COMPUTER AIDED GEOMETRIC DESIGN

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COMPUTER AIDED GEOMETRIC DESIGN

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PREFACE

The first International Conference on Computer Aided Geometric Design (CAGD) was held at The University of Utah March 18-21, 1974, for the purpose of displaying the latest advances in CAGD. The word *geometric* distinguishes this field from computer aided *logical* design. This book is the edited Proceedings of the Conference.

At the conclusion of the Navy Workshop at Annapolis in 1971, Philip J. Davis proposed that there be a conference on "graphics and mathematics", a project that Leila Bram of the Office of Naval Research encouraged. Distinguished representatives in Europe and North America from universities, industry, and government laboratories were sought. Both researchers and users of the research were invited. The conference had an informal tone and ample time for discussion, with about 120 participants. There were talks and computer graphics demonstrations at The University of Utah and at Evans and Sutherland Corporation.

P. Bézier and S. A. Coons have played fundamental roles in CAGD, as was evidenced by the fact that most of the speakers referred to their pioneering work. The principal topics covered in the Proceedings are Coons patches, Bézier curves, and splines, with their applications to CAGD.

The editors express their sincere appreciation to the contributing authors, and to A. R. Forrest, W. J. Gordon, J. A. Gregory, and R. J. McDermott for their help. Proofreading and presentations of most of the papers were carried out by graduate students in Mathematics and Computer Science. The editors also thank C. Jensen, L. Merrell, and M. Holbrook for their typing and layout work, and A. R. Barnhill and L. Williams for artwork in the Proceedings.

PREFACE

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ROBERT E. BARNHILL

SURFACE PATCHES AND B-SPLINE CURVES

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Introduction.

This paper begins with a review of the surface patch equation, which may or may not be familiar to the reader. The properties of surface patches are described, but without proof. However, it is relatively simple to verify them by actual algebraic calculations.

Following this, the notion of "uniform cubic B-spline curves" is introduced, again without either derivation or verification of the B-spline formula. Such matters are dealt with elsewhere (see for instance Riesenfeld, deBoor, and others). These (compound) curves are then used to define the boundary conditions of surface patches, and also to describe the "blending functions" which appear in the surface patch equation. An interesting consequence of B-spline curves as boundaries of a patch is that boundaries with slope discontinuities (cusps) can be introduced, without inducing sharp folds or creases in the interior of the patch. This is an interesting, seemingly paradoxical result.

Surface Patches.

A surface "patch" is a segment of a surface, and it is expedient to represent it as the locus of a point $[x\ y\ z]$ moving in space with two degrees of freedom, u and w. We say that the point is a vector function of two independent parametric variables.

We can write

$$P(u,w) = [\, P_x(u,w)\ P_y(u,w)\ P_z(u,w)]$$

where the P_x, P_y, P_z are arbitrary functions. We can restrict the variables u and w to take on values between zero and one, simply to make the arithmetic more tractable. The vector quantities P(0,w) and P(1,w) then, are point loci with a single degree of freedom, represented by w, and are thus curves.

Similarly P(u,0) and P(u,1) are curves. These four curves define the boundaries of a surface segment or patch. Now we agree upon the following simplified notation:

$$\begin{aligned} P(u,w) &= uw \\ P(0,w) &= 0w \\ P(1,w) &= 1w \\ P(u,0) &= u0 \\ P(u,1) &= u1 \end{aligned}$$

We introduce some univariate functions and their special notation: we show it for u, but it also applied to the variable w:

$$\begin{aligned} F_0(u) &= F_0u \\ F_1(u) &= F_1u \\ G_0(u) &= G_0u \\ G_1(u) &= G_1u. \end{aligned}$$

Here we simply eliminate the parentheses. These have come to be known in the trade as "blending functions", because loosely speaking they "mix" or "blend" the shapes of boundary curves, to produce internal curves that define, or delineate, the surface.

A particular surface can now be defined as follows:

$$uw = [\,F_0u\ \ F_1u\ \ G_0u\ \ G_1u\,]\begin{bmatrix} 0w \\ 1w \\ 0w_u \\ 1w_u \end{bmatrix} + [\,u0\ \ u1\ \ u0_w\ \ u1_w\,]\begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}$$

$$-\ [\,F_0u\ \ F_1u\ \ G_0u\ \ G_1u\,]\begin{bmatrix} 00 & 01 & 00_w & 01_w \\ 10 & 11 & 10_w & 11_w \\ 00_u & 01_u & 00_{uw} & 01_{uw} \\ 10_u & 11_u & 10_{uw} & 11_{uw} \end{bmatrix}\begin{bmatrix} F_0w \\ F_1w \\ G_0w \\ G_1w \end{bmatrix}.$$

Gordon calls it a "Boolean sum surface" and he also calls it a "transfinite Lagrangian interpolant in two variables". He has extended it to interpolate curve networks.

In the equation, $u0$ $u1$ $0w$ and $1w$ are simply the vector functions that define the boundary curves. $u0_w$ $u1_w$ $0w_u$ and $1w_u$ are the normal vectors "across" these boundaries. Thus in particular, for example,

$$u0_w = \left.\frac{\partial(uw)}{\partial w}\right|_{w\,=\,0}.$$

The quantities in the square matrix (actually a tensor, since every element is a 3 component vector) are constants, obtainable from the boundary curve functions and the derivative functions. The partition in the lower right hand corner contains elements that are cross derivatives of the vectors. Thus, for instance

$$00_{uw} = \left.\frac{\partial^2 (uw)}{\partial u \partial w}\right|_{\substack{u=0\\w=0}} .$$

We have come to call such quantities the "twists" of the surface at the corners, because it rather well describes the geometric implications. We call this lower right hand partition of the matrix the "twist partition". It consists of four vectors, obtainable by differentiating, for instance, $0w_u$ with respect to w, and then subsequently setting $w = 0$ (or 1, of course) in the result. This would give 00_{uw} (or 01_{uw}).

It's clear that in some sense the square matrix (tensor) is redundant, since the boundary conditions already contain sufficient information to specify the corner conditions. We need to know something (but not very much about the blending functions F_0 F_1 G_0 and G_1. Accordingly, we will make some rather weak stipulations on these functions. Consider $F_i j$, a symbol that means $F_0 0$, or $F_0 1$, or $F_1 0$, or $F_1 1$. Similarly, consider $G_i j$, $i = 0,1$, and again with $j = 0,1$. Then

$$\begin{bmatrix} F_i j & F'_i j & F''_i j \\ G_i j & G'_i j & G''_i j \end{bmatrix} = \begin{bmatrix} \delta_{ij} & 0 & 0 \\ 0 & \delta_{ij} & 0 \end{bmatrix} .$$

δ_{ij} is of course the Kronecker delta symbol. The prime marks indicate differentiation with respect to the independent variable. We can draw pictures of F and G functions that satisfy these conditions, as in Figure 1.

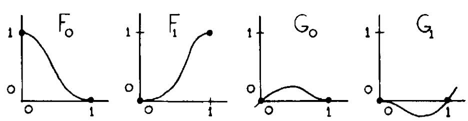

Figure 1

The "internal" shape of these functions is not immediately important; we only care about the behavior of the functions and their first and second derivatives at 0 and 1.

When the F and G functions obey the Kronecker delta conditions, it turns out that the resulting surface has the benign property of "containing" or passing through the boundary curves; with tangent vector functions "containing" the boundary derivative functions $u0_w$ $u1_w$ $0w_u$ and $1w_u$, and with <u>second</u> derivatives on the boundaries which are blended (or weighted) combinations of the second derivatives at the 0 and 1 ends of the boundary. We call such second derivative boundary vectors "intrinsic" to the surface. The implied consequence is that we can adjoin two such surfaces, and guarantee that they will be C^2 continuous (curvature continuous) across their mutually shared boundary. Two such patches, or an array, a mosaic, of such patches, thus provides a compound surface that is everywhere at least curvature continuous, provided only that the boundary curves are themselves everywhere curvature continuous.

A Special Default Condition.

If we do not want to specify the boundary tangent vectors, but wish to specify the corner twists, so as to avoid the pseudoflats that a null twist partition will yield, the surface patch equation becomes

$$uw = [\,F_0u \quad F_1u\,] \begin{bmatrix} 0w \\ 1w \end{bmatrix} + [\,u0 \quad u1\,] \begin{bmatrix} F_0w \\ F_1w \end{bmatrix}$$

$$- [\,F_0u \quad F_1u\,] \begin{bmatrix} 00 & 01 \\ 10 & 11 \end{bmatrix} \begin{bmatrix} F_0w \\ F_1w \end{bmatrix}$$

$$+ [\,G_0u \quad G_1u\,] \begin{bmatrix} 00_{uw} & 01_{uw} \\ 10_{uw} & 11_{uw} \end{bmatrix} \begin{bmatrix} G_0w \\ G_1w \end{bmatrix} .$$

The four vectors in the square tensor of the last term can be adjusted so as to remove the pseudo-flats.

B-Splines

Now it turns out that an extremely attractive curve form can define the $u0$ $u1$ $0w$ $1w$ $u0_w$ $u1_w$ $0w_u$ $1w_u$ boundary vector quantities. Since we plan to describe these curves in a somewhat simplistic way, we present the following formula to describe what we call "uniform cubic B-splines".

A point vector for a B-spline curve is:

$$P(u) = [\,s^3 \quad s^2 \quad s \quad 1\,]\ \frac{1}{6} \begin{bmatrix} -1 & 3 & -3 & 1 \\ 3 & -6 & 3 & 0 \\ -3 & 0 & 3 & 0 \\ 1 & 4 & 1 & 0 \end{bmatrix} \begin{bmatrix} v_i \\ v_{i+1} \\ v_{i+2} \\ v_{i+3} \end{bmatrix}$$