Edited by **Dr. I. M. JAMES** *The Mathematical Institute, Oxford*

Volume 3 – 'Homotopy Theory'

The Mathematical Works of J. H. C. Whitehead

THE MATHEMATICAL WORKS OF

J. H. C. WHITEHEAD

VOLUME III

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J. H. C. WHITEHEAD
1904–1960

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EDITED BY

I. M. JAMES

VOLUME III

HOMOTOPY THEORY

PERGAMON PRESS

OXFORD · LONDON · NEW YORK · PARIS

1962

PERGAMON PRESS LTD.
Headington Hill Hall, Oxford
4 & 5 *Fitzroy Square, London W.*1

PERGAMON PRESS INC.
122 *East 55th Street, New York* 22, *N.Y.*

PERGAMON PRESS S.A.R.L.
24 *Rue des Écoles, Paris Ve*

PERGAMON PRESS G.m.b.H.
Kaiserstrasse 75, *Frankfurt am Main*

Library of Congress Catalogue Card No. 62-11568

Printed in Hungary

EDITORIAL PREFACE

THESE volumes are believed to contain all the published mathematical work of J. H. C. Whitehead, excluding reviews and lecture notes. The arrangement differs from the chronological in so far as it seemed desirable to bring related papers together. No corrections or other alterations have been made except those which were, in some sense, authorised. For example, in [37] corrections circulated in mimeographed form have been partly incorporated in the text and partly listed at the end of the paper.

The introductory section contains a list of Whitehead's works, in chronological order of writing ; a biographical note by M. H. A. Newman and Barbara Whitehead ; and a mathematical appreciation by John Milnor.

Whitehead's earliest interests were in geometry, especially differential geometry. All his published work on the subject is contained in the first volume, together with some papers on algebras. Most of it was written in the period 1929–1937, but a few later articles are included.

In the second volume, most of the papers are related in some way to the classification problem for manifolds, especially the Poincaré conjecture, but towards the end one sees the gradual transition in the direction of algebraic topology. This volume, with the preceding one, includes all Whitehead's published work up to the year 1941, as well as a few later papers.

The papers in the third volume, written between 1947 and 1955, are closely linked together. The study of simple homotopy types (" nuclei " in the previous volume) is carried a stage further but most of the material relates to the realization problem for homotopy types. Here is to be found Whitehead's version of homotopy theory in terms of *CW*-complexes (originally " membrane " complexes).

The papers in the last volume fall into two groups. The first group, written between 1952 and 1957, is principally concerned with fibre spaces and the Spanier-Whitehead *S*-theory. In the second group, written between 1957 and 1960, Whitehead returns to classical topology after a long interval, and participates in the renewed assault on the problems which fascinated him most.

I. M. J.

ACKNOWLEDGMENT

THE Publishers wish to express their sincere gratitude for the kind co-operation received from the publishers of the various publications in which the articles reproduced in these volumes first appeared, and for permission to reproduce this material. The exact source of each article is given in the " Publications of J. H. C. Whitehead", page ix.

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NOTE ON A THEOREM DUE TO BORSUK

1. Introduction. Let A, $B \subset A$ and B' be compacta, which are[1] ANR's (absolute neighbourhood retracts). Let $B' \subset A'$ where A' is a compactum, and let $f:(A, B) \rightarrow (A', B')$ be a map such that $f|(A-B)$ is a homeomorphism onto $A'-B'$. Thus A' is homeomorphic to the space defined in terms of A, B, B' and the map $g=f|B$ by identifying each point $b \in B$ with $gb \in B'$. K. Borsuk [3] has shown that A' is locally contractible. It is therefore an ANR if $\dim A' < \infty$. The main purpose of this note is to prove, without this restriction on $\dim A'$:

THEOREM 1. *A' is an ANR.*

We also derive some simple consequences of this theorem. For example, it follows that the homotopy extension theorem, in the form in which the image space is arbitrary, may be extended[2] from maps of polyhedra to maps of compact ANR's, P and $Q \subset P$. That is to say, if $f_0: P \rightarrow X$ is a given map, the space X being arbitrary, and if $g_t: Q \rightarrow X$ is a deformation of $g_0 = f_0|Q$, then there is a homotopy $f_t: P \rightarrow X$, such that $f_t|Q = g_t$. For let $R = (P \times 0) \cup (Q \times I) \subset P \times I$ and let $h: R \rightarrow X$ be given by $h(p, 0) = f_0 p$, $h(q, t) = g_t q$ $(p \in P, q \in Q)$. Since $Q \times I$ is (obviously) a compact ANR it follows from Theorem 1, with $A = Q \times I$, $B = Q \times 0$, $B' = P \times 0$, $A' = R$ that R is an ANR. Therefore R is a retract of some open set $U \subset P \times I$. If $\theta: U \rightarrow R$ is a retraction, then $h\theta: U \rightarrow X$ is an extension of $h: R \rightarrow X$ throughout U. This is all we need for the homotopy extension theorem (see [5, pp. 86, 87]). Thus we have the corollary:

COROLLARY. *A given homotopy, $g_t: Q \rightarrow X$, of $g_0 = f_0|Q$, can be extended to a homotopy, $f_t: P \rightarrow X$, where P and $Q \subset P$ are compact ANR's and $f_0: P \rightarrow X$ is a given map of P in an arbitrary space X.*

We also use Theorem 1 to prove another theorem. We shall describe a map $\xi: X \rightarrow Y$ as a *homotopy equivalence* if, and only if, there is a map, $\eta: Y \rightarrow X$, such that $\eta\xi \simeq 1$, $\xi\eta \simeq 1$, where X and Y are any two spaces. Thus the statement that $\xi: X \rightarrow Y$ is a homotopy equivalence implies that X and Y are of the same homotopy type. Let A, B, A', B' and $f:(A, B) \rightarrow (A, B')$ be as in Theorem 1 and let $g = f|B$.

Received by the editors January 26, 1948.

[1] For an account of these spaces, on which this note is based, see [2]. Numbers in brackets refer to the references cited at the end of the paper.

[2] Cf. [4].

Then we shall prove:

THEOREM 2. *If $g:B\to B'$ is a homotopy equivalence so is $f:A\to A'$.*

For example B' may consist of a single point, in which case we describe the identification of B with B' as the operation of *shrinking B into a point.* Then it follows from Theorem 2 that any (compact) absolute retract, $B\subset A$, may be shrunk into a point, without altering the homotopy type of A. As another example let A and B' be cell complexes[3] and B a sub-complex of A. Then A' is also a cell complex, subject to suitable conditions on the map[4] $g=f|B$, and Theorem 2 shows that certain combinatorial operations do not alter the homotopy type of A. For example, if B is the n-section of A and if B' is any complex, of at most n dimensions, which is of the same homotopy type as B, then there is a complex, A', of the same homotopy type as B, whose n-section is B'.

2. **Another theorem.** We prove Theorem 1 by means of another theorem. Let X and $Y\subset X$ be compacta and let Y be an ANR. Given $\rho>0$ let $V_\rho\subset X$ be the subset consisting of points whose distances from Y are less than ρ. We assume that

(a) *given $\epsilon>0$ there is a $\rho(\epsilon)>0$ and an ϵ-homotopy, $\theta_t:X\to X$, such that $\theta_0=1$, $\theta_t|Y=1$, $\theta_1 V_{\rho(\epsilon)}=Y$,*

(b) *given ϵ, $\rho>0$ there is a $u(\epsilon,\rho)>0$ such that any partial realization, $g:L\to X-V_\rho$, whose mesh does not exceed $u(\epsilon,\rho)$, of a finite simplicial complex, K, can be extended to a full realization, $f:K\to X$, whose mesh does not exceed ϵ, $u(\epsilon,\rho)$ being independent of K and L.*

Then we prove:

THEOREM 3. *Subject to these conditions X is an ANR.*

For this we shall need a sharpened form of the homotopy extension theorem. Let P and $Q\subset P$ be compacta and let $f_0:P\to M$ be a given map of P in a metric space M. Let $g_t:Q\to M$ be an ϵ-deformation of $g_0=f_0|Q$. Assume that either

(1) *M is a (separable) ANR or that*

(2) *P is a finite polyhedron and Q a sub-polyhedron.*

Then we have:

LEMMA 1. *Given $\epsilon'>0$ there is an $(\epsilon+\epsilon')$-deformation, $f_t:P\to M$, such that $f_t|Q=g_t$.*

[3] That is, a complex of the sort defined in [6], and in a forthcoming book by S. Eilenberg and N. E. Steenrod.

[4] For example, $gB^n\subset B'^n$ for each $n=0, 1, \cdots$, where K^n denotes the n-section of a complex, K, or $A^n\subset B$, $gB\subset B'^n$ for a particular value of n.

By way of proof it is sufficient to add a few comments to a standard proof of the homotopy extension. (See [5, pp. 86, 87].) Let $R=(P\times 0)\cup(Q\times I)\subset P\times I$ and let $h:R\to M$ be given by $h(p,0)=f_0p$, $h(q,t)=g_tq$ $(p\in P,\ q\in Q)$. If P is a polyhedron and Q a sub-polyhedron, then R is a polyhedron and hence a neighbourhood retract of $P\times I$ (in fact R is a deformation retract of $P\times I$). Therefore h can be extended throughout some neighbourhood, $U\subset P\times I$, of R, as it can be if M is an ANR and P, Q arbitrary compacta. There is a neighbourhood, $V\subset P$, of Q such that $V\times I\subset U$. Since Q is compact we may take V to be the neighbourhood given by $\delta(p,Q)<\rho$, for some $\rho>0$, where $\delta(p,p')$ is a distance function in P. On following the argument given by Hurewicz and Wallman [5, pp. 86, 87] it is easily seen that the extension $f_t:P\to M$ is an $(\epsilon+\epsilon')$-deformation provided ρ is sufficiently small.

We now proceed to prove Theorem 3 by showing that X is LC^*, as defined by Lefschetz.[5] Given $\epsilon>0$ let $\eta'=\eta(\epsilon/2)/4$, $\rho'=\rho(\eta')/2$, where η is an extension function[5] for Y and $\rho(\eta')$ means the same as in the condition (a). Let

$$\xi_1(\epsilon)=\min(2\eta',\rho').$$

We shall prove that

$$\xi(\epsilon)=u\{\xi_1(\epsilon),\rho'\}$$

is an extension function for X. Let K be a finite simplicial complex and $L\subset K$ a sub-complex, which contains all the vertices of K. Let $g:L\to X$ be a partial relization of K, whose mesh does not exceed $\xi(\epsilon)$. We first assume that $s\subset L$ if $g(s\cap L)\subset X-V_{\rho'}$, where s is the closure of any simplex in K. Let $K_1\subset K$ be the sub-complex consisting of all the (closed) simplexes, $s\in K$, such that $g(s\cap L)$ meets $V_{\rho'}$. Then $K=K_1\cup L$. Let $L_1=K_1\cap L$, $g_1=g|L_1$. Then it is sufficient to prove that g_1 can be extended to a full realization, $f_1:K_1\to X$, whose mesh does not exceed ϵ. For since $K_1\cap L=L_1$, $f_1|L_1=g|L_1$, the desired realization, $f:K\to X$, will be given by $f|L=g$, $f|K_1=f_1$. Clearly $\xi(\epsilon)\leqq\xi_1(\epsilon)$ and we shall prove this special case on the weaker assumption that the mesh of $g:L\to X$ does not exceed $\xi_1(\epsilon)$.

Since $\xi_1(\epsilon)\leqq\rho'$ we have $g_1L_1\subset V_{2\rho'}=V_\rho$ where $\rho=\rho(\eta')$. Let $\theta_t:X\to X$ be the η'-deformation associated with V_ρ as in condition (a). Since $K_1^0\subset K^0\subset L$, $K_1^0\subset K_1$, we have $K_1^0\subset L_1$. Also $\theta_1V_\rho\subset Y$. Therefore $\theta_1g_1:L_1\to Y$ is a partial realization of K_1 in Y, whose mesh does not exceed

[5] [2, pp. 82, 83, 84] (N.B. $K^0\subset L$).

$$\xi_1(\epsilon) + 2\eta' \leqq 2^{-1}\eta(\epsilon/2) + 2^{-1}\eta(\epsilon/2) = \eta(\epsilon/2).$$

Therefore $\theta_1 g_1: L_1 \to Y$ can be extended to a full realization, $f_0: K_1 \to Y$, whose mesh does not exceed $\epsilon/2$. By Lemma 1 there is an $(\eta'+\epsilon/8)$-homotopy, $f_t: K_1 \to X$, such that $f_t | L_1 = \theta_{1-t} g_1$. Clearly $\eta(\epsilon/2) \leqq \epsilon/2$, whence $\eta'+\epsilon/8 \leqq \epsilon/8+\epsilon/8=\epsilon/4$. Therefore the mesh of $f_1: K_1 \to X$ does not exceed $\epsilon/2+2(\eta'+\epsilon/8) \leqq \epsilon$ and $f_1 | L_1 = \theta_0 g_1 = g_1$. Therefore this special case is established.

In general let $K_0 \subset K$ be the sub-complex consisting of all the closed simplexes, $s \in K$, such that $g(s \cap L) \subset X - V_{\rho'}$. Let $L_0 = K_0 \cap L$. Then $g | L_0$ is a partial realization of K_0, whose mesh does not exceed $\xi(\epsilon) = u\{\xi_1(\epsilon), \rho'\}$. By condition (b) it can be extended to a full realization, $f_0: K_0 \to X$, of mesh at most $\xi_1(\epsilon)$. Since $K_0 \cap L = L_0$, $f_0 | L_0 = g | L_0$, a map, $g': K_0 \cup L \to X$, is defined by $g' | K_0 = f_0$, $g' | L = g$ and its mesh does not exceed $\xi_1(\epsilon)$. Therefore L may be replaced by $K_0 \cup L$ and the theorem follows from what we have already proved.

3. **Proof of Theorem** 1. We shall prove Theorem 1 by showing that the conditions (a) and (b) in §2 are satisfied when $X = A'$, $Y = B'$. Let $\delta(a_1, a_2)$ be a distance function in A and let $\epsilon > 0$ be given. Since A is compact there is a $\lambda(\epsilon) > 0$ such that $\delta'(fa_1, fa_2) < \epsilon$ provided $\delta(a_1, a_2) < \lambda(\epsilon)$, where $\delta'(a_1', a_2')$ is a distance function in A'. Let $U_\gamma \subset A$ be the neighbourhood of B which consists of all points, $a \in A$, such that $\delta(a, B) < \gamma$. As shown by Borsuk [3], there is a homotopy, $\phi_t: \overline{U}_\gamma \to A$, such that $\phi_0 = 1$, $\phi_t | B = 1$, $\phi_1 \overline{U}_\gamma = B$ for some $\gamma > 0$. By uniform continuity there is a $\mu > 0$ $(\mu \leqq \gamma)$ such that $\delta(\phi_t a, b) = \delta(\phi_t a, \phi_t b) < \lambda(\epsilon)/4$ if $\delta(a, b) \leqq \mu$. Hence $\phi_t | \overline{U}_\mu$ is a $\lambda(\epsilon)/2$-deformation. By Lemma 1, $\phi_t | \overline{U}_\mu$ can be extended to a $\lambda(\epsilon)$-deformation $\psi_t: A \to A$ $(\psi_0 = 1)$. Let $\theta_t: A' \to A'$ be given by $\theta_t | B' = 1$, $\theta_t | fA = f\psi_t f^{-1} | fA$. Since $f^{-1} | (A' - B')$ is single-valued and since $f^{-1}B' = B$ and $\psi_t | B = 1$ it follows that θ_t is single-valued. It is therefore continuous.[6] Since $\theta_t | B' = 1$ and the diameter of the trajectory, $\psi_t a$, of any point $a \in A$ is less than $\lambda(\epsilon)$ it follows that θ_t is an ϵ-deformation. Also $\theta_1(fU_\mu) = f\psi_1 U_\mu = fB \subset B'$. Therefore $\theta_1(B' \cup fU_\mu) = B'$. Since $f | (A - B)$ is a homeomorphism onto $A' - B'$ and $fB \subset B'$ it follows that $B' \cup fU_\mu$ is an open subset of A'. For

$$\begin{aligned} f(A - U_\mu) &= f\{(A - B) - (U_\mu - B)\} \\ &= A' - B' - f(U_\mu - B) \\ &= A' - (B' \cup fU_\mu). \end{aligned}$$

But $A - U_\mu$ is compact, whence $f(A - U_\mu)$ is closed and $B' \cup fU_\mu$ open.

[6] See §5 below.

Therefore there is a $\rho(\epsilon)>0$ such that $V_{\rho(\epsilon)}\subset B'\cup fU_\mu$, whence $\theta_1 V_{\rho(\epsilon)} = B'$. This establishes (a).

Let $\alpha(\epsilon')$ be an extension function for A. Since $f^{-1}|(A'-B')$ is a homeomorphism and $A'-V_\rho$ is a compact subset of $A'-B'$, for a given $\rho>0$, there is a $u(\epsilon, \rho)>0$ such that, if $\delta'(a', a'')<u(\epsilon, \rho)$ $(a', a''\subset A'-V_\rho)$, then $\delta(f^{-1}a', f^{-1}a'')<\alpha\{\lambda(\epsilon)\}$. If $\psi:L\to A'-V_\rho$ is a partial realization, of mesh at most $u(\epsilon, \rho)$, of a complex K, it follows that $f^{-1}\psi:L\to A$ is of mesh at most $\alpha\{\lambda(\epsilon)\}$. The latter can therefore be extended to a full realization, $\phi:K\to A$, of mesh at most $\lambda(\epsilon)$. Then $\phi'=f\phi:K\to A'$ is a realization of K, whose mesh does not exceed ϵ. Moreover $f\phi|L=ff^{-1}\psi=\psi$. Therefore (b) is satisfied and Theorem 1 is established.

4. **Proof of Theorem** 2. We first prove a lemma. Let X, Y be topological spaces[7]: let $X_0\subset X$, $Y_0\subset Y$ be closed subsets and let $\phi:(X, X_0)\to(Y, Y_0)$ be a map such that $\phi|X-X_0$ is a homeomorphism onto $Y-Y_0$. Moreover let the topology of Y be such that a subset $F\subset Y$ is closed if, and only if, $F\cap Y_0$ and $\phi^{-1}F$ are both closed.

LEMMA 2. *If X_0 is a deformation retract[8] of X, then Y_0 is a deformation retract of Y.*

After replacing X by a homeomorph, if necessary, we assume that it has no point in common with Y_0 and we unite X, Y_0 in the space, $Q=X\cup Y_0$, of which X and Y_0, each with its own topology, are closed subspaces. Then Y has the identification topology[6] determined by the map $\psi:Q\to Y$, where $\psi|X=\phi$, $\psi|Y_0=1$. Let $\xi_t:X\to X$ be a homotopy such that $\xi_0=1$, $\xi_t|X_0=1$, $\xi_1 X=X_0$ and let ξ_t be extended throughout Q by taking $\xi_t|Y_0=1$. Let $\eta_t=\psi\xi_t\psi^{-1}:Y\to Y$. Clearly $\psi^{-1}|Y-Y_0$ is single-valued. Also $\psi^{-1}Y_0=X_0\cup Y_0$. Since $\xi_t|X_0\cup Y_0=1$ it follows that η_t is single-valued and hence continuous.[6] Obviously $\eta_0=1$, $\eta_t|Y_0=1$, $\eta_1 Y=Y_0$, which establishes the lemma.

Notice that the topology of Y certainly satisfies the above condition if X is compact (that is, bi-compact) and if Y is a Hausdorff space. For let this be so and let $F\subset Y$ be such that $\phi^{-1}F$ and $F\cap Y_0$ are both closed. Then $\phi^{-1}F$ is compact, whence $\phi\phi^{-1}F$ is also compact and hence closed. But $\phi\phi^{-1}F=F\cap\phi X$ and

$$F = (F\cap\phi X)\cup(F\cap Y_0),$$

[7] We do not need to assume that X and Y satisfy any separation axioms.

[8] Following Lefschetz [1, p. 40] we do not admit that X_0 is a deformation retract of X unless there is a retracting deformation throughout which each point of X_0 is held fixed (see [7]).

whence F is closed. The converse follows from the continuity of ϕ and the fact that Y_0 is closed.

We now turn to Theorem 2. We recall that $f:(A, B)\to(A', B')$ is such that $f|(A-B)$ is a homeomorphism onto $A'-B'$ and $g=f|B$ is a homotopy equivalence. Replacing A, A' by homeomorphs, if necessary, we assume that no two of the spaces A, A', $A\times I$ have a point in common. We form the mapping cylinder, Γ, of the map f by identifying $(a, 0)\in A\times I$ with a and $(a, 1)$ with $fa\in A'$ for each[9] $a\in A$. The theorem will follow when we have proved that A is a deformation retract[10] of Γ.

Let $C=(A\times 0)\cup(B\times I)$. Then C is an ANR, as shown in §1. Let $\delta_s:A\times I\to A\times I$ be the retracting deformation of $A\times I$ onto $A\times 0$, which is given by $\delta_s(a, t)=(a, t-st)$ $(0\leqq s\leqq 1)$. Then $\delta_s C\subset C$ and it follows that C is a deformation retract[10] of $A\times I$. Let $\phi:A\times I\to\Gamma$ be the map which is given by $\phi(a, 0)=a$, $\phi(a, 1)=fa$, $\phi(a, t)=(a, t)$ if $0<t<1$. Since $fB\subset B'$ and $f|(A-B)$ is a homeomorphism onto $A'-B'$ it follows that $\phi|(A\times I)-(B\times 1)$ is a homeomorphism onto $\Gamma-B'$. Therefore $\phi|(A\times I)-C$ is a homeomorphism onto $\Gamma-(B'\cup\phi C)$. It follows from Lemma 2 that $B'\cup\phi C$ is a deformation retract of Γ. Since $g=f|B$ is a homotopy equivalence, B is a deformation retract[10] of $\Gamma_g=B'\cup\phi(B\times I)$, which is the mapping cylinder of $g:B\to B'$. A homotopy, $\eta_s:\Gamma_g\to\Gamma_g$, such that $\eta_0=1$, $\eta_s|B=1$, $\eta_1\Gamma_g=B$, can be extended throughout $B'\cup\phi C=B'\cup\phi(B\times I)\cup A$ by writing $\eta_s|A=1$. The result is a retracting deformation of $B'\cup\phi C$ onto A. Therefore A is a deformation retract of $B'\cup\phi C$ and hence of Γ, which proves the theorem.

5. **Note on identification spaces.**[11] Let $\phi:P\to X$ be a map of a space P onto a space X, which has the *identification topology* determined by ϕ. That is to say a subset $X_0\subset X$ is closed (open) if, and only if, $\phi^{-1}X_0$ is closed (open). A subset $P_0\subset P$ is said to be *saturated* (with respect to ϕ) if, and only if, $P_0=\phi^{-1}\phi P_0$. Therefore $X_0\subset X$ is closed if, and only if, it is the image under ϕ of a saturated closed set $P_0=\phi^{-1}X_0$. If P is compact and if X is a Hausdorff space then X certainly has the identification topology determined by ϕ. For in this case, if $P_0\subset P$ is closed, and hence compact, ϕP_0 is compact, and hence closed, whether P_0 is saturated or not.

Let $f:P\to Z$ be a map of P in any space Z.

[9] The points in A and A' shall retain their individualites in Γ, so that A, $A'\subset\Gamma$.

[10] See [7, Theorems 1.4 and 3.7] and [8].

[11] Cf. [9, pp. 61 et seq.] and [10, pp. 52 et seq.]. Concerning the theorem on p. 56 of [10] and Lemma 4 below see the correction at the beginning of [11].

LEMMA 3. *If X has the identification topology determined by ϕ, then the transformation $f\phi^{-1}:X\to Z$ is continuous if it is single-valued.*

If $p\in P$, then $p\in\phi^{-1}\phi p$, whence $fp\in f\phi^{-1}\phi p$. If $f\phi^{-1}$ is single-valued it follows that $fp=f\phi^{-1}\phi p$, or that $(f\phi^{-1})\phi=f$. Therefore the lemma follows from Theorem 1 on p. 53 of [**10**].

Let X have the identification topology determined by $\phi:P\to X$ and let $h:P\times I\to X\times I$ be given by $h(p, t)=(\phi p, t)$ $(p\in P,\ 0\leqq t\leqq 1)$. Then it follows from Lemma 4 below that $X\times I$ has the identification topology determined by h. Therefore we have the following corollary to Lemma 3, with P, X, ϕ and f replaced by $P\times I$, $X\times I$, h and $f:P\times I\to Z$, where $f(p, t)=f_t p$.

COROLLARY. *If $f_t:P\to Z$ is a given homotopy in any space, Z, then $f_t\phi^{-1}:X\to Z$ is continuous if it is single-valued.*

Let $\psi:Q\to Y$ be a map of a space, Q, onto a space, Y, which has the identification topology determined by ψ and which satisfies the following condition. Each point in any saturated open set, $V\subset Q$, is contained in a saturated open set, whose closure is a compact subset of V. This condition is satisfied if, for example, Q and Y are compacta. For in this case, if $q\in V$, there is a neighbourhood, $W\subset Y$, of ψq, such that $\overline{W}\subset\psi V$. Then $\psi^{-1}W$ is a saturated open set, whose (compact) closure is contained in V. In particular the condition is satisfied if $Q=Y=I$ and $\psi=1$.

Let X, Y have the identification topologies determined by maps $\phi:P\to X$, $\psi:Q\to Y$, which are onto X and Y, and let Y satisfy the above condition. Let $h:P\times Q\to X\times Y$ be given by $h(p, q)=(\phi p, \psi q)$ $(p\in P,\ q\in Q)$. Then we have:

LEMMA 4. *The space $X\times Y$ has the identification topology determined by h.*

Let $W\subset P\times Q$ be an open subset, which is saturated with respect to h, and let (x_0, y_0) be an arbitrary point in hW. Then we have to prove that there are open sets $U\subset P$, $V\subset Q$, which are saturated with respect to ϕ, ψ and are such that

$$(x_0, y_0)\in\phi U\times\psi V\subset hW.$$

Let $p_0\in\phi^{-1}x_0$, $q_0\in\psi^{-1}y_0$ and let

$$(p_0\times Q)\cap W = p_0\times Q_0.$$

Then it is easily verified that Q_0 is an open subset of Q, which is saturated with respect to ψ. Therefore q_0 is contained in a saturated