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A Series of Monographs and Textbooks

PROBABILITY ALGEBRAS AND STOCHASTIC SPACES

Probability and Mathematical Statistics

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PREFACE

During the last twenty years, many excellent textbooks have enriched the mathematical literature on probability theory. With the help of classical set-theoretic measure and integral theory, these books introduce the fundamental concepts of probability theory, then formulate and study old and recent problems germane to the theory. There is however an alternate way to introduce the main notions of probability theory, a way which is more naturally adapted to the empirical origins of the subject. The present volume is an exploration of this alternate development.

There are three fundamental notions of probability theory: Event, probability of an event, random variable. A given set of events forms a Boolean algebra with respect to the logical connectives (considered as operations) " or ", " and ", and " not ". Probability is a normed measure on a Boolean algebra of events. It is natural to consider probability as finitely additive and strictly positive, i.e., equal to zero only for the impossible event. Therefore, a Boolean algebra $\mathfrak{A}$ endowed with a finitely additive and strictly positive probability p can be considered as a probability algebra $(\mathfrak{A}, p)$. In all empirical cases a Boolean algebra of events can be endowed with an additive and strictly positive probability. Moreover, the σ-(countable) additivity of probability, which has important mathematical consequences in the theory, can always be obtained by a metric extension of a probability algebra $(\mathfrak{A}, p)$ to a probability σ-algebra $(\tilde{\mathfrak{A}}, \tilde{p})$, in which $\tilde{\mathfrak{A}}$ is a Boolean σ-algebra and $\tilde{p}$ a countably additive and strictly positive probability.

It is well-known that a Boolean algebra $\mathfrak{A}$ is always isomorphic to a field (Boolean algebra) of subsets of a set (= space) Ω. Thus the investigation of events and their probabilities can be reduced to a study of normed measures on fields of sets. Moreover, it is always possible to represent the Boolean algebra $\mathfrak{A}$ of events by a field $\mathbf{A}$ of subsets of a set, so that the normed measure (the probability) P on $\mathbf{A}$ is set-theoretically countably additive. Let $(\Omega, \mathbf{A}, P)$ be a so-called probability space, which represents, set-theoretically, a probability algebra $(\mathfrak{A}, p)$; then $(\Omega, \mathbf{A}, P)$ can always be extended to a complete probability σ-space $(\Omega, \tilde{\mathbf{A}}, \tilde{P})$, in which $\tilde{\mathbf{A}}$ is a σ-field containing $\mathbf{A}$ and P a complete, normed, countably additive measure on $\tilde{\mathbf{A}}$. The elements of $\tilde{\mathbf{A}}$ can be considered as events; in the case, however, in which the cardinality of Ω is $> \aleph_0$ it may happen that there exist non-empty sets $E \in \tilde{\mathbf{A}}$ (i.e., events different from the impossible event) of measure (= probability) zero, which have no probabilistic interpretation. We can overcome this difficulty by considering directly the probability σ-algebra $(\tilde{\mathfrak{A}}, \tilde{p})$ instead of the probability σ-space $(\Omega, \tilde{\mathbf{A}}, \tilde{P})$.

Our aim is to develop the fundamental notions of probability theory in this " point-free " way. This, however, requires knowledge of lattice theory. We find that lattice theory also provides simplicity and generality, since it deals with classes of random variables which are the elements of the so-called stochastic spaces. The space of all elementary random variables defined over a probability algebra in a " point-free " way is a base for the stochastic space of all random variables, which can be obtained from it by lattice-theoretic extension processes. There are, however, problems in which one wants to consider individual samples and cannot work without points; then one can always assign a suitable probability σ-space to the probability algebra under consideration. Conversely, one can assign to every probability σ-space $(\Omega, \mathbf{A}, P)$ a probability σ-algebra $(\mathfrak{A}, p)$ in which $\mathfrak{A}$ is the quotient Boolean σ-algebra $\mathbf{A}/\mathbf{N}$, where $\mathbf{N}$ is the σ-ideal of all sets of probability zero. Thus the two theories are equivalent.

In the lattice-theoretic treatment of probability theory, a structural classification of all possible probability algebras (therefore an analogous classification of all possible stochastic spaces) can easily be obtained, which provides us with a representation of every probability σ-algebra by a probability σ-space $(\Omega, \mathbf{A}, P)$ in which Ω is the cartesian product of factors equal to the interval $[0, 1]$ of the real line, $\mathbf{A}$ a σ-subfield of the σ-field of all Lebesgue product measurable subsets of Ω, and P the product measure of $\mathbf{A}$. A corresponding representation of random variables by Lebesgue measurable functions defined on Ω can also be obtained.

The origin of this book is in a set of lectures which I gave in the academic year 1963–64 at the Catholic University of America, Washington, D.C. The Statistical Laboratory there issued a mimeographed version of my notes under the title " Lattices and their Applications to Probability ". The present text is a revised and expanded version of these notes, maintaining the central mathematical ideas of the lectures, namely probability algebras and stochastic spaces (i.e., spaces of random variables). The part of the notes devoted to pure lattice theory has been shortened and the most important concepts and theorems of this theory have been stated in two appendices, mostly without proofs.

In addition to the material in the mimeographed edition, the present volume contains a general way to introduce the concept of random variables taking values in spaces endowed with any algebraic or topological structure. In particular, we study the cases in which the space of the values is a lattice group, or vector lattice. When the space of the values is a Banach space, a theory of expectation and moments is stated. A theory of expectation can be easily stated in more general cases of spaces of values: for example, locally convex vector spaces or topological vector spaces, for which an integration theory is known.

We have restricted ourselves on the introduction and study only of the fundamental mathematical notions. We mention only a few facts about the concepts of independence and conditional probabilities and expectations. Certainly, it would be interesting to state the theory of stochastic processes and, especially, the theory of martingales. But this would go beyond the scope of the present monograph, or it would have to be published in a second volume.

The author wishes to express his gratitude to Dr. Eugene Lukacs of the Catholic University, who made it possible for him to give lectures and publish them. It is a pleasure to offer thanks to F. Papangelou and G. Anderson who read critically the manuscript of the lectures and made valuable suggestions during the mimeographed edition of them.

He expresses his best thanks to Miss Susan Papadopoulou who read all the manuscript of the present edition carefully and made valuable suggestions. In many cases she has helped him to give simpler formulations and proofs.

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Athens, Greece. D. A. Kappos.
April 1969.

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PROBABILITY ALGEBRAS AND STOCHASTIC SPACES

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I

PROBABILITY ALGEBRAS

1. DEFINITIONS AND PROPERTIES

1.1. A *probability algebra* (*pr algebra*) $(\mathfrak{A}, p)$ consists of a nonempty set $\mathfrak{A}$ of elements denoted by lower-case latin letters: $a, b, c, \ldots, x, y, \ldots,$ called *events* and a real-valued function p on $\mathfrak{A}$, called a *probability* (pr). In the set $\mathfrak{A}$ two binary operations $a \vee b$ (*a or b*) and $a \wedge b$ (*a and b*) and one unitary operation a^c (*not a*) are defined, which introduce in $\mathfrak{A}$ the algebraic structure of a Boolean algebra.†

The probability p satisfies the following conditions:

1.1.1. p is *strictly positive*, i.e., $p(x) \geqslant 0$, for every $x \in \mathfrak{A}$ and $p(x) = 0$ if and only if $x = \emptyset$, where $\emptyset$ is the zero of $\mathfrak{A}$.

1.1.2. p is *normed*, i.e., $p(e) = 1$, where e is the unit of $\mathfrak{A}$.

1.1.3. p is *additive* i.e., $p(a \vee b) = p(a) + p(b)$ if a and b are disjoint.‡

We shall call the unit e the *sure event* and the zero $\emptyset$ the *impossible event* of the event algebra $\mathfrak{A}$. Every $x \in \mathfrak{A}$ with $x \neq \emptyset$ and $x \neq e$ is called a *possible event* of the algebra $\mathfrak{A}$.

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† We consider the theory of Boolean algebras as known; *cf.* also Appendix 1 of this book.

‡ We say that the event a and the event b are disjoint (exclude each other, or are mutually exclusive, or are incompatible) if $a \wedge b = \emptyset$.