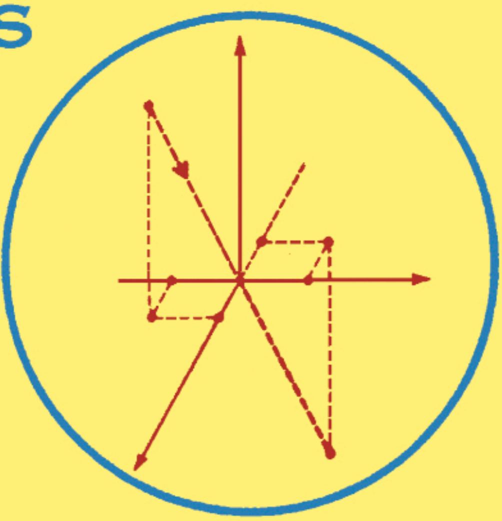
INTRODUCTORY COLLEGE MATHEMATICS

WITH LINEAR ALGEBRA AND FINITE MATHEMATICS



Harley Flanders
Justin J. Price

VECTORS

$$\mathbf{u} = (a, b), \qquad \mathbf{v} = (c, d)$$
$$\mathbf{u} + \mathbf{v} = (a + c, b + d)$$
$$\mathbf{u} \cdot \mathbf{v} = ac + bd$$
$$\|\mathbf{u}\| = \sqrt{a^2 + b^2}$$

POLAR COORDINATES

$$\begin{cases} x = r \cos \theta \\ y = r \sin \theta \end{cases} \qquad \begin{cases} x^2 + y^2 = r^2 \\ \cos \theta = \dfrac{x}{r}, \quad \sin \theta = \dfrac{y}{r} \end{cases}$$

COMPLEX NUMBERS

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i$$
$$\overline{a + bi} = a - bi$$
$$|a + bi|^2 = a^2 + b^2$$
$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

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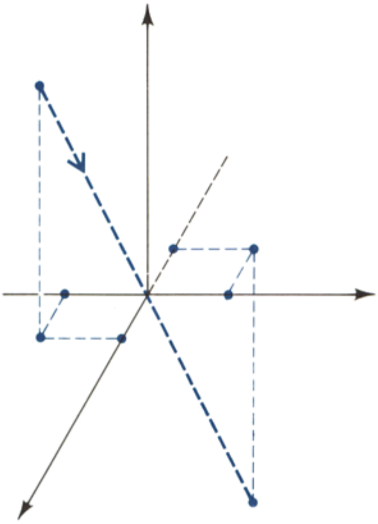

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To the Memories of Anna B. Price and Abraham White

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PREFACE

OBJECTIVES

This book is an introduction to college mathematics. It aims at providing a working knowledge of

1. basic functions: polynomial, rational, exponential, logarithmic, and trigonometric;
2. graphing techniques and the numerical aspects and applications of functions;
3. two and three dimensional vector methods and analytic geometry;
4. the fundamental ideas of linear algebra;
5. complex numbers, elementary combinatorics, the binomial theorem, and mathematical induction.

APPROACH

A solid understanding of these topics is essential for the study of calculus, statistics, computing science, the physical and biological sciences, engineering, and economics. Equally essential is the ability to apply the concepts, to solve problems, and to get answers. We have tried to achieve a balance between theory and practice, between thought problems and drill.

Our presentation is concrete; we believe that a formal theorem–proof style at this level quickly deadens the interest of most students. Accordingly, we have integrated theory with applications, always motivating the abstract by down-to-earth problems.

SUBJECT MATTER

The topics treated fall into five categories; let us remark briefly on each.

A. Elementary Functions (Chapters 1–8). We stress the graphical properties of the basic functions in analysis. For example, one section is devoted to graphing factored polynomials, something which frequently troubles even advanced students. This fits in nicely with theoretical properties of zeros of polynomials.

We stress also algebraic and numerical properties of these functions. Some sections contain review material and may be omitted according to the needs of the class. Some instructors may choose to omit the optional sections on computation with logarithms and solution of triangles.

B. Plane Analytic Geometry (Chapters 9–10). We treat the subject in modern vector style. For instance, in Chapter 9 we show that one basic vector equation $\mathbf{x} = \mathbf{x}_0 + t\mathbf{v}$ contains all the usual forms of equations for lines.

Chapter 10 deals with conics, loci, and polar coordinates. Here, as elsewhere, we have tried to present those topics needed in other courses. This chapter includes an extra section of particularly challenging exercises.

C. Space Geometry and Linear Algebra. We believe in early introduction to the tools of linear algebra, but motivated geometrically in two and three dimensions. In Chapters 11–13, we present an integrated approach to space geometry and linear algebra.

The vector ideas of Chapter 9 generalize easily to three dimensions. We introduce matrices at once and operate with them on two and three dimensional vectors. We discuss linear systems of equations, $A\mathbf{x} = \mathbf{b}$, introducing 2×2 and 3×3 determinants, Cramer's Rule, etc.

The concepts of linear independence and spanning arise naturally in the study of lines, planes, intersections, etc.

In Chapter 13 we discuss linear transformations and their associated matrices. We attempt to solidify these ideas by concrete examples: reflections, projections, rotations, etc.

D. Complex Numbers. In Chapter 14, we present the complex number system as a natural extension of the reals. We discuss the Fundamental Theorem of Algebra and its implications. For example, we include an optional section on complex matrices and the existence of complex eigenvectors.

E. Finite Mathematics. Chapter 15 is concerned with some basic skills that students are expected to acquire somewhere along the way. Painful experience shows that many do not; it is distressing that often juniors and seniors cannot shift a summation index or give a proof by induction.

The chapter begins with finite sequences and sums, providing drill in the use of subscripts and summation notation. Then it discusses basic principles of counting, the binomial and the multinomial theorems.

Finally it treats mathematical induction. To many students induction is a vague process in which "you add $n+1$ to both sides," presumably to prove something about sums of integers. We hope to break out of this rut by demonstrating various ways induction is actually used, for instance, in inductive definitions. We believe our problem set is truly representative. Often texts give little else than a list of summation formulas to verify (nothing to discover). Little wonder that the student soon begins to believe that induction is indeed "adding $n + 1$ to both sides."

FEATURES

The 130 worked examples are an essential part of the text. The more than 2100 exercises are graded in difficulty, and harder ones are marked with an asterisk. Answers to alternate exercises are provided.

There are over 250 figures. Accurate graphs and drawings are essential to this subject, and a picture often is worth more than a thousand words. We emphasize plain line drawings with uniform shading, drawings that the student can and should learn to do himself with simple tools: a couple of pencils, ruler, protractor, compass, and graph paper. Illustrations by air brush and other fancy equipment of the professional artist can be admirable and revealing to the student, but he cannot draw them himself. We think the philosophy of learning mathematics by doing mathematics extends as well to graphical skills.

Each chapter ends with two sample tests of the type a student can expect at the end of a unit.

We include some basic numerical tables, adequate for all examples and exercises in this book. There is additional useful information inside the covers. We recommend for numerical problems the use of a slide rule, and for greater accuracy a good book of tables such as the *C.R.C. Standard Mathematical Tables*.

ACKNOWLEDGMENTS

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CHAPTER 1 FUNCTIONS AND GRAPHS

1. INTRODUCTION

Everyone is familiar with the use of graphs to summarize data (Fig. 1.1). The figure shows three typical graphs. There are many others; one sees graphs concerning length, time, speed, voltage, blood pressure, supply, demand, etc.

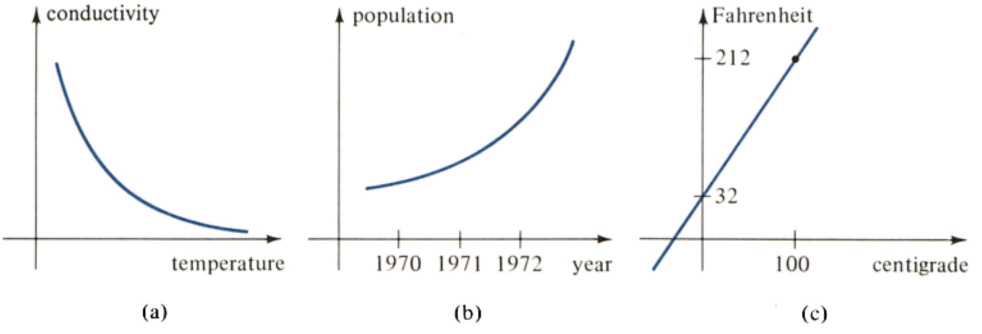

Fig. 1.1. common graphs

All graphs have an essential common feature; they illustrate visually the way one numerical quantity depends on (or varies with) another. In Fig. 1.1, (a) shows how the conductivity of a certain material depends on its temperature, (b) shows how the population depends on (varies with) time, and (c) shows how Fahrenheit readings depend on (are related to) centigrade readings.

Graphs are pictures of **functions**. Roughly speaking, a function describes the dependence of one quantity on another or the way in which one quantity varies with another. We say, for instance, that pressure is a function of temperature, or that population is a function of time, etc.

Functions lurk everywhere; they are the basic idea in almost every application of mathematics. Therefore, a great deal of study is devoted to their nature and properties. This book is largely an introduction to the more common functions in mathematics.

As Fig. 1.1 illustrates, a graph is an excellent tool in understanding the nature of a function. For it is a kind of "life history" of a function, to be seen at a glance. That is why there is so much emphasis on graphs in this book.

The functions and graphs we shall deal with concern quantities measured in the **real number system**. This consists, roughly speaking, of the familiar numbers of our experience. Before starting our study of functions and graphs, we shall devote a brief section to the real number system itself.

2. REAL NUMBERS

The real numbers are the common numbers of everyday life. Everyone is familiar with their arithmetic. In more advanced courses they are defined and developed rigorously. However, that is a deep and lengthy project, not in the spirit of this book. We shall be content to list the basic properties of real numbers for actual use.

First of all we can compute with real numbers, we can do arithmetic. In doing so we automatically make use of the basic rules listed below.

Associative laws

$$a + (b + c) = (a + b) + c \qquad a(bc) = (ab)c.$$

Commutative laws

$$a + b = b + a \qquad ab = ba.$$

Zero and unity laws

$$a + 0 = a \qquad a \cdot 1 = a.$$

Distributive laws

$$a(b + c) = ab + ac \qquad (a + b)c = ac + bc.$$

Inverse Laws

If a is any real number, then there is a unique real number $-a$ such that

$$a + (-a) = 0.$$

If a is any real number different from 0, then there is a unique real number a^{-1} such that

$$a \cdot a^{-1} = 1.$$

Recall that we write $a - b = a + (-b)$, and $a/b = ab^{-1}$ if $b \neq 0$.

Besides satisfying these arithmetic rules, the real number system also carries an **order relation**; we can say that one number is greater or less than another. We write "$a < b$" or "$b > a$" to mean "a is less than b", or equivalently, "b is greater than a". Actually it is often more convenient to write "$a \leq b$" meaning "a is less than or equal to b". Let us review the rules that govern the order relation.

Reflexivity

$$a \leq a$$

Anti-symmetry

$$\text{If } a \leq b \text{ and } b \leq a, \text{ then } a = b.$$

Transitivity

$$\text{If } a \leq b \text{ and } b \leq c, \text{ then } a \leq c.$$

There are important rules relating the algebraic operations and the order relation, but they have no common names:

$$\text{If } a \leq b, \text{ then } a + c \leq b + c.$$

$$\text{If } a \leq b \text{ and } c \geq 0, \text{ then } ac \leq bc.$$

$$\text{If } a \leq b, \text{ then } -b \leq -a.$$

$$\text{If } 0 < a < b, \text{ then } 0 < b^{-1} < a^{-1}.$$

$$\text{If } a \leq b, \text{ then } 0 \leq b - a;$$
$$\text{conversely, if } 0 \leq b - a, \text{ then } a \leq b.$$

ABSOLUTE VALUES

We often need the numerical size of a number, regardless of its sign. For instance, in some sense -10 is greater than 3, but we cannot write $-10 > 3$. For this reason we introduce the absolute value of a real number. We define the **absolute value** of a, written $|a|$, by:

$$|a| = \begin{cases} a & \text{if } \ a \geq 0 \\ -a & \text{if } \ a < 0. \end{cases}$$

Thus $|-10| = 10$, $|3| = 3$, $|0| = 0$. Now it is correct to write $|-10| > |3|$.

Absolute values satisfy the following rules:

$$|a| > 0 \text{ if } a \neq 0; \quad |a| = 0 \text{ if } a = 0.$$

$$\text{If } a \leq b \text{ and } -a \leq b, \text{ then } |a| \leq b.$$

$$|-a| = |a|.$$

$$|ab| = |a|\,|b|.$$

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}, \; b \neq 0.$$

$$|a + b| \leq |a| + |b| \qquad \text{(the triangle inequality).}$$

EXERCISES

Prove (using the rules):

1. $(a+b)(c+d) = da + cb + ac + bd$

2. $(a+b)(a-b) = a^2 - b^2$

3. $(a^2 + ab + b^2)(a-b) = a^3 - b^3$

4. $(a+a+a)(b+b+b+b) = 12ab$

5. if $a > 0$ and $b < 0$, then $ab < 0$

6. if $a < 0$ and $b < 0$, then $ab > 0$

7. if $a \neq 0$, then $a^2 > 0$

8. $-(a+b) = (-a) + (-b)$

9. $(-a)(-b) = ab$

10. $-(-a) = a$

11. if $ab \neq 0$, then $a \neq 0$ and $b \neq 0$

12. if $ab = 0$, then $a = 0$ or $b = 0$

13. if $a \neq 0$ and $b \neq 0$, then $(ab)^{-1} = a^{-1}b^{-1}$

14. if $b \neq 0$ and $d \neq 0$, then $(a/b)(c/d) = (ac)/(bd)$

15. if $b \neq 0$, $c \neq 0$, and $d \neq 0$, then $(a/b)/(c/d) = (ad)/(bc)$

16. if $b \neq 0$ and $d \neq 0$, then
$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

17. if $a \neq 0$ and $b \neq 0$, then $(a/b)^{-1} = b/a$

18. if $0 < a < b$, then $0 < a/b < 1$

19. if a, b, c, d are not all zero, then $a^2 + b^2 + c^2 + d^2 > 0$

20. $$\frac{1}{a^2+b^2+c^2+d^2} = \left(\frac{a}{a^2+b^2+c^2+d^2}\right)^2 + \cdots + \left(\frac{d}{a^2+b^2+c^2+d^2}\right)^2$$

21. $|ab| = |a|\,|b|$

22. $\left|\dfrac{a}{b}\right| = \dfrac{|a|}{|b|}$ if $b \neq 0$

23. if a and b have the same sign, or if either is zero, then $|a+b| = |a| + |b|$

24*. if $a \leq b$ and $-a \leq b$, then $|a| \leq$ b

25*. if a and b have opposite signs, then $|a+b| < |a| + |b|$

26*. (cont.) $|a| - |b| \leq |a-b| \leq |a| + |b|$.

Find all numbers a for which

27. $|a-3| = 1$

28. $|a+5| = 2$

29. $|a-5| \leq 3$

30. $|a+2| \leq 1$

31. $\left|\dfrac{a}{2}\right| \leq 1$

32. $|-4a| > 6$

33. $|a^2| = |a|^2$

34. $|a^3| = |a|^3$.

3. COORDINATES ON THE LINE AND PLANE

The real numbers provide labels for the points on a line. First, choose a point and mark it 0. Then choose a point to the right of 0 and mark it 1. In other words, choose a starting point, a unit length, and a positive direction (the direction from 0 towards 1). Then mark the points 2, 3, 4, $\cdots$ to the right and -1, -2, -3, $\cdots$ to the left (Fig. 3.1). (It is perfectly possible to take the positive direction to the left; perhaps that is the convention on some planet in some galaxy.)

Here we must make a fundamental assumption. We take it as an axiom that there is a perfect one-to-one correspondence between the points on the line and the system of real numbers. That means each point is assigned a unique real number label, and each real number labels exactly one point.

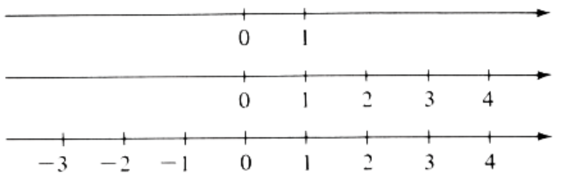

Fig. 3.1. *number line*

Because of this close association of the real number system and the set of points on a line, it is common to refer to a line "as" the real number system and to the real number system "as" a line. For instance, in a mathematical discussion, the real number 5.2 and the point labeled 5.2 might be considered the same. Although this is not correct logically, it almost never causes confusion; in fact it often sharpens our feeling for a problem.

Once the identification between real numbers and points on the line has been made, many arithmetic statements can be translated into geometric statements, and vice versa. Here are a few examples:

ARITHMETIC STATEMENT	GEOMETRIC STATEMENT
a is positive.	The point a lies to the right of the point 0.
$a > b$.	a lies to the right of b.
$a - b = c > 0$.	a lies c units to the right of b.
$a < b < c$.	b lies between a and c.
$\|3 - a\| < \frac{1}{2}$.	The point a is within $\frac{1}{2}$ unit of the point 3.
$\|a\| < \|b\|$.	The point a is closer to the origin than the point b is.

This close relationship between arithmetic and geometry is extremely important; often we can use arithmetical reasoning to solve geometrical problems or geometrical reasoning to solve arithmetical problems. Thus we may have two different ways of looking at a problem and, hence, increased chances of solving it.

If we denote a typical real number by x, we call the corresponding line the x-**axis** and draw Fig. 3.2.

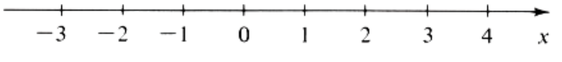

Fig. 3.2. *x-axis*

If we are measuring time, we generally choose t for a typical real number and draw the t-axis (Fig. 3.3).

Fig. 3.3. *t-axis*

Usually 0 on the t-axis represents the time when an experiment begins; negative numbers represent past time, positive numbers future time.

COORDINATES IN THE PLANE

When the points of a line are specified by real numbers, we say that the line is **coordinatized**: each point has a label or **coordinate**. It is possible also to label, or coordinatize, the points of a plane.

Draw two perpendicular lines in the plane. Mark their intersection O and coordinatize each line as shown in Fig. 3.4. By convention, call one line horizontal and name it the x-axis; call the other line vertical and name it the y-axis.

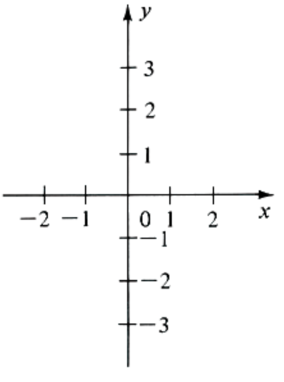

Fig. 3.4. *coordinate axes in the plane*

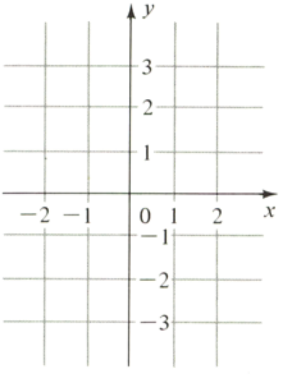

Fig. 3.5. *rectangular grid*

Fig. 3.6. *coordinates of a point*

Consider all lines parallel to the x-axis and all lines parallel to the y-axis (Fig. 3.5). These two systems of parallel lines impose a rectangular grid on the whole plane. We use this grid to coordinatize the points of the plane.

Take any point P of the plane. Through P pass one vertical line and one horizontal line (Fig. 3.6). They meet the axes in points x and y respectively. Associate with P the ordered pair (x, y); it completely describes the location of P.

Conversely, take any ordered pair (x, y) of real numbers. The vertical line through x on the x-axis and the horizontal line through y on the y-axis meet in a point P whose coordinates are precisely (x, y). Thus there is a one-to-one correspondence,

$$P \longleftrightarrow (x, y),$$

between the set of points of the plane and the set of all ordered pairs of real numbers. The numbers x and y are the **coordinates** of P. The point $(0, 0)$ is called the **origin**.

Remark 1: The pair (x, y) is also sometimes called (ungrammatically) the coordinates of P.