Statistical Decision Theory and Related Topics II

Edited by
Shanti S. Gupta and David S. Moore

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PREFACE

Decision theory and areas of research related to it have been at the heart of advances in mathematical statistics during the past generation. This volume contains the invited papers presented at an international symposium on Statistical Decision Theory and Related Topics held at Purdue University in May, 1976. Decision theory was broadly interpreted to include related areas that have been the scene of rapid progress since the preceding Purdue symposium in 1970. This symposium featured sessions on general decision theory, multiple decision theory, optimal experimental design, and robustness. The researchers invited to participate, and to author papers for this volume, are among the leaders in these fields.

We are especially grateful to Professor Felix Haas, Executive Vice President and Provost, and to Professor Allan Clark, Dean of the School of Science, for the encouragement and financial support provided by Purdue. The symposium was also supported by the National Science Foundation under grant MPS75-23196, by the Air Force Office of Scientific Research under grant AFOSR 76-2969, and by the Office of Naval Research under contract N00014-75-C-0455. We wish to thank these agencies for their assistance, and in particular Dr. I. N. Shimi of the Air Force Office of Scientific Research and Drs. Bruce McDonald and Robert Lundegard of the Office of Naval Research.

Many individuals contributed to the success of the symposium and to the preparation of this volume. The program for the symposium was developed under the guidance of an advisory committee composed of S. S. Gupta, Chairman, Purdue University, R. R. Bahadur, University of Chicago, L. Le Cam, University of California at Berkeley, J. C. Kiefer, Cornell University, H. E. Robbins, Columbia University, and J. Wolfowitz, University of Illinois. The excellence of the program was due in large part to the efforts of these colleagues. Local arrangements were coordinated by G. P. McCabe, assisted by other faculty and students in the Purdue Department of Statistics.

Numerous colleagues at Purdue and elsewhere served as referees for the papers presented here. In many cases, their comments helped to strengthen the papers. We are happy to acknowledge the encouragement and assistance of Academic Press in preparing this volume for publication. Finally, the burden of typing the entire contents accurately and attractively was borne with great skill by Norma Lucas.

The papers presented here bear witness to the vigor of research in statistical theory. We are pleased to present them to the statistical community.

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SELECTING THE LARGEST INTERACTION IN A TWO-FACTOR EXPERIMENT

By Robert E. Bechhofer, Thomas J. Santner and Bruce W. Turnbull
Cornell University

1. *Introduction.* The research described in the present paper was motivated by consideration of the following type of problem: Suppose that a medical research worker wishes to plan an experiment to study the effect of several (c) different methods of treatment on a physiological response of male and female subjects (who are otherwise matched with respect to other factors which might affect the response). It is assumed known that the effect of the treatment on the mean response is different for men than for women, and also that it varies from treatment to treatment. It is suspected that there may be a large interaction between sex and method of treatment, and it is desired to identify the sex-treatment combination for which this interaction is largest in the hope that such information might provide some clue as to the mechanism underlying the effectiveness of the methods of treatment. The statistical problem is to design the experiment on such a scale that this largest interaction can, if it is sufficiently large to be of practical importance to the experimenter, be detected with preassigned probability. The setup described above consists of 2c sex-treatment combinations, and is one of the general class of 2-factor experiments involving $r \geq 2$ levels of one qualitative factor and $c \geq 2$ levels of a second qualitative factor. More generally, one might consider multifactor experiments involving three or more qualitative factors. It should be noted that our present goal of selecting the combination associated with the

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largest interaction is quite different from the classical one of selecting the combination associated with the largest response, the later goal having been treated in [1].

The present paper considers 2-factor experiments and concentrates on the $2\times c$ case. It is organized as follows: In Section 2 we give the model and statistical assumptions that we adopt. The formulation of our selection problem for $r\times c$ factorial experiments is proposed in Section 3; this involves the statement of a goal and an associated probability requirement. A single-stage selection procedure is proposed in Section 4; the criterion for choosing sample size is introduced in this section, and this leads to a statement of the basic statistical problem that we seek to solve.

In Section 5 we derive an exact expression for the probability of a correct selection (PCS) in the $r\times c$ case when the single-stage selection procedure is used; special cases involving particular choices of r and c are studied in detail. Section 6 contains the main result of the paper. It concerns the explicit determination of the so-called least favorable (LF) configuration of the interactions (γ_{ij}) in the $2\times c$ case; this result is stated as Theorem 6.1, the proof of which is given following the theorem. Directions of future research are indicated in Section 7.

2. *Model and statistical assumptions*. We consider a 2-factor experiment, both factors qualitative, the first factor being studied at r levels and the second at c levels. We assume the usual fixed-effects linear model with observations $Y_{ijk}(1 \leqq i \leqq r, 1 \leqq j \leqq c;\ 1 \leqq k \leqq n)$ which are normal and independent with $E\{Y_{ijk}\} = \mu_{ij} = \mu + \alpha_i + \beta_j + \gamma_{ij}$ $(\sum_{i=1}^{r} \alpha_i = \sum_{j=1}^{c} \beta_j = \sum_{i=1}^{r} \gamma_{ij} = \sum_{j=1}^{c} \gamma_{ij} = 0)$, $\mathrm{Var}\{Y_{ijk}\} = \sigma^2$. We further assume that μ, the α_i, β_j, and γ_{ij} are unknown, and that σ^2 is known. Our interest is in the γ_{ij}.

If $\gamma_{ij} \equiv 0$ (all i,j) then $\mu_{ij} = \mu + \alpha_i + \beta_j$, and the "effects" of the two factors on the mean are said to be strictly additive for all factor-level combinations; if $\gamma_{ij} \not\equiv 0$ (all i,j) then interaction is said to be present. It is with this latter situation that we shall be particularly concerned in this paper.

Let $\gamma_{[1]} \leq \cdots \leq \gamma_{[rc]}$ denote the ranked values of the γ_{ij} ($1 \leq i \leq r$, $1 \leq j \leq c$). (Note that $\gamma_{[1]} \leq 0 \leq \gamma_{[rc]}$, and that for $r = c = 2$ we have $\gamma_{[1]} = \gamma_{[2]} = -\gamma_{[3]} = -\gamma_{[4]}$.) It is assumed that the experimenter has no prior knowledge concerning the pairing of the $\gamma_{[s]}$ ($1 \leq s \leq rc$) with the levels of either of the factors.

3. *Goal and probability requirement.* We shall consider a particular goal which would appear to be appropriate in these situations. In our formulation of the decision problem we adopt a new variant of the so-called indifference-zone approach (Bechhofer [1]). The goal and formulation are given below.

(3.1) GOAL I: "To select the factor-level combination associated with $\gamma_{[rc]}$."

Equivalently we might be concerned with selecting the factor-level combination associated with $\gamma_{[1]}$. Other meaningful goals could also be posed. (See Section 7.)

The experimenter restricts consideration to selection procedures which guarantee the following

PROBABILITY REQUIREMENT

$$P\{\text{Correct Selection}\} \geq P^* \quad \text{whenever } \gamma_{[rc]} \geq \Delta^* \text{ and } \gamma_{[rc]} - \gamma_{[rc-1]} \geq \delta^*. \tag{3.2}$$

The three quantities $\{\Delta^*, \delta^*, P^*\}$ ($0<\Delta^*<\infty$, $0<\delta^*< \frac{(r-1)(c-1)-1}{(r-1)(c-1)} \Delta^*$, $\frac{1}{rc} < P^* < 1$) are to be specified by the experimenter prior to the start of experimentation, the choice of their values depending on economic and cost considerations. The event "Correct Selection" (CS) in (3.2) means the selection of the levels of the two factors associated with $\gamma_{[rc]}$ when $\gamma_{[rc]} - \gamma_{[rc-1]} > 0$.

REMARK 3.1. Values of δ^*/Δ^* greater than $[(r-1)(c-1)-1]/(r-1)(c-1)$ are inappropriate. For suppose $\gamma_{[rc]} = \gamma_{11}$ and $\gamma_{11} - \gamma_{ij} \geqq \delta^*$; then $\gamma_{11} = \sum_{a=2}^{r} \sum_{b=2}^{c} \gamma_{ab} \leqq (r-1)(c-1)(\gamma_{11}-\delta^*)$. Hence $\delta^* \leqq [(r-1)(c-1)-1]\gamma_{11}/(r-1)(c-1)$ which must be true for all $\gamma_{11} \geqq \Delta^*$.

REMARK 3.2. If the traditional indifference-zone approach were used, the inequality $\gamma_{[rc]} \geqq \Delta^*$ would not be present in (3.2). However, in the present situation in which our interest is in the largest positive interaction, we are interested in this interaction only when it is sufficiently large relative to zero which is the standard which defines no interaction. The situation here is similar to the one considered in Bechhofer-Turnbull [2].

4. *Selection procedure, and the choice of sample size.* We shall employ the following "natural" single-stage selection procedure for (3.1).

> P: Take n independent observations Y_{ijk} $(1 \leqq k \leqq n)$ for each (i,j)-combination $(1 \leqq i \leqq r,\ 1 \leqq j \leqq c)$. Compute
>
> $\hat{\gamma}_{ij} = \bar{Y}_{ij.}-\bar{Y}_{i..}-\bar{Y}_{.j.}+\bar{Y}_{...}$ where $\bar{Y}_{ij.} = \sum_{k=1}^{n} Y_{ijk}/n$,

$$(4.1)\quad \bar{Y}_{i..} = \sum_{j=1}^{c} Y_{ij.}/c,\ \bar{Y}_{.j.} = \sum_{i=1}^{r} Y_{ij.}/r, \text{ and}$$

> $\bar{Y}_{...} = \sum_{i=1}^{r} \sum_{j=1}^{c} Y_{ij.}/rc$. Select the factor-level combination which produced $\hat{\gamma}_{[rc]} = \max\{\hat{\gamma}_{ij} | 1 \leqq i \leqq r,\ 1 \leqq j \leqq c\}$ as the one associated with $\gamma_{[rc]}$.

Procedure P is completely defined with the exception of the common sample size n which is under the control of the experimenter. This leads to the central problem that we address in the present paper:

PROBLEM. For given $c \geq 3$, $c \geq r \geq 2$ and specified

(4.2) $\{\Delta^*, \delta^*, P^*\}$, find the smallest value of n which will guarantee (3.2) when P is used.

The first stage in solving (4.2) is that of deriving an exact expression for the PCS when P is used.

5. *Expression for the PCS.* We assume that $\gamma_{[rc]} > 0$, i.e., $\gamma_{ij} \neq 0$, and that for $c \geq 3$, $c \geq r \geq 2$ there is a unique largest γ_{ij}; without loss of generality we assume $\gamma_{11} = \gamma_{[rc]}$. Then the PCS using P can be written as

$$P\left\{\hat{\gamma}_{11} > \max_{\substack{2\leq a\leq r\\ 2\leq b\leq c}} \hat{\gamma}_{ab},\ \hat{\gamma}_{11} > \max_{2\leq a\leq r} \hat{\gamma}_{a1},\ \hat{\gamma}_{11} > \max_{2\leq b\leq c} \hat{\gamma}_{1b}\right\}$$

$$= P\{\sum_{i=2}^{r}\sum_{j=2}^{c}(X_{ij}+\gamma_{ij})-(X_{ab}+\gamma_{ab})>0,\ (2 \leq a \leq r,\ 2 \leq b \leq c);$$

$$(5.1)\qquad \sum_{i=2}^{r}\sum_{j=2}^{c}(X_{ij}+\gamma_{ij}) + \sum_{j=2}^{c}(X_{aj}+\gamma_{aj}) > 0,\ (2 \leq a \leq r);$$

$$\sum_{i=2}^{r}\sum_{j=2}^{c}(X_{ij}+\gamma_{ij}) + \sum_{i=2}^{r}(X_{ib}+\gamma_{ib}) > 0,\ (2 \leq b \leq c)\}$$

where the $X_{ij} = \hat{\gamma}_{ij} - \gamma_{ij}$ have an $(r-1)(c-1)$-variate normal distribution with $E\{X_{ij}\} = 0$ $(2 \leq i \leq r,\ 2 \leq j \leq c)$,

$$\mathrm{Var}\{X_{ij}\} = \frac{(r-1)(c-1)}{rcn}\,\sigma^2$$
$$(2 \leq i \leq r,\ 2 \leq j \leq c)$$

$$\mathrm{Cov}\{X_{i_1,j_1},X_{i_2,j_2}\} = \frac{1}{rcn}\,\sigma^2$$

$$(5.2)\qquad (i_1 \neq i_2,\ j_1 \neq j_2;\ 2 \leq i_1,i_2 \leq r,\ 2 \leq j_1,j_2 \leq c)$$

$$\mathrm{Cov}\{X_{i,j_1},X_{i,j_2}\} = -\frac{r-1}{rcn}\,\sigma^2$$
$$(j_1 \neq j_2;\ 2 \leq i \leq r,\ 2 \leq j_1,j_2 \leq c)$$

$$\mathrm{Cov}\{X_{i_1,j},X_{i_2,j}\} = -\frac{c-1}{rcn}\,\sigma^2$$

$$(i_1 \neq i_2;\ 2 \leqq i_1, i_2 \leqq r,\ 2 \leqq j \leqq c).$$

Thus the exact PCS can be expressed in terms of rc-1 linear stochastic inequalities involving (r-1)(c-1) correlated normal variates.

In the remainder of the present paper we shall concentrate our attention on the $r = 2$, $c \geqq 3$ case, although in (5.4), Remark 6.4 and the Appendix we shall also refer to the $r = c = 3$ case. Expressions for the exact PCS for these cases are given below.

$2\times c$ *case*. When $r = 2$ the PCS of (5.1) reduces to

$$(5.3)\qquad \begin{aligned} P\{ &\sum_{j=2}^{c}(X_{2j}+\gamma_{2j}) - (X_{2b}+\gamma_{2b}) > 0,\ (2 \leqq b \leqq c); \\ &\sum_{j=2}^{c}(X_{2j}+\gamma_{2j}) + (X_{2b}+\gamma_{2b}) > 0,\ (2 \leqq b \leqq c)\} \end{aligned}$$

which involves only 2(c-1) inequalities.

3×3 *case*. When $r = c = 3$ the PCS of (5.1) reduces to

$$(5.4)\qquad P\{\sum_{i=2}^{3}\sum_{j=2}^{3}(X_{ij}+\gamma_{ij}) - (X_{ab}+\gamma_{ab}) > 0,\ (2 \leqq a,\ b \leqq 3)\}$$

which involves only 4 inequalities.

6. LF-*configuration of the* γ_{ij} *for the* $2\times c$ *case*. The most difficult step in solving (4.2) is that of determining a least favorable (LF) configuration of the γ_{ij}, i.e., a set of γ_{ij} $(1 \leqq i \leqq r,\ 1 \leqq j \leqq c)$ which minimizes (5.1) for any $n \geqq 1$, subject to $\gamma_{[rc]} \geqq \Delta^*$ <u>and</u> $\gamma_{[rc]} - \gamma_{[rc-1]} \geqq \delta^*$ where $0 < \Delta^* < \infty$ and $0 < \delta^* < \frac{(r-1)(c-1)-1}{(r-1)(c-1)}\Delta^*$. Our result for the $2\times c$ case is presented in Theorem 6.1 below.

We introduce the following notation. Let $\underset{\sim}{\gamma} = (\gamma_2,\ldots,\gamma_c)$ denote a (c-1)-vector where $\gamma_b = \gamma_{2b}$ $(2 \leqq b \leqq c)$. For each $\gamma_{11} \geqq \Delta^*$, let $E(\gamma_{11}) = \{\underset{\sim}{\gamma} \mid \sum_{j=2}^{c}\gamma_j = \gamma_{11};\ \delta^* - \gamma_{11} \leqq \gamma_b \leqq \gamma_{11} - \delta^*,$ $(2 \leqq b \leqq c)\}$ be the set of configurations in the preference zone for a CS on the plane $\sum_{j=2}^{c} \gamma_j = \gamma_{11}$; thus $E = \bigcup_{\gamma_{11} \geqq \Delta^*} E(\gamma_{11})$ is the entire preference zone.

A basic lemma. Our first result describes the infimum of the PCS over the section $E(\gamma_{11})$. Let $p = (c-3)/2$ or $(c-4)/2$ according as c is odd or even, and let j be an integer defined according to the rule

$$(6.1)\quad j = \begin{cases} p & , \text{ if } 0 < \delta^* < \frac{c-2-2p}{c-1-2p}\gamma_{11} \\ t\ (0 \leqq t < p), & \text{if } \frac{c-4-2t}{c-3-2t}\gamma_{11} \leqq \delta^* < \frac{c-2-2t}{c-1-2t}\gamma_{11}. \end{cases}$$

Clearly j always exists and is uniquely defined since $0 < \delta^* < (c-2)\Delta^*/(c-1)$.

LEMMA 6.1. <u>For the</u> 2×c <u>case</u> ($c \geqq 3$) <u>and for fixed</u> $\gamma_{11} \geqq \Delta^*$ <u>the</u> LF-<u>configuration of the</u> $\{\gamma_j\}$ <u>over the section</u> $E(\gamma_{11})$ <u>is given by</u>

$$(6.2)\quad \underset{\sim}{\gamma}(j,\gamma_{11}) = (\gamma_{11}-\delta^*,\ldots,\gamma_{11}-\delta^*,d_j,\delta^*-\gamma_{11},\ldots,\delta^*-\gamma_{11})$$

<u>where</u> $d_j = \delta^*(c-2-2j) - \gamma_{11}(c-3-2j)$, <u>and there are</u> c-2-j <u>and</u> j <u>elements</u> $\gamma_{11}-\delta^*$ <u>and</u> $\delta^*-\gamma_{11}$, <u>respectively</u>.

REMARK 6.1. $\underset{\sim}{\gamma}(j,\gamma_{11})$ completely defines the γ_{ab} ($1 \leqq a \leqq 2$, $1 \leqq b \leqq c$) matrix since $\sum_{a=1}^{2}\gamma_{ab} = \sum_{b=1}^{c}\gamma_{ab} = 0$; all matrices obtained from $\{\gamma_{ab}\}_{2\times c}$ by permuting rows and/or columns have the same associated PCS.

We next note that representation (5.3) of the PCS can equivalently be expressed as

$$(6.3)\qquad f(\underset{\sim}{\gamma}) = P\{\underset{\sim}{X} + \underset{\sim}{\gamma} \in A\}$$

where $\underset{\sim}{X} = (X_{22},\ldots,X_{2c})$ and $A = \{\underset{\sim}{w} = (w_2,\ldots,w_c) \mid \sum_{j=2}^{c} w_j - w_b > 0$, $(2 \leqq b \leqq c)$; $\sum_{j=2}^{c} w_j + w_b > 0$, $(2 \leqq b \leqq c)\}$. In the proof that follows we show that (1) $f(\underset{\sim}{\gamma})$ is log concave in $\underset{\sim}{\gamma}$, and (2) $\underset{\sim}{\gamma}(j,\gamma_{11})$ and its permutations form the extreme points of the convex set

$E(\gamma_{11})$.

Our proof that $f(\underset{\sim}{\gamma})$ is log concave is based on a characterization of the log-concavity property for probability measures which are generated by densities and is proved, for example, in Prékopa [5]. The proof that $E(\gamma_{11})$ is the convex hull of $\underset{\sim}{\gamma}(j,\gamma_{11})$ and its permutations is obtained by showing that $\underset{\sim}{\gamma}(j,\gamma_{11})$ majorizes every $\underset{\sim}{\gamma}$ in $E(\gamma_{11})$ in the sense of Hardy, Littlewood and Pólya [3] and the fact that $\underset{\sim}{a}$ majorizes $\underset{\sim}{b}$ if and only if $\underset{\sim}{b}$ is in the convex hull of $\underset{\sim}{a}$ and $\underset{\sim}{a}$'s permutations. For additional details, references, and associated ideas see Marshall and Olkin [4]. The definition of majorization and the lemma characterizing log-concavity in probability measures are stated below for convenient reference.

In n dimensions the vector $\underset{\sim}{b}$ is said to majorize the vector $\underset{\sim}{a}$ (written $\underset{\sim}{a} \prec \underset{\sim}{b}$) if upon reordering components to achieve $a_1 \geq a_2 \geq \cdots \geq a_n$, $b_1 \geq b_2 \geq \cdots \geq b_n$, it follows that

$\sum_{i=1}^{k} a_i \leq \sum_{i=1}^{k} b_i$, $1 \leq k < n$ and $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$. It is known that $\underset{\sim}{a} \prec \underset{\sim}{b}$ if and only if $\underset{\sim}{a}$ is a convex combination of $\underset{\sim}{b}$ and permutations of $\underset{\sim}{b}$.

LEMMA 6.2. _Let_ P _be a probability measure on_ R^n _generated by a density_ $g(\underset{\sim}{x})$, $\underset{\sim}{x} \in R^n$, _i.e.,_ $P\{A\} = \int_A g(\underset{\sim}{x})d\underset{\sim}{x}$ _for any Borel set_ $A \subset R^n$. _Then_ P _satisfies_

$$\text{(6.4)} \quad P\{\alpha A_0 + (1-\alpha)A_1\} \geq [P\{A_0\}]^{\alpha}[P\{A_1\}]^{1-\alpha} \text{ for all } 0 \leq \alpha \leq 1$$

if and only if $g(\underset{\sim}{x})$ _is log-concave_. (_In_ (6.4) _it is assumed that all events are measurable_.)

For a proof of Lemma 6.2 see [5]. In our problem the joint distribution of $\underset{\sim}{X}$ is non-singular (c-1)-variate multivariate normal, and is thus log concave. For any (c-1)-vectors $\underset{\sim}{\gamma}_1, \underset{\sim}{\gamma}_2$ and $0 \leq \alpha \leq 1$ let $\underset{\sim}{\gamma} = \alpha\underset{\sim}{\gamma}_1 + (1-\alpha)\underset{\sim}{\gamma}_2$. Then $\{X + \alpha\underset{\sim}{\gamma}_1 + (1-\alpha)\underset{\sim}{\gamma}_2 \in A\} = \{X \in \alpha(A-\underset{\sim}{\gamma}_1) + (1-\alpha)(A-\underset{\sim}{\gamma}_2)\}$ since A is convex. Thus from Lemma 6.2 we have

$$\begin{aligned} f(\underset{\sim}{\gamma}) &= P\{\underset{\sim}{X} + \underset{\sim}{\gamma} \in A\} \\ &= P\{X \in \alpha(A-\underset{\sim}{\gamma}_1)+(1-\alpha)(A-\underset{\sim}{\gamma}_2)\} \\ &\geqq [P\{\underset{\sim}{X} \in A-\underset{\sim}{\gamma}_1\}]^{\alpha}[P\{\underset{\sim}{X} \in A-\underset{\sim}{\gamma}_2\}]^{1-\alpha} \\ &\geqq \min\{f(\underset{\sim}{\gamma}_1),\ f(\underset{\sim}{\gamma}_2)\}. \end{aligned}$$

Hence the infimum of the PCS occurs at an extreme point of $E(\gamma_{11})$. Since $f(\underset{\sim}{\gamma})$ is constant under permutations it suffices to show that $\underset{\sim}{\gamma}(j,\gamma_{11}) \in E(\gamma_{11})$ and $\underset{\sim}{\gamma}(j,\gamma_{11}) \succ \underset{\sim}{\gamma}$ for all $\underset{\sim}{\gamma} \in E(\gamma_{11})$ to complete the proof of Lemma 6.1.

Now $\underset{\sim}{\gamma}(j,\gamma_{11}) \in E(\gamma_{11})$ since $(c-2-j)(\gamma_{11}-\delta^*) + d_j + j(\delta^*-\gamma_{11})= \gamma_{11}$ and $\gamma_{11} - \delta^* \geqq d_j \geqq \delta^* - \gamma_{11}$ provided that $\frac{c-2-2j}{c-1-2j}\gamma_{11} > \delta^* \geqq \frac{c-4-2j}{c-3-2j}\gamma_{11}$ which is the defining condition (6.1) for j. Select an arbitrary $\underset{\sim}{\gamma} \in E(\gamma_{11})$ and without loss of generality assume $\gamma_2 \geqq \cdots \geqq \gamma_c$; hence $\delta^* - \gamma_{11} \leqq \gamma_c \leqq \gamma_2 \leqq \gamma_{11}-\delta^*$ and $\sum_{j=2}^{c}\gamma_j = \gamma_{11}$. We shall show that $\underset{\sim}{\gamma} \prec \underset{\sim}{\gamma}(j,\gamma_{11})$.

a) For all m $(2 \leqq m \leqq c-1-j)$ we have

$$\sum_{i=2}^{m}\gamma(j,\gamma_{11})_i=(m-1)(\gamma_{11}-\delta^*)\geqq\sum_{i=2}^{m}\gamma_i \text{ since } \gamma_i < \gamma_{11}-\delta^*.$$

b) For all m $(c-j+1 \leqq m \leqq c)$ we have

$$\sum_{i=m}^{c}\gamma(j,\gamma_{11})_i=(c-m+1)(\delta^*-\gamma_{11}) \leqq \sum_{i=m}^{c}\gamma_i \text{ since } \gamma_i \geqq \delta^*-\gamma_{11}.$$

Thus $\gamma_{11} - \sum_{i=m}^{c}\gamma(j,\gamma_{11})_i \geqq \gamma_{11} - \sum_{i=m}^{c}\gamma_i$,

i.e., $\sum_{i=2}^{m-1}\gamma(j,\gamma_{11})_i \geqq \sum_{i=2}^{m-1}\gamma_i \quad (c-j \leqq m-1 \leqq c-1)$

and we have completed the proof that $\underset{\sim}{\gamma}(j,\gamma_{11})$ is the LF-configuration over $E(\gamma_{11})$.

REMARK 6.2. Alternatively, Lemma 6.1 could have been proved by applying results in Marshall and Olkin [4] to show that $f(\underset{\sim}{\gamma})$ is a Schur-concave function of $\underset{\sim}{\gamma}$ and hence must attain its minimum at an extreme point of $E(\gamma_{11})$. We have shown the (stronger) property that $f(\underset{\sim}{\gamma})$ is log-concave; some of our later results depend on this stronger property of $f(\underset{\sim}{\gamma})$.

Geometry of the 2×5 *case.* Consideration of the geometry of the set $E(\gamma_{11})$ over which $f(\underset{\sim}{\gamma})$ is to be minimized w.r.t $\gamma_2, \gamma_3, \ldots, \gamma_c$ gives added insight into the dependence of the LF-configuration $\underset{\sim}{\gamma}(j,\gamma_{11})$ on the relationship between δ^* and γ_{11}. We use the 2×5 case for illustrative purposes since 5 is the smallest c-value when r = 2 for which there is more than one (in this case two) distinct $\underset{\sim}{\gamma}(j,\gamma_{11})$'s.

When c = 5 the set $E(\gamma_{11})$ is isomorphic to the convex set $E'(\gamma_{11})$ in 3-space defined by

$$E'(\gamma_{11}) = \{(\gamma_2,\gamma_3,\gamma_4) | \delta^* - \gamma_{11} \leqq \gamma_2, \gamma_3, \gamma_4 \leqq \gamma_{11} - \delta^*;$$
$$\delta^* \leqq \gamma_2 + \gamma_3 + \gamma_4 \leqq 2\gamma_{11} - \delta^*\},$$

since $\sum_{i=2}^{5} \gamma_i = \gamma_{11}$ in $E(\gamma_{11})$. Now $E'(\gamma_{11})$ is the intersection of the cube C centered at (0,0,0) with faces parallel to the planes formed by the coordinates axes and with edges of length $2(\gamma_{11}-\delta^*)$, and the slab in 3-space bounded by the parallel planes $\gamma_2+\gamma_3+\gamma_4 = \delta^*$ and $\gamma_2+\gamma_3+\gamma_4 = 2\gamma_{11}-\delta^*$. There are two possible shapes for $E'(\gamma_{11})$ depending on the relationship between δ^* and γ_{11}. The differences between these two shapes as seen in the plane $\gamma_4 = \gamma_{11}-\delta^*$ are displayed as the shaded regions in Figures 6.1 and 6.2 which are valid for $0<\delta^*<\gamma_{11}/2$ and $\gamma_{11}/2<\delta^*<3\gamma_{11}/4$, respectively; when $\delta^* = \gamma_{11}/2$ the two figures are identical and the line $\gamma_2+\gamma_3 = \gamma_{11}$ passes through the point $(\gamma_{11}-\delta^*, \gamma_{11}-\delta^*)$. Note that in the plane $\gamma_4 = \gamma_{11}-\delta^*$ the planes $\gamma_2+\gamma_3+\gamma_4 = \delta^*$ and $\gamma_2+\gamma_3+\gamma_4 = 2\gamma_{11}-\delta^*$ become the lines $\gamma_2+\gamma_3 = 2\delta^*-\gamma_{11}$ and $\gamma_2+\gamma_3 = \gamma_{11}$, respectively. The plane $\gamma_2+\gamma_3+\gamma_4 = \delta^*$ always intersects the cube C, and hence always plays a role in determining $E'(\gamma_{11})$; the plane $\gamma_2+\gamma_3+\gamma_4 = 2\gamma_{11}-\delta^*$ intersects the cube only when $0 < \delta^* < \gamma_{11}/2$, and hence only plays a role in determining $E'(\gamma_{11})$ in that situation.

The key to understanding the role that the shape of $E'(\gamma_{11})$