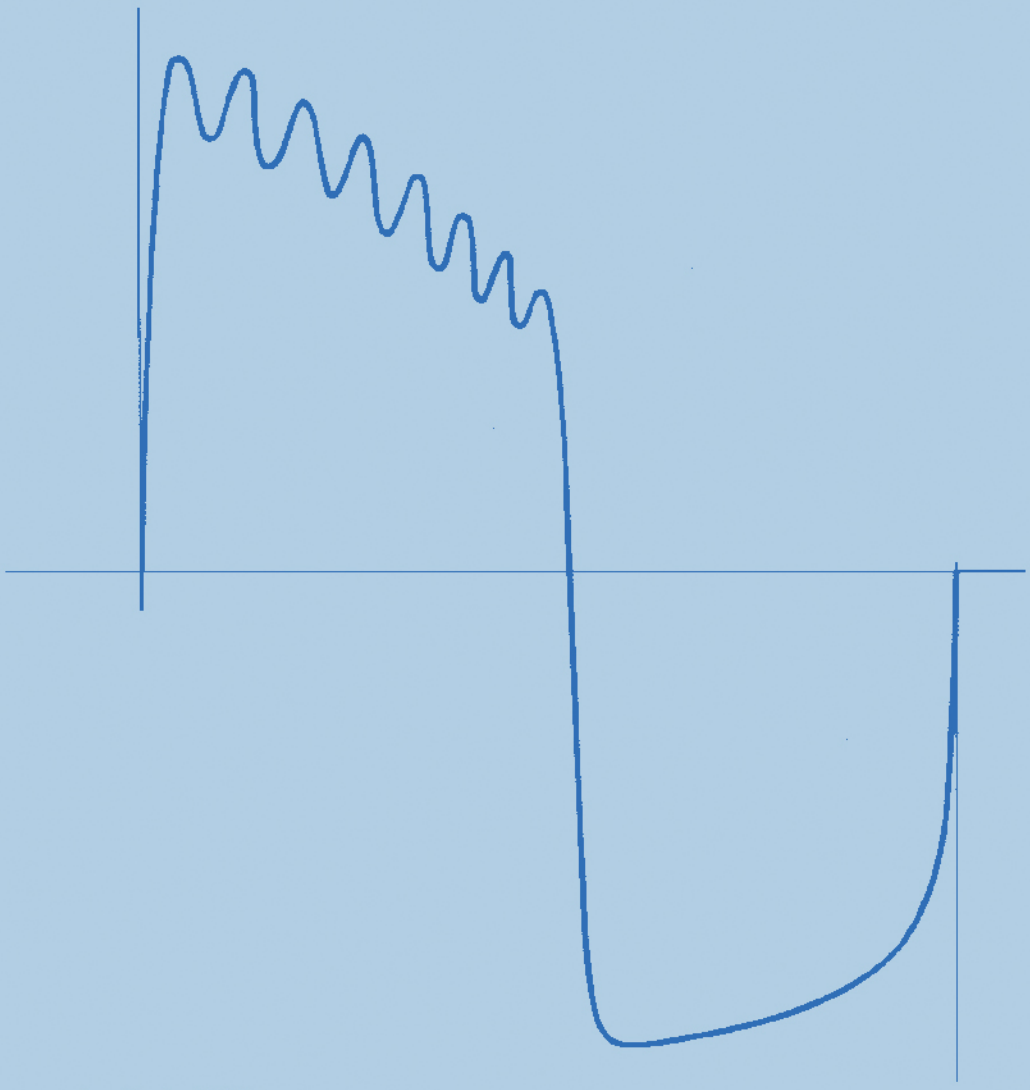
MATHEMATICS RESEARCH CENTER
SYMPOSIA AND ADVANCED SEMINAR SERIES

Singular Perturbations and Asymptotics

Edited by
Richard E. Meyer / Seymour V. Parter



ACADEMIC PRESS
A Subsidiary of Harcourt Brace Jovanovich, Publishers

SINGULAR PERTURBATIONS AND ASYMPTOTICS

Publication No. 45
of the Mathematics Research Center
The University of Wisconsin—Madison

Academic Press Rapid Manuscript Reproduction

SINGULAR PERTURBATIONS AND ASYMPTOTICS

Edited by

Richard E. Meyer
Seymour V. Parter

Mathematics Research Center
The University of Wisconsin—Madison
Madison, Wisconsin

Proceedings of an Advanced Seminar
Conducted by the Mathematics Research Center
The University of Wisconsin—Madison
May 28–30, 1980

ACADEMIC PRESS 1980
A Subsidiary of Harcourt Brace Jovanovich, Publishers

NEW YORK LONDON TORONTO SYDNEY SAN FRANCISCO

This work relates to Department of the Navy Research Grant N00014-80-G-0108 issued by the Office of Naval Research. The United States Government has a royalty-free license throughout the world in all copyrightable material contained herein.

ACADEMIC PRESS, INC.
111 Fifth Avenue, New York, New York 10003

United Kingdom Edition published by
ACADEMIC PRESS, INC. (LONDON) LTD.
24/28 Oval Road, London NW1 7DX

Library of Congress Cataloging in Publication Data

Main entry under title:

Singular perturbations and asymptotics.

Publication of the Mathematics Research Center, the University of Wisconsin—Madison ; no. 45)

Includes index.
1. Differential equations—Asymptotic theory—Congresses. 2. Perturbation (Mathematics)—Congresses. I. Meyer, Richard E., DATE II. Parter, Seymour V. III. Wisconsin. University—Madison. Mathematics Research Center. IV. Series: Wisconsin. University—Madison. Mathematics Research Center. Publication ; no. 45.
QA3.U45 no. 45 [QA371] 510s [515.3'5] 80-24946
ISBN 0-12-493260-6

PRINTED IN THE UNITED STATES OF AMERICA

80 81 82 83 9 8 7 6 5 4 3 2 1

CONTENTS

Contributors *vii*
Preface *ix*

I. Theory of Singular Layer Problems

On Some Basic Concepts in the Analysis of Singular Perturbations 1
Wiktor Eckhaus

Limit Process Expansions and Approximate Equations 19
Julian D. Cole

Some Old and New Results on Singularly Perturbed Boundary Value Problems 41
Fred Howes

On Multiple Solutions of Singularly Perturbed Systems in the Conditionally Stable Case 87
Robert E. O'Malley

II. Resonance in Singular Perturbations and Applications to Physical Chemistry

Singular Perturbations, Stochastic Differential Equations, and Applications 109
Bernard J. Matkowsky

The Singularly Perturbed Turning-Point Problem: A Spectral Approach 149
Pieter P. N. de Groen

The Singularly Perturbed Turning-Point Problem: A Geometric Approach 173
Nancy Kopell

III. Multivariate Methods and Applications

Passage through Resonance 191
Jerry Kevorkian

A Comparison of Perturbation Methods for Nonlinear Hyperbolic Waves 223
Ali H. Nayfeh

Asymptotic Calculus of Variations 277
Jacques L. Lions

IV. Turning-Point Theory and Applications

Turning-Point Problems for Ordinary Differential Equations of Hydrodynamic Type 297
William H. Reid

The General Connection-Formula Problem for Linear Differential Equations of the Second Order 317
Frank W. J. Olver

Connection Formulas and Behavior in the Large for Solutions of Linear Differential Equations Depending Singularly on a Parameter 345
Anthony W. Leung

Connection Problems in the Parameterless Case: Progress and More Problems 357
Donald A. Lutz

Some New Results on Power-Series Solutions of Algebraic Differential Equations 379
Yasutaka Sibuya and Steven Sperber

Index 405

Contributors

Numbers in parentheses indicate the pages on which the authors' contributions begin.

Julian D. Cole (19), *Department of Mathemetics, University of California at Los Angeles, Los Angeles, California 90024*

Wiktor Eckhaus (1), *Mathematisch Instituut, Rijksuniversiteit Utrecht, 3508 TA Utrecht, The Netherlands*

Pieter P. N. de Groen (149), *Department Wiskunde, Vrije Universiteit Brussel, 1050 Brussel, Belgium*

Fred A. Howes (41), *Department of Mathematics, University of California, Davis, Davis, California 95616*

Jerry Kevorkian (191), *Applied Mathematics Group, University of Washington, Seattle, Washington 98195*

Nancy Kopell (173), *Mathematics Department, Northeastern University, Boston, Massachusetts 02115*

Anthony W. Leung (345), *Department of Mathematical Sciences, University of Cincinnati, Cincinnati, Ohio 45221*

Jacques L. Lions (277), *Institut National de Recherche en Informatique et en Automatique, Voluceau-Rocquencourt, B.P. 105, 78150 Le Chesnay, France*

Donald A. Lutz (357), *Department of Mathematical Sciences, University of Wisconsin-Milwaukee, Milwaukee, Wisconsin 53201*

Bernard J. Matkowsky (109), *Department of Engineering Sciences and Applied Mathematics, Northwestern University, Evanston, Illinois, 60201*

Ali H. Nayfeh (223), *Department of Engineering Science and Mechanics, Virginia Polytechnic Institute and State University, Blacksburg, Virginia 24061*

Frank W. J. Olver (317), *University of Maryland, College Park, Maryland 20742; and National Bureau of Standards, Washington, DC 20234*

Robert E. O'Malley (87), *Program in Applied Mathematics, University of Arizona, Tuscon, Arizona 85721*

William H. Reid (297), *Department of Mathematics, The University of Chicago, Chicago, Illinois 60637*

Yasutaka Sibuya (379), *School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455*

Steven Sperber (379), *School of Mathematics, University of Minnesota, Minneapolis, Minnesota 55455*

This page intentionally left blank

Preface

This volume collects the invited lectures presented at an Advanced Seminar on Singular Perturbation and Asymptotics in Madison, Wisconsin on May 28–30, 1980 under the auspices of the Mathematics Research Center of the University of Wisconsin—Madison, sponsored by the United States Army under Contract No. DAAG29-80-C-0041 and supported by the Office of Naval Research, U.S. Navy, under grant N00014-80-G-0108.

The subject of singular perturbations, not to mention asymptotics, is too large for a single conference, and the selection of topics reflects both areas of recent research activity and advances and areas of interest to Professor Wolfgang Wasow, in whose honor this advanced seminar was organized. We are indebted to the contributors for achieving not only a high level of excellence in individual contributions, but also a coherent, cooperative survey of an influential field of applied mathematics.

We also thank Gladys Moran for the expert handling of the conference details and Elaine DuCharme for putting the volume together and compiling the index.

Richard E. Meyer
Seymour V. Parter

This page intentionally left blank

On Some Basic Concepts in the Analysis of Singular Perturbations

Wiktor Eckhaus

The methods and techniques of singular perturbations have been extremely successful in dealing with problems from many branches of sciences. Originating in fluid dynamics, the study of singular perturbations has spread into a large and diversified population of scientists, with interests ranging from engineering and biology to almost pure mathematics.

The frequent occurrence of singular perturbations in applications can be explained by the observation that whenever some basic mathematic model of some phenomena is improved by incorporating some of the effects that were first neglected, the improved model is most likely to be a problem of singular perturbations. On the other hand, the theoretical interest comes from the fact that the analysis of singular perturbations is not a straight-forward generalization and extension of classical asymptotic analysis and perturbation theory, but rather an entirely new discipline.

Every practitioner of singular perturbations uses, implicitely or explicitely, certain concepts which are commonly accepted as the basis for the method of analysis. In this lecture we shall discuss the basic concepts (in a formulation taken from [1]), which will also lead us to certain essential and still open questions.

ISBN 0-12-493260-6

1. CLASSICAL PERTURBATION ANALYSIS.

What are problems of singular perturbations? The adjective "singular" is a negation of "regular", let us therefore first look briefly at "regular" problems.

Let V and F be two linear spaces of functions and L_ε some given mapping of V into F, usually a differential operator that contains a "small" parameter ε. We are concerned with the problem of determining an element $\Phi \in V$ such that for some given $F \in F$ one has

$$L_\varepsilon \Phi = F.$$

For simplicity of exposition we suppose here that any boundary conditions, or initial conditions imposed on Φ are incorporated in the definition of the space V. This, of course, is only possible if the boundary conditions or initial conditions are linear and homogeneous. In the classical perturbation analysis L_ε is decomposed into

$$L_\varepsilon = A + P_\varepsilon$$

where A is independent of ε, and P_ε is considered as a perturbation. One thus has

$$A\Phi = F - P_\varepsilon \Phi, \quad \Phi \in V, \quad F \in F.$$

Furthermore, the decomposition must be such that the inverse $A^{-1} : F \to V$ exists, at least locally. One can then write

$$\Phi = A^{-1} [F - P_\varepsilon \Phi].$$

It is now natural to suppose that an approximation of Φ will be given by the function

$$\Phi_{as} = A^{-1} F.$$

Indeed one has

$$\Phi - \Phi_{as} = A^{-1}[F - P_\varepsilon \Phi] - A^{-1} F$$

and it should be clear that if the perturbation $P_\varepsilon \Phi$ can be expected to be small in some suitable sense, and A^{-1} is a continuous operator with sufficiently nice properties, then one should be able to prove that $\Phi - \Phi_{as}$ is small. This last step can be accomplished by proving convergence of an obvious iteration procedure, or in a more abstract setting, by using a contraction argument in a suitably defined Banach space. All that is needed for the analysis are properties of the operators A^{-1} and P_ε, which are both given by construction.

In singular perturbation problems a decomposition of the operator L_ε as described above is generally impossible and the classical procedure fails at the outset. Let us illustrate this by an example.

We consider first the problem of determining $\Phi(x,\varepsilon)$, $x \in \mathcal{D} \subset \mathbb{R}^n$, as the solution of

$$\Delta\Phi - \varepsilon q(x)\Phi = F, \qquad x \in \mathcal{D}$$

where Δ is the Laplace operator, $q(x)$ a continuous function and $\mathcal{D}$ a bounded domain, on the boundary of which we impose $\Phi = 0$. Then obviously the classical procedure holds, with $A = \Delta$, and A^{-1} is explicitely given with the aid of the Greens function.

Let us now modify the differential equation into

$$\varepsilon \Delta \Phi - q(x)\Phi = F.$$

Furthermore, let us stipulate that we look for approximations in the sense of uniform convergence, that is, for any function $f(x,\varepsilon)$:

$f(x,\varepsilon) = O(\delta(\varepsilon))$ means $\sup_{x\in\mathcal{D}} | f(x,\varepsilon) | = O(\delta(\varepsilon))$

$f(x,\varepsilon) = o(\delta(\varepsilon))$ means $\lim_{\varepsilon\to 0} \frac{1}{\delta(\varepsilon)} \sup_{x\in\mathcal{D}} | f(x,\varepsilon) | = 0.$

It should be clear that the classical procedure does not apply. The "unperturbed" equation reads

$$- q(x)\ \Phi_0 = F.$$

The solution Φ_0 will in general not satisfy the conditions imposed on the boundary, and cannot be an approximation of Φ in the whole domain $\mathcal{D}$.

2. THE GENERAL PROCEDURE IN SINGULAR PERTURBATIONS.

The elegant classical perturbation analysis combines the construction of approximations and the proof of their validity into one line of thinking. In singular perturbations the complete analysis requires various different ingredients. In general terms one can distinguish two main parts of the procedure:

I. Heuristic analysis. Using some deductive thinking, but also much induction from experience, one constructs a function which, by a reasonable expectation, should be an approximation to a solution of the problem under consideration. The heuristic reasoning employs some rigorous analysis, but also some "principles" which almost always work, yet have never been demonstrated. These principles, and other inductive evidence, will be the subject of a large part of this lecture.

II. Analysis of formal approximations.
The heuristic analysis of the problem

$$L_\varepsilon \Phi = F,\ \Phi \in V,\ F \in \mathcal{F}$$

usually produces a function Φ_{as} which satisfies

$$L_\varepsilon \Phi_{as} = F + \rho, \ \Phi_{as} \in V$$

with

$$\rho = o(1).$$

We then call Φ_{as} a *formal approximation.*

A formal approximation thus is a solution of a neighbouring problem, and the expectation that Φ_{as} will be an approximation of Φ is based on the expectation that two neighbouring problems will have neighbouring solutions. Such expectation seems reasonable in general. However, in problems of singular perturbations, there remains at this stage a nontrivial, and sometimes difficult, task to prove that a formal approximation indeed is an approximation of a solution of the problem under consideration.

Let us first show, by an example, that the problem of proving the validity of a formal approximation is not a trivial one.

Let $\Phi(x,\varepsilon)$, $x \in [0,A]$, $A > 0$, $\varepsilon \in (0,\varepsilon_0]$ be solution of

$$\varepsilon \frac{d\Phi}{dx} - \Phi = e^{-\frac{1}{x}}, \ \Phi(0,\varepsilon) = 0.$$

By an obvious iteration one obtains

$$\Phi_{as}^{(m)}(x,\varepsilon) = \sum_{n=0}^{m} \varepsilon^n \phi_n(x)$$

with

$$\phi_0(x) = -e^{-\frac{1}{x}}, \ \phi_n = \frac{d}{dx} \phi_{n-1}, \ n = 1,\ldots.$$

The function $\Phi_{as}^{(m)}(x,\varepsilon)$, for any m, satisfies

$$\varepsilon \frac{d\Phi_{as}^{(m)}}{dx} - \Phi_{as}^{(m)} = e^{-\frac{1}{x}} + o(\varepsilon^m); \ \Phi_{as}^{(m)}(0,\varepsilon) = 0.$$

On the other hand, the exact solution reads

$$\Phi(x,\varepsilon) = \frac{1}{\varepsilon} e^{\frac{1}{\varepsilon}x} \int_0^x e^{-\frac{1}{\varepsilon}x'} e^{\frac{1}{x'}} dx'.$$

An easy exercise shows that for any $x > 0$ the function $\Phi(x,\varepsilon)$ grows without bounds as $\varepsilon \downarrow 0$. Hence $\Phi_{as}^{(m)}(x,\varepsilon)$, which is bounded for $\varepsilon \downarrow 0$, cannot be an approximation of Φ.

The example given above, although disturbing, is not meant to create general mistrust in formal approximations. For large classes of problems the proof of validity of a formal approximation can be achieved, sometimes even by very simple means. This is exemplified by:

Lemma. Let L_ε *be a linear operator and suppose that for all* $\widetilde{F} \in \widetilde{\mathcal{F}} \subset \mathcal{F}$ *the solution* $\widetilde{\Phi}$ *of*

$$L_\varepsilon \widetilde{\Phi} = \widetilde{F}, \quad \widetilde{\Phi} \in V$$

is bounded for $\varepsilon \downarrow 0$.

Let Φ *satisfy*

$$L_\varepsilon \Phi = F, \quad \Phi \in V, \quad F \in \widetilde{\mathcal{F}}$$

and let $\Phi_{as}^{(m)}$ *satisfy*

$$L_\varepsilon \Phi_{as}^{(m)} = F + \rho_m, \quad \Phi_{as}^{(m)} \in V$$

with

$$\rho_m = \delta_m(\varepsilon)\bar{\rho}_m, \quad \bar{\rho}_m = O(1)$$

If $\bar{\rho}_m \in \widetilde{\mathcal{F}}$, then

$$\Phi - \Phi_{as}^{(m)} = O(\delta_m(\varepsilon)).$$

The proof of the lemma is trivial. The essential information which makes it possible to establish the result given in the lemma is an a priori estimate on the boundedness of solutions. In applications such estimates can often be deduced from some general information about the class of problems under consideration. For example, in elliptic problems one can use the Maximum Principle. In recent years many results that can be useful for proving validity of formal approximations have been established.

The interested reader can consult for a survey [1] Chapter 6, which also contains methods and results for nonlinear problems.

3. REGULAR EXPANSIONS AND LOCAL EXPANSIONS.

We return now to the heuristic analysis. For simplicity of exposition we consider for the function $\Phi(x,\varepsilon)$ a problem of the structure

$$\varepsilon L_1\Phi + L_0 = F, \quad x \in \mathcal{D} \subset \mathbb{R}^n$$

where L_1 and L_0 are linear and independent of ε. Φ is further subject to some boundary conditions or initial conditions.

In a first step, by an obvious iteration, one constructs a *regular expansion*

$$\Phi(x,\varepsilon) \approx \sum_{n=0}^{m} \delta_n(\varepsilon)\, \phi_n(x), \quad \delta_{n+1} = o(\delta_n), \ \forall n$$

which is usually called *an outer expansion*.
This expansion will manifestly fail near some manifolds (such as the boundary of the domain in the example given in section 1). Or may be suspect for some reasons near some other manifolds.

One now attempts to construct, in the neighbourhood of such a manifold, *a local expansion*, also called *boundary layer expansion*, or *inner expansion*, of the following structure:

Consider for simplicity the one dimensional situation and let x_0 be the point near which the regular expansion fails. A *stretched local variable* is defined by

$$\xi = \frac{x-x_0}{\delta_s(\varepsilon)}, \quad \delta_s(\varepsilon) = o(1).$$

The effect of this magnifying glass on the function $\Phi(x,\varepsilon)$ is given by

$$\Phi(x_0+\delta_s(\varepsilon)\xi,\varepsilon) = \Phi^*(\xi,\varepsilon)$$

This suggests an expansion

$$\Phi^*(\xi,\varepsilon) \approx \sum_{n=0}^{m} \delta^*_n(\varepsilon)\ \psi_n(\xi),\ \delta^*_{n+1} = o(\delta^*_n),\ \forall_n.$$

The question immediately arises: what is the proper choice of the stretched local variable, i.e. the proper choice of the function $\delta_s(\varepsilon)$?

In a simple situation the question may be answered by an educated guess. However, the intellectual curiosity makes one look for some basic principle. Such a guiding principle becomes a necessity in more complicated situations when one has near x_0 a multiple-layer structure, i.e. more than one different local variables (with different choices of $\delta_s(\varepsilon)$) and corresponding local expansions are needed to remove the failure of the regular expansion.

4. THE CORRESPONDENCE PRINCIPLE.

We shall need a bit of formalism, describing the effect of transformations of variables and expansions on functions and on operators.

A regular expansion of a function $\Phi(x,\varepsilon)$ in some subdomain will be denoted by:

$$E^m_x\ \Phi = \sum_{n=0}^{m} \delta_n(\varepsilon)\ \phi_n(x).$$

Similarly, considering the expansion after transformation to some local variable ξ, we shall write

$$E^m_\xi\ \Phi = \sum_{n=0}^{m} \delta^*_n(\varepsilon)\ \psi_n(\xi).$$

We now ask the question: what makes a local variable into an important one?

Consider, near some fixed x_0. two local variables:

$$\xi_1 = \frac{x-x_0}{\delta_1(\varepsilon)}, \quad \xi_2 = \frac{x-x_0}{\delta_2(\varepsilon)},$$

with $\delta_1 = o(\delta_2)$, or vice versa.
The function $E^m_{\xi_1} \Phi$ can be transformed into the local variable ξ_2 and reexpanded. The result is $E^p_{\xi_2} E^m_{\xi_1} \Phi$.

We shall say that $E^p_{\xi_2} \Phi$ *is contained in* $E^m_{\xi_1} \Phi$ if

$$E^p_{\xi_2} E^m_{\xi_1} \Phi = E^p_{\xi_2} \Phi.$$

Obviously, an approximation that is contained in another approximation, is not very important.
We shall therefore say that:

A local approximation $E^m_{\xi_\nu} \Phi$ *is significant if* $E^m_{\xi_\nu} \Phi$ *is not contained in* $E^q_x \Phi$ *for all* q, *and there exists no local approximation* $E^p_{\xi_\mu}$ *with* ξ_μ *different from* ξ_ν *which would contain* $E^m_{\xi_\nu} \Phi$.

A local variable ξ *which produces a significant approximation is called a boundary layer variable.*

We now consider the effect of transformation to local variables on the operator $\varepsilon L_1 + L_0$. The transformation produces, in an obvious way, an operator L_ε which acts on functions $\Phi^*(\xi,\varepsilon)$. This operator usually admits, in a formal way, an expansion

$$L_\varepsilon \approx \sum_n \tilde{\delta}_n(\varepsilon) L_n, \quad \tilde{\delta}_{n+1} = o(\tilde{\delta}_n)$$

where L_n are operators independent of ε. We call L_0 the *degeneration* of L in the ξ variable.

For example consider the operator

$$L = \varepsilon\frac{d^2}{dx^2} - q(x)$$

and the local variable

$$\xi = \frac{x}{\sqrt{\varepsilon}}$$

then the degeneration is

$$L_0 = \frac{d^2}{d\xi^2} - q(0).$$

The degenerations are the governing operators in the construction of the terms of the local expansions.

We now compare degenerations in two different local variables ξ_1, ξ_2 as we did before for the local approximations.

Let $L_0^{(1)}$ be the degeneration in the ξ_1 variable. Transforming to the ξ_2 variable we get an operator $L^{(1.2)}$ (acting on functions $\Phi^*(\xi_2,\varepsilon)$), which again has a degeneration $L_0^{(1.2)}$. We shall say that

The degeneration $L_0^{(2)}$ (in the ξ_2 variable) is contained in $L_0^{(1)}$ if

$$L_0^{(1.2)} = L_0^{(2)}.$$

A degeneration L_0 in a local variable ξ is significant if there exists no local variable different from ξ such that the corresponding degeneration would contain L_0.

We can now formulate the following heuristic principle:

Boundary layer variables correspond to significant degenerations.

In other words: when constructing local approximations it is sufficient to consider those local variables for which the degenerations are significant.