

Theories of Probability

An Examination
of Foundations

TERRENCE L. FINE

ACADEMIC PRESS

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TERRENCE L. FINE

*School of Electrical Engineering
Cornell University
Ithaca, New York*



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Contents

<i>Preface</i>	xi
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I.	Introduction	
IA.	Motivation	1
IB.	Types of Probability Theories	3
	1. Characteristics of Probability Theories	3
	2. Domains of Application	3
	3. Forms of Probability Statements	4
	4. Relations between Statements`	5
	5. Measurement	6
	6. Goals and Their Achievement	6
	7. Tentative Classification of Some of the Theories to Be Discussed	7
IC.	Guide to the Discussion	10
	1. Outline of Topics	10
	2. References	13
	3. Known Omissions	13
	References	14
II.	Axiomatic Comparative Probability	
IIA.	Introduction	15
IIB.	Structure of Comparative Probability	16

1. Fundamental Axioms That Suffice for the Finite Case	16
2. Compatibility with Quantitative Probability	18
3. An Archimedean Axiom	19
4. An Axiom of Monotone Continuity	20
5. Compatibility of CP Relations	21
IIC. Compatibility with Finite Additivity	22
1. Introduction	22
2. Necessary and Sufficient Conditions for Compatibility	23
3. Should CP Be Compatible with Finite Additivity?	24
4. Sufficient Conditions for Compatibility with Finite Additivity	25
IID. Compatibility with Countable Additivity	27
II E. Comparative Conditional Probability	28
1. Comparative Conditional Probability as a Ternary Relation	28
2. Comparative Conditional Probability as a Quaternary Relation	29
3. Relationship between QCCP and CP	30
4. Bayes' Theorem	31
5. Compatibility of CCP with Kolmogorov's Quantitative Probability	32
II F. Independence	32
1. Independent Events	32
2. Mutually Independent Events	35
3. Independent Experiments	36
4. Concluding Remarks	36
IIG. Application to Decision-Making	37
1. Formulation of the Decision Problem	37
2. Axioms for Rational Decision-Making	38
3. Representations of \succsim_α and \succsim	40
4. Extensions	41
IIH. Expectation in Comparative Probability	42
Appendix: Proofs of Results	43
References	56
 III. Axiomatic Quantitative Probability	
IIIA. Introduction	58
IIIB. Overspecification in the Kolmogorov Setup:	
Sample Space and Event Field	60
1. The Sample Space Ω	60
2. The σ -Field of Events \mathcal{F}	61
3. The \mathcal{A} - and π -Fields of Events	63
4. The von Mises Field of Events	64
IIIC. Overspecification in the Probability Axioms: View from Comparative Probability	65
1. Unit Normalization and Nonnegativity Axioms	65
2. Finite Additivity Axiom	66
3. Continuity Axiom	67
IIID. Overspecification in the Probability Axioms: View from Measurement Theory	68
1. Fundamentals of Measurement Theory	68
2. Probability Measurement Scale	70
3. Necessity for an Additive Probability Scale	72

III E.	Further Specification of the Event Field and Probability Measure	74
1.	Preface	74
2.	Selecting the Event Field	74
3.	Selecting P	75
III F.	Conditional Probability	76
1.	Structure of Conditional Probability	76
2.	Motivations for the Product Rule	78
IIIG.	Independence	80
1.	Role of Independence	80
2.	Structure of Independence	81
IIIH.	The Status of Axiomatic Probability	83
	References	84

IV. *Relative-Frequency and Probability*

IV A.	Introduction	85
IV B.	Search for a Physical Interpretation of Probability Based on Finite Data	86
1.	Reduction through Exchangeability	86
2.	Maturity of Chances and Practical Certainty	88
IV C.	Search for a Physical Interpretation of Probability Based on Infinite Data	89
1.	Introduction	89
2.	Definition of Apparent Convergence and Random Binary Sequence	91
3.	Relations between Apparent Convergence of Relative-Frequency and Randomness	92
4.	Conclusion	94
IV D.	Bernoulli/Borel Formalization of the Relation between Probability and Relative-Frequency: Strong Laws of Large Numbers	95
IV E.	Von Mises' Formalization of the Relation between Probability and Relative-Frequency: The Collective	97
IV F.	Role of Relative-Frequency in the Measurement of Probability	100
IV G.	Predication of Outcomes from Probability Interpreted as Relative-Frequency	102
IV H.	The Argument of the "Long Run"	103
IV I.	Preliminary Conclusions and New Directions	103
IV J.	Axiomatic Approaches to the Measurement of Probability	105
1.	Introduction	105
2.	An Approach without Explicit Probability Models	105
3.	The Approach of Statistical Decision Theory	108
4.	Bayesian Approaches	109
5.	What Has Been Accomplished	110
IV K.	Measurement of Comparative Probability: Induction by Enumeration	111
1.	Introduction	111
2.	Defining Induction by Enumeration	111
3.	Justifying Induction by Enumeration	113
IV L.	Conclusion	115
	References	116

V. *Computational Complexity, Random Sequences, and Probability*

VA.	Introduction	118
VB.	Definition of Random Finite Sequence Using Place-Selection Functions	119
VC.	Definition of the Complexity of Finite Sequences	121
	1. Background to Absolute Complexity	121
	2. A Definition of Absolute Complexity	123
	3. Properties of the Definition of Absolute Complexity	125
	4. Other Definitions of Absolute Complexity	127
	5. Conditional Complexity	127
	6. Ineffectiveness of Complexity Calculations	129
VD.	Complexity and Statistics	130
	1. Statistical Tests for Goodness-of-Fit	130
	2. Universal Statistical Tests	131
	3. Role of Complexity in Defining Probabilistic Models	132
	4. A Relation between Complexity and Critical Level	133
VE.	Definition of Random Finite Sequence Using Complexity	134
VF.	Random Infinite Sequences	136
	1. Complexity-Based Definitions	136
	2. Statistical Definition	137
	3. Relations between the Complexity and Statistical Definitions	138
VG.	Exchangeable and Bernoulli Finite Sequences	138
	1. Exchangeable Sequences	138
	2. Bernoulli Sequences	140
VH.	Independence and Complexity	141
	1. Introduction	141
	2. Relative-Frequency, Complexity, and Stochastic Independence	142
	3. Empirical Independence	144
	4. Conclusions	145
VI I.	Complexity-Based Approaches to Prediction and Probability	146
	1. Introduction	146
	2. Comparative Probability	147
	3. Solomonoff's Definitions of Probability	149
	4. Critique of the Complexity Approach to Probability	150
VI J.	Reflections on Complexity and Randomness: Determinism versus Chance	152
VK.	Potential Applications for the Complexity Approach	153
	Appendix: Proofs of Results	154
	References	164

VI. *Classical Probability and Its Renaissance*

VIA.	Introduction	166
VIB.	Illustrations of the Classical Argument and Assignments of Equiprobability	167
VIC.	Axiomatic Formulations of the Classical Approach	170
	1. The Principle of Invariance	170
	2. Information-Theoretic Principles	172

VID.	Justifying the Classical Approach and Its Axiomatic Reformulations	174
1.	Classical Probability and Decision under Uncertainty	174
2.	Principle of Invariance	175
3.	Information-Theoretic Principles	176
VIE.	Conclusions	176
	References	177

VII. *Logical (Conditional) Probability*

VIIA.	Introduction	179
VII B.	Classificatory Probability and Modal Logic	181
VII C.	Koopman's Theory of Comparative Logical Probability	183
1.	Structure of Comparative Probability	183
2.	Relation to Conditional Quantitative Probability	185
3.	Relation to Relative-Frequency	186
4.	Conclusions	186
VII D.	Carnap's Theory of Logical Probability	187
1.	Introduction	187
2.	Compatibility with Rational Decision-Making	188
3.	Axioms of Invariance	190
4.	Learning from Experience	193
5.	Selection of a Unique Confirmation Function	195
VII E.	Logical Probability and Relative-Frequency	197
VII F.	Applications of C^* and C_λ	198
VII G.	Critique of Logical Probability	201
1.	Roles for Logical Probability	201
2.	Formulation of Logical Probability	202
3.	Justifying Logical Probability	203
	Appendix: Proofs of Results	204
	References	210

VIII. *Probability as a Pragmatic Necessity: Subjective or Personal Probability*

VIIIA.	Introduction	212
VIII B.	Preferences and Utilities	214
VIII C.	An Approach to Subjective Probability through Reference to Preexisting Probability	216
1.	Axioms of Anscombe and Aumann Type	216
2.	The Associated Objective Distribution	218
VIII D.	Approaches to Subjective Probability through Decision-Making	219
1.	Formulation of Savage	219
2.	Formulation of Krantz and Luce	222
VIII E.	Subjective versus Arbitrary: Learning from Experience	226
VIII F.	Measurement of Subjective Probability	228
VIII G.	Roles for Subjective Probability	229
VIII H.	Critique of Subjective Probability	230
1.	Role of Subjective Probability	230

2. Formulation of Subjective Probability	231
3. Measurement of Subjective Probability	233
4. Justification of Subjective Probability	234
References	236

IX. Conclusions

IXA. Where Do We Stand?	238
1. With Respect to Definitions of Probability	238
2. With Respect to Definitions of Associated Concepts	240
IXB. Probability in Physics	242
1. Introduction	242
2. Statistical Mechanics	242
3. Quantum Mechanics	244
4. Conclusions	245
IXC. What Can We Expect from a Theory of Probability?	246
IXD. Is Probability Needed?	249
References	251

<i>Author Index</i>	253
<i>Subject Index</i>	256

Preface

My interest in the foundations of probability was aroused by difficulties I encountered when first faced with applying a conventional (Kolmogorov's axioms and a limit of a relative-frequency interpretation) theory of probability to actual electrical engineering problems. Efforts to understand, usefully formulate, and resolve the problems encountered in the design and analysis of inference and decision-making systems led, it now seems inexorably, to a study of the foundations of probability. As I gradually became aware of the issues and proposals that constitute the present subject of the foundations of probability, I found myself drawn to their consideration not only for the primary pragmatic reasons but also out of respect for the breadth, depth, and provocative originality of many of the contributions to this study. It is my hope that the reader will share this appreciation.

My aim is to address all who explicitly use some theory of probability but who may not be aware of the criticisms of their preferred theory or the claims of alternative theories. Many of the difficulties encountered by engineers, physical and social scientists, and philosophers are, perhaps, attributable to misapprehensions as to the nature of the concepts of probability on which they rely. It is expected that the reader has some knowledge of a particular theory of probability. The many well-written texts on probability discouraged the inclusion of enough material

to make this book self-contained as an introduction to probability. It is recommended that proofs be ignored, except by those interested in research in specific areas. To underscore this, the proofs in Chapters II, V, and VII are appended to their respective chapters, and the few proofs in the other chapters are mainly worked into the discussion.

This book can at best describe aspects of the present stage in the nascent study of theories of probability. It should raise more questions than it answers. I will be well satisfied if the critical view of probability presented stimulates the reader to the thought and research necessary to lead us all eventually to a better understanding of the work so many of us are engaged in. It is my belief that an improved understanding of the foundations of probability will induce far reaching changes in engineering and scientific practice and not merely lead to an improved justification for what we presently do.

Some years ago I set out to write a survey paper on decision theory as practiced in electrical engineering. Feeling that some of the problems of decision theory could best be understood when referred to the problems with the underlying notion of probability, I wrote an introductory section on this issue. My dissatisfaction with each draft was only temporarily allayed by an expanded redraft. This book is in fact the still unsatisfactory introduction to that as yet unwritten paper. As to its publication, I take comfort in the words attributed to Cardinal Newman that "Nothing would be done at all if a man waited till he could do it so well that no one could find fault with it."

My great debt to the many contributors to the foundations of probability, and especially to A. N. Kolmogorov, R. von Mises, and L. J. Savage, is evident throughout the book. Less evident, but no less pervasive, is the influence of discussions I have had with Messrs. Max Black, Thomas M. Cover, Zoltan Domotor, Arthur Fine, Peter C. Fishburn, Michael A. Kaplan, R. Duncan Luce, Leonard J. Savage, Herbert Shank, Georg H. von Wright, and with my wife, Susan Woodward Fine.

Introduction

IA. Motivation

Formal uses for probability and its associated concepts are found in the construction of models of random phenomena, the design of inference and decision-making systems, statements and verifications of the applicability of scientific laws, and attempts to understand knowledge and induction. Informal uses for probability include the Butlerian view of probability as a guide to life and the frequent appearance of the words “probably” and “likely” in ordinary discourse. Notwithstanding the importance of probability in the explication of knowledge and induction, its roles in the verification of laws or “as a guide to life,” or its prevalence in discourse, we leave the analyses of these issues to suitably trained philosophers. Our concern is primarily with those concepts of probability that are important for the modeling of random phenomena and the design of information-processing systems.

Methods for modeling the random phenomena of chance and uncertainty and the design of inference and decision-making systems are of great importance in fields as diverse as engineering and the physical and social sciences. In electrical engineering, areas such as communications, detection, pattern classification, and stochastic control owe their very formulation to concepts of probability theory. The fundamental work