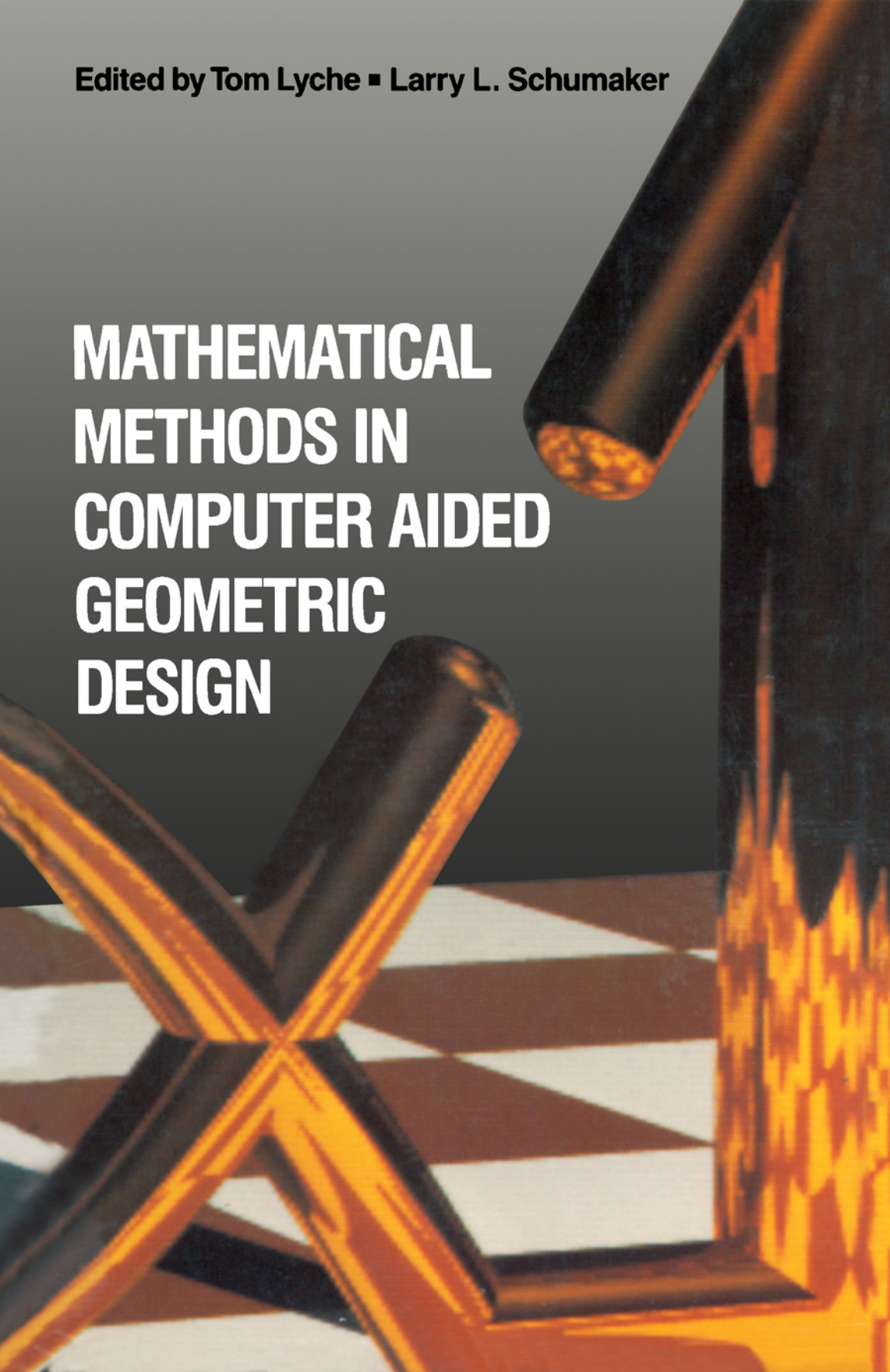


Edited by Tom Lyche ■ Larry L. Schumaker

MATHEMATICAL METHODS IN COMPUTER AIDED GEOMETRIC DESIGN



**MATHEMATICAL
METHODS IN
COMPUTER AIDED
GEOMETRIC DESIGN**

This page intentionally left blank

MATHEMATICAL METHODS IN COMPUTER AIDED GEOMETRIC DESIGN

Edited by

TOM LYCHE

Institutt for informatikk
Universitetet i Oslo
Oslo, Norway

LARRY L. SCHUMAKER

Department of Mathematics
Vanderbilt University
Nashville, Tennessee



ACADEMIC PRESS, INC.

Harcourt Brace Jovanovich, Publishers

Boston San Diego New York
Berkeley London Sydney
Tokyo Toronto

Copyright © 1989 by Academic Press, Inc.

All rights reserved.

No part of this publication may be reproduced or transmitted in any form or by any means, electronic or mechanical, including photocopy, recording, or any information storage and retrieval system, without permission in writing from the publisher.

ACADEMIC PRESS, INC.

1250 Sixth Avenue, San Diego, CA 92101

United Kingdom Edition published by

ACADEMIC PRESS INC. (LONDON) LTD.

24-28 Oval Road, London NW1 7DX

Library of Congress Cataloging-in-Publication Data

Mathematical methods in computer aided geometric design / edited by

Tbm Lyche, Larry L. Schumaker.

p. cm.

Papers originally presented at an international conference held June 16-22, 1988, at the University of Oslo, Norway.

Bibliography: p.

ISBN 0-12-460515-X (alk. paper)

1. Geometry--Data processing--Congresses. I. Lyche, Tbm.

II. Schumaker, Larry L., Date

QA448.D38M38 1989

516'.15'028--dc20

89-32842

CIP

Printed in the United States of America

89 90 91 92 9 8 7 6 5 4 3 2 1

CONTENTS

Preface	viii
Participants	ix
Scattered Data Interpolation in Three or More Variables <i>Peter Alfeld</i>	1
Some Applications of Discrete D^m Splines <i>R. Arcangéli</i>	35
Spline Elastic Manifolds <i>M. Attéia and M. N. Benbourhim</i>	45
Geometry Processing: Curvature Analysis and Surface-Surface Intersection <i>Robert E. Barnhill</i>	51
Three Examples of Dual Properties of Bézier Curves <i>P. J. Barry and R. N. Goldman</i>	61
What is the Natural Generalization of a Bézier Curve? <i>P. J. Barry and R. N. Goldman</i>	71
Convexity and a Multidimensional Version of the Variation Diminishing Property of Bernstein Polynomials <i>M. Beška</i>	87
Gröbner Basis Methods for Multivariate Splines <i>Louis J. Billera and Lauren L. Rose</i>	93
On Finite Element Interpolation Problems <i>J. M. Carnicer and M. Gasca</i>	105
The Design of Curves and Surfaces by Subdivision Algorithms <i>Alfred S. Cavaretta and Charles A. Micchelli</i>	115
A Data Dependent Parametrization for Spline Approximation <i>E. Cohen and C. L. O'Dell</i>	155
On the Evaluation of Box Splines <i>Morten Dæhlen</i>	167
Smooth piecewise quadric surfaces <i>Wolfgang Dahmen</i>	181

Inserting New Knots Into Beta-Spline Curves <i>P. Dierckx and B. Tytgat</i>	195
Recursive Subdivision and Iteration in Intersections and Related Problems <i>Tor Dokken, Vibeke Skytt, and Anne-Marie Ytrehus</i>	207
Rational Curves and Surfaces <i>Gerald Farin</i>	215
Hierarchical Segmentations of Algebraic Curves and Some Applications <i>Rida T. Farouki</i>	239
An Algorithm for Shape Preserving Parametric Interpolating Curves with G^2 Continuity <i>T. A. Foley, T. N. T. Goodman and K. Unsworth</i>	249
Knot Selection for Parametric Spline Interpolation <i>Thomas A. Foley and Gregory M. Nielson</i>	261
Splines and Estimation of Nonlinear Parameters <i>D. Girard and P. J. Laurent</i>	273
On Beta-continuous Functions and Their Application to the Construction of Geometrically Continuous Curves and Surfaces <i>R. N. Goldman and B. A. Barsky</i>	299
Algebraic Aspects of Geometric Continuity <i>R.N. Goldman and C.A. Micchelli</i>	313
Shape Preserving Representations <i>T. N. T. Goodman</i>	333
Geometric Continuity <i>John A. Gregory</i>	353
Curvature Continuous Triangular Interpolants <i>Hans Hagen and Helmut Pottmann</i>	373
Box-Spline Surfaces <i>Klaus Höllig</i>	385
Parallelization of the Subdivision Algorithm for Intersection of Bézier Curves on the FPS T20 <i>A. Kaufmann</i>	403
Composite Quadrilateral Finite Elements of Class C^r <i>M. Laghchim-Lahlou and P. Sablonniere</i>	413
A Knot Removal Strategy for Scattered Data in \mathbb{R}^2 <i>A. J. Y. Le Méhauté and Y. Lafranche</i>	419

Interpolation Systems and the Finite Element Method <i>J. Lorente and V. Ramirez</i>	427
Uniform Bivariate Hermite Interpolation <i>R. A. Lorentz</i>	435
A Survey of Applications of an Affine Invariant Norm <i>Gregory M. Nielson and Thomas A. Foley</i>	445
An Algorithm for Smooth Interpolation to Scattered Data in \mathbb{R}^2 <i>P. R. Pfluger and R. H. J. Gmelig Meyling</i>	469
Some Remarks on Three B-Spline Constructions <i>Hartmut Prautzsch</i>	481
Modified B-Spline Approximation for Quasi- Interpolation or Filtering <i>Christophe Rabut</i>	489
Design Tools for Shaping Spline Models <i>R. F. Riesenfeld</i>	499
A Process Oriented Design Method for Three- dimensional CAD Systems <i>D. Roller and E. Gschwind</i>	521
Open Questions in the Application of Multivariate B-splines <i>Malcolm Sabin</i>	529
On Global GC^2 Convexity Preserving Interpolation of Planar Curves by Piecewise Bézier Polynomials <i>R. Schaback</i>	539
Best Interpolation with Free Nodes by Closed Curves <i>Karl Scherer</i>	549
Segmentation Operators on Coons' Patches <i>Guido Schulze</i>	561
A General Subdivision Theorem for Bézier Triangles <i>H.-P. Seidel</i>	573
Cardinal Interpolation with Translates of Shifted Bivariate Box-Splines <i>Joachim Stoeckler</i>	583
Approximation of Surfaces Constrained by a Differential Equation Using Simplex Splines <i>C. R. Traas</i>	593
A Construction for VC^1 Continuity of Rational Bézier Patches <i>A. Vinacua and P. Brunet</i>	601

PREFACE

During the week of June 16 - 22, 1988, an international conference on *Mathematical Methods in Computer Aided Geometric Design* was held at the University of Oslo, Norway. Twelve one-hour survey lectures were presented along with 49 shorter research talks. The conference was attended by over 120 mathematicians from fifteen countries. This volume contains papers based on the survey lectures, along with 33 full-length research papers.

The conference was supported by grants from a number of international scientific organizations as well as several industrial concerns. Major support came from the Royal Norwegian Research Council for Science and the Humanities (NAVF), the Royal Norwegian Research Council for Scientific and Industrial Research (NTNF), the US Army European Research Office (London), the US Office of Naval Research (London), the US Air Force (EOARD), and the Center for Industrial Research, Oslo. We are also grateful for support from the Digital Equipment Corp. A/S, the Norsk Data A/S, the Veritas Sesam Systems A/S, the Moelven Bygg Gruppen A/S, and the Norwegian Contractors.

We would like to thank Morten Dæhlen and Knut Mørken for their help with organizing the program, accommodations, and entertainment as well as assisting with the preparation of the proceedings. We would also like to thank Erlend Arge, Sigurd Glærum, Ky Van Ha, Arne Lakså, Dag Nylund, Kyrre Strøm, Inger Ann Faye, Evy Madsen, and Lise Opsahl (all of the University of Oslo) for their help.

Our thanks are also due to the authors of the papers in this volume, both for submitting a paper, and for preparing it in \TeX . Finally, we would like to express our appreciation to Ewald Quak and Gerda Schumaker who assisted in the preparation of the final manuscript.

Nashville, March 1, 1989

PARTICIPANTS

Asle Aasen, *Moelven Bygg Gruppen A/S, P.O. Box 116, 2391 Moelv, NORWAY.*

Peter Alfeld, *University of Utah, Department of Mathematics, Salt Lake City, UT, 84112, USA.*

A. Andreev, *Bulgarian Academy of Sciences, Center of Mathematics and Mechanics, 1090 Sofia, P.O. Box 373, BULGARIA.*

Rémi Arcangéli, *Université de Pau, Laboratoire d'Analyse Numérique, Avenue de l'Université, 64000 Pau, FRANCE.*

Erlend Arge, *Universitetet i Oslo, Institutt for informatikk, P.O. Box 1080, Blindern, 0316 Oslo 3, NORWAY.*

Lise Wenstøp Arneberg, *Senter for Industriforskning, P.O. Box 350 Blindern, 0314 Oslo 3, NORWAY.*

Marc Atteia, *Université Paul Sabatier, Laboratoire d'Analyse Numérique, 118 Route de Narbonne, 31062 Toulouse Cédex, FRANCE.*

Lothar Bamberger, *Universität München, Mathematisches Institut, Siemens AG, Otto-Hahn-Ring 6, DAP 412, 8 München 83, W. GERMANY.*

Robert E. Barnhill, *Arizona State University, Dept. of Computer Science, Tempe, AZ 85287-5406, USA.*

Phillip Barry, *University of Waterloo, Dept. of Computer Science, Waterloo, Ontario, N2L 3G1, CANADA.*

Brian A. Barsky, *Univ. of California Berkeley, Computer Science Division, Elec. Eng. & Comp. Sci. Dept., Berkeley, CA 94720, USA.*

Richard Bartels, *University of Waterloo, Dept. of Computer Science, Waterloo, Ontario N2L 3G1, CANADA.*

Günter Baszenski, *Ruhr University, Computer Centre, Universitätstr. 150, 4630 Bochum 1, W. GERMANY.*

Marek Beška, *Technical University of Gdańsk, Institut of Mathematics, 80-952 Gdańsk, Majakowskiego 11/12, POLAND.*

- Louis J. Billera, *Rutgers University, Department of Mathematics, New Brunswick, NJ 08903, USA.*
- Wolfgang Boehm, *Universität Braunschweig, Angewandte Geom. & Comp. Graphic, Pockelstr. 14, D-3300 Braunschweig, W. GERMANY.*
- Mats Boholm, *Volvo Data AB, Dept 2930 DVF2, S-405 08 Gothenburg, SWEDEN.*
- P. Brunet, *Univ. Polit. de Catalunya, ETSEIB, Dept. de Llenguatges i Sistemes Inform., Av. Diagonal 647, 8ena planta, 08028 Barcelona, SPAIN.*
- M. D. Buhmann, *University of Cambridge, Dept. of Appl. Math. & Theor. Phys., Silver Street, Cambridge CB3 9EW, UK.*
- Rosemary E. Chang, *Silicon Graphics Comp. Systems, M/S 3U-550, 2011 Stierlin Rd., Mountain View, CA 94043, USA.*
- Elaine Cohen, *University of Utah, Dept. of Computer Science, Salt Lake City, UT 84112, USA.*
- Morten Dæhlen, *Senter for Industriforskning, P.O. Box 124 Blindern, 0314 Oslo 3, NORWAY.*
- Wolfgang Dahmen, *Freie Universität Berlin, Institut für Mathematik III, Arnimallee 2-6, 1000 Berlin 33, W. GERMANY.*
- Anthony DeRose, *University of Washington, Dept. of Computer Science, FR-35, Seattle, WA 98195, USA.*
- Flavia De Tisi, *Università di Milano, Dipartimento di Matematica, Via C. Saldini, 50-20133 Milano, ITALY.*
- Paul Dierckx, *Katholieke Universiteit Leuven, Departement Computerwetenschappen, Celestijnenlaan 200 A, B-3030 Heverlee, BELGIUM.*
- Tor Dokken, *Senter for Industriforskning, P.O. Box 124 Blindern, 0314 Oslo 3, NORWAY.*
- Per Evensen, *Senter for Industriforskning, P.O. Box 124 Blindern, 0314 Oslo 3, NORWAY.*
- Gerald Farin, *Arizona State University, Dept. of Computer Science, Tempe, AZ 85287-5406, USA.*
- R. T. Farouki, *IBM T. J. Watson Research Center, P.O. Box 218, Yorktown Heights, NY 10598, USA.*
- Thomas A. Foley, *Arizona State University, Dept. of Computer Science, Tempe, AZ 85287-5406, USA.*

Ferruccio Fontanella, Firenze, Dept. of Energetica, Facolta' di Ingegneria,
Via di S. Marta 3, I 50139 Firenze, ITALY.

Richard Franke, Naval Postgrad. School, Monterey, Dept. of Mathematics,
CA 93943-5100, USA.

Jean Gaches, Laboratoire d'Analyse Num., U.F.R./M.I.G., Université
P. Sabatier, 118, Route de Narbonne, 31062 Toulouse Cédex, FRANCE.

Mariano Gasca, Universidad de Zaragoza, Departamento de Matematica
Aplicada, 50009 Zaragoza, SPAIN.

Didier Girard, Université J. Fourier (Grenoble), TIM3. IMAG, B.P 68, 38402
St. Martin d' Hères Cédex, FRANCE.

Ronald Goldman, University of Waterloo, Dept. of Computer Science, Wa-
terloo, Ontario N2L 3G1, CANADA.

Manfred v. Golitschek, Universität Würzburg, Institut für Angewandte
Mathematik, Am Hubland, 8700 Würzburg, W. GERMANY.

T. N. T. Goodman, University of Dundee, Dept. of Maths. & Comp. Science,
DD1 4HN, Scotland, UK.

John A. Gregory, Brunel University, Mathematics and Statistics Dept.,
Uxbridge, Middlesex UB8 3PH, UK.

Dag Gravningsbråten, Norsk Data A/S, Schwabesgt. 5, P.O. Box 489, 3601
Kongsberg, NORWAY.

Per Kristian Gurrik, Norsk Data A/S, Schwabesgt. 5, P.O. Box 489, 3601
Kongsberg, NORWAY.

Ernst Gschwind, Hewlett-Packard GmbH, Herrenbergerstrasse 130, 7030
Böblingen, W. GERMANY.

Monika Haase, Norsk Data A/S, Schwabesgt. 5, P.O. Box 489, 3601 Kongs-
berg, NORWAY.

Nils Terje Haavie, Raufoss A/S, P.B. 1, 2831 Raufoss, NORWAY.

Ky Van Ha, Universitetet i Oslo, Institutt for informatikk, P.O. Box 1080,
Blindern, 0316 Oslo 3, NORWAY.

Hans Hagen, Universität Kaiserslautern, Institut für graph. Datenverarbeit-
ung und Computergeometrie, D-6750 Kaiserslautern, W. GERMANY.

Øyvind Hjelle, Senter for Industriforskning, P.B. 124 Blindern, 0314 Oslo 3,
NORWAY.

Klaus Hölbig, *Universität Stuttgart, Mathematisches Institut A, Pfaffenwaldring 57, 7000 Stuttgart 80, W. GERMANY.*

G. L. Iliev, *Bulgarian Academy of Sciences, Center of Mathematics and Mechanics, 1090 Sofia, P.O. Box 373, BULGARIA.*

Thomas Jensen, *Evans and Sutherland, Interactive Systems, P.O. Box 8700, Salt Lake City, UT 84108, USA.*

Bo Johansson, *Volvo Data AB, Dept 2930 DVF2, S-405 08 Gothenburg, SWEDEN.*

Alain Kaufmann, *Université J. Fourier (Grenoble), Lab. TIM3-IRMA, B.P. 53X, 38401 Grenoble Cédex, FRANCE.*

Per Erik Koch, *Norges Tekniske Høgskole, Institutt for Matematiske fag, 7034 Trondheim NTH, NORWAY.*

Linde Wittmeyer-Koch, *Linköping University, Department of Mathematics, S 58183 Linköping, SWEDEN.*

Jernej Kozak, *University of Ljubljana, Department of Mathematics, Jadranska 19, 61000 Ljubljana, YUGOSLAVIA.*

Ulf Krystad, *IDA, P.B 1163 Sentrum, 0107 Oslo 1, NORWAY.*

Johannes Kåsa, *Senter for Industriforskning, P.O. Box 124 Blindern, 0314 Oslo 3, NORWAY.*

Dieter Lasser, *Technische Hochschule Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, 6100 Darmstadt, W. GERMANY.*

Pierre-Jean Laurent, *Université J. Fourier (Grenoble), TIM 3. IMAG, B.P. 68, 38402 Saint Martin d'Hères Cédex, FRANCE.*

Alain Le Méhauté, *Université de Lille, Laboratoire d'Analyse Numérique, et Optimisation, 59655 Villeneuve d'Ascq Cédex, FRANCE.*

Charles Loop, *University of Washington, Computer Science Dept., FR- 35 Seattle, WA 98195, USA.*

J. Lorente, *Universidad de Granada, Dpto. Matematica Aplicada, Facultad de Ciencias, Avda. Fuentenueva s/n, 18071 Granada,, SPAIN.*

Rudolph Lorentz, *Gesellschaft für Math. und Datenv., Postfach 1240, 5205 St. Augustin 1, W. GERMANY.*

Tom Lyche, *Universitetet i Oslo, Institutt for informatikk, P.O. Box 1080, Blindern, 0316 Oslo 3, NORWAY.*

Carla Manni, *Istituto Matematico "U.Dini", Firenze, Viale Morgagni 67/A, Firenze, ITALY.*

Harry McLaughlin, *Rensselaer Polytechnic Inst., Department of Mathematical Sciences, Troy, NY 12181, USA.*

Steinar Meen, *Norsk Data A/S, Schwabesgt. 5, P.O. Box 489, 3601 Kongsberg, NORWAY.*

Ingrid Melinder, *Kungliga Tekniska Högskolan, NADA, 10044 Stockholm, SWEDEN.*

Charles Micchelli, *IBM T. J. Watson Research Center, Mathematical Sciences Department, P.O. Box 218, Yorktown Heights, NY 10598, USA.*

Ernest J. Mintel, *East Hartford, 15 Grant St., CT 06118, USA.*

Truls Engebret Moe, *Norsk Data A/S, Schwabesgt. 5, P.O. Box 489, 3601 Kongsberg, NORWAY.*

Knut Mørken, *Universitetet i Oslo, Institutt for informatikk, P.O. Box 1080, Blindern, 0316 Oslo 3, NORWAY.*

G. Müllenheim, *Katholische Universität Eichstätt, Mathematische Geographische Fakultät, Ostenstr. 18, 8078 Eichstätt, W. GERMANY.*

Udo Müller, *Technische Hochschule Darmstadt, Fachbereich Mathematik, Schloßgartenstraße 7, D-6100 Darmstadt, W. GERMANY.*

Edmond Nadler, *Wayne State University, Department of Mathematics, Detroit, MI 48202, USA.*

Norman W. Naugle, *Texas A&M University, Department of Mathematics, College Station, TX 77843-3368, USA.*

Gregory M. Nielson, *Arizona State University, Dept. of Computer Science, Tempe, AZ 85287-5406, USA.*

Geir Nilsen, *Raufoss A/S, P.B. 1, 2831 Raufoss, NORWAY.*

Carl S. Petersen, *Evans and Sutherland, Interactive Systems, P.O. Box 8700, Salt Lake City, UT 84108, USA.*

Pia Pfluger, *Univ. v. Amsterdam, Math. Institut, Roeterstr. 15, 1018 WB Amsterdam, THE NETHERLANDS.*

Hartmut Prautzsch, *Rensselaer Polytechnic Inst., Dept. of Math. Sciences, Troy, NY 12180-3590, USA.*

- Ewald Quak, *Vanderbilt University, Department of Mathematics, Nashville, TN 37235, USA.*
- Christophe Rabut, *Université Paul Sabatier, Service de Mathématiques, I.N.S.A., Avenue de Rangueil, 31077 Toulouse Cédex, FRANCE.*
- V. T. Rajan, *T. J. Watson Research Center, Manufacturing Research, IBM Research Division, Yorktown Heights, NY 10510, USA.*
- Ole Magnus Raff Reinemo, *Norsk Data A/S, Schwabesgt. 5, P.O. Box 489, 3601 Kongsberg, NORWAY.*
- Richard F. Riesenfeld, *University of Utah, Dept. of Computer Science, Salt Lake City, UT 84112, USA.*
- Dieter Roller, *Hewlett-Packard GmbH, Herrenberger Strasse 130, 7030 Boeblingen, W. GERMANY.*
- Hans Amund Rosbach, *Veritas Sesam Systems, P.O. Box 300, Veritasveien 1, 1322 Høvik, NORWAY.*
- Malcolm Sabin, *FEGS Ltd., Oakington, Cambridge, CB4 5BA, UK.*
- Paul Sablonnière, *INSA Rennes, Laboratoire LANS, 20 Av. des Buttes de Coësmes, 35043 Rennes Cédex, FRANCE.*
- K. Salkauskas, *University of Calgary, Dept. of Mathematics & Statistics, Calgary, Alberta, T2N 1N4, CANADA.*
- Paolo Santarelli, *Universität Kaiserslautern, Inst. für graph. Datenv. & Comp.-geom., D-6750 Kaiserslautern, W. GERMANY.*
- Ramon F. Sarraga, *General Motors Research Labs., Computer Science Department, Warren, MI 48090, USA.*
- Robert Schaback, *Universität Göttingen, Inst. für Num. und Angew. Math., Lotzestrasse 16-18, D-3400 Göttingen, W. GERMANY.*
- Karl Scherer, *Universität Bonn, Inst. für Angew. Math., Wegelerstrasse 6, 5300 Bonn, W. GERMANY.*
- Gerd Schmeltz, *Technische Hochschule Darmstadt, Fachbereich Mathematik, Schloßgartenstraße 7, D-6100 Darmstadt, W. GERMANY.*
- Guido Schulze, *Universität Kaiserslautern, Inst. für graph. Datenv. & Comp.-geom., D-6750 Kaiserslautern, W. GERMANY.*
- Larry L. Schumaker, *Vanderbilt University, Department of Mathematics, Nashville, TN 37235, USA.*

Hans-Peter Seidel, *Universität Tübingen, W.-Schickard-Inst. für Informatik, Graphisch-Interaktive Systeme, Auf der Morgenstelle 10 C9, D-7400 Tübingen, W. GERMANY.*

Vibeke Skytt, *Senter for Industriforskning, P.O. Box 124 Blindern, 0314 Oslo 3, NORWAY.*

Joachim Stoeckler, *Department of Mathematics, Texas A&M University, College Station, TX 77843, USA.*

Kyrre Strøm, *Universitetet i Oslo, Institutt for informatikk, P.O. Box 1080, Blindern, 0316 Oslo 3, NORWAY.*

J. P. Thiran, *University of Namur, Depart. Math. Fac., Rempart de la Vierge 8, B 5000 Namur, BELGIUM.*

Maria Bozzini Tirani, *Università di Milano, Dipartimento di Matematica, Via C. Saldini, 50-20133 Milano, ITALY.*

Romeo Tirani, *Università di Milano, Dipartimento di Matematica, Via C. Saldini, 50-20133 Milano, ITALY.*

C. R. Traas, *University of Twente, Faculty of Mathematics, P.O. Box 217, 7500 AE Enschede, THE NETHERLANDS.*

Keith Unsworth, *University of Dundee, Dept. of Maths. & Comp. Science, DD1 4HN, Scotland, UK.*

Alvar Vinacua, *Univ. Polit. de Catalunya, ETSEIB, Dept. de Llenguatges i Sistemes Inform., Av. Diagonal 647, 8ena planta, 08028 Barcelona, SPAIN.*

Mark Watkins, *Evans and Sutherland, Interactive Systems, P.O. Box 8700, Salt Lake City, UT 84108, USA.*

Andrew J. Worsey, *Univ. of N. Carolina Wilmington, Dept. of Mathematical Sciences, Wilmington, NC 28403-3297, USA.*

Anne-Marie Ytrehus, *Senter for Industriforskning, P.O. Box 124 Blindern, 0314 Oslo 3, NORWAY.*

This page intentionally left blank

Scattered Data Interpolation in Three or More Variables

Peter Alfeld

Abstract. This is a survey of techniques for the interpolation of scattered data in three or more independent variables. It covers schemes that can be used for any number of variables as well as schemes specifically designed for three variables. Emphasis is on breadth rather than depth, but there are explicit illustrations of different techniques used in the solution of multivariate interpolation problems.

§1. Introduction

In this paper, we consider the following:

Problem (*Scattered data interpolation problem*). Given

$$(x_i, y_i) \in \Omega \times \mathbb{R}, \quad i = 1, 2, \dots, N, \quad \Omega \subset \mathbb{R}^k, \quad (1)$$

find $s \in S(\Omega)$ such that

$$s(x_i) = y_i, \quad i = 1, 2, \dots, N. \quad (2)$$

The points (x_i, y_i) are the *data points* (or just *data*). We refer to the x_i as *data sites* to separate them from the *function values* y_i . The integer k is the number of independent variables. Our case of interest is

$$k > 2. \quad (3)$$

The integer N is the number of data to be interpolated to, and Ω is a suitable domain containing the data sites. $S(\Omega)$ is the *interpolating space*; i.e., a (usually finite dimensional) linear space of functions defined on Ω . Usually, $S(\Omega) \subset C^1(\Omega)$.

It is often convenient (e.g., for describing an interpolation scheme) to think of the data as having been generated by a *primitive function* f ; i.e.,

$$y_i = f(x_i), \quad i = 1, 2, \dots, N. \quad (4)$$

However, we do not assume that any information about f is available other than the data themselves.

A completely specified procedure of prescribing an interpolation space S and an interpolant s (for general Ω and general data) constitutes an interpolation *scheme*. This is distinguished from a *technique* which is used to build an interpolation scheme. An *algorithm* is a scheme that has been implemented in a commercially or freely available computer program. Emphasis in this paper is on techniques, and not on schemes or algorithms.

Multivariate scattered data interpolation problems arise in many different ways, as suggested by the following examples (others can be found in the literature listed at the end of this paper):

1. **Gravitational Field of the Earth [53].** Earth's gravitational field constitutes a vector-valued function of three components, which is the gradient of a scalar valued function, the potential function. This is generally regarded as being of significance when modelling the gravitational field. In particular, the problem is generally approached by modelling the potential function itself, but the approximation should be differentiable in order to be able to compute the force field. In fact, it should satisfy the 3-D Laplace Equation. The field has been measured with great accuracy at many points on the surface of the earth and in space. Yet no comprehensive model is currently available that is sufficiently accurate for all applications. Major difficulties in modeling the field are caused by the presence and significance of disparate spatial scales; on the surface of the planet the field may change appreciably within a few miles. It is affected—both in strength and direction—e.g., by the presence of mountain ranges. Gravitational effects on a larger scale affect a satellite in a typically low orbit of a few hundred kilometers. Finally, for the purposes of navigating a space craft in the vicinity of another planet, the earth can be thought of as a point source of gravity.
2. **Color Film Processing [48].** In generating color prints or slides, a processing system has to be corrected for effects, e.g., of film, chemistry, lenses, and filters. The correction is a (vector-valued) function of the intensities of three colors (Red, Green, Blue; i.e., RGB) on certain test slides. Appropriate corrections can be determined for specific RGB values. Corrections for other RGB values can then be obtained by interpolation.
3. **Implicitly Defined Surfaces [73].** Tom Sederberg has explored the design and representation of (two-dimensional) surfaces as the contour surfaces of trivariate piecewise polynomial functions. Such functions can be constructed by interpolation. The data then serve as *Design Handles*.
4. **Oil exploration.** In exploratory oil drilling an abundance of data is obtained at each bore hole. The data are distributed in a peculiar fashion: they are dense along bore holes and widely scattered otherwise.

To make the scope of this paper manageable, a number of topics and issues had to be omitted:

1. **Approximation Problems.** Here the *residuals* $e_i = s(x_i) - y_i$ are to be made small in some appropriate sense. For practical applications, this is probably a more important objective than actual interpolation (which renders the residuals zero) since realistic data usually are comprised of measured values and include errors whose exact reproduction is pointless. Sometimes an approximation scheme can be turned into an interpolation scheme by choosing the dimension of the approximating space equal to the number of data. Used frequently, and related to some of the techniques described in this paper, are *smoothing splines* [82]; *i.e.*, functions that minimize

$$I(s) = \sum_{i=1}^N (s(x_i) - y_i)^2 + \lambda J(s), \quad (5)$$

where λ is a parameter and $J(s)$ is a suitable semi-norm, measuring for example the strain energy in a clamped elastic plate (see Section 4.2.2). There are many additional references on smoothing splines in the bibliography [41].

2. **Interpolation Error.** This is the difference $\|s - f\|$ for some primitive function f (and a suitable norm $\|\cdot\|$), usually considered a function of the density of the data sites. This is an issue in classical approximation theory.
3. **Multivariate B-Splines.** This is a large and fascinating area in its own right. However, to date multivariate B-splines do not appear to have been particularly competent in solving multivariate scattered data interpolation problems. There is a large number of papers on multivariate B-splines in the bibliography [41].
4. **Geometric Continuity.** In Computer Aided Geometric Design (of one and two dimensional surfaces embedded in \mathbb{R}^3) a central objective is to recognize, obtain, and represent properties of a surface that are independent of any particular parameterization. This allows a great degree of flexibility in the choice of the parameterization, and in the structure of the surfaces that can be designed. For higher dimensional surfaces this issue has not yet played a prominent role. The only reference on this topic known to the author is Dieter Lasser's recent thesis [50]. In this paper, we restrict ourselves to the functional case (1) (which of course includes *vector-valued functions* defined on Ω).

I have attempted to cover existing techniques as widely as possible. However, only primary references are listed at the end of this paper. Their reference lists should be consulted for further information. For a recent extensive bibliography of 1107 entries on multivariate approximation and interpolation, see [41]. Recent collections of papers include [31,36], and the five volumes of the CAGD journal that have appeared to date. For a rich source of information and ideas, see the Winter, 1984 issue of *Rocky Mountain Journal of Mathemat-*

ics which is entirely devoted to multivariate interpolation and approximation problems.

It is unlikely that any multivariate interpolation schemes are going to emerge that are truly *general purpose* in the sense that they will solve satisfactorily the bulk of realistic problems. This observation applies to problems in three or more variables even more so than to the case of just two independent variables. There is a tremendous degree of variety and peculiarity present in multivariate interpolation problems. Thus, it is important to be aware of many different techniques for designing interpolation schemes, and for obtaining certain effects. Many such techniques are illustrated explicitly in this paper. The purpose of these illustrations is always the technique itself, and not a detailed description of the particular interpolation scheme to which it is applied.

In spite of the complexity of real problems, most of the schemes defined in this paper are *general purpose* in the sense that they are designed for generic situations rather than specific applications. (The exceptions are: tensor product schemes, interpolation on the sphere and other surfaces, and hypercubic methods). Calling a method *general purpose* requires certain underlying assumptions that should be made explicit:

1. There is no readily utilizable structure in the data that calls for the use of a more specific interpolation scheme. (In particular, it is hard to conceive of a situation where one would want to interpolate to tensor product data with a scattered data scheme.)
2. On the other hand, there is no *peculiar* structure that would upset a *general purpose method*. For example, a scheme based on a (k -dimensional) triangulation of the data sites is likely to perform poorly in a situation where data arise densely along scattered lines.
3. There is no appreciable noise in the data. (Otherwise one would use an approximation rather than an interpolation scheme.)
4. The representation of the data by a smooth (e.g., C^1) function is appropriate. This is often not the case. For example, in Computer Aided Tomography one is particularly interested in surfaces across which the density of the human body changes discontinuously (*i.e.*, the boundary of organs and bones). Similarly, in oil exploration a major issue is the location of *faults* (where subterranean layers of rock have been ruptured). The prediction and modeling of unknown discontinuities is an open and difficult problem that most likely will have to be addressed in the context of individual applications.
5. We will always assume that the data are such that the existence and uniqueness of an interpolant is assured. In several variables there are many settings where this is not the case. For example, it is impossible to interpolate to three data by a bivariate linear function if the data sites are situated along a straight line. Similarly, one cannot interpolate with a bivariate quadratic to six points whose sites lie on a conic section. Typically, such situations are *singular* in the sense that existence and uniqueness of the interpolant can be restored by an arbitrarily small perturbation of the

data sites. In this context, the opposite of *singular* is *generic*. For all of our interpolation schemes, existence and uniqueness are assured at least generically.

In designing, choosing, or using a multivariate interpolation scheme several issues may have to be considered:

1. How can a function of more than three variables be rendered?
2. Is the scheme *local* or *global*; *i.e.*, is the value of the interpolant at a given point dependent on all data or only a *small* subset of data sites close to the point of evaluation? A precise and general definition of these terms is not possible, but for any particular scheme it usually makes sense to classify it as local or global. Local schemes are more attractive for applications, while global schemes are easier to build and to design so as to possess other desirable properties (which depend on the particular problems that are to be solved).
3. What is the *degree of precision* of a scheme? This is the largest number m such that $s = p$ whenever p is polynomial of degree up to m and $y_i = p(x_i)$, $i = 1, 2, \dots, N$. The degree of precision is loosely related to the *shape fidelity* of a scheme. No precise statements can be made, but it is hard to envision viable schemes that do not have at least linear precision. On the other hand, a high degree of precision has no value in itself.
4. What is the *degree of smoothness* of the interpolant, *i.e.*, how often is it differentiable? Usually, first order differentiability is required.
5. What kind of data are required for the construction of the interpolant? As a rule, only functional data are provided by the problem, as indicated in (1). However, other data, *e.g.*, values of derivatives at data sites, or function and derivative values at points other than data sites, may be required by the scheme to obtain properties such as localness, or increased precision. See Section 11 for a discussion of derivative generation techniques.
6. Is the scheme invariant under affine transformations? Let A be a non-singular $k \times k$ matrix, γ a non-zero real number, and let \bar{s} be the solution of the modified interpolation problem with the data $(Ax_i, \gamma y_i)$, $i = 1, 2, \dots, N$. The interpolation scheme is *affinely invariant* if, for all data sets, and all points $x \in \Omega$, $\bar{s}(Ax) = \gamma s(x)$. Affine invariance implies for example that interpolated physical quantities are independent of the units chosen for expressing the data.
7. Does the interpolant depend continuously on the data sites? This property is clearly desirable, yet it is not present, *e.g.*, for any triangulation based scheme since the triangulation itself depends discontinuously on the data sites.
8. A variety of properties may be desirable depending on the particular interpolation problem. For example, the value of the interpolant may be constrained to be bracketed by the minimum and maximum function values; the interpolant may have to be convex, or monotone with respect to some or all of the independent variables; or it may be required to minimize

a physically relevant variational principle over an affine space of potential interpolants.

We conclude this section by introducing a few terms and some notation that will be used throughout the paper. When it is necessary we will denote the components of $x \in \mathbb{R}^k$ by ξ_i , $i = 1, 2, \dots, k$, to avoid confusion with the data sites. The norm $\|\cdot\|$ will be the *Euclidean Norm*; i.e.,

$$\|x\| = \sqrt{\sum_{i=1}^k \xi_i^2}. \quad (6)$$

The symbol \mathbb{P}_d^k denotes the $\binom{d+k}{k}$ -dimensional linear space of polynomials of k variables of total degree d .

The interpolant s will be sought in the finite dimensional linear interpolation space

$$S(\Omega) = \text{span} \{w_i \in C^1(\Omega) : i = 1, 2, \dots, M\}, \quad (7)$$

where $M \geq N$. Thus, it will be of the form

$$s(x) = \sum_{i=1}^M \alpha_i w_i(x), \quad (8)$$

where the α_i are *coefficients* that are determined by the interpolation conditions (2) and, if $M > N$, by additional auxiliary conditions that are imposed to enforce certain useful properties of s . The w_i are the *basis functions*.

We define the *Kronecker delta* as usual by

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j. \end{cases} \quad (9)$$

A frequently arising special case is given when $M = N$ and

$$w_i(x_j) = \delta_{ij}. \quad (10)$$

In that case,

$$s(x) = \sum_{i=1}^N y_i w_i(x) \quad (11)$$

and the interpolant is said to be in *cardinal form*.

The generalization of a triangle to k -dimensional space is a *k-simplex* or just a *simplex*. A simplex S is the convex hull of $k+1$ points called the *vertices* of the simplex. S is *non-degenerate* if its k -dimensional volume is non-zero, and *degenerate* otherwise. The convex hull of any subset of $\kappa+1$ vertices of a simplex is called a κ -*face* of the simplex. A face that is the convex hull of precisely k vertices is also called a *facet*. A 1-face is also called an *edge*. Note

that a κ -face of a simplex is itself a κ -dimensional simplex. (In [7] and [84], the meanings of the terms *facet* and *face* are interchanged. On the other hand, the usage proposed here is also employed, e.g., in [27] and [52]. There seem to be no linguistic reasons to prefer one usage over the other.)

The location of a point $x \in \mathbb{R}^k$ can be expressed uniquely in terms of *barycentric coordinates* b_i with respect to a non-degenerate simplex \mathcal{S} with vertices V_i , $i = 1, 2, \dots, k+1$, as

$$x = \sum_{i=1}^{k+1} b_i V_i, \quad \text{where} \quad \sum_{i=1}^{k+1} b_i = 1. \quad (12)$$

A polynomial p of degree d can be expressed uniquely in *Bernstein-Bézier* form as

$$p(x) = \sum_{i_1+i_2+\dots+i_{k+1}=d} c_{i_1 i_2 \dots i_{k+1}} \frac{d!}{i_1! i_2! \dots i_{k+1}!} b_1^{i_1} b_2^{i_2} \dots b_{k+1}^{i_{k+1}}. \quad (13)$$

The Bernstein-Bézier form of a polynomial has many useful properties and is virtually indispensable in the design of piecewise polynomial multivariate interpolation schemes. For a recent survey of the properties of the Bernstein-Bézier form in a general number of variables, see [27].

§2. Rendering of Trivariate Functions

The graphs of bivariate functions are frequently displayed as contour plots, three dimensional line drawings with or without hidden line elimination, or as shaded surfaces. Such surfaces reside in \mathbb{R}^3 and can be processed by the human mind as the familiar concept of a three dimensional object projected onto the two-dimensional retina. There is a fundamental biologic limit to similar renderings of higher dimensional objects. This seems to be the reason that pictorial displays of functions of more than two variables do not play nearly the same prominent role as for functions of two variables. However, some efforts have been made to render functions *s* of precisely *three variables*. Two obvious approaches suggest themselves: 1.) Display a number of *cross sections* where one of the variables is held constant, or, 2.) display *contour surfaces* where the value of $s(x)$ equals some constant.

The latter approach is illustrated, e.g., in [16]. Particular schemes of displaying contour surfaces of a function s are described in [61] and [74]. Petersen *et al* assume that s is piecewise polynomial (of any degree) in Bernstein-Bézier form on a tetrahedral *tessellation* (but not necessarily a *triangulation*) of Ω . Their scheme comes with an interpolation scheme of this character which is described in Section 8.1. The contouring scheme proceeds by approximating a polynomial p on a given tetrahedron t by another polynomial of degree one less than that of p . If the approximation is sufficiently accurate, it is accepted. Otherwise, t is split (about its longest side) into two subtetrahedra t_1 and t_2 , p is restricted to each of t_1 and t_2 , and the process is repeated on both t_1 and t_2 . Eventually the approximation becomes a continuous piecewise linear

function on a tetrahedral tessellation. The corresponding planar segments are then drawn to generate a *continuous piecewise linear* contour surface. The scheme has the advantage of spending computational effort only where it is needed, *i.e.*, it is *adaptive*. In an alternative approach, Sewell assumes that function values are given on a uniform three dimensional rectangular grid. (Of course, any interpolating function can be evaluated on such a grid.) The contour surfaces are drawn as frameworks of overlapping *bands* (*i.e.*, opaque lines of non-zero thickness). Bands in the background can be seen through gaps between bands in the foreground. Sewell's scheme has the advantage of being available as a FORTRAN algorithm [75].

§3. Tensor Product Schemes

This section comprises a deviation from the main theme of this paper in that it assumes a very special distribution of the data sites. Suppose for simplicity that $k = 3$. Deviating momentarily from the notation defined in (1), suppose the data sites are of the form $(\zeta_i, \eta_j, \theta_\ell)$, $i = 1, \dots, N_\zeta$, $j = 1, \dots, N_\eta$, $\ell = 1, \dots, N_\theta$, and we wish to find an interpolant of the form

$$s(\zeta, \eta, \theta) = \sum_{\mu=1}^{N_\zeta} \sum_{\nu=1}^{N_\eta} \sum_{\kappa=1}^{N_\theta} \alpha_{\mu\nu\kappa} p_\mu(\zeta) q_\nu(\eta) r_\kappa(\theta) \quad (14)$$

satisfying the interpolation conditions

$$s(\zeta_i, \eta_j, \theta_\ell) = y_{ij\ell}, \quad i = 1, \dots, N_\zeta, \quad j = 1, \dots, N_\eta, \quad \ell = 1, \dots, N_\theta. \quad (15)$$

This is a *tensor product* problem. It can be reduced to a set of *univariate* interpolation problems. Suppose for simplicity that the basis functions for each variable are given in *cardinal form*; *i.e.*,

$$p_i(\zeta_j) = \delta_{ij}, \quad q_i(\eta_j) = \delta_{ij}, \quad r_i(\theta_j) = \delta_{ij}. \quad (16)$$

The interpolant can then be written simply as

$$s(\zeta, \eta, \theta) = \sum_{\mu=1}^{N_\zeta} \sum_{\nu=1}^{N_\eta} \sum_{\kappa=1}^{N_\theta} y_{\mu\nu\kappa} p_\mu(\zeta) q_\nu(\eta) r_\kappa(\theta). \quad (17)$$

Any set of (univariate) non-cardinal basis functions can be converted to cardinal form by calculating the inverse of the corresponding Vandermonde matrix, *e.g.*, the matrix $[p_i(\zeta_j)]_{i,j=1,\dots,N_\zeta}$ for the variable ζ . In this manner, the effort of finding the coefficients of the interpolant is reduced from $O((N_\zeta N_\eta N_\theta)^3)$ to $O(N_\zeta^3 + N_\eta^3 + N_\theta^3)$ which constitutes a substantial saving. Tensor products do not have to be implemented in cardinal form, but in any case they can be reduced to univariate rather than truly multivariate interpolation problems. More detailed discussions of tensor product schemes are in [25] and [26]. See also [55] for a more abstract discussion of tensor product approximation.

§4. Point Schemes

The term *Point Schemes* refers to interpolation schemes that are not based on a tessellation of the underlying domain Ω .

4.1. Shepard's Methods

Shepard's method [76] may be the best known among all scattered data interpolants in a general number of variables. In its simplest form, it is given by

$$s(x) = \sum_{i=1}^N w_i(x) f(x_i) \quad \text{where} \quad w_i(x) = \frac{\|x - x_i\|^{-p}}{\sum_{j=1}^N \|x_i - x_j\|^{-p}}. \quad (18)$$

In this form, the evaluation of s at a data site leads to a division by zero. However, the definition of s can be extended continuously by the interpolation requirement $s(x_i) = y_i$.

Shepard's method is a particular example of a *convex combination* based scheme; *i.e.*, the weights w_i are non-negative and sum to 1. The interpolant is arbitrarily often differentiable if $p > 1$. The most frequent choice is $p = 2$, in which case the basis functions w_i are *rational*. If $p > 1$, the first derivatives vanish at the data points; *i.e.*, the interpolant has *flat spots*. The scheme is obviously global and has only constant precision. It does have the property

$$\min_{1 \leq i \leq N} \{y_i\} \leq s(x) \leq \max_{1 \leq i \leq N} \{y_i\} \quad \forall x \in \mathbb{R}^k \quad (19)$$

which is sometimes required. However, for most purposes Shepard's method in its unmodified form yields unacceptable interpolants. This is perhaps best illustrated by the fact that in the *univariate* case, where many other techniques are readily available, it has never been proposed as an interpolation scheme.

The deficiencies of Shepard's method can be overcome in various ways. We now describe some of them:

1. **Mollifying Basis Functions.** To localize Shepard's method one can multiply the basis functions w_i by *mollifying functions* μ_i satisfying

$$\mu_i(x_i) = 1 \quad (20)$$

and having local support in some appropriate sense. For example, the *Franke-Little weights* [22] are given by

$$\mu_i(x) = \left(1 - \frac{\|x - x_i\|}{R_i}\right)_+^\nu \quad (21)$$

where

$$x_+^\nu = \begin{cases} x^\nu, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0, \end{cases} \quad (22)$$

and the R_i are suitably chosen radii of circles around the data sites that constitute the support of the modified basis function. The smoothness of the multiplying factor is $\nu - 1$. This technique yields an interpolant

$$s(x) = \sum_{i=1}^N y_i \mu_i(x) w_i(x) \quad (23)$$

which is still in cardinal form. However, the (constant) precision of the scheme has been destroyed.

2. **Interpolation to Taylor operators.** To increase the precision of Shepard's method one can interpolate to Taylor expansions instead of function values. For example, the interpolant defined by

$$s(x) = \sum_{i=1}^N w_i(x) [f(x_i) + \nabla f(x_i)^T (x - x_i)] \quad (24)$$

has linear precision (since the weights w_i add to 1 and the individual Taylor operators have linear precision). Since $\nabla w_i(x_j) = 0$ for all i and j , it also follows that s interpolates to $\nabla f(x_i)$. The precision can be increased further by interpolating to Taylor operators of order greater than linear. However, for Shepard's method, if higher order Taylor operators were to be used, higher order derivatives would not be interpolated. Obviously, the technique calls for additional derivative values that are not normally available as data.

3. **Boolean sums.** Let P and Q be two linear operators defined on a suitable function space, and such that the composition PQ is defined. The *Boolean sum* of P and Q is defined by

$$S = P \oplus Q = P + Q - PQ. \quad (25)$$

Barnhill and Gregory show in [18] that S has (at least) the interpolation properties of P and the precision properties of Q . Thus, one can obtain an interpolation scheme with an arbitrarily high degree of precision m , say, by letting Pf be Shepard's interpolant and $Qf \in \mathbb{P}_m^k$ be a least squares approximation.

4. **Delta sums** [38,39]. A Boolean sum can be formed only if the operator Q can be applied when f is represented solely by the given data. This fact rules out, e.g., tensor product based operators. This motivated Foley to define a *delta sum* of two operators P and Q as

$$S = P \triangle Q = P \oplus (QP) = QP + P(I - QP).$$

The process can be repeated to yield a *delta iteration*. Several Boolean sum and delta sum based interpolation schemes are described and discussed in [22].

For a recent survey of Shepard's method and analysis of its approximation order, see [37]. Other papers on implementations, modifications, and tests of Shepard's method include [15,17,40,42,49,56].

4.2. Radial Interpolants

The term *radial* is due to Rippa [68]. Radial interpolants are of the form

$$s(x) = \sum_{i=1}^N \alpha_i g(\|x - x_i\|) + p_m(x), \quad (26)$$

where g is a given univariate so-called *radial function*, and $p_m \in \mathbb{P}_m^k$. The coefficients of s are determined by the interpolation condition (2) and the additional requirement that

$$\sum_{i=1}^N \alpha_i q(x_i) = 0 \quad (27)$$

for all polynomials $q \in \mathbb{P}_m^k$. Obviously, if $y_i = p(x_i)$, $i = 1, 2, \dots, N$ for some polynomial $p \in \mathbb{P}_m^k$, then the choice $\alpha_i = 0$, $i = 1, 2, \dots, N$ and $p_m = p$ will satisfy all requirements. Thus, radial interpolants have degree of precision m .

Recent discussions of radial schemes can be found in [35,57,62]. Micchelli's paper [57] establishes, in particular, that for many radial schemes, including those described here, the interpolant exists and is unique for all data sets (provided only that the interpolation problem with \mathbb{P}_m^k as interpolation space has at most one solution for all sets of function values). Remarkably, this fact was established only after Multiquadrics had been used with great success for fifteen years. Dyn and her co-workers in [35] and some of the references listed there develop particular radial schemes and numerical techniques to deal with the large and ill-conditioned linear system defining the interpolants.

The following subsections describe two specific choices of the radial function g .

4.2.1. Hardy Multiquadrics

These interpolants are defined by

$$g(t) = (h^2 + t^2)^\alpha, \quad \alpha = \pm \frac{1}{2}, \quad h > 0, \quad m = 0, \quad (28)$$

and were first proposed by Hardy [44] in 1971. The parameter h is at the user's disposal and its best choice is related to the distance between data points. Multiquadrics constitute one of the most successful and widely used general purpose interpolants for multivariate data. Until recently, that success was based entirely on numerical experiments. Further insight into the effectiveness of multiquadrics was gained by Buhmann [28,29], and Jackson [47] who study their approximation properties.

4.2.2. Duchon Thin Plate Splines

These interpolants were originally introduced by Duchon [34] for the case $k = 2$ as solutions of the variational problem

$$I_\mu s = \int_{\mathbb{R}^k} \|D^\mu(s)\|^2 dx = \min, \quad (29)$$

where

$$D^\mu = \left(\frac{\partial^\mu}{\partial \xi_{i_1} \partial \xi_{i_2} \cdots \partial \xi_{i_\mu}} : (i_1, i_2, \dots, i_\mu) \in [1, d]^\mu \right), \quad \mu = m + 1 \quad (30)$$

over all admissible functions, subject to the interpolation conditions (2). In the special case $\mu = k = 2$,

$$I_2 s = \int_{\mathbb{R}^2} (s_{\xi_1 \xi_1}^2 + 2s_{\xi_1 \xi_2}^2 + s_{\xi_2 \xi_2}^2) d\xi_1 d\xi_2 \quad (31)$$

and the functional I_2 measures the strain energy in a clamped elastic plate, giving rise to the name *thin plate spline*.

The solution of the minimization problem (29) are radial interpolants, called *surface splines*, with

$$g(t) = \begin{cases} t^{(2\mu-k)} \log t, & \text{if } k \text{ is even} \\ t^{(2\mu-k)}, & \text{if } k \text{ is odd.} \end{cases} \quad (32)$$

§5. Natural Neighbor Interpolation

This scheme was introduced by Sibson [77]. It is based on the *Dirichlet* (or *Thiessen* or *Voronoi*) *Tessellation* of Ω . This is the collection of *tiles* of the form

$$\tau_i = \left\{ x \in \mathbb{R}^k : \|x - x_i\| \leq \|x - x_j\|, \quad \forall j = 1, 2, \dots, N \right\} \cap \Omega. \quad (33)$$

Obviously,

$$\Omega = \bigcup_{i=1}^N \tau_i, \quad (34)$$

and the tiles are disjoint except for parts of their boundaries.

We assume that the domain $\Omega \subset \mathbb{R}^k$ has finite volume. In order to evaluate the interpolant at a point $x \in \Omega$ we can think of adding x to the set of data sites and carving its tile from the tiles of neighboring data sites. Thus, we let

$$\tau(x) = \left\{ z \in \mathbb{R}^k : \|z - x\| \leq \|z - x_j\|, \quad \forall j = 1, 2, \dots, N \right\}, \quad (35)$$

$$\tau_i(x) = \tau(x) \cap \tau_i \quad (36)$$

and

$$v(\tau) = \text{volume}(\tau). \quad (37)$$

Sibson's natural neighbor interpolant is then defined as

$$s(x) = \frac{\sum_{i=1}^N \lambda_i(x) \|x - x_i\|^{-1} \xi_i(x)}{\sum_{i=1}^N \lambda_i(x) \|x - x_i\|^{-1}}, \quad (38)$$

where

$$\begin{aligned} \xi_i(x) &= f(x_i) + \nabla f(x_i)^T (x - x_i) \\ \lambda_i(x) &= \frac{v(\tau_i(x))}{v(\tau(x))}. \end{aligned} \quad (39)$$

Note that if we replace $\lambda_i(x)$ by $\|x - x_i\|^{-1}$, we obtain a linearly precise version of Shepard's method. However, $\lambda_i(x) \neq 0$ only for points close to x , which causes the scheme to be local. As for Shepard's method, we have to define $s(x_i) = y_i$ explicitly. The natural neighbor interpolant has some remarkable properties:

1. it has linear precision. In fact it also reproduces *spherical quadratics* (i.e., quadratic functions whose matrix of second derivatives is a multiple of the identity matrix);
2. it is local;
3. in the special case $k = 1$ the function s is piecewise cubic;
4. $s \in C^1(\Omega)$;
5. the value $s(x)$ depends continuously on the data sites x_i .

The proof of the last two properties is involved. Obviously, in order to construct the interpolant (38), gradient values have to be supplied. Sibson's paper describes a scheme in the spirit of natural neighbor interpolation that generates derivative values from given function values.

§6. k-dimensional Triangulations

Many bivariate interpolation schemes are based on triangulations of the data set, and it is natural to use the same ideas in more than two variables. Let $\mathcal{T} = \{t_i : i = 1, 2, \dots, T\}$ be a set of non-degenerate simplices, and let $\Omega = \bigcup_{i=1}^T t_i$. It is useful to allow for the possibility that Ω is not convex. The following definition is a slight variation of the definition in [52] (where Ω is the convex hull of the data sites). Denote the set of data sites by $\mathcal{X} = \{x_i : i = 1, 2, \dots, N\}$. Then the set \mathcal{T} is a *triangulation* of \mathcal{X} if the following conditions are satisfied:

1. all vertices of each simplex are members of \mathcal{X}
2. the interiors of the simplices are pairwise disjoint

3. each facet of a simplex is either on the boundary of Ω or else is a common facet of exactly two simplices
4. each simplex contains no points of \mathcal{X} other than its vertices
5. Ω is homeomorphic to $[0, 1]^k$.

The last of the above conditions rules out degenerate triangulations consisting, for example, of two tetrahedra touching in just one vertex or edge.

There are some significant differences between bivariate and higher dimensional triangulations. These include for example:

1. Specification of \mathcal{X} and Ω does not determine the number of simplices. Counterexamples are given in [19] and [52].
2. If $k > 3$, possible triangulations may not be distinguishable by information about connectivity of points; *i.e.*, by the specification of edges. For examples, see [52].
3. A standard technique for building 2-dimensional triangulations consists of adding one triangle at a time, maintaining at each stage a valid triangulation. For some triangulations, this is not possible if $k > 2$. A simple counterexample (consisting of 14 points and 41 tetrahedra) is given in [69]. A more complicated one for many different triangulations of a particular domain is given in [24] and also described in [70], p. 17. If the construction is possible, the triangulation is said to be *shellable*.

Many criteria have been proposed for selecting particular 2-dimensional triangulations. Perhaps the easiest to generalize is that of the *Delaunay triangulation*. This is the dual of the Thiessen tessellation: two points are connected if their tiles share a line segment. An equivalent requirement is that for any two neighboring triangles the circumcircle of one triangle does not contain the opposite vertex of the other triangle. This is the *local circle test*. It implies the stronger *global circle test*: the circumcircle of any triangle in the triangulation does not contain any other data site at all. (We ignore special cases where four or more points lie on the circumference of a circle.) Lawson [52] generalizes these concepts. He shows:

1. a set of $k + 2$ points in \mathbb{R}^k may be triangulated in at most 2 different ways;
2. the (local) sphere test (the obvious generalization of the local circle test) selects a preferred one of these two triangulations;
3. a triangulation that satisfies the local sphere test also satisfies the global sphere test;
4. a triangulation satisfying the global sphere test is dual to the Thiessen tessellation; *i.e.*, the tiles of two points connected by an edge share a $k - 1$ dimensional boundary polygon (again we consider only the generic case).

Pascal codes for constructing k -dimensional triangulations are given in [45]. They contain facilities for forcing certain facets to be contained in the triangulation. This option can be used *e.g.*, to generate non-convexities, holes, and cavities.

We now give a precise definition of the localness of triangulation-based schemes that is suitable for the purposes of this paper. More specifically, we

generalize the term *minimally supported* defined in [13] for triangular schemes.

The *star* of a κ -face φ is the union of all simplices that contain φ . A function s defined on a k -dimensional triangulation is said to be *minimally supported* if there is a κ -face φ in the triangulation such that the support of s is contained in the star of φ . A linear space $S(\Omega)$ is *minimally supported* if it has a basis consisting of minimally supported functions. An interpolation scheme is *minimally supported* if the interpolation space S is minimally supported.

Note that in spite of the term's negative connotation, it is highly desirable for a space or a scheme to be minimally supported.

If $S \subset C(\Omega)$ contains all constant functions, then for any face φ there must be a function $s \in S$ whose support comprises all simplices containing φ . Otherwise, all functions $s \in S$ would have to vanish on φ . This fact gave rise to the term *minimally supported*. It is reasonable to consider a minimally supported scheme to be local. On the other hand, a scheme might also be called *local* without being minimally supported. It is, however, no easy task to find a reasonable more general definition of the word *local*. For example, in the case of 2-dimensional triangulations an easy fallacy would be to define a scheme to be local if for each basis function w there exists a vertex V such that the support of w consists of the union of the stars of all vertices contained in the star of V . However, simple examples show that such a support set may not even be simply connected, which renders the utility of the definition doubtful. The problem is compounded for triangulations of dimension greater than two.

§7. Tetrahedral Schemes

In this chapter, we consider schemes that have been specifically designed for interpolants on a *three-dimensional triangulation*. Interpolants that apply to triangulations of a general dimension are described in the next chapter.

The original motivation for defining interpolants piecewise on tetrahedra is the prospect of obtaining local schemes. Indeed, all schemes described in this chapter are minimally supported. However, with the increasing availability of *parallel processing*, tetrahedral (and, more generally, simplicial) schemes are likely to become even more attractive since individual simplices can be processed independently and simultaneously!

7.1. Polynomial Schemes

It is natural to consider the use of finite elements as an interpolant, particularly since there is a large and sophisticated machinery available for handling them. If a piecewise defined interpolant is to be polynomial on each tetrahedron and globally differentiable, then ([86]) its polynomial degree must be at least 9. Rescorla [67] gives an explicit description of such a scheme. It requires 220 (*i.e.*, the dimension of the space of trivariate nonic polynomials) data per tetrahedron. These include derivatives through fourth order at the vertices of the tetrahedron, as well as function and derivative values at various points on the faces and the centroid of the tetrahedron. The degree of precision of