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The Theory of Splines
and Their Applications

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The Theory of Splines and Their Applications

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PREFACE

Spline functions constitute a relatively new subject in analysis. During the past decade both the theory of splines and experience with their use in numerical analysis have undergone a considerable degree of development. Discoveries of new and significant results are of frequent occurrence.

It is useful at this juncture, nevertheless, to make some serious effort to organize and present material already developed up to this time. Much of this has become standardized. On the other hand, there are several areas where the theory is not yet complete. This book contains much of the material published since 1956 together with a considerable amount of the authors' own research not previously presented; it also reflects a considerable amount of practical experience with splines on the part of the authors.

In the interests of holding the present volume to a reasonable size, certain areas related to splines have been omitted. Thus the work of Schoenberg and his associates on the use of splines in the smoothing of equidistant data has not been included, nor is there any treatment of the theory of splines of complex argument. We hope, nevertheless, that the material presented will provide the reader with the necessary background for both theoretical and applied work in what promises to be a very active and extensive area.

In Chapter I there is a brief description of what is meant by a spline; this is followed by a survey of the development of spline theory since 1946 when Schoenberg first introduced the concept of a mathematical spline. We develop in Chapters II and IV, respectively, the theory of cubic splines and polynomial splines of higher degree from an algebraic point of view; the methods employed depend heavily on the equations used to define the spline. In particular, these chapters contain much of the material basic for applications. In Chapters III and V we reconsider cubic and polynomial splines of higher degree from a different point of view which reveals more clearly their deeper structure. Although the resulting theorems are not so sharp as their counterparts in Chapters II and IV, they are more easily carried over to new settings. This is done in Chapters VI, VII, and VIII, in which we consider in turn generalized splines, doubly cubic splines, and two-dimensional generalized splines.

We wish to express our deep gratitude to all those who have contributed to making this book a reality. Specifically, we wish to thank the United Aircraft Research Laboratories, the Pratt & Whitney Division of the United Aircraft Corporation, Harvard University, and the University of Maryland, whose support has made possible much of our research in spline theory.

May, 1967

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E. N. NILSON
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CONTENTS

Chapter IV The Polynomial Spline

Chapter V Intrinsic Properties of Polynomial Splines of Odd Degree

Chapter VI Generalized Splines

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CHAPTER I

Introduction

1.1. What Is a Spline?

It seems appropriate to begin a book on spline theory by defining a spline in its simplest and most widely used form, and also to indicate the motivation leading to this definition. For many years, long, thin strips of wood or some other material have been used much like French curves by draftsmen to fair in a smooth curve between specified points. These strips or splines are anchored in place by attaching lead weights called "ducks" at points along the spline. By varying the points where the ducks are attached to the spline itself and the position of both the spline and the duck relative to the drafting surface, the spline can be made to pass through the specified points provided a sufficient number of ducks are used.

If we regard the draftsman's spline as a thin beam, then the Bernoulli-Euler law

$$M(x) = EI[1/R(x)]$$

is satisfied. Here $M(x)$ is the bending moment, E is Young's modulus, I is the geometric moment of inertia, and $R(x)$ is the radius of curvature of the elastica, i.e., the curve assumed by the deformed axis of the beam. For small deflections, $R(x)$ is replaced by $1/y''(x)$, where $y(x)$ denotes the elastica. Thus we have

$$y''(x) = (1/EI)M(x).$$

Since the ducks act effectively as simple supports, the variation of $M(x)$ between duck positions is linear.

The mathematical spline is the result of replacing the draftsman's spline by its elastica and then approximating the latter by a piecewise cubic (normally a different cubic between each pair of adjacent ducks) with certain discontinuities of derivatives permitted at the junction points (the ducks) where two cubics join.

In its simple form, the mathematical spline is continuous and has both a continuous first derivative and a continuous second derivative. Normally, however, there is a jump discontinuity in its third derivative at the junction points. This corresponds to the draftsman's spline having continuous curvature with jumps occurring in the rate of change of curvature at the ducks. For many important applications, this mathematical model of the draftsman's spline is highly realistic.

In practice, the draftsman does not place the ducks at the specified points through which his spline must pass. Moreover, there is not usually a one-to-one correspondence between the specified points and the ducks. On the other hand, when the mathematical analog is used, it is common practice to interpolate to the specified points at the junction points and to keep the number of specified points and junction points (including the endpoints) the same.

In the next section, we outline the recent history of the mathematical spline approximation. From this history, some of the properties of the mathematical spline become evident. Also, a considerable extension of the concept of a spline from that approximating the draftsman's tool is apparent.

1.2. Recent Developments in the Theory of Splines

The spline approximation in its present form first appeared in a paper by Schoenberg [1946].* As indicated in Section 1.1, there is a very close relationship between spline theory and beam theory. Sokolnikoff [1956, pp. 1–4] provides a brief but very readable account of the development of beam theory. From the latter, one might anticipate some of the recent developments in the theory of splines, particularly the *minimum curvature property*. As suggested in Schoenberg's paper [1946], approximations employed in actuarial work also frequently involve concepts that relate them closely to the spline.

After 1946, Schoenberg, together with some of his students, continued these investigations of splines and monosplines. In particular, Schoenberg and Whitney [1949; 1953] first obtained criteria for the existence of certain splines of interpolation. For the case of splines of even order with interpolation at the junction points, a simpler approach to the question of existence due to Ahlberg, Nilson, and Walsh [1964; 1965] is now possible; it makes use of a basic integral relation obtained for cubic

* Data in square brackets refer to items in the Bibliography.