ELASTICITY

Theory and Applications

by Adel S. Saada

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Elasticity: Theory and Applications

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Elasticity
Theory and Applications

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Contents

Preface

This book is an outgrowth of notes used by the author during the past few years in a course on solid mechanics. It is intended to give advanced undergraduate and graduate students sound foundations on which to build advanced courses such as mathematical elasticity, plasticity, plates and shells, and those branches of mechanics which require the analysis of strain and stress. The book is divided into three parts: *Part I* is concerned with the kinematics of continuous media, *Part II* with the analysis of stress, and *Part III* with the theory of elasticity and its applications to engineering problems.

In *Part I*, the use of the notion of linear transformation of points makes it possible to present the geometry of deformation in a language that is easily understood by the majority of engineering students. It is agreed that tensor calculus is the most elegant tool available to mechanicists, but experience has shown that most engineering students are not ready to accept it without a reasonable amount of preparation. The study of finite and linear strains, using the notion of linear transformation, gradually introduces the tensor concept and removes part of the abstraction commonly associated with it. Orthogonal curvi-linear coordinates are examined in detail and the results extensively used throughout the text.

In *Part II*, the study of stress proceeds along the same lines as that of strain, and the similarities between the two are pointed out. All seven chapters of Parts I and II are essential to the understanding of Part III and serve as a common base for all branches of mechanics.

In *Part III*, Chapter 8 covers the three-dimensional theory of linear elasticity and the requirements for the solution of elasticity problems. The method of potentials is presented in Chapter 9. Torsion is discussed in Chapter 10 and topics related to cylinders, disks, and spheres are treated in Chapter 11. Straight and curved beams are analysed in Chapters 12 and 13 respectively, and the answers of the elementary theories are compared to the more rigorous results of the theory of elasticity. In Chapter 14, the semiinfinite elastic medium and some of its related problems are studied using the results of Chapter 9.

Energy principles and variational methods are presented in Chapter 15 and their application illustrated by a large number of simple examples. Columns and beam-columns are discussed in Chapter 16 and the bending of thin flat plates in Chapter 17. Chapter 18 is more than an introduction to the theory of thin shells. It includes a relatively detailed presentation of the theory of surfaces which is necessary for the full understanding of the analysis of thin shells. In this Chapter, as well as throughout this text, geometry and the relations between strain and displacement are emphasized since it is my conviction that once geometry is mastered most of the difficulties in studying the mechanics of solids will have disappeared.

The material in this text is suitable for two successive courses on solid mechanics and elasticity. A first course would include Chapters 1 to 5, some results from Chapter 6 and Chapter 7 to 13. A second course would include Chapter 6 and Chapters 14 to 18. Chapters 10 to 18 can be read independently from one another.

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Last but not least, I wish to acknowledge the encouragement and understanding of my wife Nancy during the various stages of writing this book.

Adel S. Saada

About the Author

Adel S. Saada (Ph.D., Princeton University) is presently Professor of Civil Engineering at the Case Institute of Technology of Case Western Reserve University, Cleveland, Ohio. Dr. Saada received his *Ingénieur des Arts et Manufactures* degree from École Centrale des Arts et Manufactures de Paris, France and the equivalent of a Master of Science degree from the University of Grenoble, France. Before coming to Princeton University the author was a practicing structural engineer in France. Dr. Saada's teaching activities are in two major areas: the first is that of the mechanics of solids and in particular elasticity; the second is that of mechanics applied to soils and foundations. His research activities are primarily in the area of stress-strain relations and failure of transversely isotropic materials, in particular clay soils. Much of his research work has been supported by personal grants from the National Science Foundation. Dr. Saada is a member of several professional societies, a consulting engineer, and the author of many papers on soil mechanics published in both national and international journals.

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PART I

KINEMATICS OF CONTINUOUS MEDIA
(Displacement, Deformation, Strain)

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Chapter 1

INTRODUCTION TO THE KINEMATICS OF CONTINUOUS MEDIA

1.1 Formulation of the Problem

The theory of deformation of continuous media is a purely mathematical one. It is concerned with the study of the intrinsic properties of the deformations independent of their physical causes. It is most conveniently expressed by the notion of transformation, which implies displacement and change in shape. The problem is formulated as follows: Given the positions of the points of a body in its initial state (i.e., before transformation) and in its final state (i.e., after transformation), it is required to determine the change in length and in direction of a line element joining two arbitrary points originally at an infinitesimal distance from one another.

In the following, we shall make use primarily of orthogonal sets of cartesian coordinates. Let x_1, x_2, x_3 be the coordinates of a point M of a body B before transformation. After transformation, this point becomes M^* with coordinates ξ_1, ξ_2, ξ_3 :

$$\begin{aligned} \xi_1 &= x_1 + u_1 \\ \xi_2 &= x_2 + u_2 \\ \xi_3 &= x_3 + u_3, \end{aligned} \tag{1.1.1}$$

where u_1, u_2, u_3 are the projections of $\overline{MM^*}$ on the three axes OX_1, OX_2, OX_3 (Fig. 1.1). We shall assume that u_1, u_2, u_3, as well as their

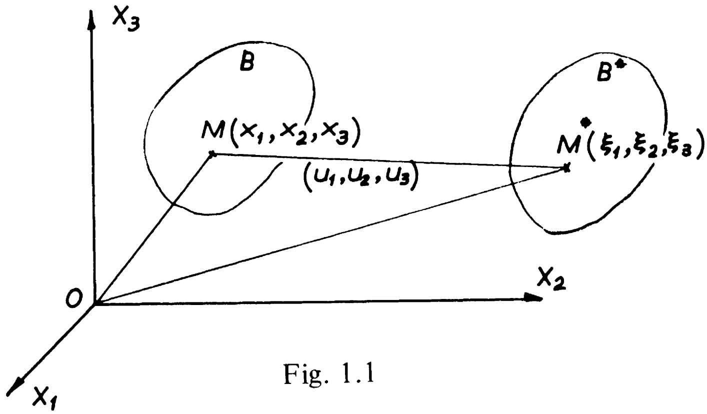

Fig. 1.1

partial derivatives with respect to x_1, x_2, x_3, are continuous functions of x_1, x_2, x_3. Eqs. (1.1.1) can therefore be written as:

$$\begin{aligned}\xi_1 &= x_1 + u_1(x_1, x_2, x_3)\\ \xi_2 &= x_2 + u_2(x_1, x_2, x_3)\\ \xi_3 &= x_3 + u_3(x_1, x_2, x_3).\end{aligned} \tag{1.1.2}$$

Let us consider two points, $M(x_1, x_2, x_3)$ and $N(x_1 + dx_1, x_2 + dx_2, x_3 + dx_3)$, infinitesimally near one another. As a result of the transformation, M is displaced to $M^*(\xi_1, \xi_2, \xi_3)$ and N is displaced to $N^*(\xi_1 + d\xi_1, \xi_2 + d\xi_2, \xi_3 + d\xi_3)$ (Fig. 1.2). The coordinates of N^* are

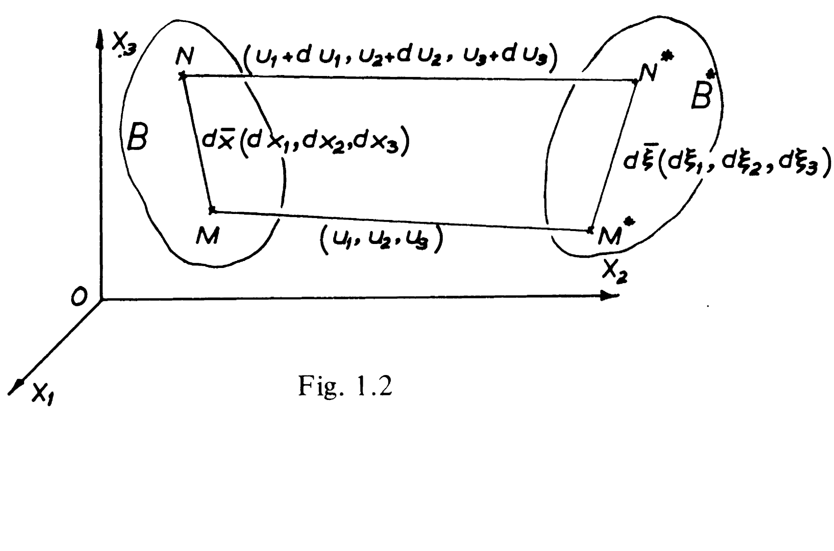

Fig. 1.2

given by:

$$\begin{aligned}
\xi_1 + d\xi_1 &= x_1 + dx_1 + u_1 + du_1 \\
\xi_2 + d\xi_2 &= x_2 + dx_2 + u_2 + du_2 \\
\xi_3 + d\xi_3 &= x_3 + dx_3 + u_3 + du_3 .
\end{aligned} \tag{1.1.3}$$

Because of the assumptions on u_1, u_2, u_3, we can write the displacement of N under the form of a Taylor series in the neighborhood of M:

$$\begin{aligned}
u_1 + du_1 &= (u_1)_M + \left(\frac{\partial u_1}{\partial x_1}\right)_M dx_1 + \left(\frac{\partial u_1}{\partial x_2}\right)_M dx_2 + \left(\frac{\partial u_1}{\partial x_3}\right)_M dx_3 + \ldots \\
u_2 + du_2 &= (u_2)_M + \left(\frac{\partial u_2}{\partial x_1}\right)_M dx_1 + \left(\frac{\partial u_2}{\partial x_2}\right)_M dx_2 + \left(\frac{\partial u_2}{\partial x_3}\right)_M dx_3 + \ldots \\
u_3 + du_3 &= (u_3)_M + \left(\frac{\partial u_3}{\partial x_1}\right)_M dx_1 + \left(\frac{\partial u_3}{\partial x_2}\right)_M dx_2 + \left(\frac{\partial u_3}{\partial x_3}\right)_M dx_3 + \ldots
\end{aligned} \tag{1.1.4}$$

If we substitute Eqs. (1.1.4) in Eqs. (1.1.3), and subtract Eqs. (1.1.1) from the resulting equations, we obtain:

$$\begin{aligned}
d\xi_1 &= \left[1 + \left(\frac{\partial u_1}{\partial x_1}\right)_M\right] dx_1 + \left(\frac{\partial u_1}{\partial x_2}\right)_M dx_2 + \left(\frac{\partial u_1}{\partial x_3}\right)_M dx_3 + \ldots \\
d\xi_2 &= \left(\frac{\partial u_2}{\partial x_1}\right)_M dx_1 + \left[1 + \left(\frac{\partial u_2}{\partial x_2}\right)_M\right] dx_2 + \left(\frac{\partial u_2}{\partial x_3}\right)_M dx_3 + \ldots \\
d\xi_3 &= \left(\frac{\partial u_3}{\partial x_1}\right)_M dx_1 + \left(\frac{\partial u_3}{\partial x_2}\right)_M dx_2 + \left[1 + \left(\frac{\partial u_3}{\partial x_3}\right)_M\right] dx_3 + \ldots,
\end{aligned} \tag{1.1.5}$$

If, in Eqs. (1.1.5), we neglect the higher-order terms of Taylor's series, the relations between $d\xi_1$, $d\xi_2$, $d\xi_3$ and dx_1, dx_2, dx_3 become linear. The system of equations can be looked upon as an operation which transforms a vector $d\bar{x}$ (dx_1, dx_2, dx_3) of length ds to a vector $d\bar{\xi}$ $(d\xi_1, d\xi_2, d\xi_3)$ of length ds^*. This type of transformation is called *linear transformation*. It is the linearization of Eqs. (1.1.5) that allows us to assume that the vector $d\bar{x}$ is transformed to a vector $d\bar{\xi}$ and not to a curve. The properties of linear transformations are discussed in Chapter 3. If we omit the subscript M, Eqs. (1.1.5) are written as:

$$\begin{aligned}
d\xi_1 &= \left(1 + \frac{\partial u_1}{\partial x_1}\right)dx_1 + \frac{\partial u_1}{\partial x_2}dx_2 + \frac{\partial u_1}{\partial x_3}dx_3 \\
d\xi_2 &= \frac{\partial u_2}{\partial x_1}dx_1 + \left(1 + \frac{\partial u_2}{\partial x_2}\right)dx_2 + \frac{\partial u_2}{\partial x_3}dx_3 \\
d\xi_3 &= \frac{\partial u_3}{\partial x_1}dx_1 + \frac{\partial u_3}{\partial x_2}dx_2 + \left(1 + \frac{\partial u_3}{\partial x_3}\right)dx_3,
\end{aligned} \tag{1.1.6}$$

provided we keep in mind that the partial derivatives of the functions u_1, u_2, u_3 are taken at the point M.

In essentially static problems, while little consideration is given to rigid body displacements, particular attention is given to the changes in length and in orientation of elements like ds. These changes are described by the three components of the relative displacement vector du_1, du_2, du_3 (Fig. 1.3):

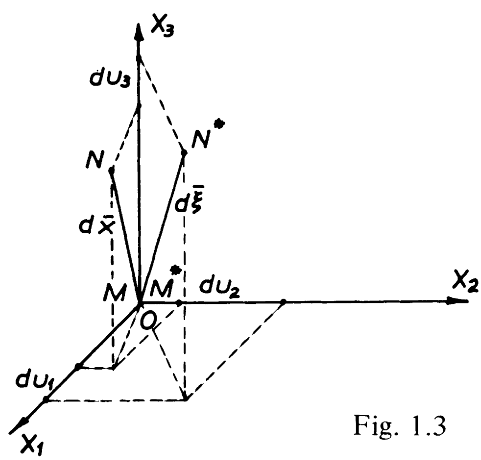

Fig. 1.3

$$
\begin{aligned}
du_1 &= d\xi_1 - dx_1 = \frac{\partial u_1}{\partial x_1}dx_1 + \frac{\partial u_1}{\partial x_2}dx_2 + \frac{\partial u_1}{\partial x_3}dx_3 \\
du_2 &= d\xi_2 - dx_2 = \frac{\partial u_2}{\partial x_1}dx_1 + \frac{\partial u_2}{\partial x_2}dx_2 + \frac{\partial u_2}{\partial x_3}dx_3 \\
du_3 &= d\xi_3 - dx_3 = \frac{\partial u_3}{\partial x_1}dx_1 + \frac{\partial u_3}{\partial x_2}dx_2 + \frac{\partial u_3}{\partial x_3}dx_3 .
\end{aligned} \tag{1.1.7}
$$

The kinematics of continuous media is centered on the two sets of Eqs. (1.1.6) and (1.1.7). Within the scope of this text, the necessary mathematical tool required to study these equations is the notion of linear transformation. Since matrix algebra was developed primarily to express linear transformations in a concise and lucid manner, it is natural that it should be employed in the formulation and the solution of kinematics problems. A brief review of matrix algebra is given in Chapter 2.

1.2 Notation

The following system of notation will be adhered to throughout this text:

$$
\begin{array}{lll}
\dfrac{\partial u_1}{\partial x_1} = e_{11} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_2} + \dfrac{\partial u_2}{\partial x_1}\right) = e_{12} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_1} - \dfrac{\partial u_1}{\partial x_2}\right) = \omega_{21} \\
\dfrac{\partial u_2}{\partial x_2} = e_{22} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} + \dfrac{\partial u_3}{\partial x_1}\right) = e_{13} & \dfrac{1}{2}\left(\dfrac{\partial u_1}{\partial x_3} - \dfrac{\partial u_3}{\partial x_1}\right) = \omega_{13} \\
\dfrac{\partial u_3}{\partial x_3} = e_{33} & \dfrac{1}{2}\left(\dfrac{\partial u_2}{\partial x_3} + \dfrac{\partial u_3}{\partial x_2}\right) = e_{23} & \dfrac{1}{2}\left(\dfrac{\partial u_3}{\partial x_2} - \dfrac{\partial u_2}{\partial x_3}\right) = \omega_{32}
\end{array} \tag{1.2.1}
$$

In Eqs. (1.2.1), the e's remain unchanged and the ω's change sign when the indices are interchanged. Thus,

$$
\begin{aligned}
e_{12} &= e_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) \\
e_{13} &= e_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \\
e_{23} &= e_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)
\end{aligned} \tag{1.2.2}
$$

and

$$
\begin{aligned}
-\omega_{12} &= +\omega_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)\\
-\omega_{31} &= +\omega_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)\\
-\omega_{23} &= +\omega_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right).
\end{aligned}
\tag{1.2.3}
$$

With these notations, Eqs. (1.1.6) become:

$$
\begin{aligned}
d\xi_1 &= (1 + e_{11})dx_1 + (e_{12} - \omega_{21})dx_2 + (e_{13} + \omega_{13})dx_3\\
d\xi_2 &= (e_{12} + \omega_{21})dx_1 + (1 + e_{22})dx_2 + (e_{23} - \omega_{32})dx_3\\
d\xi_3 &= (e_{13} - \omega_{13})dx_1 + (e_{23} + \omega_{32})dx_2 + (1 + e_{33})dx_3.
\end{aligned}
\tag{1.2.4}
$$

Eqs. (1.1.7) become:

$$
\begin{aligned}
du_1 &= e_{11}\,dx_1 + (e_{12} - \omega_{21})dx_2 + (e_{13} + \omega_{13})dx_3\\
du_2 &= (e_{12} + \omega_{21})dx_1 + e_{22}\,dx_2 + (e_{23} - \omega_{32})dx_3\\
du_3 &= (e_{13} - \omega_{13})dx_1 + (e_{23} + \omega_{32})dx_2 + e_{33}\,dx_3.
\end{aligned}
\tag{1.2.5}
$$

In all the previous equations, the coordinates of the points of the body in the transformed state are expressed in terms of their coordinates in the initial state. This is known as the Lagrangian Method of describing the transformation of a continuous medium. Another method, the Eulerian Method, expresses the coordinates in the initial state in terms of the coordinates in the final state. Each method has its advantages. It is, however, more convenient in the study of the mechanics of solids to use the Lagrangian approach because the initial state of the body often possesses symmetries which make it susceptible to description in a simple system of coordinates. The Lagrangian Method is exclusively used in this text.

Chapter 2

REVIEW OF MATRIX ALGEBRA

2.1 Introduction

The use of matrices in mechanics introduces a notation that enables one to see the components of the entities being studied in their totality, while providing great conciseness. In this chapter, the basic definitions and the operations of matrix algebra which will be needed in this text are given.

2.2 Definition of a Matrix. Special Matrices

A matrix is an array of elements arranged in rows and columns. For instance, a matrix of m rows and n columns is written:

$$[a] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \tag{2.2.1}$$

and is called an $(m \times n)$ matrix. The first subscript i of each element a_{ij} represents the number of the row, and the second subscript j represents the number of the column. The a_{ij}'s can be pure numbers, functions, instructions to a computer, or other matrices. In this text, the elements are all real. A square matrix with n rows and columns is said to be of order n.

A symmetric matrix has elements which satisfy the condition $a_{ij} = a_{ji}$. This means that elements symmetrically located with respect to the

main diagonal of the matrix are equal in magnitude and sign.
An antisymmetric or skew symmetric matrix has elements which satisfy the condition $a_{ij} = -a_{ji}$. This means that elements symmetrically located with respect to the main diagonal are equal in magnitude and opposite in sign, and that the elements of the diagonal are equal to zero.
A diagonal matrix is a matrix whose elements a_{ij} vanish except for $i = j$. These non-vanishing elements constitute the main diagonal of the matrix.
A unit matrix is a diagonal matrix whose elements are equal to unity. It is written [1].
A null matrix has all its elements equal to zero. It is written [0].
A column matrix has m rows and one column. it is also called a column vector and is written:

$$\{\bar{a}\} = \begin{bmatrix} a_{11} \\ a_{21} \\ \cdot \\ \cdot \\ a_{m1} \end{bmatrix}.$$

A row matrix is a matrix with one row and n columns. It is also called a row vector and is written:

$$[\bar{a}] = [a_{11} \ldots a_{1n}].$$

The transpose of a matrix $[a]$ is a matrix $[a]'$, whose rows are the same as the columns of $[a]$. Thus, a symmetric matrix is its own transpose and the transpose of a column matrix is a row matrix.
A scalar matrix is a diagonal matrix whose elements are identical.

2.3 Index Notation and Summation Convention

The introduction of numerical subscripts in Chapter 1 to denote the reference axes makes the use of indices in writing the components of vectors quite natural. When writing relations between vectors or other directional quantities (such as tensors), a great deal of space is saved when a shorthand notation is introduced. In this text, the only indices to be used are subscripts and the following conventions will be adhered to:

The range convention: Whenever a subscript is repeated in a term, it is understood to represent a summation over the range 1, 2, 3 unless otherwise stated. Also, an index never appears more than twice in the same term. For example, the expression

$$\xi_i = a_{ij} x_j \tag{2.3.1}$$

contains, in the right-hand term, the index j which is repeated. Therefore, taking the values of $i = 1, 2, 3$ in turn, we obtain the three linear equations:

$$\begin{aligned} \xi_1 &= a_{11} x_1 + a_{12} x_2 + a_{13} x_3 \\ \xi_2 &= a_{21} x_1 + a_{22} x_2 + a_{23} x_3 \\ \xi_3 &= a_{31} x_1 + a_{32} x_2 + a_{33} x_3 . \end{aligned} \tag{2.3.2}$$

i is the *identifying index* and j is the *summation index*. We notice that the summation index can be changed at will and is therefore called a *dummy index*. Thus, Eqs. (2.3.2) can also be written:

$$\xi_i = a_{ik} x_k .$$

The index k is similar to the dummy variable of integration in a definite integral and can be changed freely.

For convenience, it is sometimes useful to introduce the two following symbols:

The Kronecker delta, δ_{ij}, which by definition is such that:

$$\begin{aligned} \delta_{ij} &= 1, \text{ when } i = j \text{ and} \\ \delta_{ij} &= 0, \text{ when } i \neq j. \end{aligned} \tag{2.3.3}$$

The alternating symbol, ε_{ijk}, which by definition is such that:

$$\begin{aligned} \varepsilon_{ijk} &= 0, \text{ when any two of } i, j, k \text{ are equal} \\ \varepsilon_{ijk} &= 1, \text{ when } i, j, k \text{ are different and in cyclic order } (1,2,3, \\ &\qquad 1,2,3, \ldots) \\ \varepsilon_{ijk} &= -1, \text{ when } i, j, k \text{ are different and not in cyclic order } (1, \\ &\qquad 3,2,1,3,2, \ldots). \end{aligned} \tag{2.3.4}$$

Examples

1). $\delta_{ik} x_k$ for $i = 1$ is equal to:

$$\delta_{11} x_1 + \delta_{12} x_2 + \delta_{13} x_3 = x_1 = x_i.$$

2). $\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$

3). A vector $\bar{x}$ whose components are x_1, x_2, x_3, has a magnitude $|\bar{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2} = \sqrt{x_i x_i}$. Its direction cosines are given by $\ell_i = x_i / \sqrt{x_j x_j}$.

4). The sum of the diagonal elements of a matrix $[a]$ is called the trace of $[a]$ and is written a_{ii}.

5). The determinant of the matrix $[a]$ is written $\varepsilon_{ijk} a_{1i} a_{2j} a_{3k}$.

2.4 Equality of Matrices. Addition and Subtraction

Let us turn now to the rules governing the manipulation of the arrays of elements forming a matrix. Two matrices $[a]$ and $[b]$ of the same order are said to be equal if, and only if, their corresponding elements are identical; that is, we have:

$$[a] = [b], \tag{2.4.1}$$

provided that

$$a_{ij} = b_{ij} \text{ for all } i \text{ and } j. \tag{2.4.2}$$

If $[a]$ and $[b]$ are matrices of the same order, then the sum of $[a]$ and $[b]$ is defined to be a matrix $[c]$, the typical element of which is $c_{ij} = a_{ij} + b_{ij}$. In other words, by definition:

$$[c] = [a] + [b], \tag{2.4.3}$$

provided

$$c_{ij} = a_{ij} + b_{ij}. \tag{2.4.4}$$

In a similar manner, we have:

$$[d] = [a] - [b], \tag{2.4.5}$$

provided

$$d_{ij} = a_{ij} - b_{ij}. \tag{2.4.6}$$

From the above definitions, it can be shown that the following operations are valid:

$$[a] + [b] = [b] + [a] \tag{2.4.7}$$

$$([a] + [b]) + [c] = [a] + ([b] + [c]). \tag{2.4.8}$$

An important property of square matrices, which follows from the laws of addition and subtraction, is that any square matrix may be given as the sum of a symmetric and of an antisymmetric matrix. Indeed, if $[a]$ is a square matrix, then

$$[a] = \frac{[a] + [a]'}{2} + \frac{[a] - [a]'}{2}. \tag{2.4.9}$$

2.5 Multiplication of Matrices

The product of a matrix $[a]$ by a matrix $[b]$ is defined by the equation

$$[a][b] = [c], \tag{2.5.1}$$

where the elements of $[c]$ are given by:

$$c_{ij} = a_{ik} b_{kj}. \tag{2.5.2}$$

Thus

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} & a_{11}b_{13} + a_{12}b_{23} + a_{13}b_{33} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} & a_{21}b_{13} + a_{22}b_{23} + a_{23}b_{33} \\ a_{31}b_{11} + a_{32}b_{21} + a_{33}b_{31} & a_{31}b_{12} + a_{32}b_{22} + a_{33}b_{32} & a_{31}b_{13} + a_{32}b_{23} + a_{33}b_{33} \end{bmatrix}.$$

Two matrices can be multiplied by each other only if they are *conformable*, which means that the number of the columns of the first is equal to the number of the rows of the second. Thus, if $[a]$ is an $(m \times p)$ matrix and $[b]$ is a $(p \times n)$ matrix, then $[c]$ is an $(m \times n)$ matrix.

Two nonzero matrices can be multiplied by each other and result in a zero matrix. For example,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

A permutation of the matrices will lead to a different result:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}.$$

The product $[b]\,[a]$ is, in general, not equal to $[a]\,[b]$. Therefore, it is necessary to differentiate between *premultiplication*, as when $[b]$ is premultiplied by $[a]$ to yield the product $[a]\,[b]$, and *postmultiplication*, as when $[b]$ is postmultiplied by $[a]$ to yield $[b]\,[a]$. If we have two matrices which are such that

$$[a][b] = [b][a], \tag{2.5.3}$$

these matrices are said to *commute* or to be permutable.

Of particular importance is the *associative law* of continued products,

$$[d] = ([a][b])[c] = [a]([b][c]), \tag{2.5.4}$$

which allows one to dispense with parentheses and to write $[a]\,[b]\,[c]$ without ambiguity since the double summation

$$d_{ij} = a_{ik}\,b_{kl}\,c_{lj} \tag{2.5.5}$$

can be carried out in either of the orders indicated. It must be noticed that the product of a chain of matrices will have meaning only if the adjacent matrices are conformable.

The product of matrices is *distributive*, that is

$$[a]([b] + [c]) = [a][b] + [a][c]. \tag{2.5.6}$$

The multiplication of a matrix $[a]$ by a scalar k is defined by:

$$k[a] = [b], \tag{2.5.7}$$

where

$$b_{ij} = ka_{ij}.$$

Using the definition of the transpose and the laws of addition and multiplication of matrices, it can be shown that:

$$([a] + [b])' = [a]' + [b]' \tag{2.5.8}$$

$$(k[a])' = k[a]' \tag{2.5.9}$$

$$([a][b])' = [b]'[a]' \text{ (note the order).} \tag{2.5.10}$$

For the case of the unit matrix, we have:

$$[a][1] = [1][a] = [a] \tag{2.5.11}$$

and, if k is a constant,

$$[a]k[1] = k[a][1] = k[a] = k[1][a]. \tag{2.5.12}$$

An important result in the theory of matrices is that the determinant of the product of two square matrices is equal to the product of their determinants. Thus,

$$\|[a][b]\| = \Big(\|[a]\|\Big)\Big(\|[b]\|\Big) = \Big(\|[b]\|\Big)\Big(\|[a]\|\Big). \tag{2.5.13}$$

Among the special matrices defined in Sec. 2.2, the diagonal matrix plays an important part in operations involving matrices. The premultiplication of a matrix $[a]$ by a diagonal matrix $[d]$ produces a matrix whose rows are those of $[a]$ multiplied by the element in the corresponding row of $[d]$:

$$\begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_1 a_{12} & d_1 a_{13} \\ d_2 a_{21} & d_2 a_{22} & d_2 a_{23} \\ d_3 a_{31} & d_3 a_{32} & d_3 a_{33} \end{bmatrix}. \tag{2.5.14}$$

The postmultiplication of $[a]$ by $[d]$ produces a matrix whose columns are those of [a] multiplied by the element in the corresponding column of $[d]$:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} = \begin{bmatrix} d_1 a_{11} & d_2 a_{12} & d_3 a_{13} \\ d_1 a_{21} & d_2 a_{22} & d_3 a_{23} \\ d_1 a_{31} & d_2 a_{32} & d_3 a_{33} \end{bmatrix}. \tag{2.5.15}$$