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WAVES ON FLUID INTERFACES

Edited by
Richard E. Meyer

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PREFACE

This volume collects the invited addresses given at a symposium on fluid interfaces of the Mathematics Research Center of the University of Wisconsin in Madison in October 1982. The articles survey many different aspects of recent research developments, from nonlinear instabilities of classical interfaces to the physical structure of real interfaces and the new challenges they pose to our intuition about fluids. They concern theory and experiment, and touch on many applications of acute interest in technology and medicine. Collectively, they illuminate a multifaceted subject in rapid progress.

I am greatly indebted to the authors for the excellence of their articles outlining so many recent advances in which they have played a decisive part. The Mathematics Research Center also wishes to thank the United States Army, which sponsored the conference under its Contract No. DAAG29-80-C-0041, the National Science Foundation, which supported it by Grants MEA-8212157 and MCS-7927062(2) of its Mechanical Engineering and Applied Mechanics and its Mathematical and Computer Sciences Divisions, and to the United States Department of Energy, which supported it by Grant DE-FG02-82ER13020 of its Applied Mathematical Sciences Division in the Office of Basic Energy Sciences. My personal thanks go to Gladys Moran for the expert handling of yet another symposium and to Elaine DuCharme for assembling the volume and index.

Richard E. Meyer

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FINITE-AMPLITUDE INTERFACIAL WAVES

P. G. Saffman

1. INTRODUCTION.

We consider gravity waves at the interface between two uniform, unbounded fluids of different densities in the presence of a current or relative horizontal velocity U. The fluids are supposed to be immiscible, incompressible and inviscid, and the motion is assumed to be irrotational. We are concerned with the properties and existence of finite amplitude two-dimensional, periodic waves of permanent form which propagate steadily without change of shape. By two-dimensional, we mean that the flow field depends only on the horizontal direction of propagation, which will be the x-axis, and the vertical y-direction. In the field of surface gravity waves, which is the limit of the present study when the density of the upper fluid is zero, it has been found recently that three-dimensional waves of permanent form exist and are observed experimentally (see e.g. [5]). It is expected that such waves will also exist and be important for interfacial waves, but they will not be considered in the present work.

For the purpose of calculating steady waves, there is no loss of generality in taking the speed of propagation c parallel to the current U, as an arbitrary constant transverse velocity may be linearly superposed on any

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two-dimensional steady wave without affecting its properties (the stability characteristics would, however, be affected). The wave can be reduced to rest by choosing a frame of reference moving with the wave. The problem is then to calculate steady irrotational solutions of the Euler equations which satisfy continuity of pressure across a common streamline. It follows from dimensional analysis that apart from scaling factors all flow variables will depend upon three dimensionless parameters:

$$\frac{h}{L}, \quad \frac{\rho_2}{\rho_1}, \quad \frac{\rho_2 U^2}{\rho_1 gL}, \tag{1.1}$$

where h is the height of the wave defined as the vertical distance between crest and trough, L is the wavelength (the horizontal distance over which the flow field repeats itself which in the present work will be the distance between crests), ρ_2 and ρ_1 are the densities of the upper and lower fluid respectively, and g is the acceleration due to gravity. For example, the speed of the waves is given by

$$c = (gL/2\pi)^{1/2} C\left(\frac{h}{L}, \frac{\rho_2}{\rho_1}, \frac{\rho_2 U^2}{\rho_1 gL}\right) \tag{1.2}$$

where C is a dimensionless function of its arguments. For surface waves, where $\rho_2 = 0$ and there is dependence on only one parameter, namely h/L, it is known that many interesting and unexpected phenomena exist, especially when the wave steepness becomes large. When there is dependence on three parameters, it is to be expected that many more phenomena are likely. However, in the absence of exact solutions for large h/L, it is a highly non-trivial task to search a three-dimensional parameter space. The results to be presented below are limited to those phenomena which seem currently to be of the most interest.

In contrast to the voluminous work on surface waves, relatively little seems to have been done on interfacial waves of permanent form, and that work seems to have been confined to the case of zero current, i.e. $U = 0$. Tsuji and Nagata [7] calculated Stokes type expansions to order $(h/L)^5$, and Holyer [3] used the computer to compute the coefficients in such an expansion to order $(h/L)^{37}$, and then used Padé approximants to estimate the behavior for large h/L. We are not aware of any work for waves with current.

For the mathematical formulation, there is no loss of generality in taking $g = 1$, $L = 2\pi$, and $\rho_1 = 1$. The mathematical problem is to determine the x-periodic velocity potentials and stream functions, for the lower and upper fluid respectively, which satisfy Laplace's equation and are harmonic conjugate pairs, so that at the unknown interface $y = Y(x)$,

$$\psi_1(x,Y(x)) = 0, \quad \psi_2(x,Y(x)) = 0 , \tag{1.3}$$

$$\frac{1}{2} (\nabla\phi_1)^2 + Y(x) + b = \frac{1}{2} \rho_2(\nabla\phi_2)^2 + \rho_B Y(x) . \tag{1.4}$$

In general, $\rho_B = \rho_2$, but we allow for the possibility of Boussinesq waves (in which the inertia of the two fluids is the same and density differences only matter when multiplied by g) by setting $\rho_2 = 1$ and $\rho_B = 0$. Surface tension is neglected throughout. The quantity b is the Bernoulli constant, which by suitable choice of the origin of pressure may be set equal to zero in the lower fluid. Infinitely far from the interface, we have

$$\phi_1 \sim -Cx, \quad \phi_2 \sim (U - c)x . \tag{1.5}$$

The vertical origin is set by requiring that the mean elevation of the interface is zero and the horizontal origin can be fixed by placing the crest at $x = 0$. This problem now appears to be free of arbitrary constants and the wave is determined by the crest to trough height h. It is expected that isolated families of solutions exist in connected regions in (h,ρ_2,U) space, although this does not yet appear to have been proved.

One question of considerable interest is the domain of parameter space in which solutions exist. Suppose that we consider a fixed value of ρ_2 and vary h and U. It is found that as U increases with h kept constant the system of equations describing steady solutions fails to have a solution, even though the 'limiting' wave profile is smooth and exhibits no singular properties. For $U > 0$ there are, when solutions exist, at least two physically distinct waves corresponding to the two wave speeds for propagation with and against the current. As U increases, the wave propagating against the current is 'entrained' by the current and at a certain value of U, which depends on h and ρ_2, the two waves become identical and for larger U there are no real solutions of the equations. Mathematically, this is like the disappearance of roots of a quadratic). We shall term this factor which limits existence a 'dynamical limit'.

The second factor is what we term a 'geometrical limit'. The mathematical formulation remains well-behaved but the solutions cease to make physical sense as the wave profiles cross themselves. This occurs for fixed U and increasing h. Examples of this phenomenon are found in pure capillary and capillary-gravity waves [2,1] for which the wave profile crosses itself at a critical value of h. If $U \neq 0$, this limit is going to be different for the two solutions of waves moving with and against the current. In the case of surface waves, this limit corresponds to a 120° cusp. It is easy to see that except for two special cases (see §4), this cannot happen for interfacial waves. Holyer [3] identified the geometrical limit for $U = 0$ with the existence of a vertical tangent. We shall present evidence that waves can exist with a vertical tangent and significant overhang, and the evidence indicates that the geometrical limit is associated with the wave crossing itself when it is sufficiently high for $U > 0$.

2. <u>WEAKLY NONLINEAR WAVES.</u>

The properties of weakly nonlinear steady waves may be obtained by using the Stokes expansion in which all variables are expanded as power series in h/L. However, the algebra can be simplified somewhat by using Whitham's variational approach. Proceeding in the usual manner, one finds after some algebra that the average Lagrangian is

$$L = \frac{1}{4}(\rho_B - 1)(\frac{1}{4}h^2 + a_2^2)$$

$$+ \frac{h^2}{16k}[\omega^2 + \rho_2(Uk-\omega)^2] - (\frac{a_2^2}{2k} + \frac{kh^4}{256})[\omega^2 + \rho_2(Uk-\omega)^2]$$

$$- \frac{h^2 a_2}{16}[\omega^2 - \rho_2(Uk-\omega)^2] + O(h^6) \qquad (2.1)$$

for the wave with interface shape

$$Y(x) = \frac{h}{2}\cos(kx - \omega t) + a_2 \cos 2(kx - \omega t) \qquad (2.2)$$

The value of a_2 is found from $\partial L/\partial a_2 = 0$ to be

$$a_2 = \frac{h^2}{8(1 - \rho_B)}[C^2 - \rho_2(U - C)^2] + O(h^4) ,$$

where $C = \omega/k$ is the phase speed. The dispersion relation for the weakly nonlinear wave then follows from $\partial L/\partial h = 0$:

$$C^2 + \rho_2(U - C)^2$$

$$= (1 - \rho_B)[1 + \frac{h^2}{8}(\frac{2C^2}{1 - \rho_B} - 1)^2 + \frac{h^2}{8}] + O(h^4) . \qquad (2.4)$$

For $U = 0$, the values of C agree with those in [7]

The values of the energy, momentum and action densities and fluxes follow from the expression (2.1) for L in the usual way. In particular, the total energy density E is given by

$$E = kCL_\omega - L . \qquad (2.5)$$

It is to be noted that for $U > 0$, the energy is measured relative to the energy of the uniform state with a flat interface. Negative energies may therefore exist and mean that the energy of the state with waves is less than that of the undisturbed flow.

It follows from the dispersion relation (2.4) that for linear waves $(h \to 0)$ and given values of ρ_2 and U, there are two solutions corresponding to the two roots of the quadratic equation for C in terms of ρ_2 and U. We denote these two solutions by C_+ and C_-, where

$C_+ > C_-$. For the linear case, steady solutions cease to exist when U exceeds a critical value U_{c0} given by

$$U_{c0} = [(1 + \rho_2)(1 - \rho_B)/\rho_2]^{1/2} \tag{2.6}$$

for which the two wave speeds are equal with the value $C_+ = C_- = \rho_2 U_{c0}/(1 + \rho_2)$.

The values of C_+ and C_- are

$$\rho_2 U \pm [\rho_2^2 U^2 - (1 + \rho_2)(\rho_2 U^2 - 1 + \rho_B)]^{1/2} . \tag{2.7}$$

For $U = 0$, the values are equal and opposite. As U increases, the speed of the wave propagating with the current originally increases but eventually decreases. The speed of the wave propagating against the current increases monotonically (in the algebraic sense), becomes zero when $U = [(1 - \rho_B)/\rho_2]^{1/2}$ and then increases to equal C_+ when U is given by (2.6). According to the linear approximation, the energy density E equals $\frac{1}{8} h^2[C^2(1+r) - rCU]$, and it is interesting that the energy becomes negative when the direction of the C_- waves changes.

For finite amplitude waves, the two solutions corresponding to C_+ and C_- waves continue into two families of solutions marked by wave speeds $C_+(h,\rho_2,U)$ and $C_-(h,\rho_2,U)$. For any given value of h and ρ_2, there will again be a critical current U_c beyond which steady solutions no longer exist. For the weakly nonlinear approximation, this value is given by

$$U_c = U_{c0}\left[1 + \frac{h^2}{4} \frac{(1 + \rho_2^2)}{(1 + \rho_2)^2}\right]^{1/2} . \tag{2.9}$$

It is noteworthy that increasing h increases U_c.

3. NUMERICAL METHODS.

For values of h that are not small, it is necessary to employ numerical methods. Three different techniques were employed. The first was to compute in physical space, i.e. the interface, potentials and stream function were expanded as Fourier series in x with coefficients which are exponential in y. The series were truncated to N modes and the boundary conditions were then satisfied at $N + 1$ equally horizontally spaced points on the interface. This procedure gives $3N + 4$ equations for $3N + 4$ unknowns. These equations were solved by Newton's method, using continuation in either U or h to give the first guesses. Note that this formulation is essentially equivalent to calculating numerically the coefficients of the Stokes expansion as done in [3].

The second method used the potential and stream function as the independent variables and expands the physical coordinates as series in these. The boundary conditions are now satisfied at equally spaced values of the velocity potential and the resulting system of $3N + 3$ equations in $3N + 3$ variables, the expansions being truncated to N modes, was also solved by Newton's method with continuation in U and h employed to give a first guess.

The third method used a vortex sheet representation in which the unknowns are the shape of the interface and the dipole strength of the equivalent double layer. This gives a nonlinear integrodifferential equation, which was solved by discretization and collocation, the resulting system of nonlinear equations again being solved by Newton's method with continuation.

For details, see [4,5]. All methods worked extremely well for small values of h/L, which generally meant $h < 0.6$, with some dependence on ρ_2 and U. (With our scaling, the surface wave of greatest height has $h = 0.89$). The first method was the first to fail as h increased. It is of course clear that this approach of working in physical space must fail when the wave becomes very steep, but the failure, marked by the apparent failure of the Fourier series to converge, seemed to be due to other causes. What actually happened was that the singularities of the analytic continuation of the lower velocity potential, say, into the upper half plane moved down below the crest. In this case, the expansion of the velocity potential would have to diverge near the crest, even though the solution was perfectly well behaved and physically meaningful. This difficulty would not affect the other two methods which were used for values of h up to 1.2 for various values of ρ_2 and U. For large values of h, 100 modes were used in the second method and this seemed adequate except for the largest h. The vortex sheet method with 65 intervals was employed for this case. This method offers in principle the advantage of being able to concentrate points near regions of high curvature, although this was not done.

The accuracy of the calculations was checked by comparing the results of the somewhat different methods with each other in regions of apparent validity and by performing the usual tests of internal consistency by investigating the dependence on number of retained modes. The calculations were carried out on a PRIME 750 and the CRAY-1 at NCAR.

4. A SPECIAL CLASS OF SOLUTIONS.

It is interesting to note that a special class of solutions exist which are simple transformations of the well-known surface permanent wave solutions, which have been extensively studied both numerically and theoretically by many authors. For each value of ρ_2, these solutions describe the shape of the interface for the C_+ case when

$$C_+ = U = (1 - \rho_B)^{1/2} c_s(h) \tag{4.1}$$

where $c_s(h)$ is the wave speed of the surface wave of permanent frm for the given wave height h. Since $C_+ = U$, the upper fluid is stagnant in the wave-fixed coordinates. The dynamic boundary condition for the motion in the lower fluid then becomes that for surface waves with a reduced gravity $g(1 - \rho_B)^{1/2}$. The velocities and wave speed are therefore those of the surface wave multiplied by the factor $(1 - \rho_B)^{1/2}$.

For the C_- branch, special solutions exist with

$$C_- = 0, \quad U = [(1 - \rho_B)/\rho_2]^{1/2} c_s(h) . \tag{4.2}$$

In this case, the lower fluid is stagnant and the dynamic boundary condition on the motion of the upper fluid is that with a reduced upside down gravity. The wave profiles are