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ASYMPTOTICS AND SPECIAL FUNCTIONS

F. W. J. Olver

Institute for Fluid Dynamics and Applied Mathematics
University of Maryland
College Park, Maryland

and

National Bureau of Standards
Washington, D.C.

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To the memory of
my daughter Linda (1953–1965)

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PREFACE

Classical analysis is the backbone of many branches of applied mathematics. The purpose of this book is to provide a comprehensive introduction to the two topics in classical analysis mentioned in the title. It is addressed to graduate mathematicians, physicists, and engineers, and is intended both as a basis for instructional courses and as a reference tool in research work. It is based, in part, on courses taught at the University of Maryland.

My original plan was to concentrate on asymptotics, quoting properties of special functions as needed. This approach is satisfactory as long as these functions are being used as illustrative examples. But the solution of more difficult problems in asymptotics, especially ones involving uniformity, necessitate the use of special functions as approximants. As the writing progressed it became clear that it would be unrealistic to assume that students are sufficiently familiar with needed properties. Accordingly, the scope of the book was enlarged by interweaving asymptotic theory with a systematic development of most of the important special functions. This interweaving is in harmony with historical development and leads to a deeper understanding not only of asymptotics, but also of the special functions. Why, for instance, should there be four standard solutions of Bessel's differential equation when any solution can be expressed as a linear combination of an independent pair? A satisfactory answer to this question cannot be given without some knowledge of the asymptotic theory of linear differential equations.

A second feature distinguishing the present work from existing monographs on asymptotics is the inclusion of error bounds, or methods for obtaining such bounds, for most of the approximations and expansions. Realistic bounds are of obvious importance in computational applications. They also provide theoretical insight into the nature and reliability of an asymptotic approximation, especially when more than one variable is involved, and thereby often avoid the need for the somewhat unsatisfactory concept of generalized asymptotic expansions. Systematic methods of error analysis have evolved only during the past decade or so, and many results in this book have not been published previously.

The contents of the various chapters are as follows. Chapter 1 introduces the basic concepts and definitions of asymptotics. Asymptotic theories of definite integrals containing a parameter are developed in Chapters 3, 4, and 9; those of ordinary linear differential equations in Chapters 6, 7, 10, 11, 12, and 13; those of sums and sequences in Chapter 8. Special functions are introduced in Chapter 2 and developed in most of the succeeding chapters, especially Chapters 4, 5, 7, 8, 10, 11, and 12. Chapter 5 also introduces the analytic theory of ordinary differential equations. Finally, Chapter 14 is a brief treatment of methods of estimating (as opposed to bounding) errors in asymptotic approximations and expansions.

An introductory one-semester course can be based on Chapters 1, 2, and 3, and the first parts of Chapters 4, 5, 6, and 7.† Only part of the remainder of the book can be covered in a second semester, and the selection of topics by the instructor depends on the relative emphasis to be given to special functions and asymptotics. Prerequisites are a good grounding in advanced calculus and complex-variable theory. Previous knowledge of ordinary differential equations is helpful, but not essential. A course in real-variable theory is not needed; all integrals that appear are Riemannian. Asterisks (*) are attached to certain sections and subsections to indicate advanced material that can be bypassed without loss of continuity. Worked examples are included in almost all chapters, and there are over 500 exercises of considerably varying difficulty. Some of these exercises are illustrative applications; others give extensions of the general theory or properties of special functions which are important but straightforward to derive. On reaching the end of a section the student is strongly advised to read through the exercises, whether or not any are attempted. Again, a warning asterisk (*) is attached to exercises whose solution is judged to be unusually difficult or time-consuming.

All chapters end with a brief section entitled *Historical Notes and Additional References*. Here sources of the chapter material are indicated and mention is made of places where the topics may be pursued further. Titles of references are collected in a single list at the end of the book. I am especially indebted to the excellent books of de Bruijn, Copson, Erdélyi, Jeffreys, Watson, and Whittaker and Watson, and also to the vast compendia on special functions published by the Bateman Manuscript Project and the National Bureau of Standards.

Valuable criticisms of early drafts of the material were received from G. F. Miller (National Physical Laboratory) and F. Stenger (University of Utah), who read the entire manuscript, and from R. B. Dingle (University of St. Andrews), W. H. Reid (University of Chicago), and F. Ursell (University of Manchester), who read certain chapters. R. A. Askey (University of Wisconsin) read the final draft, and his helpful comments included several additional references. It is a pleasure to acknowledge this assistance, and also that of Mrs. Linda Lau, who typed later drafts and assisted with the proof reading and indexes, and the staff of Academic Press, who were unfailing in their skill and courtesy. Above all, I appreciate the untiring efforts of my wife Grace, who carried out all numerical calculations, typed the original draft, and assisted with the proof reading.

† For this reason, the first seven chapters have been published by Academic Press as a separate volume, for classroom use, entitled *Introduction to Asymptotics and Special Functions*.

1

INTRODUCTION TO ASYMPTOTIC ANALYSIS

1 Origin of Asymptotic Expansions

1.1 Consider the integral

$$F(x) = \int_0^\infty e^{-xt} \cos t \, dt \tag{1.01}$$

for positive real values of the parameter x. Let us attempt its evaluation by expanding $\cos t$ in powers of t and integrating the resulting series term by term. We obtain

$$F(x) = \int_0^\infty e^{-xt}\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cdots\right) dt \tag{1.02}$$

$$= \frac{1}{x} - \frac{1}{x^3} + \frac{1}{x^5} - \cdots. \tag{1.03}$$

Provided that $x > 1$ the last series converges to the sum

$$F(x) = \frac{x}{x^2+1}.$$

That the attempt proved to be successful can be confirmed by deriving the last result directly from (1.01) by means of two integrations by parts; the restriction $x > 1$ is then seen to be replaceable by $x > 0$.

Now let us follow the same procedure with the integral

$$G(x) = \int_0^\infty \frac{e^{-xt}}{1+t} \, dt. \tag{1.04}$$

We obtain

$$G(x) = \int_0^\infty e^{-xt}(1 - t + t^2 - \cdots) \, dt$$

$$= \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \frac{3!}{x^4} + \cdots. \tag{1.05}$$

This series diverges for all finite values of x, and therefore appears to be meaningless.

Why did the procedure succeed in the first case but not in the second? The answer is not hard to find. The expansion of $\cos t$ converges for all values of t; indeed it converges uniformly throughout any bounded t interval. Application of a standard theorem concerning integration of an infinite series over an infinite interval† confirms that the step from (1.02) to (1.03) is completely justified when $x > 1$. In the second example, however, the expansion of $(1+t)^{-1}$ diverges when $t \geqslant 1$. The failure of the representation (1.05) may be regarded as the penalty for integrating a series over an interval in which it is not uniformly convergent.

1.2 If our approach to mathematical analysis were one of unyielding purity, then we might be content to leave these examples at this stage. Suppose, however, we adopt a heuristic approach and try to sum the series (1.05) numerically for a particular value of x, say $x = 10$. The first four terms are given by

$$0.1000 - 0.0100 + 0.0020 - 0.0006, \tag{1.06}$$

exactly, and the sum of the series up to this point is 0.0914. Somewhat surprisingly this is very close to the correct value $G(10) = 0.09156\ldots$.‡

To investigate this unexpected success we consider the difference $\varepsilon_n(x)$ between $G(x)$ and the nth partial sum of (1.05), given by

$$\varepsilon_n(x) = G(x) - g_n(x),$$

where

$$g_n(x) = \frac{1}{x} - \frac{1!}{x^2} + \frac{2!}{x^3} - \cdots + (-)^{n-1}\frac{(n-1)!}{x^n}.$$

Here n is arbitrary, and $\varepsilon_n(x)$ is called the *remainder term*, *error term*, or *truncation error* of the partial series, or, more precisely, the nth such term or error. Since

$$\frac{1}{1+t} = 1 - t + t^2 - \cdots + (-)^{n-1}t^{n-1} + \frac{(-)^n t^n}{1+t},$$

substitution in (1.04) yields

$$\varepsilon_n(x) = (-)^n \int_0^\infty \frac{t^n e^{-xt}}{1+t}\,dt. \tag{1.07}$$

Clearly,

$$|\varepsilon_n(x)| < \int_0^\infty t^n e^{-xt}\,dt = \frac{n!}{x^{n+1}}. \tag{1.08}$$

In other words, the partial sums of (1.05) approximate the function $G(x)$ with an error that is numerically smaller than the first neglected term of the series. It is also

† Bromwich (1926, §§175–6). This theorem is quoted fully later (Chapter 2, Theorem 8.1).

‡ Obtainable by numerical quadrature of (1.04) or by use of tables of the exponential integral; compare Chapter 2, §3.1.

clear from (1.07) that the error has the same sign as this term. Since the next term in (1.06) is 0.00024, this fully explains the closeness of the value 0.0914 of $g_4(10)$ to that of $G(10)$.

1.3 Thus the expansion (1.05) has a hidden meaning: it may be regarded as *constituting a sequence of approximations* $\{g_n(x)\}$ *to the value of* $G(x)$. In this way it resembles a convergent expansion, for example (1.03). For in practice we cannot compute an infinite number of terms in a convergent series; we stop the summation when we judge that the contribution from the tail is negligibly small compared to the accuracy required. There are, however, two important differences. First, $\varepsilon_n(x)$ cannot be expressed as the sum of the tail. Secondly, by definition the partial sum of a convergent series becomes arbitrarily close to the actual sum as the number of terms increases indefinitely. With (1.05) this is not the case: for a given value of x, successive terms $(-)^s s!/x^{s+1}$ diminish steadily in size as long as s does not exceed $[x]$, the integer part of x. Thereafter they increase without limit. Correspondingly, the partial sums $g_n(x)$ at first approach the value of $G(x)$, but when n passes $[x]$ errors begin to increase and eventually oscillate wildly.†

The essential difference, then, is that whereas the sum of a convergent series can be computed to arbitrarily high accuracy with the expenditure of sufficient labor, the accuracy in the value of $G(x)$ computed from the partial sums $g_n(x)$ of (1.05) is restricted. For a prescribed value of x, the best we can do is to represent $G(x)$ by $g_{[x]}(x)$. The absolute error of this representation is bounded by $[x]!/x^{[x]+1}$, and the relative error by about $[x]!/x^{[x]}$.

Although the accuracy is restricted, it can be extremely high. For example, when $x = 10$, $[x]!/x^{[x]} \risingdotseq 0.36 \times 10^{-3}$.‡ Therefore when $x \geqslant 10$, the value of $G(x)$ can be found from (1.05) to at least three significant figures, which is adequate for some purposes. For $x \geqslant 100$, this becomes 42 significant figures; there are few calculations in the physical sciences that need accuracy remotely approaching this.

So far, we have considered the behavior of the sequence $\{g_n(x)\}$ for fixed x and varying n. If, instead, n is fixed, then from (1.08) we expect $g_n(x)$ to give a better approximation to $G(x)$ than any other partial sum when x lies in the interval $n < x < n+1$.§ Thus, no single approximation is "best" in an overall sense; each has an interval of special merit.

1.4 The expansion (1.05) is typical of a large class of divergent series obtained from integral representations, differential equations, and elsewhere when rules governing the applicability of analytical transformations are violated. Nevertheless, such expansions were freely used in numerical and analytical calculations in the eighteenth century by many mathematicians, particularly Euler. In contrast to the foregoing analysis for the function $G(x)$ little was known about the errors in approximating functions in this way, and sometimes grave inaccuracies resulted.

† For this reason, series of this kind used to be called *semiconvergent* or *convergently beginning*.

‡ Here and elsewhere the sign $\risingdotseq$ denotes approximate equality.

§ Since (1.08) gives a bound and not the *actual* value of $|\varepsilon_n(x)|$, the interval in which $g_n(x)$ gives the best approximation may differ slightly from $n < x < n+1$.

Early in the nineteenth century Abel, Cauchy, and others undertook the task of placing mathematical analysis on firmer foundations. One result was the introduction of a complete ban on the use of divergent series, although it appears that this step was taken somewhat reluctantly.

No way of rehabilitating the use of divergent series was forthcoming during the next half century. Two requirements for a satisfactory general theory were, first, that it apply to most of the known series; secondly, that it permit elementary operations, including addition, multiplication, division, substitution, integration, differentiation, and reversion. Neither requirement would be met if, for example, we confined ourselves to series expansions whose remainder terms are bounded in magnitude by the first neglected term.

Both requirements were satisfied eventually by Poincaré in 1886 by defining what he called *asymptotic expansions*. This definition is given in §7.1 below. As we shall see, Poincaré's theory embraces a wide class of useful divergent series, and the elementary operations can all be carried out (with some slight restrictions in the case of differentiation).

2 The Symbols $\sim$, o, and O

2.1 In order to describe the behavior, as $x \to \infty$, of a wanted function $f(x)$ in terms of a known function $\phi(x)$, we shall often use the following notations, due to Bachmann and Landau.† At first, we suppose x to be a real variable. At infinity $\phi(x)$ may vanish, tend to infinity, or have other behavior—no restrictions are made.

(i) If $f(x)/\phi(x)$ tends to unity, we write

$$f(x) \sim \phi(x) \qquad (x \to \infty),$$

or, briefly, when there is no ambiguity, $f \sim \phi$. In words, *f is asymptotic to ϕ, or ϕ is an asymptotic approximation to f.*

(ii) If $f(x)/\phi(x) \to 0$, we write

$$f(x) = o\{\phi(x)\} \qquad (x \to \infty),$$

or, briefly, $f = o(\phi)$; in words, f is of order less than ϕ.‡

(iii) If $|f(x)/\phi(x)|$ is bounded, we write

$$f(x) = O\{\phi(x)\} \qquad (x \to \infty),$$

or $f = O(\phi)$; again, in words, *f is of order not exceeding ϕ.*

Special cases of these definitions are $f = o(1)$ $(x \to \infty)$, meaning simply that f vanishes as $x \to \infty$, and $f = O(1)$ $(x \to \infty)$, meaning that $|f|$ is bounded as $x \to \infty$.

† Landau (1927, Vol. 2, pp. 3–5).

‡ In cases in which $\phi(x)$ is not real and positive, some writers use modulus signs in the definition, thus $f(x) = o(|\phi(x)|)$. Similarly in Definition (iii) which follows.

As simple examples

$$(x+1)^2 \sim x^2, \qquad \frac{1}{x^2} = o\left(\frac{1}{x}\right), \qquad \sinh x = O(e^x).$$

2.2 Comparing (i), (ii), and (iii), we note that (i) and (ii) are mutually exclusive. Also, each is a particular case of (iii), and when applicable each is more informative than (iii).

Next, the symbol O is sometimes associated with an interval $[a, \infty)$† instead of the limit point ∞. Thus

$$f(x) = O\{\phi(x)\} \qquad \text{when} \quad x \in [a, \infty) \tag{2.01}$$

simply means that $|f(x)/\phi(x)|$ is bounded throughout $a \leqslant x < \infty$. Neither the symbol $\sim$ nor o can be used in this way, however.

The statement (2.01) is of existential type: it asserts that there is a number K such that

$$|f(x)| \leqslant K|\phi(x)| \qquad (x \geqslant a), \tag{2.02}$$

without giving information concerning the actual size of K. Of course, if (2.02) holds for a certain value of K, then it also holds for every larger value; thus there is an infinite set of possible K's. The least member of this set is the supremum (least upper bound) of $|f(x)/\phi(x)|$ in the interval $[a, \infty)$; we call it the *implied constant* of the O term for this interval.

2.3 The notations $o(\phi)$ and $O(\phi)$ can also be used to denote the *classes* of functions f with the properties (ii) and (iii), respectively, or *unspecified* functions with these properties. The latter use is generic, that is, $o(\phi)$ does not necessarily denote the same function f at each occurrence. Similarly for $O(\phi)$. For example,

$$o(\phi) + o(\phi) = o(\phi), \qquad o(\phi) = O(\phi).$$

It should be noted that many relations of this kind, including the second example, are not reversible: $O(\phi) = o(\phi)$ is false. Relations involving $\sim$ are always reversible, however.

An instructive relation is supplied by

$$e^{ix}\{1+o(1)\} + e^{-ix}\{1+o(1)\} = 2\cos x + o(1). \tag{2.03}$$

This is easily verified by expressing $e^{\pm ix}$ in the form $\cos x \pm i \sin x$ and recalling that the trigonometric functions are bounded. The important point to notice is that the right-hand side of (2.03) cannot be rewritten in the form $2\{1+o(1)\}\cos x$, for this would imply that the left-hand side is *exactly* zero when x is an odd multiple of $\frac{1}{2}\pi$. In general this is false because the functions represented by the $o(1)$ terms differ.

† Throughout this book we adhere to the standard notation (a,b) for an open interval $a < x < b$; $[a,b]$ for the corresponding closed interval $a \leqslant x \leqslant b$; $(a,b]$ and $[a,b)$ for the partly closed intervals $a < x \leqslant b$ and $a \leqslant x < b$, respectively.

Ex. 2.1† If ν has any fixed value, real or complex, prove that $x^\nu = o(e^x)$ and $e^{-x} = o(x^\nu)$. Prove also that‡ $\ln x = o(x^\nu)$, provided that $\operatorname{Re}\nu > 0$.

Ex. 2.2 Show that

$$x + o(x) = O(x), \qquad \{O(x)\}^2 = O(x^2) = o(x^3).$$

Ex. 2.3 Show that

$$\cos\{O(x^{-1})\} = O(1), \qquad \sin\{O(x^{-1})\} = O(x^{-1}),$$

and

$$\cos\{x+\alpha+o(1)\} = \cos(x+\alpha) + o(1),$$

where α is a real constant.

Ex. 2.4 Is it true that

$$\{1+o(1)\}\cosh x - \{1+o(1)\}\sinh x = \{1+o(1)\}e^{-x}?$$

Ex. 2.5 Show that

$$O(\phi)O(\psi) = O(\phi\psi), \qquad O(\phi)o(\psi) = o(\phi\psi), \qquad O(\phi) + O(\psi) = O(|\phi|+|\psi|).$$

Ex. 2.6 What are the implied constants in the relations

$$(x+1)^2 = O(x^2), \qquad (x^2-\tfrac{1}{2})^{1/2} = O(x), \qquad x^2 = O(e^x),$$

for the interval $[1, \infty)$?

Ex. 2.7 Prove that if $f \sim \phi$, then $f = \{1+o(1)\}\phi$. Show that the converse holds provided that infinity is not a limit point of zeros of ϕ.

Ex. 2.8 Let $\phi(x)$ be a positive nonincreasing function of x, and $f(x) \sim \phi(x)$ as $x \to \infty$. By means of the preceding exercise show that

$$\sup_{t \in (x,\infty)} f(t) \sim \phi(x) \qquad (x \to \infty).$$

3 The Symbols ~, o, and O (continued)

3.1 The definitions of §2.1 may be extended in a number of obvious ways. To begin with, there is no need for the asymptotic variable x to be continuous; it can pass to infinity through any set of values. Thus

$$\sin\left(\pi n + \frac{1}{n}\right) = O\left(\frac{1}{n}\right) \qquad (n \to \infty),$$

provided that n is an integer.

Next, we are not obliged to concern ourselves with the behavior of the ratio $f(x)/\phi(x)$ solely as $x \to \infty$; the definitions (i), (ii), and (iii) of §2.1 also apply when x tends to any finite point, c, say. For example, if $c \neq 0$, then as $x \to c$

$$\frac{x^2-c^2}{x^2} \sim \frac{2(x-c)}{c} = O(x-c) = o(1).$$

† In Exercises 2.1–2.5 it is assumed that large positive values of the independent variable x are being considered.

‡ $\ln x \equiv \log_e x$.

We refer to c as the *distinguished point* of the asymptotic or order relation.

3.2 The next extension is to complex variables. Let S be a given infinite sector $\alpha \leqslant \operatorname{ph} z \leqslant \beta$, $\operatorname{ph} z$ denoting the *phase* or *argument* of z. Suppose that for a certain value of R there exists a number K, *independent of* $\operatorname{ph} z$, such that

$$|f(z)| \leqslant K|\phi(z)| \qquad (z \in \mathbf{S}(R)), \tag{3.01}$$

where $\mathbf{S}(R)$ denotes the intersection of S with the annulus $|z| \geqslant R$. Then we say that $f(z) = O\{\phi(z)\}$ as $z \to \infty$ in S, or, equivalently, $f(z) = O\{\phi(z)\}$ in $\mathbf{S}(R)$. Thus the *symbol O automatically implies uniformity with respect to* $\operatorname{ph} z$.† *Similarly for the symbols* $\sim$ *and o.*

For future reference, the point set $\mathbf{S}(R)$ just defined will be called an *infinite annular sector* or, simply, *annular sector*. The vertex and angle of S will also be said to be the *vertex* and *angle* of $\mathbf{S}(R)$.

The least number K fulfilling (3.01) is called the *implied constant* for $\mathbf{S}(R)$. Actually there is no essential reason for considering annular sectors, the definitions apply equally well to any *region* (that is, point set in the complex plane) having infinity or some other distinguished point as a limit point; compare Exercise 3.2 below.

3.3 An important example is provided by the tail of a convergent power series:

Theorem 3.1 *Let* $\sum_{s=0}^{\infty} a_s z^s$ *converge when* $|z| < r$. *Then for fixed* n,

$$\sum_{s=n}^{\infty} a_s z^s = O(z^n)$$

in any disk $|z| \leqslant \rho$ *such that* $\rho < r$.

To prove this result, let ρ' be any number in the interval (ρ, r). Then $a_s \rho'^s \to 0$ as $s \to \infty$; hence there exists a constant A such that

$$|a_s| \rho'^s \leqslant A \qquad (s = 0, 1, 2, \ldots).$$

Accordingly,

$$\left|\sum_{s=n}^{\infty} a_s z^s\right| \leqslant \sum_{s=n}^{\infty} A \frac{|z|^s}{\rho'^s} = \frac{A\rho'^{(1-n)}|z|^n}{\rho' - |z|} \leqslant \frac{A\rho'^{(1-n)}}{\rho' - \rho}|z|^n.$$

This establishes the theorem.

A typical illustration is supplied by

$$\ln\{1 + O(z)\} = O(z) \qquad (z \to 0).$$

3.4 An asymptotic or order relation may possess uniform properties with respect to other variables or parameters. For example, if u is a parameter in the interval $[0, a]$, where a is a positive constant, then

$$e^{(z-u)^2} = O(e^{z^2})$$

† Not all writers use O and the other two symbols in this way.

as $z \to \infty$ in the right half-plane, uniformly with respect to u (and ph z). Such regions of validity are often interdependent: $u \in [-a, 0]$ and the left half of the z plane would be another admissible combination in this example.

Ex. 3.1 If δ denotes a positive constant, show that $\cosh z \sim \frac{1}{2}e^z$ as $z \to \infty$ in the sector $|\mathrm{ph}\, z| \leqslant \frac{1}{2}\pi - \delta$, but not in the sector $|\mathrm{ph}\, z| < \frac{1}{2}\pi$.

Ex. 3.2 Show that $e^{-\sinh z} = o(1)$ as $z \to \infty$ in the half-strip $\mathrm{Re}\, z \geqslant 0$, $|\mathrm{Im}\, z| \leqslant \frac{1}{2}\pi - \delta < \frac{1}{2}\pi$.

Ex. 3.3 If p is fixed and positive, calculate the implied constant in the relation $e^{-z} = O(z^{-p})$ for the sector $|\mathrm{ph}\, z| \leqslant \frac{1}{2}\pi - \delta < \frac{1}{2}\pi$, and show that it tends to infinity as $\delta \to 0$.

Ex. 3.4 Assume that $\phi(x) > 0$, p is a real constant, and $f(x) \sim \phi(x)$ as $x \to \infty$. With the aid of Theorem 3.1 show that $\{f(x)\}^p \sim \{\phi(x)\}^p$ and $\ln\{f(x)\} \sim \ln\{\phi(x)\}$, provided that in the second case $\phi(x)$ is bounded away from unity.

Show also that $e^{f(x)} \sim e^{\phi(x)}$ may be false.

Ex. 3.5 Let x range over the interval $[0, \delta]$, where δ is a positive constant, and $f(u, x)$ be a positive real function such that $f(u, x) = O(u)$ as $u \to 0$, uniformly with respect to x. Show that

$$\{x + f(u, x)\}^{1/2} = x^{1/2} + O(u^{1/2})$$

as $u \to 0$, uniformly with respect to x.

4 Integration and Differentiation of Asymptotic and Order Relations

4.1 As a rule, asymptotic and order relations may be *integrated*, subject to obvious restrictions on the convergence of the integrals involved. Suppose, for example, that $f(x)$ is an integrable function of the real variable x such that $f(x) \sim x^\nu$ as $x \to \infty$, where ν is a real or complex constant. Let a be any finite real number. Then as $x \to \infty$, we have

$$\int_x^\infty f(t)\, dt \sim -\frac{x^{\nu+1}}{\nu+1} \qquad (\mathrm{Re}\, \nu < -1), \tag{4.01}$$

and

$$\int_a^x f(t)\, dt \sim \begin{cases} \text{a constant} & (\mathrm{Re}\, \nu < -1), \\ \ln x & (\nu = -1), \\ x^{\nu+1}/(\nu+1) & (\mathrm{Re}\, \nu > -1). \end{cases} \tag{4.02}$$

To prove, for example, the third of (4.02), we have $f(x) = x^\nu\{1 + \eta(x)\}$, where $|\eta(x)| < \varepsilon$ when $x > X > 0$, X being assignable for any given positive number ε. Hence if $x > X$, then

$$\int_a^x f(t)\, dt = \int_a^X f(t)\, dt + \frac{1}{\nu+1}(x^{\nu+1} - X^{\nu+1}) + \int_X^x t^\nu \eta(t)\, dt,$$

and so

$$\frac{\nu+1}{x^{\nu+1}} \int_a^x f(t)\, dt - 1 = \frac{\nu+1}{x^{\nu+1}} \int_a^X f(t)\, dt - \frac{X^{\nu+1}}{x^{\nu+1}} + \frac{\nu+1}{x^{\nu+1}} \int_X^x t^\nu \eta(t)\, dt.$$

The first two terms on the right-hand side of the last equation vanish as $x \to \infty$, and the third term is bounded by $|\nu+1|\,\varepsilon/(1+\operatorname{Re}\nu)$. The stated result now follows.

The results (4.01) and (4.02) may be extended in a straightforward way to complex integrals.

4.2 *Differentiation* of asymptotic or order relations is not always permissible. For example, if $f(x) = x+\cos x$, then $f(x) \sim x$ as $x \to \infty$, but it is not true that $f'(x) \to 1$. To assure the legitimacy of differentiation further conditions are needed. For real variables, these conditions can be expressed in terms of the monotonicity of the derivative:

Theorem 4.1† *Let $f(x)$ be continuously differentiable and $f(x) \sim x^p$ as $x \to \infty$, where p ($\geqslant 1$) is a constant. Then $f'(x) \sim px^{p-1}$, provided that $f'(x)$ is nondecreasing for all sufficiently large x.*

To prove this result, we have $f(x) = x^p\{1+\eta(x)\}$, where $|\eta(x)| \leqslant \varepsilon$ when $x > X$, assignable and positive, ε being an arbitrary number in the interval $(0, 1)$. If $h > 0$, then

$$\begin{aligned} hf'(x) &\leqslant \int_x^{x+h} f'(t)\,dt = f(x+h) - f(x) \\ &= \int_x^{x+h} pt^{p-1}\,dt + (x+h)^p\eta(x+h) - x^p\eta(x) \\ &\leqslant hp(x+h)^{p-1} + 2\varepsilon(x+h)^p. \end{aligned}$$

Set $h = \varepsilon^{1/2}x$. Then we have

$$f'(x) \leqslant px^{p-1}\{(1+\varepsilon^{1/2})^{p-1} + 2p^{-1}\varepsilon^{1/2}(1+\varepsilon^{1/2})^p\} \qquad (x > X).$$

Similarly,

$$f'(x) \geqslant px^{p-1}\{(1-\varepsilon^{1/2})^{p-1} - 2p^{-1}\varepsilon^{1/2}\} \qquad (x > X/(1-\varepsilon^{1/2})).$$

The theorem now follows.

Another result of this type is stated in Exercise 4.4 below. It should be appreciated, however, that monotonicity conditions on $f'(x)$ are often difficult to verify in practice because $f'(x)$ is the function whose properties are being sought.

4.3 In the complex plane, differentiation of asymptotic or order relations is generally permissible in subregions of the original region of validity. An important case is the following:

Theorem 4.2‡ *Let $f(z)$ be holomorphic§ in a region containing a closed annular sector* **S**, *and*

$$f(z) = O(z^p) \qquad (or\quad f(z) = o(z^p)) \tag{4.03}$$

† de Bruijn (1961, §7.3).
‡ Ritt (1918).
§ That is, analytic and free from singularity.

as $z \to \infty$ in **S**, *where p is any fixed real number. Then*

$$f^{(m)}(z) = O(z^{p-m}) \qquad (or\ \ f^{(m)}(z) = o(z^{p-m})) \tag{4.04}$$

as $z \to \infty$ in any closed annular sector **C** *properly interior to* **S** *and having the same vertex.*

The proof depends on Cauchy's integral formula for the mth derivative of an analytic function, given by

$$f^{(m)}(z) = \frac{m!}{2\pi i}\int_{\mathscr{C}} \frac{f(t)\,dt}{(t-z)^{m+1}}, \tag{4.05}$$

in which the path $\mathscr{C}$ is chosen to be a circle enclosing $t = z$. The essential reason z is restricted to an interior region in the final result is to permit inclusion of $\mathscr{C}$ in **S**.

Since $|z - \text{constant}|^p \sim |z|^p$, the vertex of **S** may be taken to be the origin without loss of generality. Let **S** be defined by $\alpha \leqslant \operatorname{ph} z \leqslant \beta$, $|z| \geqslant R$, and consider the annular sector **S**$'$ defined by

$$\alpha + \delta \leqslant \operatorname{ph} z \leqslant \beta - \delta, \qquad |z| \geqslant R',$$

where δ is a positive acute angle and $R' = R/(1 - \sin\delta)$; see Fig. 4.1. By taking δ small enough we can ensure that **S**$'$ contains **C**. In (4.05) take $\mathscr{C}$ to be $|t - z| = |z|\sin\delta$. Then

$$|z|(1 - \sin\delta) \leqslant |t| \leqslant |z|(1 + \sin\delta).$$

Hence $t \in$ **S** whenever $z \in$ **S**$'$. Moreover, if K is the implied constant of (4.03) for **S**, then

$$|f^{(m)}(z)| \leqslant \frac{m!}{(|z|\sin\delta)^m} K|z|^p(1 \pm \sin\delta)^p,$$

the upper or lower sign being taken according as $p \geqslant 0$ or $p < 0$. In either event $f^{(m)}(z)$ is $O(z^{p-m})$, as required. The proof in the case when the symbol O in (4.03) and (4.04) is replaced by o is similar.

We have shown, incidentally, that the implied constant of (4.04) in **S**$'$ does not

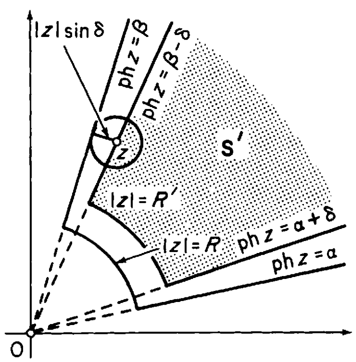

Fig. 4.1 Annular sectors **S**, **S**$'$.

exceed $m!(\csc\delta)^m(1\pm\sin\delta)^p K$, but because this bound tends to infinity as $\delta\to 0$, we cannot infer that (4.04) is valid in **S**.

Ex. 4.1 Show that if $f(x)$ is continuous and $f(x)=o\{\phi(x)\}$ as $x\to\infty$, where $\phi(x)$ is a positive nondecreasing function of x, then $\int_a^x f(t)\,dt = o\{x\phi(x)\}$.

Ex. 4.2 It may be expected that in the case $\operatorname{Re}\nu=-1$, $\operatorname{Im}\nu\neq 0$, the result corresponding to (4.02) would be $\int_a^x f(t)\,dt = O(1)$. Show that this is false by means of the example $f(x)=x^{i\mu-1}+(x\ln x)^{-1}$, where μ is real.

Ex. 4.3 If u and x lie in $[1,\infty)$, show that

$$\int_x^\infty \frac{dt}{t(t^2+t+u^2)^{1/2}} = \frac{1}{x}+O\left(\frac{1}{x^2}\right)+O\left(\frac{u^2}{x^3}\right).$$

Ex. 4.4 Suppose that $f(x)=x^2+O(x)$ as $x\to\infty$, and $f'(x)$ is continuous and nondecreasing for all sufficiently large x. Show that $f'(x)=2x+O(x^{1/2})$. [de Bruijn, 1961.]

Ex. 4.5 In place of (4.03) assume that $f(z)\sim z^\nu$, where ν is a nonzero real or complex constant. Deduce from Theorem 4.2 that $f'(z)\sim \nu z^{\nu-1}$ as $z\to\infty$ in **C**.

Ex. 4.6 Let **T** and **T′** denote the half-strips

$$\mathbf{T}\ :\quad \alpha\leqslant \operatorname{Im} z\leqslant\beta,\qquad \operatorname{Re} z\geqslant\rho,$$
$$\mathbf{T}'\ :\quad \alpha+\delta\leqslant \operatorname{Im} z\leqslant\beta-\delta,\qquad \operatorname{Re} z\geqslant\rho,$$

where $0<\delta<\frac{1}{2}(\beta-\alpha)$. Suppose that $f(z)$ is holomorphic within **T**, and $f(z)=O(e^z)$ as $z\to\infty$ in **T**. Show that $f'(z)=O(e^z)$ as $z\to\infty$ in **T′**.

Ex. 4.7 Show that the result of Exercise 4.6 remains valid if both terms $O(e^z)$ are replaced by $O(z^p)$, where p is a real constant.

Show further that $f'(z)=O(z^{p-1})$ is false by means of the example z^pe^{iz}.

5 Asymptotic Solution of Transcendental Equations: Real Variables

5.1 Consider the equation

$$x+\tanh x = u,$$

in which u is a real parameter. The left-hand side is a strictly increasing function of x. Hence by graphical considerations there is exactly one real root $x(u)$, say, for each value of u. What is the asymptotic behavior of $x(u)$ for large positive u?

When x is large, the left-hand side is dominated by the first term. Accordingly, we transfer the term $\tanh x$ to the right and treat it as a "correction":

$$x = u-\tanh x.$$

Since $|\tanh x|<1$, it follows that

$$x(u)\sim u\qquad (u\to\infty). \tag{5.01}$$

This is the first approximation to the root. An immediate improvement is obtained by recalling that $\tanh x = 1+o(1)$ as $x\to\infty$; thus

$$x = u-1+o(1)\qquad (u\to\infty). \tag{5.02}$$

To derive higher approximations we expand $\tanh x$ in a form appropriate for large x, given by

$$\tanh x = 1 - 2e^{-2x} + 2e^{-4x} - 2e^{-6x} + \cdots \qquad (x > 0),$$

and repeatedly substitute for x in terms of u. From (5.02). it is seen that $e^{-2x} = O(e^{-2u})$.† Hence with the aid of Theorem 3.1 we obtain

$$x = u - 1 + O(e^{-2x}) = u - 1 + O(e^{-2u}).$$

The next step is given by

$$\begin{aligned} x &= u - 1 + 2\exp\{-2u+2+O(e^{-2u})\} + O(e^{-4x}) \\ &= u - 1 + 2e^{-2u+2} + O(e^{-4u}). \end{aligned} \tag{5.03}$$

Continuation of the process produces a sequence of approximations with errors of steadily diminishing asymptotic order. Whether the sequence converges as the number of steps tends to infinity is not discernible from the analysis, but the numerical potential of the process can be perceived by taking, for example, $u = 5$ and ignoring the error term $O(e^{-4u})$ in (5.03). We find that $x = 4.0006709\ldots$, compared with the correct value $4.0006698\ldots$, obtained by standard numerical methods.‡

5.2 A second example amenable to the same approach is the determination of the large positive roots of the equation

$$x \tan x = 1.$$

Inversion produces

$$x = n\pi + \tan^{-1}(1/x),$$

where n is an integer and the inverse tangent has its principal value. Since the latter is in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$, we derive $x \sim n\pi$ as $n \to \infty$.

Next, when $x > 1$,

$$\tan^{-1}\frac{1}{x} = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \frac{1}{7x^7} + \cdots.$$

Hence $x = n\pi + O(x^{-1}) = n\pi + O(n^{-1})$. The next two substitutions produce

$$x = n\pi + \frac{1}{n\pi} + O\left(\frac{1}{n^3}\right), \qquad x = n\pi + \frac{1}{n\pi} - \frac{4}{3(n\pi)^3} + O\left(\frac{1}{n^5}\right).$$

And so on.

5.3 A third example is provided by the equation

$$x^2 - \ln x = u, \tag{5.04}$$

in which u is again a large positive parameter. This differs from the preceding

† It should be observed that this relation cannot be deduced directly from (5.01).

‡ Error bounds for (5.03) are stated in Exercise 5.3 below.

examples in that the "correction term" $\ln x$ is unbounded as $x \to \infty$. To assist with (5.04) and similar equations we establish the following simple general result:

Theorem 5.1 *Let $f(\xi)$ be continuous and strictly increasing in an interval $a < \xi < \infty$, and*

$$f(\xi) \sim \xi \qquad (\xi \to \infty). \tag{5.05}$$

Denote by $\xi(u)$ the root of the equation

$$f(\xi) = u \tag{5.06}$$

which lies in (a, ∞) when $u > f(a)$. Then

$$\xi(u) \sim u \qquad (u \to \infty). \tag{5.07}$$

Graphical considerations show that $\xi(u)$ is unique, increasing, and unbounded as $u \to \infty$. From (5.05) and (5.06) we have $u = \{1+o(1)\}\xi$ as $\xi \to \infty$, and therefore, also, as $u \to \infty$. Division by the factor $1+o(1)$ then gives $\xi = \{1+o(1)\}u$, which is equivalent to (5.07).

5.4 We return to the example (5.04). Here $\xi = x^2$ and $f(\xi) = \xi - \frac{1}{2}\ln\xi$. Therefore $f(\xi)$ is strictly increasing when $\xi > \frac{1}{2}$, and the theorem informs us that $\xi \sim u$ as $u \to \infty$; equivalently,

$$x = u^{1/2}\{1+o(1)\} \qquad (u \to \infty).$$

Substituting this approximation into the right-hand side of

$$x^2 = u + \ln x, \tag{5.08}$$

and recalling that $\ln\{1+o(1)\}$ is $o(1)$, we see that

$$x^2 = u + \tfrac{1}{2}\ln u + o(1),$$

and hence (Theorem 3.1)

$$x = u^{1/2}\left\{1 + \frac{\ln u}{4u} + o\left(\frac{1}{u}\right)\right\}.$$

As in §§5.1 and 5.2, the resubstitutions can be continued indefinitely.

Ex. 5.1 Prove that the root of the equation $x \tan x = u$ which lies in the interval $(0, \frac{1}{2}\pi)$ is given by

$$x = \tfrac{1}{2}\pi(1-u^{-1}+u^{-2}) - (\tfrac{1}{2}\pi - \tfrac{1}{24}\pi^3)u^{-3} + O(u^{-4}) \qquad (u \to \infty).$$

Ex. 5.2 Show that the large positive roots of the equation $\tan x = x$ are given by

$$x = \mu - \mu^{-1} - \tfrac{2}{3}\mu^{-3} + O(\mu^{-5}) \qquad (\mu \to \infty),$$

where $\mu = (n+\frac{1}{2})\pi$, n being a positive integer.

Ex. 5.3 For the example of §5.1, show that when $u > 0$

$$x = u - 1 + 2\vartheta_1 e^{-2u+2},$$

and hence that

$$x = u - 1 + 2e^{-2u+2} - 10\vartheta_2 e^{-4u+4},$$

where ϑ_1 and ϑ_2 are certain numbers in the interval $(0, 1)$.