ADVANCED STUDIES IN PURE MATHEMATICS 15

Automorphic Forms and Geometry of Arithmetic Varieties

Automorphic Forms and Geometry of Arithmetic Varieties

ADVANCED STUDIES IN PURE MATHEMATICS 15

Chief Editor: H. Morikawa (Nagoya University)

Automorphic Forms and Geometry of Arithmetic Varieties

Edited by

K. Hashimoto (Waseda University) and

Y. Namikawa (Nagoya University)

ACADEMIC PRESS, INC.

Harcourt Brace Jovanovich, Publishers

Boston San Diego New York

Berkeley London Sydney

Tokyo Toronto

KINOKUNIYA

KINOKUNIYA COMPANY LTD.

TOKYO, JAPAN

Originally Published by Kinokuniya Company Ltd. Tokyo, Japan.
Distributed outside Japan by Academic Press Inc. Orlando, Florida, U.S.A.

ISBN 0-12-330580-2

Library of Congress Catalog Card Number 89-85239

PRINTED IN JAPAN
by Tokyo Press Co., Ltd.

Advanced Studies in Pure Mathematics

A SERIES OF UP-TO-DATE GUIDES OF LASTING INTEREST
TO ADVANCED MATHEMATICS

Volume 1	Algebraic Varieties and Analytic Varieties. Edited by S. Iitaka. February, 1983	
Volume 2	Galois Groups and their Representations. Edited by Y. Ihara. December, 1983	
Volume 3	Geometry of Geodesics and Related Topics. Edited by K. Shiohama. June, 1984	
Volume 4	Group Representations and Systems of Differential Equations. Edited by K. Okamoto. March, 1985	
Volume 5	Foliations. Edited by I. Tamura. February, 1986	
Volume 6	Algebraic Groups and Related Topics. Edited by R. Hotta. March, 1985	
Volume 7	Automorphic Forms and Number Theory. Edited by I. Satake. February, 1986	
Volume 8	Complex Analytic Singularities. Edited by T. Suwa and P. Wagreich. February, 1987	
Volume 9	Homotopy Theory and Related Topics. Edited by H. Toda. February, 1987	
Volume 10	Algebraic Geometry, Sendai, 1985. Edited by T. Oda. July, 1987	
Volume 11	Commutative Algebra and Combinatorics. Edited by M. Nagata and H. Matsumura. October, 1987	
Volume 12	Galois Representations and Arithmetic Algebraic Geometry. Edited by Y. Ihara. November, 1987	
Volume 13	Investigations in Number Theory. Edited by T. Kubota. June, 1988	
Volume 14	Representations of Lie Groups, Kyoto, Hiroshima, 1986. Edited by K. Okamoto and T. Oshima. December, 1988	

Volume 15	Automorphic Forms and Geometry of Arithmetic Varieties. Edited by K. Hashimoto and Y. Namikawa. July, 1989
Volume 16	Conformal Field Theory and Solvable Lattice Models. Edited by M. Jimbo, T. Miwa and A. Tsuchiya. June, 1988
Volume 17	Algebraic Number Theory—in honor of K. Iwasawa Edited by J. Coates, R. Greenberg, B. Mazur and I. Satake. Forthcoming
Volume 18	Recent Topics in Differential and Analytic Geometry Edited by T. Ochiai. Forthcoming
Volume 19	Integrable Systems in Quantum Field Theory and Statistical Mechanics Edited by M. Jimbo, T. Miwa and A. Tsuchiya. Forthcoming

TO BE CONTINUED

All volumes are published in Japan by
KINOKUNIYA COMPANY LTD.

Volume 1 ~ 12 Distributed outside Japan by
ELSEVIER SCIENCE PUBLISHERS B.V.
(NORTH-HOLLAND)

P.O. Box 1991
1000 BZ Amsterdam
The Netherlands

Volume 13 ~ Published in the United States by
ACADEMIC PRESS INC.

Orlando, Florida 32887
U.S.A.

Foreword

A new era in mathematics is in dawning, as fields continue to cross-fertilize each other and successive challenges consequently mount to cultivate new basic ideas and to formulate and solve new problems.

Our purpose is to bring significant momentum to this new period, by creating conditions that will encourage researchers in the pioneering spirit and action that are needed to explore the mathematical frontiers. Accordingly, symposia will be organized on important topics in pure mathematics, bringing mathematicians of various specialities together in a consciousness of the new directions that are defining themselves.

The organizational activity for the symposia will be provisionally based at the Department of Mathematics of Nagoya University, and the proceedings of the symposia will be published under the title *Advanced Studies in Pure Mathematics.* The board of editors of each volume will consist of the core members listed below plus some of the members of the program committee of each particular symposium.

Efforts are being made at the present time to obtain government approval for establishing a new institute for advanced studies in pure mathematics. At such time as this institute comes into being, the activities described above will be transferred thereto.

H. MORIKAWA and M. NAGATA

This page intentionally left blank

This volume is warmly dedicated to

Prof. Ichiro Satake

and

Prof. Friedrich Hirzebruch

on their sixtieth birthdays in 1987
with sincere gratitude
from us all who owe so much to them mathematically
through both their works and their personal guidance

This page intentionally left blank

Preface

Since Gauß, automorphic forms have played one of the most important role in number theory together with other but deeply related functicons such as zeta functions.

In this century, the theory of automorphic forms has been extensively enlarged and enriched by relating it to the geometry of symmetric spaces. This generalization allows us to use huge techniques in the Lie group theory, especially the representation theory with harmonic analysis, differential geometry and algebraic geometry. In particular, it has become one of the central problems to study the geometry of the quotients of the Siegel upper half-space or its generalization, the hermitian bounded symmetric domains, by arithmetic subgroups.

The importance of automorphic forms seems to become still greater when we see the very recent development of the theory of strings in quantum field theory where they appear unexpectedly as an essential tool to describe fundamental, physically important quantities. This new tendency would suggest a completely new theory of automorphic forms, namely the one over the moduli space of curves, which would be closely related to non-abelian class field theory.

Here in Japan intensive study on automorphic forms has been made, and it has given, we believe, a not small historical contribution to the theory. It would be our great pleasure if this volume would also contribute something.

This book grew out of a series of symposia held in 1985–86 where the main topic was to study the dimension formulas of various automorphic forms. (For these conferences we obtained financial support from the Research Institute for Mathematical Sciences, Kyoto University, and the Japan Society for the Promotion of Science, to which we together with all participants would express our gratitude.) One of our main concerns was to clarify the relation between two fundamental methods to obtain dimension formulas (for cusp forms), the Selberg trace formula and the index theorem (Riemann-Roch's theorem and the Lefschetz fixed point formula). This relation is well understood when the quotient is compact. But in the case of non-compact quotients, one should analyse the term corresponding to the so-called η-invariants which are related with special values of some zeta functions, and here the relation between the above two methods is not clear. Though this aim is still to be attained (cf. the articles by Satake-Ogata, Sczech, Tsushima), several important results were obtained which we might consider as fruits of these meetings.

This volume is divided into two parts. Part I consists of survey articles, most of which originated from the lectures given in these meetings but were newly written to be up-to-date. We regret that we could not include here some other interesting survey talks given there, which treat much wider related topics such as those in differential and algebraic geometry.

Part II consists of original papers which have enlarged the subjects treated in the final form of this volume. These subjects cover a large part of those which are now studied in Japan. We also refer the reader to Vol. 13 in this series to understand the whole trend of research in this area in Japan.

This volume is dedicated to Prof. Ichiro Satake and Prof. Friedrich Hirzebruch with our hearty congratulations and thanks on the occasions of their sixtieth birthdays in 1987.

Prof. Satake has long been one of our most important leaders through his work. His theory of compactifications, now known by his name together with Baily and Borel, of quotients of hermitian bounded symmetric domains by arithmetic subgroups, is a fundamental tool to the study of automorphic forms by the geometric method (cf. Tsuyumine's and Tsushima's articles in this volume). Our main aim to clarify the relation between this geometric method and the group-theoretic one (using the trace formula), also originated from his recent work (cf. Satake-Ogata here). Moreover, after his return to Japan, we are enjoying his personal warm encouragements and appropriate advice. The series of meetings mentioned above as the origin of this book was also held under his guidance. Therefore we would like to dedicate this whole volume to Prof. Satake from all the participants of the meetings.

To Prof. Hirzebruch also we owe much, of course, through his Riemann-Roch formula and Lefschetz fixed point formula (which are the main tools to calculate the dimension of cusp forms in geometric method) as well as his deep work on Hilbert modular varieties. But this is not the only reason to dedicate this volume to him. We, the two editors of this volume, ask the reader to allow us to recount our personal experience. Our joint work began with a private seminar at the Max-Planck-Institut für Mathematik in Bonn at the beginning of 1984 by six Japanese members at that time (Prof. Ibukiyama was also one of them) to read Atiyah-Donnelly-Singer's article on the proof of Hirzebruch's conjecture on Hilbert modular varieties. In the course of this seminar we were led to our main motivation for the series of meetings mentioned above. Without this prehistory at MPI, whose director has been Prof. Hirzebruch since its foundation in 1982, we would never have been coeditors of this volume. Already nearly a hundred Japanese mathematicians, including most of the authors in this book, have stayed at Bonn when we consider SFB "Theoretische Mathematik" at Bonn University as MPI's antecedent. Prof. Hirzebruch has always taken the initiative to invite Japanese mathematicians, especially young ones, and given them an opportunity to study in active and international circumstances. His contribution to mathematics in Japan is hence immeasurable, for which we would like to thank him cordially on this occasion.

Lastly, the editors would ask pardon for the long delay of publishing this volume, which is entirely due to their laziness. They are afraid that this delay might have caused the authors some disadvantage.

May 1988

Ki-ichiro Hashimoto
Yukihiko Namikawa

This page intentionally left blank

CONTENTS

Advanced Studies in Pure Mathematics 15, 1989
Automorphic Forms and Geometry of Arithmetic Varieties
pp. 1–27

Zeta Functions Associated to Cones and their Special Values

I. Satake and S. Ogata

Introduction

The purpose of this paper is to give a survey on zeta functions associated to (self-dual homogeneous) cones and their special values, including some recent results of ours on this subject.

In § 1 we summarize basic facts on self-dual homogeneous cones and the associated Γ-functions. § 2 is concerned with the zeta functions. Let V be a real vector space, $\mathscr{C}$ a self-dual homogeneous cone in V, and let G be the automorphism group $\mathrm{Aut}(V, \mathscr{C})^{\circ}$. We fix a $\boldsymbol{Q}$-simple $\boldsymbol{Q}$-structure on $(V, \mathscr{C})$. As is well-known, the pair (G, V) is a "prehomogeneous vector space" in the sense of Sato-Shintani [SS]. Following the general idea in [SS], we define a set of zeta functions $\{\xi_I\}$, each one of which is associated to a connected component V_I of $V^{\times}=V-S$, S denoting the singular set; in particular, $\xi_{(0)}=Z_{\mathscr{C}}$ is the zeta function associated to the cone $V_{(0)}=\mathscr{C}$. Then we give an explicit expression for the system of functional equations (Theorems 2.2.2, 2.3.3). Under the assumption that *d is even*, taking suitable linear combinations of these zeta functions, we define a new kind of L-functions L_I, which are shown to satisfy individually (or two in a pair, according to the cases) a functional equation of ordinary type (see (2.3.5)). We give some (new) results (Theorems 2.3.9, 2.4.1) on the residues and special values of these zeta and L-functions, where two extreme L-functions $L_{(0)}$ and $L_{(r_1)}$ play an essential role. These extreme L-functions, which generalize the (partial) Dedekind zeta function and the Shimizu L-function in the Hilbert modular case, seem to be of particular importance from the number-theoretic view point.

In § 3, we consider the corresponding (rational) symmetric tube domain $\mathscr{D}=V+\sqrt{-1}\mathscr{C}$ and, under an additional assumption that the *$\boldsymbol{Q}$-rank of G is one*, study the geometric invariants (χ_∞, τ_∞, etc.) associated to the cusp singularities appearing in the (standard) compactification of the arithmetic quotient space $\tilde{\Gamma}\backslash\mathscr{D}$ ([S3, 4]). A typical example is the Hilbert

Received February 16, 1987.
*) Partially supported by NSF Grand DMS-8502341.

modular singularities, which were studied extensively by Hirzebruch and others ([H2], [HG]). In [H2] Hirzebruch gave a conjecture relating the "signature defects" of the cusps with the zero-values of the corresponding Shimizu L-functions, which was later proved by Atiyah-Donnelly-Singer and Müller ([ADS 1, 2], [M 4, 5]). In view of our results on these invariants and special values, we state in 3.3 some conjectures ((C1), (C2), (C3)) which may be regarded as a natural generalization of the Hirzebruch conjecture.

In § 4, we define a more general zeta function $Z_{\mathscr{C}}$ associated to a "Tsuchihashi singularity" and give a formula for the zero-value $Z_{\mathscr{C}}(0)$ (Theorem 4.2.5) by modifying a method due to Zagier [Z]. Recently, using this formula, Ishida [I3] proved the rationality of $Z_{\mathscr{C}}(0)$ in general. It is hoped that our approach might suggest a new possibility of attacking the generalized Hirzebruch conjecture.

Our study on this subject has been largely inspired by the fundamental works of Professor F. Hirzebruch, to whom this paper is respectfully dedicated. The paper was prepared during a stay at the MSRI, Berkeley in 1986–87, of the first-named author, who would like to thank the staffs of the Institute for superb service and hospitalities.

Notations. The symbols $\boldsymbol{Z}$, $\boldsymbol{Q}$, $\boldsymbol{R}$, $\boldsymbol{C}$ are used in the usual sense, e.g. $\boldsymbol{Q}$ is the field of rational numbers. $\boldsymbol{H}$ is the Hamilton quaternion algebra. We use the symbol like $\boldsymbol{R}_{\geq 0}=\{\lambda \in \boldsymbol{R} \mid \lambda \geq 0\}$, and write $\boldsymbol{R}_+$ for $\boldsymbol{R}_{>0}$. For $\xi \in \boldsymbol{C}$, $\boldsymbol{e}(\xi)$ stands for $\exp(2\pi\sqrt{-1}\xi)$. Let V be a real vector space, $v_1, \cdots, v_r \in V$ and let S be a subset of $\boldsymbol{R}$. Then we write $\{v_1, \cdots, v_r\}_S$ for $\{\sum_{i=1}^r \lambda_i v_i \mid \lambda_i \in S\}$; e.g. $\{v_1, \cdots, v_r\}_{\boldsymbol{R}_{\geq 0}}$ is a closed polyhedral cone generated by $v_1, \cdots, v_r$. For a cone $\mathscr{C}$ and a lattice M in V, $\mathscr{C}^*$ and M^* denote, respectively, the dual cone and the dual lattice in the dual space V^*. For a topological group G, G° denotes the identity connected component of G. For a finite set S, $|S|$ denotes the cardinality of S.

Let F be a subfield of $\boldsymbol{R}$, $\mathscr{G}$ a (connected) algebraic group defined over F, and $G=\mathscr{G}(\boldsymbol{R})^\circ$. By an abuse of notations, we write G_F for $\mathscr{G}(F) \cap G$ and F-rk G for F-rk $\mathscr{G}$ (i.e. the dimension of maximal F-split tori in $\mathscr{G}$). If σ is an imbedding, $F \hookrightarrow \boldsymbol{R}$, then G^σ stands for $\mathscr{G}^\sigma(\boldsymbol{R})^\circ$ and, if F_0 is a subfield of F with $[F:F_0]<\infty$, then $R_{F/F_0}(G)$ stands for $R_{F/F_0}(\mathscr{G})(\boldsymbol{R})^\circ$. When $\mathscr{G}$ is reductive, $G=\mathscr{G}(\boldsymbol{R})^\circ$ is called "reductive", and we write G^s for $\mathscr{G}^s(\boldsymbol{R})^\circ$, $\mathscr{G}^s$ denoting the semisimple part of $\mathscr{G}$.

§ 1. Self-dual homogeneous cones ([BK], [S1], [V])

1.1. Let V be a real vector space of dimension $n>0$. By a *convex cone* in V, we mean a subset $\mathscr{C}$ of V with the following property:

$$x, y \in \mathscr{C},\ \lambda,\ \mu>0 \Longrightarrow \lambda x+\mu y \in \mathscr{C}.$$

The *dual* of $\mathscr{C}$ is defined by

$$\mathscr{C}^* = \{x^* \in V^* \mid \langle x, x^* \rangle > 0 \quad \text{for all } x \in \overline{\mathscr{C}} - \{0\}\}.$$

Then $\mathscr{C}^*$ is an open convex cone in the dual space V^*. It is clear that for a (non-empty) convex cone $\mathscr{C}$ the following three conditions are equivalent:

(i) $\mathscr{C}$ does not contain a line in V;

(ii) $\overline{\mathscr{C}} \cap (-\overline{\mathscr{C}}) = \{0\}$;

(iii) $\mathscr{C}^*$ is non-empty.

When these conditions are satisfied, $\mathscr{C}$ is called *non-degenerate*. In what follows, a non-degenerate open convex cone will simply be called a "cone". For a cone $\mathscr{C}$, one has $\mathscr{C}^{**} = \mathscr{C}$.

A cone $\mathscr{C}$ is called *self-dual* if there exists a linear isomorphism $S : (V, \mathscr{C}) \rightleftarrows (V^*, \mathscr{C}^*)$, which is symmetric and positive definite. A cone $\mathscr{C}$ is called *homogeneous* if the automorphism group

$$G = \mathrm{Aut}(V, \mathscr{C})^\circ = \{g \in GL(V) \mid g\mathscr{C} = \mathscr{C}\}^\circ.$$

($\circ$ denoting the identity connected component) is transitive on $\mathscr{C}$.

In §§ 1–3, unless otherwise specified, we always assume that $\mathscr{C}$ is self-dual and homogeneous, and fix a positive definite inner product $\langle\ \rangle$ on V defining an isomorphism S mentioned above. Then $(V, \mathscr{C})$ is identified with its dual $(V^*, \mathscr{C}^*)$. In this case, the automorphism group G is the identity connected component of a reductive algebraic group and for any $c_0 \in \mathscr{C}$ the isotropy subgroup

$$K = G_{c_0} = \{g \in G \mid gc_0 = c_0\}$$

is a maximal compact subgroup of G. Thus $\mathscr{C} \approx G/K$ has a structure of Riemannian symmetric space (with a flat part).

1.2. In 1957–58, M. Koecher made an observation that the category of self-dual homogeneous cones $(V, \mathscr{C})$ with a base point $c_0 \in \mathscr{C}$ is equivalent to that of "formally real" Jordan algebras by the correspondence given as follows ([BK], [S1]). Let $\mathscr{C}$ be a self-dual homogeneous cone in V with a base point c_0 and let G, K be as above. Let $\mathfrak{g} = \mathrm{Lie}\ G$, $\mathfrak{k} = \mathrm{Lie}\ K$ and let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. Then by the homogeneity assumption there exists a unique linear isomorphism

$$V \ni x \longmapsto T_x \in \mathfrak{p}$$

such that $x = T_x c_0$. The Jordan product in V is then defined by

$$x \circ y = T_x y \quad (x, y \in V).$$

In particular, one has $T_{c_0}=\mathrm{id}_V$, i.e. c_0 is the unit element of the Jordan algebra.

By virtue of this equivalence, the classification of self-dual homogeneous cones is reduced to that of formally real Jordan algebras, which was given (by a collaboration of phisicists) as early as in 1934 ([JNW]). A self-dual homogeneous cone $\mathscr{C}$ is decomposed uniquely into the direct product of the "irreducible" ones, for which one has $G=\boldsymbol{R}_+\times G^s$ with G^s $\boldsymbol{R}$-simple (or $=\{1\}$). The irreducible self-dual homogeneous cones are classified into the following five types:

$$\begin{cases} \mathscr{P}_1(\boldsymbol{R})=\boldsymbol{R}_+, \quad \mathscr{P}_r(\boldsymbol{F}) \quad (r\geq 2, \boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}), \\ \mathscr{P}_3(\boldsymbol{O}) \quad (\boldsymbol{O} \text{ denotes the Cayley octonion algebra}), \\ \mathscr{P}(1, n-1)=\{(\xi_i)\in \boldsymbol{R}^n \mid \xi_1>0,\ \xi_1^2-\sum_{i=2}^n \xi_i^2>0\} \quad (n\geq 3), \end{cases}$$

where $\mathscr{P}_r(\boldsymbol{F})$ denotes the cone of positive definite hermitian matrices of size r with entries in $\boldsymbol{F}$. $\mathscr{P}_2(\boldsymbol{R})$, $\mathscr{P}_2(\boldsymbol{C})$, $\mathscr{P}_2(\boldsymbol{H})$ are isomorphic to the "quadratic cones" $\mathscr{P}(1, n-1)$ with $n=3, 4, 6$, respectively. For $\mathscr{C}=\mathscr{P}_3(\boldsymbol{O})$, G^s is an excetional group of type (E_6).

A more general study on "homogeneous cones" was done by Vinberg [V]] in the early 60's. In the study of general cones, the characteristic function plays an essential role. For any (non-degenerate, open convex) cone $\mathscr{C}$, the *characteristic function* $\phi(x)=\phi_{\mathscr{C}}(x)$ is defined by

$$\phi_{\mathscr{C}}(x)=\int_{\mathscr{C}^*} e^{-\langle x,x^*\rangle}dx^*.$$

Clearly one has

$$\phi_{\mathscr{C}}(x)>0, \quad \phi_{\mathscr{C}}(gx)=\det(g)^{-1}\phi_{\mathscr{C}}(x) \qquad \text{for } x\in\mathscr{C},\ g\in G,$$

and $\log\phi_c(x)$ is a convex function, which tends to infinity when $x\in\mathscr{C}$ converges to a boundary point of $\mathscr{C}$. The characteristic function will be used later in §4.

1.3. Quasi-irreducible cones. Let $\mathscr{C}$ be a self-dual homogeneous cone in V. $\mathscr{C}$ is called *quasi-irreducible* if in its irreducible decomposition all irreducible components are isomorphic.

Lemma 1.3.1. *Suppose $(V, \mathscr{C})$ has a $\boldsymbol{Q}$-simple $\boldsymbol{Q}$-structure; this means that there is a $\boldsymbol{Q}$-vector space $V_{\boldsymbol{Q}}$ such that $V=V_{\boldsymbol{Q}}\otimes_{\boldsymbol{Q}}\boldsymbol{R}$, for which G is (the identity connected component of) an algebraic group defined over $\boldsymbol{Q}$ and that, if $(V, \mathscr{C})=\prod_{\mu=1}^m (V_\mu, \mathscr{C}_\mu)$ is the irreducible decomposition, no partial product of V_μ's is defined over $\boldsymbol{Q}$, or equivalently, that the center of G is of $\boldsymbol{Q}$-rank one. Then $\mathscr{C}$ is quasi-irreducible.*

In fact, under this assumption, there exists a totally real number field F_1 of degree m such that

$$G=G_{F_1/Q}(G_1)=\prod_{\mu=1}^{m} G_\mu, \quad G_\mu=G_1^{\sigma_\mu},$$

where $G_1=\boldsymbol{R}_+\times G_1^s$ with G_1^s $\boldsymbol{R}$-simple (which may reduce to $\{1\}$), defined over F_1, and $\{\sigma_\mu\ (1\leq\mu\leq m)\}$ is the totality of the imbeddings $F_1\hookrightarrow\boldsymbol{R}$. Then the G_μ^s's are all $\boldsymbol{C}$-isomorphic and hence, by the classification theory, are also $\boldsymbol{R}$-isomorphic except for the case when there exists an even integer r_1 such that every G_μ^s is isogeneous either to $SL(r_1, \boldsymbol{R})$ or to $SL(r_1/2, \boldsymbol{H})$ and when both types $SL(r_1, \boldsymbol{R})$ and $SL(r_1/2, \boldsymbol{H})$ occur in the G_μ^s's. But actually such a "mixed type" can not occur for the following reason. Since the $\boldsymbol{Q}$-rational points are dense in $\mathscr{C}$, one may take c_0 to be $\boldsymbol{Q}$-rational; then the maximal compact subgroup K is also defined over $\boldsymbol{Q}$. One then has the corresponding decomposition

$$K=R_{F_1/Q}(K_1)=\prod_{\mu=1}^{m} K_\mu, \quad K_\mu=K_1^{\sigma_\mu}$$

and hence all K_μ's are also $\boldsymbol{C}$-isomorphic. But the dimension of the maximal compact subgroups of $SL(r_1, \boldsymbol{R})$ and $SL(r_1/2, \boldsymbol{H})$ is equal to $\frac{1}{2}r_1(r_1-1)$, $\frac{1}{2}r_1(r_1+1)$, respectively. Therefore no mixture of these two types can occur, which proves our assertion.

1.4. The norm and trace. The rank of a self-dual homogeneous cone $\mathscr{C}$ is by definition the $\boldsymbol{R}$-rank of the Lie algebra $\mathfrak{g}$, which also coincides with the (absolute) rank of the formally real Jordan algebra (V, c_0). Let $n=\dim V$ and $r=\operatorname{rank}\mathscr{C}$. If $\mathscr{C}$ is irreducible, one has (from the Peirce decomposition of (V, c_0))

$$n=r+\frac{d}{2}r(r-1), \tag{1.4.1}$$

where d is a non-negative integer. For $\mathscr{C}=\boldsymbol{R}_+$, one puts $d=0$. For $\mathscr{C}=\mathscr{P}_r(\boldsymbol{F})$ $(r\geq 2)$, one has actually rank $\mathscr{C}=r$ and $d=\dim_{\boldsymbol{R}}\boldsymbol{F}=1, 2, 4, 8$ according as $\boldsymbol{F}=\boldsymbol{R}, \boldsymbol{C}, \boldsymbol{H}, \boldsymbol{O}$. For a quadratic cone $\mathscr{C}$, one has rank $\mathscr{C}=2$ and $d=n-2$. Thus the pair (r, d) is a complete invariant for an isomorphism class of irreducible self-dual homogeneous cones.

In the Jordan algebra (V, c_0), one can define the (reduced) *norm* $N: V\to\boldsymbol{R}$ as the (unique) homogeneous polynomial function of degree r on V such that, for a "general element" x in V, $N(tc_0-x)(\in\boldsymbol{R}[t])$ is the minimal polynomial for x in the usual sense. When $\mathscr{C}$ is irreducible, the norm is uniquely characterized by the property

$$N(c_0)=1, \quad N(gx)=\det(g)^{r/n}N(x) \quad \text{for } g\in G,\ x\in V. \tag{1.4.2}$$

(Note that $\chi(g)=\det(g)^{r/n}$ is a rational character on G.) Hence one has the relation

$$N(x)=(\phi_{\mathscr{C}}(c_0)^{-1}\phi_{\mathscr{C}}(x))^{-r/n} \quad (x\in\mathscr{C}). \tag{1.4.3}$$

The (reduced) trace $\operatorname{tr}(x)$ is defined by

$$N(tc_0-x)=t^r-\operatorname{tr}(x)t^{r-1}+\cdots+(-1)^rN(x).$$

The trace is K-invariant. It follows that, when $\mathscr{C}$ is irreducible, one has

$$\operatorname{tr} x=\frac{r}{n}\operatorname{tr}(T_x), \tag{1.4.4}$$

where $T_x: y\mapsto xy$ is the multiplication in the Jordan algebra (V, c_0). (Note also, putting $P(x)=2T_x^2-T_{x^2}$, one has the relations $P(gx)=gP(x)^tg$, $\det(P(x))=N(x)^{2n/r}$.)

In what follows, we assume that $\mathscr{C}$ is quasi-irreducible. Let $(V,\mathscr{C})=\prod_{\mu=1}^m(V_\mu, \mathscr{C}_\mu)$ be the irreducible decomposition,

$$G=\prod_{\mu=1}^m G_\mu, \quad G_\mu=\operatorname{Aut}(V_\mu, \mathscr{C}_\mu)^\circ,$$

and put $n_1=\dim V_1$, $r_1=\operatorname{rank}\mathscr{C}_1=\boldsymbol{R}\text{-rk}\, G_1$. Then one has $n=mn_1$, $r=mr_1$ and

$$\frac{n}{r}=\frac{n_1}{r_1}=1+\frac{d}{2}(r_1-1).$$

Hence the formulae (1.4.2–4) remain valid. We normalize the inner product on V in such a way that $\langle c_{0,\mu}, c_{0,\mu}\rangle=r_1$ $(1\le\mu\le m)$, where $c_0=(c_{0,\mu})$. Then one has

$$\langle x, y\rangle=\operatorname{tr}(xy). \tag{1.4.5}$$

The Euclidean (i.e. self-dual) measure on V for this $\langle\ \rangle$ will be denoted as dx. A G-invariant measure on $\mathscr{C}$ is then given by $N(x)^{-n/r}dx$.

1.5. The Γ-function. Let $\mathscr{C}$ be a quasi-irreducible self-dual homogeneous cone in V. The "Γ-function" of $\mathscr{C}$ (introduced by Koecher) is defined by the integral

$$\Gamma_{\mathscr{C}}(s)=\int_{\mathscr{C}} e^{-\operatorname{tr}(x)}N(x)^{s-(n/r)}dx \quad (s\in\boldsymbol{C}), \tag{1.5.1}$$

which converges absolutely for $\mathrm{Re}\, s>n/r-1$. By a change of variable, one gets

$$N(x)^{-s}\Gamma_{\mathscr{C}}(s)=\int_{\mathscr{C}} e^{-\langle x,y\rangle}N(y)^{s-n/r}dy \quad (x\in\mathscr{C}). \tag{1.5.2}$$

On V one can define a (unique) differential operator of degree r, denoted as $N(\nabla_x)$, with the property

$$N(\nabla_x)e^{\langle x,y\rangle}=N(y)e^{\langle x,y\rangle}$$

(cf. [R], [SS]). Then $N(\nabla_x)$ is relatively invariant in the sense that one has

$$L_g^{-1}N(\nabla_x)L_g=\det(g)^{-r/n}N(\nabla_x) \quad (g\in G,\ x\in V),$$

where $(L_g f)(x)=f(g^{-1}x)$ for any function f on V. The associated "b-function" is defined by

$$N(\nabla_x)N(x)^s=b(s)N(x)^{s-1} \quad (x\in\mathscr{C},\ s\in \boldsymbol{C})$$

(cf. [SS]).*) Then, applying $N(\nabla_x)$ on the both sides of (1.5.2), one gets

$$b(s)=(-1)^r\frac{\Gamma_{\mathscr{C}}(1-s)}{\Gamma_{\mathscr{C}}(-s)}.$$

By a direct computation from (1.5.1) (see e.g. [S2]), one obtains

$$\Gamma_{\mathscr{C}}(s)=(2\pi)^{(n-r)/2}\left(\prod_{i=1}^{r_1}\Gamma\left(s-\frac{d}{2}(i-1)\right)\right)^m, \tag{1.5.3}$$

$$b(s)=\left(\prod_{i=1}^{r_1}\left(s+\frac{d}{2}(i-1)\right)\right)^m. \tag{1.5.4}$$

(For the Γ-function of a more general cone, see [G].)

§ 2. Zeta functions associated to a self-dual homogeneous cone

2.1. We assume in this section that $(V,\mathscr{C})$ is endowed with a $\boldsymbol{Q}$-simple $\boldsymbol{Q}$-structure in the sense stated in Lemma 1.3.1. Then G^s is $\boldsymbol{Q}$-simple (or reduces to $\{1\}$), the center of G is of $\boldsymbol{Q}$-rank one and $\mathscr{C}$ is quasi-irreducible. The $\boldsymbol{Q}$-rank of G, which we denote by r_0, is a divisor of $r_1=\boldsymbol{R}$-rk G_1: hence we set $\delta=r_1/r_0$. We fix a base point c_0 in $\mathscr{C}\cap V_{\boldsymbol{Q}}$; then the norm, trace and the (normalized) inner product $\langle\ \rangle$ are all defined over $\boldsymbol{Q}$. We choose a lattice M in $V_{\boldsymbol{Q}}$ and an arithmetic subgroup Γ of G such

*) Note that in some recent literature (e.g. [I1]) it has become more customary to denote our $b(s)$ by $b(s-1)$.

that $\Gamma M=M$. We define a zeta function by

$$(2.1.1) \qquad Z_{\mathscr{C},c_0}(\Gamma, M; s)=\sum_{x:\Gamma\backslash\mathscr{C}\cap M}|\Gamma_x|^{-1}N(x)^{-s} \quad (s \in \boldsymbol{C}),$$

where $\Gamma_x=\{\gamma \in \Gamma \mid \gamma x \in x\}$ (which is finite) and the summation is taken over a complete set of representatives of the Γ-orbits in $\mathscr{C}\cap M$. When c_0 is kept fixed, we write $Z_{\mathscr{C}}$ for $Z_{\mathscr{C},c_0}$. It is known that the series on the right hand side of (2.1.1) is absolutely convergent for $\operatorname{Re} s>n/r$ and has an analytic continuation to a meromorphic function on the whole plane $\boldsymbol{C}$. It is clear that, if Γ' is a subgroup of Γ of finite index, then one has

$$Z_{\mathscr{C}}(\Gamma', M; s)=[\Gamma : \Gamma']Z_{\mathscr{C}}(\Gamma, M; s).$$

Hence it suffices to consider the zeta function for the full stabilizer $\Gamma_M=\{\gamma \in G \mid \gamma M=M\}$. In that case, we write $Z_{\mathscr{C}}(M; s)$ for $Z_{\mathscr{C}}(\Gamma_M, M; s)$.

In the simplest case where G^s reduces to $\{1\}$, one obtains essentially the (partial) Dedekind zeta function of the totally real number field F_1 (see the Example 2.1.2 below). The case where $\mathscr{C}$ is a quadratic cone was studied by Siegel [S9]. Our zeta function is a special case of the zeta function associated to a (real) "prehomogeneous vector space" in the sense of Sato-Shintani [SS], who treated as examples the cases of $\mathscr{P}_r(\boldsymbol{R})$, $\mathscr{P}_r(\boldsymbol{C})$ and the quadratic cones (see [S7], [SS], pp. 160–168, pp. 155–157). For other cases, see [M3] (cf. also [SF]).

Example 2.1.2. Let F_1 be a totally real number field of degree m and let $V=F_1\otimes_{\boldsymbol{Q}}\boldsymbol{R}\cong\boldsymbol{R}^m$. Then the "angular domain" $\mathscr{C}=\boldsymbol{R}_+^m$ is a self-dual homogeneous cone with respect to the standard inner product in $\boldsymbol{R}^m$. G is identified with the multiplicative group $\boldsymbol{R}_+^m$ and G^s and K reduce to the identity. If one takes c_0 to be 1 (the unit element of F_1), then the norm and the trace are given by

$$N(x)=\prod \xi_\mu, \quad \operatorname{tr}(x)=\sum \xi_\mu \qquad \text{for } x=(\xi_\mu) \in V,$$

and the standard inner product in $\boldsymbol{R}^m$ is normalized. V has a natural $\boldsymbol{Q}$-structure for which $V_{\boldsymbol{Q}}=F_1$, G is defined over $\boldsymbol{Q}$, of $\boldsymbol{Q}$-rank 1, and $G_{\boldsymbol{Q}}=\{\alpha \in F_1^\times \mid \alpha^{\sigma_\mu}>0\,(1\leq\mu\leq m)\}$. Hence the above assumptions are all satisfied and one has $n_1=r_1=r_0=1$, $n=r=m$. Let $\mathcal{O}_{F_1}$ be the ring of integers in F_1 and choose M to be an ideal $\mathfrak{a}_1$ in $\mathcal{O}_{F_1}$. Then Γ_M is the group of totally positive units of $\mathcal{O}_{F_1}$ and one has

$$Z_{\mathscr{C}}(M; s)=\sum_{x:\Gamma_M\backslash\mathscr{C}\cap M} N(x)^{-s}=N(\mathfrak{a}_1)^{-s}\sum_{\mathfrak{a}\sim\mathfrak{a}_1^{-1}} N(\mathfrak{a})^{-s},$$

where the summation in the last expression is taken over all integral ideals $\mathfrak{a}$ "equivalent" to $\mathfrak{a}_1^{-1}$ in the narrow sense. Thus essentially $Z_{\mathscr{C}}(M; s)$ is nothing but a "parital" Dedekind zeta function of F_1 corresponding to the "ray class' of $\mathfrak{a}_1^{-1}$.

2.2. Functional equations. According to the general theory of Sato-Shintani, the functional equations for $Z_{\mathscr{C}}(M; s)$ are obtained as follows. Let

$$V^{\times}=\{x \in V \mid N(x)\neq 0\}=\prod_{\mu=1}^{m} V_{\mu}^{\times}$$

and let

$$V_{\mu}^{\times}=\coprod_{i=0}^{r_1} V_{\mu,i}$$

be the decomposition of $V_{\mu}^{\times}$ into the disjoint union of the connected components, or what amounts to the same thing, into that of the G_{μ}-orbits. (If $c_0=(c_{0,\mu})$ and if $c_{0,\mu}=\sum_{i=1}^{r_1} e_i^{(\mu)}$ is primitive decomposition, then $V_{\mu,i}$ is defined to be the G_{μ}-orbit of $-\sum_{k=1}^{i} e_k^{(\mu)}+\sum_{k=i+1}^{r_1} e_k^{(\mu)}$.) Thus one has

$$V^{\times}=\coprod\nolimits_{I\in\mathscr{I}_{r_1}^{m}} V_I,$$

where $\mathscr{I}_{r_1}^{m}$ denotes the set of all m-tuples $I=(i_1, \cdots, i_m)$ with $0\leq i_{\mu}\leq r_1$ $(1\leq\mu\leq m)$ and for $I=(i_{\mu})$, one sets $V_I=\prod_{\mu=1}^{m} V_{\mu,i_{\mu}}$. Hence $V^{\times}$ consists of $(r_1+1)^m$ connected components. We write (k) for $(k, \cdots, k)\in\mathscr{I}_{r_1}^{m}$; then $V_{(0)}=\mathscr{C}$ and $V_{(r_1)}=-\mathscr{C}$.

For each $I\in\mathscr{I}_{r_1}^{m}$, we define a zeta function

$$\xi_I(M; s)=\sum_{x: \Gamma_M\backslash V_I\cap M}\frac{\mu(x)}{|N(x)|^s}, \tag{2.2.1}$$

where the summation is taken over a complete set of representatives of Γ_M-orbits in $V_I\cap M$ and $\mu(x)$ is a "density" defined as follows. For $x\in V_I$, let U_x be a relatively compact neighbourhood of x in V_I and let

$$W_x=\{g\in G \mid gx\in U_x\},$$
$$G_x=\{g\in G \mid gx=x\},\quad \Gamma_x=G_x\cap\Gamma.$$

Then one has

$$\mu(x)=\int_{\Gamma_x\backslash W_x} dg\Big/\int_{U_x}|N(x)|^{-n/r}dx,$$

where dg is a Haar measure on G normalized in such a way that for any non-negative continuous function f on $\mathscr{C}$ one has

$$\int_G f(gc_0)\,dg = \int_{\mathscr{C}} f(x) N(x)^{-n/r}\,dx.$$

Then, except for the case $r_1 = r_0 = 2$, $d=1$ (treated in [S7], [S9]), $\mu(x)$ is finite and coincides with the volume of $\Gamma_x \backslash G_x$ with respect to a suitably normalized Haar measure on G_x ([SS], Lemma 2.4) and hence depends only on the Γ-equivalence class of x. In what follows, we omit the above-mentioned exceptional case. Then the series (2.2.1) is absolutely convergent for $\mathrm{Re}\, s > n/r$ and has an analytic continuation to a meromorphic function on $\boldsymbol{C}$ ([SS], [S7]). Clearly one has $\xi_{(0)} = \xi_{(r_1)} = Z_{\mathscr{C}}$. For $I = (i_\mu) \in \mathscr{I}^m_{r_1}$, we set $I^* = (r_1 - i_\mu)$. Then it is clear that

$$V_{I^*} = -V_I \quad \text{and} \quad \xi_{I^*} = \xi_I.$$

Thus essentially we get $[(r_1+1)^m/2]$ zeta functions.

Theorem 2.2.2. *The functions $\xi_I(M; s)$ satisfy the functional equations of the following form:*

$$\xi_J\left(M^*; \frac{n}{r} - s\right) = v(M)(2\pi)^{-rs}\Gamma_{\mathscr{C}}(s)\,\boldsymbol{e}\left(\frac{rs}{4}\right) \sum_{I \in \mathscr{I}^m_{r_1}} \xi_I(M; s) u_{IJ}(s),$$

where M^* is the dual lattice of M, $v(M) = \mathrm{vol}(V/M)$ and, for $I=(i_\mu)$, $J=(j_\mu)$, $u_{IJ}(s) = \prod_{\mu=1}^m u_{i_\mu, j_\mu}(s)$, u_{ij} $(0 \le i, j \le r_1)$ being integral polynomials in $\boldsymbol{e}(-s/2)$ of degree $\le r_1$.

For an explicit expression of u_{ij}, see [SF]*).

2.3. To obtain more precise results, we assume in the rest of this section that $r \ge 2$ *and d is even.* (Note that, if $r=1$, $Z_{\mathscr{C}}$ is essentially the Riemann zeta function. If d is odd, then one has either $r_1 = 2$ (quadratic cones) or $d=1$ ($\mathscr{P}_{r_1}(\boldsymbol{R})^m (r_1 \ge 2)$).)

Under this assumption, n/r is an integer and there are two cases:

(a) $d \equiv 0 \pmod 4$, or $d \equiv 2 \pmod 4$ and r_1 is odd. In this case, n/r is odd.

(a′) $d \equiv 2 \pmod 4$ and r_1 is even. In this case, n/r is even.

Applying the methods in [SS] and [SF], one obtains

Theorem 2.3.1. *Under the above assumption, the function $\xi_I(M; s)$ has at most r_0 simple poles at $s = n/r - (d/2)\rho$ for $0 \le \rho \le r_1 - 1$, $\delta \mid \rho$ $(\delta = r_1/r_0)$ and one has*

*) Note that $u_{ij}(s)$ in [SS] is in our notation (and in [SF]) given by $c(2\pi)^{(n-r)/2} u_{r-i, r-j}(s)$, if the measure on V in [SS] is equal to $c\,dx$ with dx self-dual. For instance, in the case $V = \mathrm{Her}_r(\boldsymbol{C})$ ([SS], pp. 160–168), one has $c = 2^{-(n-r)/2}$.