefly reviewing the previous results in this new context, elliptic equations in divergence form are treated. Chapter XI deals with the essentials of the Calderón–Zygmund theory of singular integral operators and Chapter XII with its vector-valued version, Littlewood–Paley theory.

Chapter XIII covers the good λ inequalities of Burkholder–Gundy, Chapter XIV the Fefferman–Stein theory of Hardy spaces of several real variables, and Chapter XV Carleson measures. Chapter XVI contains the Coifman–McIntosh–Meyer real variable approach to Calderón's commutator theorem and Chapter XVII one of its interesting applications, namely, the solution to the Dirichlet and Neumann problems on a C^1 domain by means of the layer potential methods. This second half of the book is easily adapted to a one- or two-semester topics course in harmonic analysis.

A word about where the material covered in the book fits into the existing literature: The first part of the book is essentially contained in Zygmund's treatise, where the so-called complex method is emphasized, and precedes Stein's book on singular integrals and differentiability properties of functions; the second half continues with the material discussed in Stein's book. These are the two basic sources of reference that my generation of analysts grew up with.

The notations used are standard, and we remark here only that c denotes a constant which may differ at different occurrences, even in the same chain of inequalities. "Theorem 3.2" means that the result alluded to appears as the second item in Section 3 of the same chapter, and "Theorem 3.2 in Chapter X" means that it appears as the second item in the third section of Chapter X. The same convention is used for formulas.

In order to encourage the active participation of the reader, numerous hints are provided for the problems; I hope the book will be "user friendly." It is not meant, however, to make the learning of the material effortless; many of the ideas discussed lie at the very heart of harmonic analysis and as such require some thought.

It is always a pleasure to acknowledge the contribution of those who make a project of this nature possible. A. P. Calderón, a singular analyst and teacher, has always been a source of inspiration to me; his decisive influence in contemporary harmonic analysis and its applications should be apparent to anyone browsing these pages. My colleague B. Jawerth shared with me his ideas on how results should, and should not, be presented. My largest debt, though, is to the students who attended the course and kept me honest when a simple "the proof is easy" was tempting. They are Alp Eden, Don Krug, Hung-Ju Kuo, Paul McGuire, Mohammad Rammaha, Edriss Titi, and Sung Hyun Yoon. The manuscript was cheerfully typed by Storme Day. The staff at Academic Press handled all my questions promptly and efficiently.

CHAPTER

I

Fourier Series

1. FOURIER SERIES OF FUNCTIONS

A trigonometric polynomial $p(t)$ is an expression of the form

$$p(t) = \sum_{|j| \leq n} c_j e^{ijt}, \qquad |c_n| + |c_{-n}| \neq 0. \tag{1.1}$$

n is the degree of p and the c_j's are (possibly complex) constants. Thus p is a continuous function of period 2π and is therefore determined by its values on $T = (-\pi, \pi]$, or any other interval of length 2π for that matter. On the other hand, given a trigonometric polynomial p of degree $\leq n$, we can easily compute the constants c_j by means of

$$c_j = \frac{1}{2\pi} \int_T p(t) e^{-ijt}\, dt, \qquad |j| \leq n.$$

This observation follows at once from the fact that

$$\frac{1}{2\pi} \int_T e^{ijt}\, dt = \begin{cases} 0 & \text{if } \ j \neq 0, \\ 1 & \text{if } \ j = 0. \end{cases} \tag{1.2}$$

A trigonometric series is an expression of the form

$$\sum_{j=-\infty}^{\infty} c_j e^{ijt}. \tag{1.3}$$

Since we make no assumption concerning the convergence of this series, (1.3) only formally represents a function of period 2π.

A Fourier series is a trigonometric series for which there is a periodic, Lebesgue summable function f such that

$$c_j = c_j(f) = \frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt, \qquad \text{all} \quad j. \tag{1.4}$$

In this case we call the constants c_j the Fourier coefficients of f and denote this correspondence by

$$f \sim \sum_j c_j e^{ijt}. \tag{1.5}$$

A word about the class of functions involved in this definition. It is denoted by $L(T)$ and it consists of those periodic, Lebesgue measurable functions f with finite L^1 norm, i.e.,

$$\|f\|_1 = \frac{1}{2\pi}\int_T |f(t)|\,dt < \infty.$$

Endowed with this norm and modulo functions which coincide a.e., $L(T)$ becomes a Banach space, one in the scale of $L^p(T)$ spaces, where for $1 \le p < \infty$

$$L^p(T) = \left\{ f \text{ periodic, measurable: } f: T \to C, \text{ and} \right.$$

$$\left. \|f\|_p = \left(\frac{1}{2\pi}\int_T |f(t)|^p\,dt\right)^{1/p} < \infty \right\}. \tag{1.6}$$

When $p = \infty$, we define the norm in $L^\infty(T)$ as the limiting expression in (1.6) as $p \to \infty$. It turns out that $L^\infty(T)$ is also a Banach space with norm $\|f\|_\infty = \operatorname{ess\,sup}_T |f(t)|$. By a mild abuse of notation we also denote by $\|f\|_p$ the quantity appearing in (1.6) when $0 < p < 1$, although in this case the triangle inequality is not satisfied and elements in $L^p(T)$ are not necessarily locally integrable functions.

Still in the case of Fourier series no assumption concerning the convergence of the series (1.5) is made. More specifically, if $s_n(f, t)$ denotes the trigonometric polynomial of degree $\le n$ corresponding to the symmetric partial sum of (1.5) of order n, i.e.,

$$s_n(f, t) = \sum_{|j| \le n} c_j e^{ijt}, \tag{1.7}$$

then nothing is known or assumed about the existence of the $\lim_{n\to\infty} s_n(f, t)$ for any $t \in T$.

At times, and especially when dealing with examples, it is convenient to work with the so-called Fourier cosine and sine series of f; this is interchangeable with (1.5). Indeed, suppose, as we often do, that f is real. Then

$$\begin{aligned} s_n(f, t) &= c_0 + \sum_{j=1}^{n} (c_j e^{ijt} + c_{-j} e^{-ijt}) \\ &= c_0 + \sum_{j=1}^{n} (c_j + c_{-j}) \cos jt + i(c_j - c_{-j}) \sin jt \\ &= \frac{a_0}{2} + \sum_{j=1}^{n} a_j \cos jt + b_j \sin jt, \end{aligned} \tag{1.8}$$

say, where the a_j's and b_j's are real since $c_{-j} = \overline{c_j}$. Conversely, given a cosine and sine series we may recover (1.5) by letting $2c_j = a_j - ib_j$, and

$$a_j = \frac{1}{\pi} \int_T f(t) \cos jt \, dt, \qquad b_j = \frac{1}{\pi} \int_T f(t) \sin jt \, dt. \tag{1.9}$$

If the function f is even, that is $f(t) = f(-t)$, the coefficients b_j vanish and the integral defining a_j may be replaced by twice the integral over $(0, \pi)$. If f is odd, that is $f(t) = -f(-t)$, then $a_j = 0$ and the second integral above may be replaced by twice the integral over $(0, \pi)$.

Harmonic analysis studies, in a broad sense, properties of the series (1.3) and (1.5). For instance, since the sign $\sim$ in (1.5) only means that the constants c_j and the function f are connected by the formula (1.4), an important problem is to determine if, and how, the Fourier series of a function represents, or converges to, that function. We address the problem of pointwise convergence in this chapter, that of Cesàro summability in Chapter II, Abel summability in Chapter VII, and norm convergence in Chapters III and V.

We begin our discussion with some general observations concerning Fourier series. In first place note that if the partial sums of a trigonometric series (1.3) converge, in some general sense, to a function $f \in L(T)$, then actually $c_j = c_j(f)$. More precisely,

Proposition 1.1. If the symmetric partial sums $s_n(t)$ of the trigonometric series (1.3) converge in L^1 norm to $f \in L(T)$, then $c_j = c_j(f)$.

Proof. Fix an integer j and observe that the sequence $f_n(t) = (f(t) - s_n(t))e^{-ijt}$, $n = 0, 1, \ldots$ converges to 0 in $L(T)$. Moreover

$$c_j(f) = \frac{1}{2\pi} \int_T (f(t) - s_n(t))e^{-ijt} \, dt + \frac{1}{2\pi} \int_T s_n(t) e^{-ijt} \, dt.$$

By (1.2) we readily see that the second integral above is c_j as soon as $|n| \geq j$. Therefore $|c_j(f) - c_j| \leq \|f_n\|_1 \to 0$, as $n \to \infty$, and consequently, $c_j(f) = c_j$. ■

The reader will observe that the conclusion of Proposition 1.1 also obtains from a weaker assumption, namely, the existence of a sequence $n_j \to \infty$ such that $s_{n_j}(t)$ converges to f in L^1; because such extensions are trivial we prefer to omit them unless they are clearly important. On the other hand, in the course of the above proof we have made use of the interesting fact that

$$|c_j(f)| \leq \|f\|_1, \qquad \text{all} \quad j, \tag{1.10}$$

and there is more we can say in this direction.

Theorem 1.2 (Riemann-Lebesgue). Let $f \in L(T)$. Then $c_j \to 0$ as $|j| \to \infty$.

Proof. We invoke the well-known fact that trigonometric polynomials are dense in $L(T)$; a proof of this is given in Proposition 2.4 of Chapter II. Now, given $\varepsilon > 0$ we show that $|c_j| \leq \varepsilon$ provided $|j| > n_0$ is large enough. Let p be a trigonometric polynomial such that $\|f - p\|_1 \leq \varepsilon$, and let $n_0 =$ degree of p. Then for $|j| > n_0$ we have

$$c_j = \frac{1}{2\pi}\int_T (f(t) - p(t))e^{-ijt}\,dt$$

and consequently $|c_j| \leq \|f - p\|_1 \leq \varepsilon$. ■

Now that there is some hope that the Fourier series of $f \in L(T)$ may converge, we take a closer look at $s_n(f, x)$. It can also be written as

$$\begin{aligned} s_n(f, x) &= \sum_{|j| \leq n}\left(\frac{1}{2\pi}\int_T f(t)e^{-ijt}\,dt\right)e^{ijx} \\ &= \frac{1}{\pi}\int_T f(t)\left(\frac{1}{2}\sum_{|j|\leq n} e^{ij(x-t)}\right)dt = \frac{1}{\pi}\int_T f(t)D_n(x-t)\,dt, \end{aligned} \tag{1.11}$$

say, where we have denoted by

$$D_n(t) = \frac{1}{2}\sum_{|j|\leq n} e^{ijt}, \qquad n = 0, 1, \ldots \tag{1.12}$$

the Dirichlet kernel of order n. We list some properties of these kernels. In the first place by summing the geometric series in (1.12) we get

$$D_n(t) = \frac{1}{2}e^{-int}\frac{(e^{i(2n+1)t} - 1)}{(e^{it} - 1)}$$

$$= \frac{1}{2} \frac{e^{i(n+1)t} - e^{-int}}{e^{it/2}(e^{it/2} - e^{-it/2})}$$

$$= \frac{1}{2} \frac{e^{i(n+1/2)t} - e^{-i(n+1/2)t}}{e^{it/2} - e^{-it/2}}$$

$$= \frac{1}{2} \frac{\sin(n+1/2)t}{\sin(t/2)}, \qquad n = 0, 1, \ldots. \tag{1.13}$$

Thus D_n is an even function, and by (1.2)

$$\frac{1}{\pi} \int_T D_n(t)\, dt = \frac{2}{\pi} \int_{[0,\pi]} D_n(t)\, dt = 1, \qquad \text{all} \quad n. \tag{1.14}$$

It is also possible to estimate $D_n(t)$. In fact by (1.12),

$$|D_n(t)| \leq \frac{1}{2} \sum_{|j| \leq n} |e^{ijt}| = \frac{2n+1}{2} = n + \frac{1}{2}, \qquad \text{all} \quad n. \tag{1.15}$$

Moreover, since as is readily seen

$$1/(2\sin(t/2)) \leq \pi/2t \qquad \text{for} \quad 0 < t < \pi, \tag{1.16}$$

by (1.13) it follows at once that

$$|D_n(t)| \leq \pi/2|t|, \qquad 0 < |t| < \pi, \qquad \text{all} \quad n. \tag{1.17}$$

This is all we need to know about this kernel.

Returning to (1.11), it is useful to replace D_n there by the symmetric expression $D_n^* = (D_{n-1} + D_n)/2$, which equals

$$D_n^*(t) = \frac{\sin((n-1/2)t) + \sin((n+1/2)t)}{2\sin(t/2)} = \frac{\sin nt}{2\tan(t/2)}, \qquad \text{all} \quad n. \tag{1.18}$$

Also note that since $D_n(t) - D_n^*(t) = (D_n(t) - D_{n-1}(t))/2 = \cos(nt/2)$, we can rewrite

$$s_n(f, x) = \frac{1}{\pi} \int_T f(t) D_n^*(x-t)\, dt + \frac{1}{2\pi} \int_T f(t) \cos n(x-t)\, dt$$

$$= s_n^*(f, x) + A_n, \tag{1.19}$$

say. We claim that the term A_n above is an "error term," in the sense that it tends to 0 as $n \to \infty$, uniformly in x, and may therefore be disregarded. This is easy to see since A_n equals

$$\cos(nx) \frac{1}{\pi} \int_T f(t) \cos nt\, dt + \sin(nx) \frac{1}{\pi} \int_T f(t) \sin nt\, dt,$$

and by the Riemann-Lebesgue theorem both integrals go to 0 as $n \to \infty$ and the factors in front of them are uniformly bounded by 1 for x in T.

A shorthand notation is useful to express this situation. We write $u_n = o(v_n)$ as $n \to \infty$, provided that $v_n > 0$ and $|u_n|/v_n \to 0$ as $n \to \infty$. Thus $u_n = o(1)$ means that $\lim_{n\to\infty}|u_n| = 0$. If on the other hand $|u_n|/v_n$ remains bounded as $n \to \infty$, we write $u_n = O(v_n)$. So $u_n = O(1)$ means that for some constant c, $|u_n| \le c$, all large n. With this notation (1.19) becomes $s_n(f, x) = s_n^*(f, x) + o(1)$, uniformly for x in T.

It is also possible to introduce another expression closely related to $s_n(f, x)$. First observe that the function

$$\phi(t) = \left(\frac{1}{2\tan(t/2)} - \frac{1}{t}\right) \in L^\infty(T). \tag{1.20}$$

So if $f \in L(T)$, then $f\phi \in L(T)$, and, consequently, again by the Riemann-Lebesgue Theorem and (1.19),

$$s_n(f, x) = \frac{1}{\pi}\int_T f(t)\frac{\sin n(x-t)}{x-t}\,dt + o(1). \tag{1.21}$$

Returning to D_n^* we list some of its properties. From (1.18) it readily follows that it also is an even function, and from (1.14) that

$$\frac{1}{\pi}\int_T D_n^*(t)\,dt = \frac{2}{\pi}\int_{[0,\pi]} D_n^*(t)\,dt = 1, \qquad \text{all} \quad n. \tag{1.22}$$

Also estimates (1.15) and (1.17) have a counterpart, to wit

$$|D_n^*(t)| \le \frac{(n-1)+\frac{1}{2}+n+\frac{1}{2}}{2} = n, \qquad \text{all} \quad n \tag{1.23}$$

and

$$|D_n^*(t)| \le \pi/2|t|, \qquad 0 < |t| < \pi. \tag{1.24}$$

As for (1.11), since D_n^* is even we also have that $s_n^*(f, x)$ equals either

$$\frac{1}{\pi}\int_T f(t)D_n^*(x-t)\,dt \quad \text{or} \quad \frac{1}{\pi}\int_T \frac{(f(x+t)+f(x-t))}{2}D_n^*(t)\,dt. \tag{1.25}$$

Moreover since the integrand in the last integral above is an even function of t we also have that it equals

$$\frac{1}{\pi}\int_{[0,\pi]} (f(x+t)+f(x-t))D_n^*(t)\,dt. \tag{1.26}$$

We are now ready to prove our first convergence result.

Theorem 1.3 (Dini). Let $f \in L(T)$ and suppose there is a constant A such that for an x in T

$$\int_{[0,\pi]} \left| \frac{f(x+t)+f(x-t)}{2} - A \right| \frac{dt}{t} < \infty. \tag{1.27}$$

Then $\lim_{n\to\infty} s_n(f, x) = A$.

Proof. By (1.19) it suffices to prove the assertion with $s_n(f, x)$ replaced by $s_n^*(f, x)$. Moreover, by (1.26), (1.22), and (1.18), we may write

$$s_n^*(f, x) - A = \frac{2}{\pi} \int_{[0,\pi]} \left(\frac{f(x+t)+f(x-t)}{2} - A \right) \frac{\sin nt}{2\tan(t/2)}\, dt.$$

Now, assumption (1.27) is clearly equivalent to the fact that the function

$$F_x(t) = \left(\frac{f(x+t)+f(x-t)}{2} - A \right) \frac{1}{2\tan(t/2)} \in L(T),$$

since $\tan(t/2) \sim t$ near 0. Therefore $s_n^*(f, x) - A$ is nothing but the nth Fourier sine coefficient of the function $F_x(t) \in L(T)$, which by the Riemann-Lebesgue theorem tends to 0 as $n \to \infty$. ■

A word about the value of A above. If x is a point of a removable discontinuity, or a jump, of f, then A is necessarily $f(x)$ or $(f(x+0) + f(x-0))/2$. Moreover, since functions $f \in L(T)$ are only determined a.e. we may always assume that $A = f(x)$ by changing the value of f at that point if necessary. Also notice that if $f(x+t) - f(x) = O(|t|^\eta)$, $\eta > 0$, then the Fourier series of f converges to $f(x)$ at that x. In particular, this is true if $f'(x)$ exists and is finite. If any of these conditions is satisfied uniformly for x in a closed subinterval of T, then $s_n(f, x)$ converges uniformly to $f(x)$ in that interval.

To state another simple criterion, this one of a.e. nature, we need a definition. We denote by $w_1(f, x)$ the L-modulus of continuity of f, namely,

$$w_1(f, x) = \frac{1}{2\pi} \int_T |f(x+t) - f(t)|\, dt. \tag{1.28}$$

We then have

Theorem 1.4 (Marcinkiewicz). Suppose that $f \in L(T)$ and that

$$\int_{[0,\pi]} w_1(f, t) \frac{dt}{t} < \infty.$$

Then $\lim s_n(f, x) = f(x)$, a.e. in T.

Proof. Let

$$I(x) = \int_{[0,\pi]} |f(x+t) - f(x)| \frac{dt}{t} \geq 0.$$

By Tonelli's theorem we have

$$\frac{1}{2\pi}\int_T I(x)\,dx = \int_{[0,\pi]} \frac{1}{2\pi}\int_T |f(x+t) - f(x)|\,dx \frac{dt}{t}$$
$$= \int_{[0,\pi]} w_1(f, t)\frac{dt}{t} < \infty.$$

Consequently, $I(x) < \infty$ a.e. in T, and therefore also

$$\int_{[0,\pi]} |f(x+t) + f(x-t) - 2f(x)| \frac{dt}{t} < \infty \qquad \text{a.e. in} \quad T.$$

This implies that Dini's theorem applies with $A = f(x)$ a.e. in T. ∎

2. FOURIER SERIES OF CONTINUOUS FUNCTIONS

Although at this point we may intuitively guess that Dini's theorem suffices to assure the convergence of $s_n(f, x)$ to $f(x)$ at a point of continuity of f, nothing could be further from the truth. Indeed, the expression

$$\int_{[0,\pi]} \left| \frac{f(x+t) + f(x-t)}{2} - f(x) \right| \frac{dt}{t}$$

may diverge everywhere in T, even for a continuous function f (and even with the absolute values removed from the integral); a closely related result will be discussed in Proposition 5.1 of Chapter III. Now we turn around and guess that there may exist a continuous function whose Fourier series does not converge at a point. Statements of this nature are supported in one of two ways: either by constructing a specific function with the desired property or else by assuming that no such function exists and reaching a contradiction. Since each method has its appeal and usefulness, we present both here in our successful quest for a continuous function with a nonconvergent Fourier series at $x = 0$.

We begin by considering the so-called Lebesgue constants L_n. They are given by

$$L_n = 2\|D_n\|_1 = \frac{1}{\pi}\int_T |D_n(t)|\,dt, \qquad n \geq 0.$$

This is why. By (1.11) it is plain that

$$|s_n(f,0)| \leq \frac{1}{\pi}\int_T |f(t)||D_n(t)|\,dt \leq \|f\|_\infty L_n.$$

Therefore by setting $f(t) = \operatorname{sgn} D_n(t)$, $t \in T$, i.e.,

$$f_n(t) = \begin{cases} 1 & D_n(t) > 0 \\ 0 & D_n(t) = 0 \\ -1 & D_n(t) < 0 \end{cases}$$

we readily see that $\|f_n\|_\infty = 1$ and

$$\sup_{f \in L^\infty(T), \|f\|_\infty \leq 1} |s_n(f,0)| = L_n.$$

Since the function $f_n(t)$ is real valued and discontinuous at a finite number of points, it is easy to modify its values in small neighborhoods of those points to obtain, now, that also for continuous functions

$$\sup_{f \in C(T), \|f\|_\infty \leq 1} |s_n(f,0)| = L_n. \tag{2.1}$$

It becomes, then, important to study the behavior of L_n for large n.

Proposition 2.1. $L_n \sim (4/\pi^2)\ln n$, as $n \to \infty$.

Proof. Since $D_n(t)$ is even and $\sin(t/2) > 0$ for $0 < t < \pi$, we have that

$$\begin{aligned} L_n &= \frac{2}{\pi}\int_{[0,\pi]} \left|\sin\left(\left(n+\frac{1}{2}\right)t\right)\right| \left(\frac{1}{2\sin(t/2)} - \frac{1}{t}\right) dt \\ &\quad + \frac{2}{\pi}\int_{[0,\pi]} \left|\sin\left(\left(n+\frac{1}{2}\right)t\right)\right| \frac{dt}{t} = A_n + B_n, \end{aligned}$$

say. By a statement similar to (1.20) we see at once that $A_n = O(1)$. We take a look at B_n now. The change of variables $(n+\frac{1}{2})t = s$ gives

$$\begin{aligned} B_n &= \frac{2}{\pi}\int_{[0,(n+1/2)\pi]} |\sin s| \frac{ds}{s} \\ &= \frac{2}{\pi}\int_{[\pi,n\pi]} |\sin s| \frac{ds}{s} + O(1) = B_n' + O(1), \end{aligned}$$

say. Thus we will be done once we show that

$$B_n' \cong \frac{4}{\pi^2}\ln n + O(1). \tag{2.2}$$

We rewrite

$$B'_n = \frac{2}{\pi}\sum_{k=1}^{n-1}\int_{[k\pi,(k+1)\pi]}\frac{|\sin s|}{s}\,ds$$
$$= \frac{2}{\pi}\sum_{k=1}^{n-1}\int_{[0,\pi]}\frac{|\sin(k\pi+t)|}{k\pi+t}\,dt$$
$$= \frac{2}{\pi}\int_{[0,\pi]}(\sin t)\left\{\sum_{k=1}^{n-1}\frac{1}{k\pi+t}\right\}dt.$$

The expression in $\{\cdot\}$ in the above integral can be estimated below and above, uniformly for $t \in (0, \pi]$, by

$$\frac{1}{\pi}\sum_{k=1}^{n-1}\frac{1}{k+1} = \frac{1}{\pi}\sum_{k=1}^{n}\frac{1}{k} - \frac{1}{\pi} \quad \text{and} \quad \frac{1}{\pi}\sum_{k=1}^{n-1}\frac{1}{k}, \tag{2.3}$$

respectively. By (2.3) then, and since $\int_{[0,\pi]}\sin t\,dt = 2$, we finally obtain

$$\frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} - O(1) \leq B'_n \leq \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}.$$

In other words (2.2) holds and we are done. ■

Corollary 2.2. If $f \in L^\infty(T)$, then $s_n(f, x) = O(\ln n)$.

We now know that for each (large) n there is a continuous function f, $|f(x)| \leq 1$, and

$$|s_n(f, 0)| \sim \frac{4}{\pi^2}\ln n. \tag{2.4}$$

It is natural then to search for a single continuous function f whose Fourier series has large partial sums at 0. Assuming that no such function exists we will reach a contradiction. Suppose, then, that the Fourier series of every continuous function converges at 0; in particular, the partial sums will be bounded there, i.e., $|s_n(f, 0)| \leq c_f < \infty$, all n, each $f \in C(T)$. By (1.11) this is equivalent to

$$\left|\int_T f(t)D_n(t)\,dt\right| \leq c_f < \infty, \qquad \text{all} \quad n, \qquad \text{each} \quad f \in C(T). \tag{2.5}$$

We now show that (2.5) cannot hold. Since the idea needed to do this can also be used in other settings, we prefer to cast the statement in a general context. In the application of this general result we will make use of the well known fact that $C(T)$ is a complete metric space, and that therefore any decreasing sequence of closed balls with radius approaching 0, has a nonempty intersection (consisting of a single point).

We state and prove the Uniform Boundedness Principle.

Theorem 2.3 (Banach-Steinhaus). Let X be a complete metric space and let Y be a normed linear space. Furthermore, let $\{T_\alpha\}_{\alpha\in A}$ be a family of bounded linear operators from X into Y with the property that for each $x \in X$ the family $\{T_\alpha x\}_{\alpha\in A}$ is bounded in Y, i.e., $\|T_\alpha x\|_Y \leq c_x < \infty$, all $\alpha \in A$. Then the family T_α is uniformly bounded, in other words there is a constant c such that

$$\sup_{\|x\|_X\leq 1} \|T_\alpha x\|_Y \leq c, \qquad \text{all} \quad \alpha \in A.$$

Proof. Suppose we can show that for some $x_0 \in X$, $\varepsilon > 0$ and a constant K we have $\|T_\alpha x\|_Y \leq K$ whenever $\|x_0 - x\|_X \leq \varepsilon$, i.e., the family $\{T_\alpha x\}_{\alpha\in A}$ is uniformly bounded at a ball $B(x_0, \varepsilon)$ about x_0. Then we are done. Indeed, for $x \neq 0$, $\|x\|_X \leq 1$, we put $z = \varepsilon x/\|x\|_X + x_0 \in B(x_0, \varepsilon)$. Then $\|T_\alpha z\|_X \leq K$ and by the triangle inequality

$$\frac{\varepsilon}{\|x\|_X}\|T_\alpha x\|_Y - \|T_\alpha x_0\|_Y \leq \|T_\alpha z\|_Y \leq K. \tag{2.6}$$

Letting $c = K + \sup_\alpha \|T_\alpha x_0\|_Y < \infty$, we may rewrite (2.6) as $\|T_\alpha x\|_Y \leq c/\varepsilon$, c independent of α, which is precisely what we wanted to prove. So, to complete the proof we must show that such a ball exists. We argue by contradiction and assume no such ball exists.

Fix a ball $B_0 = B(x_0, 2)$, then there exist $x_1 \in B_0$ and $\alpha_1 \in A$ such that $\|T_{\alpha_1} x_1\|_Y > 1$. Also by continuity $\|T_{\alpha_1} x\|_Y > 1$, $x \in B_1 = B(x_1, \varepsilon_1) \subseteq B_0$, $\varepsilon_1 < 1$. The family $\{T_\alpha x\}$ is still not uniformly bounded on B_1. So recursively, and after $B_0 \supseteq B_1 \supseteq \cdots \supseteq B_{k-1}$ have been chosen, with radius $\varepsilon_j < 1/j$ and centers x_j such that $\|T_{\alpha_j} x_j\|_Y > j$, $1 \leq j \leq k-1$, we then select a ball $B_k = B(x_k, \varepsilon_k)$, $B_k \subseteq B_{k-1}$, $\varepsilon_k < 1/k$, $\alpha_k \neq \alpha_j$, $j < k$, and $\|T_{\alpha_k} x\|_Y > k$ for x in B_k. Since X is complete there is a point $z \in \bar{B}\,(x_k, \varepsilon_k)$ for all k. The fact that $\|T_{\alpha_k} z\|_Y \geq k$, all k, contradicts the assumption that $\{T_\alpha z\}_{\alpha\in A}$ is bounded. ■

As anticipated we apply the theorem with $X = C(T)$, $Y = C$, and put

$$T_n f = s_n(f, 0) = \frac{1}{\pi}\int_T f(t) D_n(t)\, dt, \; n \geq 0.$$

By (2.1) and (2.4),

$$\sup_{\|f\|_{C(T)}\leq 1} |T_n f| \sim \frac{4}{\pi^2}\ln n. \tag{2.7}$$

Now, were (2.5) to hold, then by the Uniform Boundedness Principle, (2.7) would imply there is a constant c such that $(4/\pi^2)\ln n \leq c$, all n, which is impossible. Therefore there is a continuous function f such that