# Introduction to Asymptotics and Special Functions

F. W. J. Olver

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# PREFACE

This book comprises the first seven chapters of the author's *Asymptotics and Special Functions*. It is being published separately for the benefit of students needing only an introductory course to the subject. Since the chapters are self-contained they are reprinted without change of pagination; each of the few forward references to Chapters 8 to 14 that occur may be ignored, because the referenced matter is of a supplementary nature and does not affect the logical development. The *Answers to Exercises*, *References*, *Index of Symbols*, and *General Index* have been curtailed by omission of entries not pertaining to the first seven chapters, and to avoid confusion the letter A has been added to the page numbers assigned to these sections.

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# PREFACE TO ASYMPTOTICS AND SPECIAL FUNCTIONS

Classical analysis is the backbone of many branches of applied mathematics. The purpose of this book is to provide a comprehensive introduction to the two topics in classical analysis mentioned in the title. It is addressed to graduate mathematicians, physicists, and engineers, and is intended both as a basis for instructional courses and as a reference tool in research work. It is based, in part, on courses taught at the University of Maryland.

My original plan was to concentrate on asymptotics, quoting properties of special functions as needed. This approach is satisfactory as long as these functions are being used as illustrative examples. But the solution of more difficult problems in asymptotics, especially ones involving uniformity, necessitate the use of special functions as approximants. As the writing progressed it became clear that it would be unrealistic to assume that students are sufficiently familiar with needed properties. Accordingly, the scope of the book was enlarged by interweaving asymptotic theory with a systematic development of most of the important special functions. This interweaving is in harmony with historical development and leads to a deeper understanding not only of asymptotics, but also of the special functions. Why, for instance, should there be four standard solutions of Bessel's differential equation when any solution can be expressed as a linear combination of an independent pair? A satisfactory answer to this question cannot be given without some knowledge of the asymptotic theory of linear differential equations.

A second feature distinguishing the present work from existing monographs on asymptotics is the inclusion of error bounds, or methods for obtaining such bounds, for most of the approximations and expansions. Realistic bounds are of obvious importance in computational applications. They also provide theoretical insight into the nature and reliability of an asymptotic approximation, especially when more than one variable is involved, and thereby often avoid the need for the somewhat unsatisfactory concept of generalized asymptotic expansions. Systematic methods of error analysis have evolved only during the past decade or so, and many results in this book have not been published previously.

The contents of the various chapters are as follows. Chapter 1 introduces the basic concepts and definitions of asymptotics. Asymptotic theories of definite integrals containing a parameter are developed in Chapters 3, 4, and 9; those of ordinary linear differential equations in Chapters 6, 7, 10, 11, 12, and 13; those of sums and sequences in Chapter 8. Special functions are introduced in Chapter 2 and developed in most of the succeeding chapters, especially Chapters 4, 5, 7, 8, 10, 11, and 12. Chapter 5 also introduces the analytic theory of ordinary differential equations. Finally, Chapter 14 is a brief treatment of methods of estimating (as opposed to bounding) errors in asymptotic approximations and expansions.

An introductory one-semester course can be based on Chapters 1, 2, and 3, and the first parts of Chapters 4, 5, 6, and 7.† Only part of the remainder of the book can be covered in a second semester, and the selection of topics by the instructor depends on the relative emphasis to be given to special functions and asymptotics. Prerequisites are a good grounding in advanced calculus and complex-variable theory. Previous knowledge of ordinary differential equations is helpful, but not essential. A course in real-variable theory is not needed; all integrals that appear are Riemannian. Asterisks (*) are attached to certain sections and subsections to indicate advanced material that can be bypassed without loss of continuity. Worked examples are included in almost all chapters, and there are over 500 exercises of considerably varying difficulty. Some of these exercises are illustrative applications; others give extensions of the general theory or properties of special functions which are important but straightforward to derive. On reaching the end of a section the student is strongly advised to read through the exercises, whether or not any are attempted. Again, a warning asterisk (*) is attached to exercises whose solution is judged to be unusually difficult or time-consuming.

All chapters end with a brief section entitled *Historical Notes and Additional References*. Here sources of the chapter material are indicated and mention is made of places where the topics may be pursued further. Titles of references are collected in a single list at the end of the book. I am especially indebted to the excellent books of de Bruijn, Copson, Erdélyi, Jeffreys, Watson, and Whittaker and Watson, and also to the vast compendia on special functions published by the Bateman Manuscript Project and the National Bureau of Standards.

Valuable criticisms of early drafts of the material were received from G. F. Miller (National Physical Laboratory) and F. Stenger (University of Utah), who read the entire manuscript, and from R. B. Dingle (University of St. Andrews), W. H. Reid (University of Chicago), and F. Ursell (University of Manchester), who read certain chapters. R. A. Askey (University of Wisconsin) read the final draft, and his helpful comments included several additional references. It is a pleasure to acknowledge this assistance, and also that of Mrs. Linda Lau, who typed later drafts and assisted with the proof reading and indexes, and the staff of Academic Press, who were unfailing in their skill and courtesy. Above all, I appreciate the untiring efforts of my wife Grace, who carried out all numerical calculations, typed the original draft, and assisted with the proof reading.

† For this reason, the first seven chapters have been published by Academic Press as a separate volume, for classroom use, entitled *Introduction to Asymptotics and Special Functions*.

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# 1

## INTRODUCTION TO ASYMPTOTIC ANALYSIS

### 1 Origin of Asymptotic Expansions

**1.1** Consider the integral

$$F(x) = \int_0^\infty e^{-xt} \cos t \, dt \tag{1.01}$$

for positive real values of the parameter $x$. Let us attempt its evaluation by expanding $\cos t$ in powers of $t$ and integrating the resulting series term by term. We obtain

$$F(x) = \int_0^\infty e^{-xt} \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) dt \tag{1.02}$$

$$= \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} - \cdots. \tag{1.03}$$

Provided that $x > 1$ the last series converges to the sum

$$F(x) = \frac{x}{x^2+1}.$$

That the attempt proved to be successful can be confirmed by deriving the last result directly from (1.01) by means of two integrations by parts; the restriction $x > 1$ is then seen to be replaceable by $x > 0$.

Now let us follow the same procedure with the integral

$$G(x) = \int_0^\infty \frac{e^{-xt}}{1+t} dt. \tag{1.04}$$

We obtain

$$G(x) = \int_0^\infty e^{-xt}(1-t+t^2-\cdots)\, dt$$

$$= \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots. \tag{1.05}$$