The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations

Edited by

A. K. Aziz

The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations

ACADEMIC PRESS RAPID MANUSCRIPT REPRODUCTION

Proceedings of a Symposium
Held at the University of Maryland
Baltimore, Maryland
June 26–30, 1972

The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations

Edited by

A. K. Aziz

Division of Mathematics
University of Maryland
Baltimore, Maryland

and

Institute for Fluid Dynamics
and Applied Mathematics
University of Maryland
College Park, Maryland

Academic Press
New York and London
1972

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1

LIBRARY OF CONGRESS CATALOG CARD NUMBER: 72-82668

PRINTED IN THE UNITED STATES OF AMERICA

CONTENTS

CONTRIBUTORS ix
PREFACE xiii

Part I. Survey Lectures on the Mathematical Foundations of the Finite Element Method
Ivo Babuška and A. K. Aziz

Foreword 3
1. Preliminary Remarks 5
2. The Fundamental Notions 15
3. Properties of Solutions of Elliptic Boundary Value Problems 47
4. Theory of Approximation 83
5. Variational Principles 111
6. Rate of Convergence of the Finite Element Method 185
7. One Parameter Families of Variational Principles 243
8. Finite Element Method for Non-Smooth Domains and Coefficients 265
9. The Problems of Perturbations in the Finite Element Method 285
10. The Eigenvalue Problem 303
11. The Finite Element Method for Time Dependent Problems 345

Part II. Invited Hour Lectures

Piecewise Analytic Interpolation and Approximation in Triangulated Polygons 363
Garrett Birkhoff

Approximation of Steklov Eigenvalues of Non-Selfadjoint Second Order Elliptic Operators 387
James H. Bramble and John E. Osborn

The Combined Effect of Curved Boundaries and Numerical Integration in Isoparametric Finite Element Methods 409
P. G. Ciarlet and P.-A. Raviart

A Superconvergence Result for the Approximate Solution of the Heat Equation by a Collocation Method 475
Jim Douglas, Jr.

Some L^2 Error Estimates for Parabolic Galerkin Methods 491
Todd Dupont

Computational Aspects of the Finite Element Method 505
S. C. Eisenstat and M. H. Schultz

Effects of Quadrature Errors in Finite Element Approximation of Steady State, Eigenvalue, and Parabolic Problems 525
George J. Fix

Experience with the Patch Test for Convergence of Finite Elements 557
Bruce M. Irons and Abdur Razzaque

Higher Order Singularities for Interface Problems 589
R. B. Kellogg

On Dirichlet Problems Using Subspaces with Nearly Zero Boundary Conditions 603
J. Nitsche

Generalized Conjugate Functions for Mixed Finite Element Approximations of Boundary Value Problems 629
J. T. Oden

Finite Element Formulation by Variational Principles with Relaxed Continuity Requirements 671
Theodore H. H. Pian

Variational Crimes in the Finite Element Method 689
Gilbert Strang

Spline Approximation and Difference Schemes for the Heat Equation 711
Vidar Thomée

Part III. Short Communications

The Extension and Application of Sard Kernel Theorems
to Compute Finite Element Error Bounds 749
R. E. Barnhill, J. A. Gregory, and J. R. Whiteman

Two Types of Piecewise Quadratic Spaces and Their
Order of Accuracy for Poisson's Equation 757
Alan E. Berger

A Method of Galerkin Type Achieving Optimum L^2 Accuracy
for First Order Hyperbolics and Equations
of Schrödinger Type 763
J. E. Dendy, Jr.

Richardson Extrapolation for Parabolic Galerkin Methods 767
G. Fairweather and J. P. Johnson

Geometric Aspects of the Finite Element Method 769
William J. Gordon and Charles A. Hall

The Use of Interpolatory Polynomials for a Finite Element
Solution of the Multigroup Diffusion Equation 785
H. G. Kaper, G. K. Leaf, and A. J. Lindeman

A "Local" Basis of Generalized Splines over Right Triangles
Determined from a Nonuniform Partitioning of the Plane 791
Lois Mansfield

Least Square Polynomial Spline Approximation 793
Paul D. Patent

Subspaces with Accurately Interpolated
Boundary Conditions 797
Ridgway Scott

This page intentionally left blank

CONTRIBUTORS

Numbers in parentheses indicate the pages on which authors' contributions begin.

A. K. Aziz (3), Division of Mathematics, University of Maryland, Baltimore, Maryland 21228

Ivo Babuška (3), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742

R. E. Barnhill (749), Department of Mathematics, University of Utah, Salt Lake City, Utah 84112

Alan E. Berger (757), U. S. Naval Ordnance Laboratory, White Oak, Maryland 20910

Garrett Birkhoff (363), Department of Mathematics, Harvard University, Cambridge, Massachusetts 02138

James H. Bramble (387), Department of Mathematics, Cornell University, Ithaca, New York 14850

P. G. Ciarlet (409), Service de Mathématiques, Laboratoire Central des Ponts et Chaussées, 58, Boulevard Lefebvre, 75732 Paris Cedex 15, France

J. E. Dendy, Jr. (763), Department of Mathematics, University of Denver, Denver, Colorado 80220

Jim Douglas, Jr. (475), Department of Mathematics, University of Chicago, Chicago, Illinois 60637

Todd Dupont (491), Department of Mathematics, University of Chicago, Chicago, Illinois 60637

S. C. Eisenstat (505), Department of Computer Science, Yale University, New Haven, Connecticut 06520

G. Fairweather (767), Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

George J. Fix (525), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742

William J. Gordon (769), Mathematics Department, General Motors Research Laboratories, Warren, Michigan 48090

J. A. Gregory (749), Mathematics Department, Brunel University, Uxbridge, England

Charles A. Hall (769), Department of Mathematics, University of Pittsburgh, Pittsburgh, Pennsylvania 15213

Bruce M. Irons (557), Department of Civil Engineering, University of Wales, Swansea SA2 8PP, United Kingdom.

J. P. Johnson (767), Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506

H. G. Kaper (785), Applied Mathematics Division, Argonne National Laboratory, Argonne, Illinois 60439

R. B. Kellogg (589), Institute for Fluid Dynamics and Applied Mathematics, University of Maryland, College Park, Maryland 20742

G. K. Leaf (785), Applied Mathematics Division, Argonne National Laboratory, Argonne, Illinois 60439

A. J. Lindeman (785), Applied Mathematics Division, Argonne National Laboratory, Argonne, Illinois 60439

Lois Mansfield (791), Department of Computer Science, University of Kansas, Lawrence, Kansas 66044

J. Nitsche (603), Institute für Angewandte Mathematik der A-Ludwige-Universität, 78 Freiburg, West Germany

J. T. Oden (629), Department of Engineering Mechanics, University of Alabama, Huntsville, Alabama 35807

John E. Osborn (387), Department of Mathematics, University of Maryland, College Park, Maryland 20742

Paul D. Patent (793), Naval Undersea Research and Development Center, Pasadena, California 91107

Theodore H. H. Pian (671), Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

P.-A. Raviart (409), Analyse Numérique, Tour 55, Université de Paris VI, 11, Quai Saint-Bernard, 75230 Paris Cedex 05, France

Abdur Razzaque (557), Engineering College, Rajsahi, Bangladesh

*__A. H. Schatz__, Department of Mathematics, Cornell University, Ithaca, New York 14850

M. H. Schultz (505), Department of Computer Science, Yale University, New Haven, Connecticut 06520

Ridgway Scott (797), Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Gilbert Strang (689), Department of Mathematics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Vidar Thomée (711), Department of Mathematics, Chalmers Institute of Technology and the University of Göteborg, Göteborg, Sweden

J. R. Whiteman (749), Mathematics Department, Brunel University, Uxbridge, England

*Manuscript not received in time to be included.

This page intentionally left blank

PREFACE

The Symposium on Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations was held June 26-30, 1972, at the University of Maryland, Baltimore County Campus. Its purpose was to bring together a number of active numerical analysts currently involved in research in both theoretical and practical aspects of the finite element method. Among 250 participants were scientists from Canada, England, France, Germany, Ireland, Israel, Japan, Sweden, and Switzerland, thus providing the conference with a definite international flavor.

In recent years the scientific community, in particular the engineers, have focused considerable attention on the use of the finite element method. This is evidenced by the numerous national and international conferences held on this topic. In the last two years alone, 15 principal international conferences have been devoted to the finite element method, with the main emphasis on engineering applications.

As can be seen from the table of contents of these proceedings, the present symposium aims at bridging the gap between the mathematical and the practical aspects of the finite element method.

These proceedings consist of three parts. Part I gives the content of the 10 one-hour lectures given by Professor I. Babuška on the mathematical foundations of the field, while Part II contains all but one of the 16 one-hour lectures given by the invited speakers. These papers cover a large number of important results of both a theoretical and a practical nature. Part III contains the abstracts of 15-minute contributed talks.

The Division of Mathematics of the University of Maryland, Baltimore County Campus, and the U. S. Office of Naval Research were the joint sponsors of the symposium. The generous financial assistance of the U. S. Navy and the combined hard work of many members of the University of Maryland, faculty and staff, contributed immeasurably to the success of this meeting.

The editor wishes to express his sincere thanks to all these contributors. The advice and encouragement given by Professors I. Babuška, R. B. Kellogg, and G. J. Fix have been particularly helpful.

The Mathematical Foundations of the Finite Element Method with Applications to Partial Differential Equations

This page intentionally left blank

PART I

SURVEY LECTURES ON THE MATHEMATICAL FOUNDATIONS OF THE FINITE ELEMENT METHOD

Ivo Babuška and A. K. Aziz
with the collaboration of
G. Fix and R. B. Kellogg

This page intentionally left blank

FOREWORD

As indicated in the preface, numerous meetings and conferences in the last few years have been devoted to recent developments in theory and applications of the finite element method. The symposium at the University of Maryland, Baltimore County Campus, differed from the others in its orientation which was exclusively directed toward the theoretical foundations of the method.

To this end the organizing committee, which consisted of A.K. Aziz (Chairman), I. Babuška, N.P. Bhatia, and R.B. Kellogg, included in the program a series of ten lectures dealing exclusively with basic theoretical concepts.

The notes which follow are an attempt to focus on some of the most important of these principles. In their preparation, some constraints were imposed by the nature and goals of the conference. Therefore, they should be considered as notes describing the content of the ten lectures and not as a monograph on the finite element method.

The notes were prepared by I. Babuška[1] and A.K. Aziz[2] with significant contributions and help provided by R.B. Kellogg[3] and G. Fix.[4] In particular, Chapter 3 on regularity of the solution was written by R.B. Kellogg; Chapter 11 and parts of Chapter 10 were written jointly with G. Fix.

This work was supported in part by:
[1]The Atomic Energy Commission, Contract No. AEC AT (40-1) 3443/5.
[2]The Naval Ordnance Laboratory Independent Research fund, and the Office of Naval Research, Contract No. NR-044-453.
[3]The National Science Foundation, Grant No. NSF GP 20555.
[4]The National Science Foundation, Grant No. NSF GP 18064.

We would like to express our gratitude to Professors Kellogg and Fix for their valuable assistance and understanding. We must not fail also to thank Mrs. L. Lau and Mrs. M. Small for their generous services, including the typing of the manuscript.

I. BABUŠKA and A. K. AZIZ

College Park
September 1972

CHAPTER 1. PRELIMINARY REMARKS

1.1. Introduction.

The development of approximate methods for the numerical solution of partial differential equations has attracted attention of mathematicians, physicists and engineers for a long time. The methods, their mathematical foundation and their implementation have had a considerable impact on theoretical mathematics and on the level of sophistication of computational aids. Roughly speaking we may distinguish two stages in this development, namely the precomputer period and the computer era.

An excellent survey of the state of the art concerning the numerical solution of differential equations, up to the end of the precomputer period may be found in [1] and [2].

The application to partial differential equations is exemplified by the work of Southwell (see [3]) and others.

The advent of the computer served as a stimulus for new ideas in theory and in applications. Engineers, physicists and others have suggested many sophisticated, but often not theoretically founded, methods; nonetheless a great deal of experience has been gained. Mathematics has played a very central and essential role in this development, in particular in the understanding of the theoretical foundation of these methods, their applications and further very effective and new computer oriented methods have been devised by the numerical analysts. It can be stated without exaggeration that the present day numerical methods are essentially different from those of the precomputer period, and are primarily based on the achievements in the last 15 to 20 years.

The development of numerical methods for partial differential equations was particularly influenced by the innovations brought about by the computer era. One of the typical by-product is the present day's frequent use of the finite element method for the numerical solution of partial differential equations especially as far as the elliptic partial differential equations are concerned.

In recent years considerable attention has been focused on the finite element method primarily in the engineering literature. In the last few years an increasing number of papers on the finite element methodhas also appeared in the numerical analysis literature.

An extensive bibliography may be found in [4], [10] and [11]. For an excellent account of the interplay of mathematical and engineering ideas in the finite element method the reader is referred to [5]. [6] may be consulted for a mathematical formulation and [12] for a survay of the method.

The attention accorded to the finite element method by the scientific community is further evidenced by numerous national and international conferences held on this topic. In the last two years alone, 15 principal international conferences have been devoted to the finite element method, not counting a large number of small symposia, workshops and short courses in this field.

1.2. Numerical solution of partial differential equations.

In numerically solving a partial differential equation one first expresses approximately the solution by a finite number of parameters. Since, in general the solution is sought in a given class of functions, it is essential that one be able to express any function of this class in terms of a finite number of parameters, with a reasonable accuracy. Further details in this connection may be found in [7] and [8].

Secondly, we need to transform the given differential operator to expressions relating these parameters. For a general abstract approach the reader may consult chapter 14 of [9]. If the differential operator is linear then, in general, the relations among the parameters expressing the solution is also linear i.e., we are led to a linear system of algebraic equations. However, as supported by the general theory (see [7], [8]), in this process one cannot avoid dealing with a large number of parameters of order of at least a hundred or a thousand. Moreover, we need to determine the coefficients of the matrix, the number of which may be of order 10^4 or 10^6.

To avoid this complication it is necessary (but not sufficient) to choose the parameters in such a way that the resulting matrix is sparse. If we are seeking an approach which is applicable to the problems encountered in general practical cases, there are other restrictions too.

One of the most successful methods reflecting these features and other important aspects is the finite element method.

1.3. Finite Element Method.

The name finite element method was invented by engineers (see [4]). We will be interested in a slightly more general version which we feel reflects all essential theoretical features of the classical finite element method. We shall call it the general finite element method. For the sake of brevity we will not in the sequel emphasize the word "general" nevertheless we shall always interpret it in this general context.

A cursory analysis of the so called finite element method reveals that there are two principles which appear to be essential from a mathematical point of view.

1) The choice of local parameters of the solution.
2) The use of various types of variational principles for transforming the given equations to relations among the parameters of the solution.

By the choice of local parameters it is understood that if the approximate solution u is expressed in terms of the parameters α_i as $u = \sum \alpha_i \phi_i$, then the base functions ϕ_i have only small supports and if the number of the parameters is increased then the support of ϕ_i is decreased. A typical case is the introduction of the parameters which describe the set S_h of the piecewise linear function (see Fig. 1.3.1).

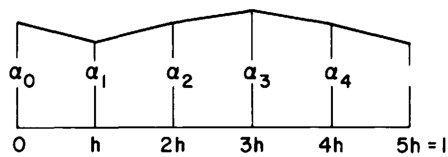

Fig. 1.3.1

It is clear that a change in one parameter, say α_3, changes the function only in the interval $\langle 2h,4h \rangle$. Obviously, for the set of piecewise linear functions, parameters may be introduced which are not local. There is a question as to whether one can find local parameters for every finite dimensional subspace. To answer this question consider the subspace $S_h = \sum_0^{1/h} \alpha_i \cos ix$ of all trigonometric polynomials. Indeed here we have quite a different situation. Thus, the assumption concerning the possibility of the introduction of local parameters for the solution means the possibility of a suitable selection of a finite dimensional subspace and a proper choice of the basis.

We remark that the term variational principle, in general, is understood in a broad sense. We interpret this term in a narrow sense. More precisely, we use the variational principle to mean that its application to a linear differential equation, transforms the given equation to a system of linear algebraic equations with the matrix M, provided the space S_h of possible approximate solutions is linear. The elements of M are given by the bilinear form $B(\phi_i,\phi_j)$, where ϕ_i and ϕ_j are basis functions. Moreover, we assume that the bilinear form is such that whenever ϕ_i and ϕ_j have disjoint support then $B(\phi_i,\phi_j) = 0$.

This is what we generally understand when we say that the differential equation has been transformed to relations among the parameters of the solution, by the use of the variational principle. Under our assumptions the matrix M is sparse and the number of entries in the rows of the matrix is relatively very small, in fact is most frequently independent of the size of the matrix.

The introduction of the parameters and the construction of the matrix (variational principle) have very important physical significance in every stage in the application of the finite element method to engineering problems. This feature has greatly contributed to the widespread use of the finite element method. However, from the theoretical point of view, this aspect does not appear to be so significant.

1.4. The sources of the theory of the finite element method.

The theory of the finite element method in all its complexity is based on the fundamental knowledge in different fields. We feel that the heart of the theory of the finite element method lies in the following sources.

1) Functional analytic theory of partial differential equations.
2) Theory of approximation by piecewise

polynomial functions.

3) Computer science.
4) Concrete applications in different fields such as mechanics etc.

It is clear that all these sources have contributed greatly to the present day development of the finite element method. It would be impossible (and would not serve a useful purpose) if we tried to trace the different sources with important impact on the method, other than the advent of computers.

1.5. The mathematical foundations of the finite element method.

The main purpose of these notes is to assemble some basic results, and to treat them from a unified point of view.

These notes do not pretend to be complete, nevertheless they aim to develop the fundamentals of the method in such a way that it may be applied to a fairly large number of problems in different fields. For obvious limitations the discussion has been confined to some important preliminaries and basic ideas.

The aim of these notes is the development of a general and complete theory. Our goal is to underline the basic ideas and illustrate them by considering a selective number of examples. The chosen examples are very simple in appearance; however, they exhibit the typical characteristic of the problems encountered in general applications and at the same time, because of the simplicity of the form of the equations, many difficulties of a technical nature are avoided. For the above reason we have confined our discussion to two dimensional problems only. The setting of the problems is in a general framework and the approach utilized admits in all cases immediate and far reaching generalizations. In some cases a more special treatment could prove to be shorter and easier, but without a general character.

The model problems are selected not merely because of their mathematical interest, but also for their relevance from the point of view of applications. Therefore we have not restricted ourselves to the case of self-adjoint problems. In mechanics in the case of the second order equations it is natural to consider the self-adjoint form, but this is not the case in the diffusion problems e.g., diffusion in a moving medium etc.

References.

[1]. L. Collatz, "Numerische Behandlung von Differentialgleichungen, Springer-Verlag, 1951.

[2]. L. V. Kantorovich, V. J. Krylov, "Näherungsmethoden der Höheren Analysis, Berlin 1956.

[3]. F. S. Show, "An introduction to relaxation methods," New York, Dover Publ. Inc., 1953.

[4]. O. C. Zienkiewicz, "The finite element method: from intuition to generality," Applied Mechanics Reviews 23 (1970), pp. 249-256.

[5]. G. Strang, G. Fix, "An analysis of the finite element method," Prentice Hall, New York (to appear 1973).

[6]. J. P. Aubin, "Approximation of elliptic boundary-value problems," Wiley-Interscience, New York, London, Sydney, Toronto, 1972.

[7]. I. Babuška, S. L. Sobolev, "The Optimization of Numerical Processes," Apl. Mat. 10 (1965), pp. 96-129 (in Russian).

[8]. A. G. Vituskin, "Estimates from Tabulation Theory (in Russian), Moscow 1959.

[9]. L. V. Kantorovich, G. P. Akilov, "Functional analysis in Normed Spaces," The MacMillian Co., New York 1964.

[10]. J. E. Abin, D. L. Fenton, W. C. T. Stoddart, "The Finite Element Method -- A bibliography of its Theory and Applications." Report EM 72-1, Department of Engineering Mechanics, the University of Tennessee Knoxville, 1971.

[11]. J. R. Whiteman, "A Bibliography for Finite Element Method," Brunel University. Report TR/9, Department

of Mathematics, Brunel University, Uxbridge, 1972.

[12]. J. T. Oden, "Some Aspects of the Mathematical Theory of Finite Elements", pp. 3-38, Advances in Computation Computational Methods and Structural Mechanics and Design, edited by J. T. Oden, R. W. Clough, Y. Yamamoto, UAH Press, the University of Alabama in Huntsville, Huntsville, Alabama.

CHAPTER 2. THE FUNDAMENTAL NOTIONS.

2.1. Introduction.

In these notes for the sake of technical simplicity we shall be concerned primarily with problems in two dimensions, but the approach used will be of quite general character. The properties of the domains considered play a fundamental role as far as the behavior of the solutions of partial differential equations are concerned. In the sequel we will be mainly interested in bounded domains, and of course in the entire plane R_2 . First, we shall classify more precisely the type of domains which will be considered in this sequel.

2.2 Domains.

Let R_2 denote the two dimensional Euclidean space. For $x \equiv (x_1,x_2) \in R_2$ we use the notations:

$$\|x\| = \sqrt{x_1^2+x_2^2} \quad \text{and} \quad |x| = \max(|x_1|,|x_2|) \quad .$$

Further let Ω be a given bounded domain, with the boundary $\Omega^{\bullet}$ which fulfills the following assumptions. There exists a system of local coordinates $x_i^{[s]}$, $i = 1,2$, $s = 1,2,\cdots,\nu$ and open intervals $I_s \in R_1$ and $\overline{I}'_s \subset I_s$, and functions ϕ_s defined on I_s which induce a mapping

χ_s of I_s such that

$$(2.2.1) \qquad \chi_s(I_s) = \Omega_s^{\bullet} = \left\{\left(x_1^{[s]}, \phi_s(x_1^{[s]})\right) \middle| x_1^{[s]} \in I_s\right\}$$

and
$$\bigcup_{s=1}^{\nu} \Omega_s^{\bullet\prime} = \Omega^{\bullet} ,$$

where

$$(2.2.2) \qquad \Omega_s^{\bullet\prime} = \left\{\left(x_1^{[s]}, \phi_s(x_1^{[s]})\right) \middle| x_1^{[s]} \in I_s'\right\} .$$

with Ω locally on one side of $\Omega^{\bullet}$'s.
We shall say that the domain Ω is smooth if all functions ϕ_s have derivatives of all orders. The domain Ω is called Lipschitzian if every function ϕ_s satisfies a Lipschitz condition.

We remark that in many of our considerations we do not need that ϕ_s be infinitely differentiable, we merely require that they be sufficiently smooth. Nevertheless in order to avoid technical difficulties we shall always assume that ϕ_s are infinitely differentiable, whenever we use the term smooth domain. The Lipschitzian domain would be the most general domain with which we shall deal. In this setting inevitably we exclude some important domains such as those shown in Fig. 2.2.1.

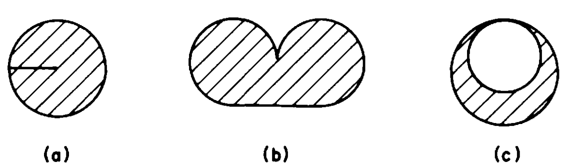

Fig. 2.2.1

or a more general domain.

We shall also deal with some special domains where the functions ϕ_s are merely piecewise smooth.

2.3. Sobolev spaces $H^\ell(\Omega)$.

Consider a Lipschitz domain Ω and let $\mathcal{E}(\overline{\Omega})$ be the space of all [real] infinitely differentiable functions on Ω such that all the derivatives have continuous extensions to $\Omega^\bullet$. Furthermore, denote by $\mathcal{D}(\Omega) \subset \mathcal{E}(\overline{\Omega})$ the subspace consisting of all functions with compact support in Ω . As usual let $L_2(\Omega)$ be the space of square integrable functions u on Ω with the norm

$$\|u\|^2_{L_2(\Omega)} = \int_\Omega u^2 \, dx \quad , \tag{2.3.1}$$

where $dx = dx_1 dx_2$. The scalar product will be denoted as $(u,v)_{L_2(\Omega)}$. Sometimes we shall use the notation

$L_2(\Omega) = H^\circ(\Omega)$.

Suppose now that $\ell \geq 1$ is an integer. The Sobolev space $H^\ell(\Omega)$ (respectively $H^\ell_0(\Omega)$) will be defined as the closure of $\mathcal{E}(\overline{\Omega})$ (respectively $\mathcal{D}(\Omega)$) in the norm $\|\cdot\|_{H^\ell(\Omega)}$, where

$$\|u\|^2_{H^\ell(\Omega)} = \sum_{0 \leq |\alpha| \leq \ell} \|D^\alpha u\|^2_{L_2(\Omega)} \tag{2.3.2}$$

and

$$(2.3.3) \qquad D^\alpha = \frac{\partial^{\alpha_1+\alpha_2}}{\partial x_1^{\alpha_1}\partial x_2^{\alpha_2}} , \quad \alpha = (\alpha_1, \alpha_2)$$

$$|\alpha| = \sum_i \alpha_i \; .$$

(α_i are non-negative integers.)

We now define the Sobolev spaces with fractional derivatives which were introduced by Aronszajn [1] and Slobodetskiĭ [2] (see also [3]). For $0 < \alpha = [\alpha] + \sigma$, $0 < \sigma < 1$ and $[\alpha]$ = integral part of α, we define for $u \in \mathcal{E}(\bar{\Omega})$

$$(2.3.4) \quad \|u\|^2_{H^\alpha(\Omega)} = \|u\|^2_{H^{[\alpha]}(\Omega)} + \sum_{|K|=[\alpha]} \|D^K u\|^2_{H^\sigma(\Omega)} ,$$

where

$$(2.3.5) \qquad \|u\|^2_{H^\sigma(\Omega)} = \int_\Omega\int_\Omega \frac{[u(t)-u(T)]^2}{\|t-T\|^{2+2\sigma}}\, dt\,dT ,$$

and $H^\alpha(\Omega)$ is defined as the closure of $\mathcal{E}(\bar{\Omega})$ in the norm (2.3.4). Similary $H_0^\alpha(\Omega)$ will be defined as the closure of $\mathcal{D}(\Omega)$ in the norm defined by (2.3.4). In this definition of $H_0^\alpha(\Omega)$ the value $\alpha = n + \frac{1}{2}$, where n is an integer is excluded. For this value of α the norm must be defined differently.

The spaces $H^\alpha(\Omega)$ respectively $H_0^\alpha(\Omega)$ are obviously Hilbert spaces. We may define the space $H^{\alpha}(\Omega)$ and $H_0^{\alpha}(\Omega)$ also for negative value of α as the closure of $E(\overline{\Omega})$ in the norm $\|\cdot\|_{H^{\alpha}(\Omega)}$ where for $\alpha < 0$

$$(2.3.6) \qquad \|u\|_{H^{\alpha}(\Omega)} = \sup_{v \in H^{-\alpha}(\Omega)} \frac{\int_{\Omega} uv\,dx}{\|v\|_{H^{-\alpha}(\Omega)}} ,$$

and the norm in $H_0^{\alpha}(\Omega)$ is similarly define. Obviously we may identify $H^{\alpha}(\Omega)$ and $H_0^{\alpha}(\Omega)$, $\alpha < 0$ with the dual spaces to $H^{-\alpha}(\Omega)$ and $H_0^{-\alpha}(\Omega)$ respectively.

We define the Sobolev spaces for a bounded domain Ω , our definition is also valid for $\Omega = R_2$. In this case we may introduce the norm also by means of Fourier transform.

Let $u \in L_2(R_2)$ be given and

$$(Fu)(\sigma) = \hat{u}(\sigma) = \int_{R_2} e^{i\langle\sigma, x\rangle} u(x)\,dx ,$$

where $\sigma = (\sigma_1, \sigma_2)$ and $\langle\sigma, x\rangle = \sigma_1 x_1 + \sigma_2 x_2$. It is well known that

$$\|Fu\|^2_{L_2^c(R_2)} = (2\pi)^2 \|u\|^2_{L_2(R_2)} ,$$

where $L_2^c(R_2)$ is the usual space of complex square integrable functions.

Therefore for ℓ an integer

$$(2.3.7) \quad \|u\|^2_{H^\ell(\Omega)} = (2\pi)^{-2} \int_{R_2} |(Fu)(\sigma)|^2 \left[1+\mu^2+\cdots+\mu^{2\ell}\right] d\sigma ,$$

with $\mu^2 = \sigma_1^2 + \sigma_2^2$.

This norm is obviously equivalent to the norm $\|u\|_{H_1^\ell(\Omega)}$ given by

$$(2.3.8) \quad \|u\|^2_{H_1^\ell(\Omega)} = (2\pi)^{-2} \int_{R_2} |(Fu)(\sigma)|^2 \left[1+\mu^2\right]^\ell d\sigma .$$

We may define also the space $H_1^\ell(R_2)$ for non-integral values of ℓ by formula (2.3.8). By direct computation it is possible to verify that

$\|u\|_{H^\ell(R_2)} \approx \|u\|_{H_1^\ell(R_2)}$ [1), for all ℓ. (We note that the

1) Norms $\|\cdot\|_{H_1}$ and $\|\cdot\|_{H_2}$ are said to be equivalent if there exist constants $0 < C_1 < C_2 < \infty$ such that

$$C_1\|u\|_{H_1} \leq \|u\|_{H_2} \leq C_2\|u\|_{H_1} .$$

We abreviate this by writing $\|\cdot\|_{H_1} \approx \|\cdot\|_{H_2}$. Throughout these notes C denotes a generic constant with possibly different values in different contexts.

constants C_1 and C_2 appearing in the definition of the equivalent norms are not uniformly bounded in the case of the value ℓ lies between two integers.)

We shall list some basic well known properties of the Sobolev spaces. For a survey of the recent results in the theory of Sobolev spaces the reader is referred to [4].

Obviously for $\ell_1 \leq \ell_2$, $H^{\ell_1}(\Omega) \supset H^{\ell_2}(\Omega)$, with $\|\cdot\|_{H^{\ell_1}(\Omega)} \leq c\|\cdot\|_{H^{\ell_2}(\Omega)}$, i.e., the imbedding of $H^{\ell_2}(\Omega)$ into $H^{\ell_1}(\Omega)$ is continuous. Moreover for $\ell_1 < \ell_2$ and Ω bounded this imbedding is also compact, i.e., the unit sphere in $H^{\ell_2}(\Omega)$ is compact in $H^{\ell_1}(\Omega)$.

If $u \in H^{\ell}(\Omega)$, $\ell > 1$, then u is continuous and $\|u\|_c \leq C\|u\|_{H^{\ell}(\Omega)}$ where $\|u\|_c = \max_{x\in\Omega}|u(x)|$ [1]. [In n dimensional case we need to require that $\ell > \frac{n}{2}$]. The continuity assertion of u does not hold for $\ell = 1$.

The spaces $H^{\ell}(\Omega)$ as we have defined generate a Hilbert scale, more precisely they are equivalent to a Hilbert scale. For a detailed study of Hilbert scales the reader may consult [5].

---

1) Since the norm in H^{ℓ} is given by an integral in the sense of Lebesgue function u is continuous after possible change of its values on the set of measure zero.

In what follows we shall describe briefly the main ideas involved in a Hilbert scale which connects the space $H^{\ell_1}(\Omega)$ with the space $H^{\ell_2}(\Omega)$, $\ell_2 > \ell_1$.

For $v \in H^{\ell_1}(\Omega)$ we define the functional $\ell \in [H^{\ell_2}(\Omega)]'$ so that for $w \in H^{\ell_2}(\Omega)$ we have

(2.3.8) $$\ell(w) = (w,v)_{H^{\ell_1}(\Omega)} ,$$

and may write

(2.3.9) $$\ell(w) = (w,z)_{H^{\ell_2}(\Omega)} = (w,Vv)_{H^{\ell_2}(\Omega)}$$

The operator V may be considered as an operator in $H^{\ell_2}(\Omega)$ or in $H^{\ell_1}(\Omega)$. In both cases the operator is bounded, self-adjoint, positive, and in $H^{\ell_1}(\Omega)$, V is a compact operator.

Denoting by $V^{\frac{1}{2}}$ the positive square root (which exists and it is uniquely determined), it is easy to show that $V^{\frac{1}{2}}$ induces an isometric isomorphism of the spaces $H^{\ell_1}(\Omega)$ and $H^{\ell_2}(\Omega)$, the operator $V^{\frac{1}{2}}$ maps $H^{\ell_1}(\Omega)$ into $H^{\ell_2}(\Omega)$, with $(u,v)_{H^{\ell_1}(\Omega)} = (V^{\frac{1}{2}}u,V^{\frac{1}{2}}v)_{H^{\ell_2}(\Omega)}$ and its inverse $V^{-\frac{1}{2}}$ maps $H^{\ell_2}(\Omega)$ onto $H^{\ell_1}(\Omega)$ with

$$(u,v)_{H^{\ell_2}(\Omega)} = (V^{-\frac{1}{2}}u, V^{-\frac{1}{2}}v)_{H^{\ell_1}(\Omega)} .$$

By denoting $T = V^{-\frac{1}{2}}$ we may write

$$V = T^{-2} = \sum_{k=1}^{\infty} \lambda_k^2 P_k , \tag{2.3.10}$$

where λ_k are the eigenvalues and P_k are the projection operators onto the subspaces of eigenfunctions g_k of the equation

$$Vu = \lambda_k u .$$

We know that $\lambda_{k+1} \leq \lambda_k$ and $0 < \lambda_k \to 0$ as $k \to \infty$. Thus we have

$$T^2 = \sum_{k=1}^{\infty} \lambda_k^{-2} P_k . \tag{2.3.11}$$

We have

$$\|u\|_{H^{\ell_2}(\Omega)} = \|Tu\|_{H^{\ell_1}(\Omega)} , \tag{2.3.12}$$

and for $\ell_1 \leq \gamma \leq \ell_2$, we define

$$\|u\|_{H_\gamma(\Omega)} = \|T^{\frac{\gamma-\ell_1}{\ell_2-\ell_1}} u\|_{H^{\ell_1}(\Omega)} . \tag{2.3.13}$$

Now if

$$u = \sum_{k=1}^{\infty} a_k g_k \quad . \tag{2.3.14}$$

then

$$\|u\|^2_{H^{\ell_1}(\Omega)} = \sum_{k=1}^{\infty} |a_k|^2 \quad , \tag{2.3.15}$$

$$\|u\|^2_{H^{\ell_2}(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{-2}|a_k|^2 \quad , \tag{2.3.16}$$

$$\|u\|^2_{H_\gamma(\Omega)} = \sum_{k=1}^{\infty} \lambda_k^{-2\frac{\gamma-\ell_1}{\ell_2-\ell_1}}|a_k|^2 \quad , \tag{2.3.17}$$

and hence we see that

$$\|u\|_{H_\gamma(\Omega)} \leq \|u\|^{\frac{\ell_2-\gamma}{\ell_2-\ell_1}}_{H^{\ell_1}(\Omega)} \cdot \|u\|^{\frac{\gamma-\ell_1}{\ell_2-\ell_1}}_{H^{\ell_2}(\Omega)} \tag{2.3.18}$$

It is possible to show that $|\cdot|_{H_\gamma(\Omega)}$ is equivalent to the norm $|\cdot|_{H^\gamma(\Omega)}$ (see [5]) introduced earlier.

We notice that when connecting any two spaces $H^{\gamma_1}(\Omega)$ and $H^{\gamma_2}(\Omega)$, $\ell_1 < \gamma_1 < \gamma_2 < \ell_2$, by a Hilbert scale we obtain equivalent spaces to those described above.