ORDINARY DIFFERENTIAL EQUATIONS

By

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Translated from the Russian

by

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PREFACE

This book has been written on the basis of lectures which I delivered at the department of mathematics and mechanics of Moscow State University. In drawing up the program for my lectures, I proceeded on the belief that the selection of material must not be random nor must it rest exclusively on established tradition. The most important and interesting applications of ordinary differential equations to engineering are found in the theory of oscillations and in the theory of automatic control. These applications were chosen to serve as guides in the selection of material. Since oscillation theory and automatic control theory without doubt also play a very important role in the development of our contemporary technical culture, my approach to the selection of material for the lecture course is, if not the only possible one, in any case a reasonable one. In attempting to give the students not only a purely mathematical tool suitable for engineering applications, but also to demonstrate the applications themselves, I included certain engineering problems in the lectures. In the book they are presented in §13, 27, and 29. I consider that these problems constitute an integral organic part of the lecture course and, accordingly, of this book.

In addition to the material presented in the lectures, I have included in the book more difficult problems which were investigated in student seminars. They are contained in §19 and 31. The material contained in §24, 25, and 30 was only partially presented in the lectures. For the convenience of the reader, in the last chapter, the sixth, are presented certain facts from linear algebra in the form in which they are used in this book.

In closing, I wish to express my gratitude to my students and to my closest co-workers V. G. Boltyanskiy, R. V. Gamkrelidze, and E. F. Mishchenko, who helped me in the preparation and delivery of the lectures and in writing and editing this book. I want also to note the decisive influence upon my scientific interests exerted by the outstanding Soviet specialist in the field of oscillation theory and automatic control theory, Aleksandr Aleksandrovich Andronov, with whom for many years I have had a friendly relationship. His influence has substantially affected the character and direction of this book.

Moscow
16 July 1960

L. S. Pontryagin

FOREWORD

This book constitutes a mildly radical departure from the usual one-semester first course in differential equations. There was a time when almost all first courses in differential equations were devoted to an exhaustive treatment of the methods and artifices by which certain elementary equations can be solved explicitly. Fortunately, much of the material of the "classical" first course in differential equations has been moved back into the elementary calculus, where it finds its proper place among the so-called techniques of integration.

The disadvantages of offering a methods course in differential equations immediately following a second course in the calculus have been recognized for a long time: Students, whose interest and tastes in mathematics had begun to take shape, are confronted with the least challenging course in the curriculum, and even the engineering students, to whose interests the methods course is supposedly dedicated, are given little idea of the theory of differential equations. In recent years, a number of books have done much to improve the level of the first course in differential equations, though many of these books are in the nature of treatises, more suitable for the two-semester course at the graduate level.

The present volume is designed for a one-semester course in the junior or senior year, preferably after the course in calculus. It makes no pretense of being a treatise; the methods and artifices which belong to the calculus are omitted, though a new importance is given the linear equation with constant coefficients. An important feature of this book is the chapter on stability theory (Chapter 5) and the introduction to the Lyapunov theory. The engineering student will be challenged by the nontrivial treatment of such topics as the Watt regulator for a steam engine and the vacuum-tube circuit. The use of matrices and linear algebra will complement the one-semester course in linear algebra which is appearing more frequently at the junior level.

The Publisher

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CHAPTER 1

INTRODUCTION

This chapter is devoted primarily to the definition of those concepts which will be studied subsequently. What is a system of ordinary differential equations, what do we mean by a solution of it, and how many of these solutions exist? These are the basic questions which we shall attempt to answer in this chapter. The number of solutions is determined by theorems of existence and uniqueness, which will not be proved here, but only formulated. The proofs of these and of a number of other theorems of the same type are given in the fourth chapter, but theorems previously formulated in the first chapter are repeatedly used, so that their meaning is thus clarified. In addition to such basic information, solutions of differential equations of several of the simplest types are given in the first chapter. At the end of the chapter complex differential equations and their complex solutions are studied, and elementary facts concerning systems of linear differential equations are given.

1. First-order differential equations. Equations in which the unknowns are functions of one or several variables and which contain not only the functions themselves, but also their derivatives, are called *differential equations*. If the unknown functions are functions of several variables, then the equations are called *partial* differential equations; in the opposite case, i.e., for the case of functions of only one independent variable, the equations are called *ordinary* differential equations. In this book we shall deal only with the latter.

In applications to physics the time is taken as the independent variable, which is conventionally designated by the letter t; throughout this book the independent variable will be designated by t. Unknown functions will be designated by x, y, z, and so on. Derivatives of functions with respect to t will as a rule be designated by dots: $\dot{x} = dx/dt$, $\ddot{x} = d^2x/dt^2$, and so on. When this is inconvenient or impossible, we shall denote the order of a derivative by an upper index in parentheses; for example, $x^{(n)} = d^nx/dt^n$.

First we shall study the *first-order differential equation*. This equation may be written in the form

$$F(t, x, \dot{x}) = 0. \tag{1}$$

Here t is the independent variable, x the unknown function, $\dot{x} = dx/dt$ the derivative, and F a given function of three variables. The function F need not be defined for all values of its arguments; therefore we speak of the *domain of definition* B of the function F or simply, the *domain* B of F; here we have in mind a domain in the space of the three variables x, y, z. Equation (1) is called a *first-order* equation because it contains only the first derivative $\dot{x}$ of the unknown function x. A function $x = \varphi(t)$ of the independent variable t, defined on a certain interval $r_1 < t < r_2$ (the cases $r_1 = -\infty, r_2 = +\infty$ are not excluded), which, when substituted for x in equation (1), reduces (1) to an identity on the entire interval $r_1 < t < r_2$, is called a *solution* of equation (1). The interval $r_1 < t < r_2$ is called the *interval of definition* of the solution $\varphi(t)$. It is evident that substitution of $x = \varphi(t)$ in (1) is possible only when the function $\varphi(t)$ has a first derivative (and, in particular, is continuous) on the entire interval $r_1 < t < r_2$. For the substitution of $x = \varphi(t)$ into equation (1) to be possible, it is also necessary that the point with coordinates $(t, \varphi(t), \dot{\varphi}(t))$ belong to the domain B of the function F for any value of t in the interval $r_1 < t < r_2$.

Relation (1) connects the three variables $t, x, \dot{x}$. In certain cases it determines $\dot{x}$ as a single-valued, implicit function of the independent variables t, x. In this case (1) is equivalent to a differential equation of the form

$$\dot{x} = f(t, x). \tag{2}$$

Equation (2) is said to be *solved explicitly for the derivative;* in certain respects it is more amenable to study than the general differential equation (1). It is such explicit equations which we shall now study. We shall not assume that (2) has been obtained as a result of solving (1) for $\dot{x}$, but shall proceed from the function $f(t, x)$ as a given function of the two independent variables t, x.

In order to visualize the situation geometrically, we introduce for study the tx-plane P. We shall plot t, as an independent variable, along the axis of abscissas, and x, as a dependent variable, along the axis of ordinates. The function f appearing in (2) need not be defined for all values of t and x, or, in geometric language, need not be defined at all points of the plane P, but only at points of a certain set Γ of P (Fig. 1). We shall assume that the set Γ is a domain. This means that for every point p in Γ there is some circle of positive radius with center at p also contained in Γ. Concerning the function f, it will be assumed that both the function itself and its partial derivative, $\partial f/\partial x$, are continuous functions of t and x in Γ. A solution $x = \varphi(t)$ of equation (2) may be thought of geometrically in P as a curve with the equation $x = \varphi(t)$. This curve has a tangent at every point and lies entirely in the domain Γ; it is called an *integral curve* of the differential equation (2).

Existence and uniqueness theorem. In algebra it is known that a large role is played by theorems which give the number of solutions to a given equation or system of equations. One such example is the *fundamental*

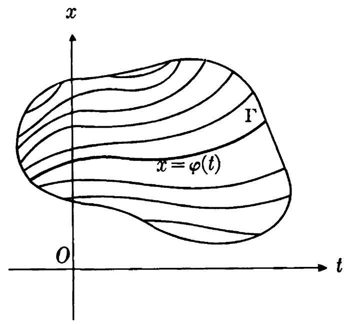

FIGURE 1

theorem of algebra, which asserts that a polynomial of the nth degree always has exactly n roots (counted according to multiplicity). In exactly the same way, in the theory of differential equations the important theoretical problem is how many solutions the differential equation has. It turns out that every differential equation has a *continuum* of solutions and this is why the question to be posed does not concern the *number* of solutions, but rather *how* the set of all solutions of a given differential equation can be described. The answer to this question is given by the *existence and uniqueness theorem* (Theorem 1), which is presented without proof in this section. The proof will be given considerably later (see §20).

THEOREM 1. Let

$$\dot{x} = f(t, x) \tag{3}$$

be a differential equation. We shall assume that the function $f(t, x)$ is defined in a certain domain Γ of the plane P of the variables t, x. We shall assume that the function f and its partial derivative $\partial f/\partial x$ are continuous in the entire domain Γ. The theorem asserts that

(1) For every point (t_0, x_0) of the domain Γ there exists a solution $x = \varphi(t)$ of equation (3) which satisfies the condition

$$\varphi(t_0) = x_0; \tag{4}$$

(2) If two solutions $x = \varphi(t)$ and $x = \chi(t)$ of equation (3) coincide for one value $t = t_0$, that is, if

$$\varphi(t_0) = \chi(t_0),$$

then these solutions are identically equal for all values of t for which they are defined.

The numbers t_0, x_0 are called the *initial values* for the solution $x = \varphi(t)$, the relation (4) represents the *initial conditions* for this solution, and we

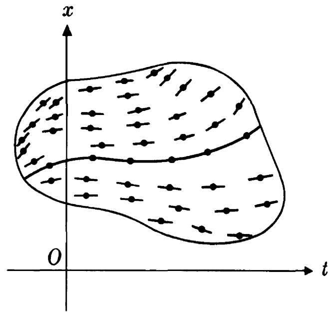

FIGURE 2

shall also say that the solution $x = \varphi(t)$ *satisfies the initial conditions* (4) or that it has *initial values* t_0, x_0. The assertion that the solution $x = \varphi(t)$ satisfies the initial conditions (4) (or has initial values t_0, x_0) assumes that the interval $r_1 < t < r_2$, where the solution $x = \varphi(t)$ is defined, contains the point t_0.

Thus Theorem 1 asserts that the coordinates of any point (t_0, x_0) of the domain Γ are initial values for some solution of equation (3) and that two solutions with common initial values coincide.

The geometrical meaning of Theorem 1 consists in the fact that *through every point* (t_0, x_0) *of* Γ *passes one and only one integral curve of equation* (3) (see Fig. 1).

We have interpreted geometrically every solution $x = \varphi(t)$ of equation (3) in the form of the graph of the function $\varphi(t)$. We now give a geometric interpretation of equation (3) itself. Through every point (t, x) of Γ we shall draw a straight line $l_{t,x}$ with slope $f(t, x)$. We obtain the *direction field* (or tangent field) corresponding to equation (3) and thus the geometric interpretation of this equation.

The connection between the geometrical interpretation of the equation and the geometrical interpretation of its solutions consists in the fact (Fig. 2) that *any integral curve* $x = \varphi(t)$ *is tangent to the straight line* $l_{t,\varphi(t)}$ at each of its points $(t, \varphi(t))$.

EXAMPLES

1. To illustrate the significance of Theorem 1 (in this case, of its second part), we shall solve the differential equation

$$\dot{x} = \alpha x, \tag{5}$$

where α is a real number. Here

$$f(t, x) = \alpha x,$$

so that the function f in fact depends only on the variable x. The domain of f coincides with the entire plane P. Both the function $f(t, x) = \alpha x$ and its derivative $\partial f(t, x)/\partial x = \alpha$ are continuous functions of t and x in the entire plane P. Thus Theorem 1 is applicable to equation (5). By direct substitution into equation (5) it is verified that each of the functions

$$x = ce^{\alpha t}, \tag{6}$$

where c is an arbitrary real number, is a solution of equation (5). We shall show that by assigning all possible values for c, we shall obtain *all* solutions of equation (5). Let $x = \varphi(t)$ be an arbitrary solution of this equation. We shall show that by proper choice of the number c we have $\varphi(t) = ce^{\alpha t}$. Let t_0 be a certain point of the interval of existence of the solution $x = \varphi(t)$, and let $x_0 = \varphi(t_0)$. Let us assume that $c = x_0 e^{-\alpha t_0}$. Then the solutions $x = \varphi(t)$ and $x = ce^{\alpha t} = x_0 e^{\alpha(t-t_0)}$ of equation (5) have the same initial values (t_0, x_0), and therefore coincide by virtue of the second part of Theorem 1. Thus, formula (6) exhausts the set of all solutions of differential equation (5).

2. We shall give a mathematical description of the process of decay of radioactive matter. The quantity of matter not yet decayed at the instant t we shall denote by $x(t)$. Then the quantity of matter which has decayed over the small interval of time t to $t + h$ is determined by the formula $\alpha h x(t)$, where α is a coefficient which depends on the properties of the radioactive matter and is slightly dependent on h; more accurately, it tends to a definite limit β as $h \to 0$. Thus we have

$$x(t) - x(t + h) = \alpha h x(t).$$

Dividing this relation by h and passing to the limit as $h \to 0$, we obtain

$$\dot{x}(t) = -\beta x(t).$$

We see that the function $x(t)$ satisfies the very simple differential equation examined in Example 1, so that

$$x(t) = ce^{-\beta t}.$$

To determine the constant c it is sufficient to specify any initial values. If, for example, it is known that at the instant $t = 0$ there was a quantity of matter x_0, then $c = x_0$, and we have

$$x(t) = x_0 e^{-\beta t}.$$

The rate of decay is expressed here by the value β having the dimension 1/sec or $(\sec)^{-1}$. Instead of the value β, the rate of decay is often characterized by the so-called *half-life*, i.e., the time required for half of the existing matter to decay. We shall designate the half-life by T and establish the connection between the values β and T. We have

$$\frac{x_0}{2} = x_0 e^{-\beta T},$$

whence

$$T = \frac{1}{\beta} \ln 2.$$

2. **Some elementary integration methods.** The main problem facing us when we deal with a differential equation is the problem of finding its solutions. In the theory of differential equations, just as in algebra, the question of what it means to find the solution of an equation may be understood in various ways. In algebra the original aim was to find a general formula involving radicals for the solution of equations of any degree. Such were the formulae for the solution of a quadratic equation, Cardan's formula for the solution of a cubic equation, and Ferrari's formula for the solution of an equation of the fourth degree. Later, it was established that for equations of degree higher than the fourth, a general formula for solution in radicals does not exist. The possibility remained of an approximate solution of equations with numerical coefficients and also the possibility of relating the dependence of the roots of an equation on its coefficients. The evolution of the concept of solution in the theory of differential equations was approximately the same. The original aim was to solve, or, as it was said, "to integrate" differential equations by means of "quadratures," i.e., the attempt was to write the solution in terms of the elementary functions and their integrals. Later, when it became clear that a solution in this sense exists only for very few types of equations, main emphasis of the theory was transferred to the study of general laws of the behavior of solutions. In this section we shall develop integration methods by quadratures for certain first-order differential equations.

(A) We shall solve the equation

$$\dot{x} = f(t), \tag{1}$$

the right-hand side of which depends only on the independent variable t. We shall assume that the function $f(t)$ is defined and continuous on the interval $r_1 < t < r_2$. Under this assumption, equation (1) satisfies the conditions of Theorem 1, and the domain Γ for this equation is a strip in the tx-plane P which is determined by the inequalities $r_1 < t < r_2$. Let t_0 be an arbitrary point of the interval $r_1 < t < r_2$; we assume

$$\varphi_0(t) = \int_{t_0}^{t} f(\tau)\, d\tau.$$

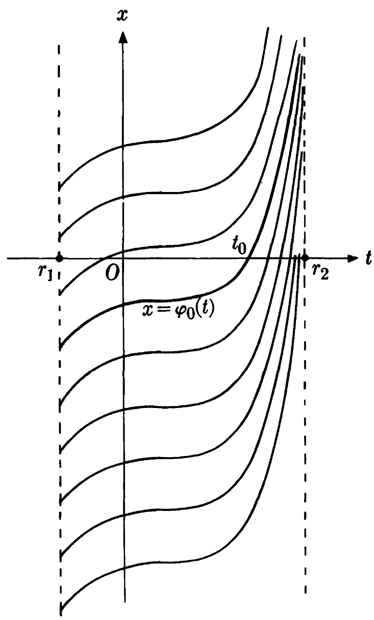

FIGURE 3

The function $\varphi_0(t)$ is defined on the interval $r_1 < t < r_2$. By inspection, an arbitrary solution of the equation (1) is given by the formula

$$x = \varphi(t) = \varphi_0(t) + c, \tag{2}$$

where c is an arbitrary constant. The right-hand side of (2) is, as is known, the *indefinite integral* of the function $f(t)$, so that (2) may be written in the form

$$x = \int f(t)\, dt.$$

It is seen by direct inspection that the function (2) satisfies equation (1). Further, the graph of every solution (2) for an arbitrary c is obtained from the graph of the solution $x = \varphi_0(t)$ by using a vertical-parallel translation by the quantity c (Fig. 3). From this it is evident that through every point of Γ passes a curve defined by formula (2). Hence, by Theorem 1 it follows that (2) actually encompasses the set of *all* solutions of (1).

(B) We shall solve the equation

$$\dot{x} = g(x), \tag{3}$$

the right-hand side of which depends only on the unknown function x. We shall assume that the function $g(x)$ is defined and has a continuous derivative on the interval $a_1 < x < a_2$. Then Theorem 1 is applicable to equation (1), and a strip in the tx-plane P which is determined by the inequalities $a_1 < x < a_2$ serves as the domain Γ. For the sake of simplicity, we assume in addition that on the interval $a_1 < x < a_2$ the function $g(x)$ does not vanish and consequently does not change sign. Let x_0 be an arbitrary point of the interval $a_1 < x < a_2$; we assume

$$G_0(x) = \int_{x_0}^{x} \frac{d\xi}{g(\xi)}. \tag{4}$$

The function $G_0(x)$ is defined on the interval $a_1 < x < a_2$, and its derivative on this interval is never zero; therefore the function $G_0(x)$ has an inverse, i.e., there exists a function $\psi_0(t)$ such that

$$G_0(\psi_0(t)) = t. \tag{5}$$

Consequently, an arbitrary solution of equation (3) is given by the formula

$$x = \psi(t) = \psi_0(t - c), \tag{6}$$

where c is an arbitrary constant. The function $\psi(t)$ is monotonic and assumes all values belonging to the interval $a_1 < x < a_2$.

We shall first prove that the function (6) is a solution of equation (3). From (5) it follows that

$$G_0(\psi(t)) = G_0(\psi_0(t - c)) = t - c. \tag{7}$$

Differentiating this relation with respect to t, we obtain

$$G_0'(\psi(t))\dot{\psi}(t) = 1,$$

hence [see (4)]

$$\dot{\psi}(t) = g(\psi(t)).$$

Since the function $\psi_0(t)$ is obtained as the inverse of the monotonic function $G_0(x)$, which is defined on the *entire* interval $a_1 < x < a_2$, the function $\psi_0(t)$ [and consequently $\psi(t)$] is monotonic and assumes all values on the interval $a_1 < x < a_2$. Since, further, the integral curve (6) is obtained from the curve $x_0 = \psi_0(t)$ by a horizontal-parallel translation (Fig. 4), a curve of the form (6) passes through every point of the strip Γ. Thus by Theorem 1, (6) contains the set of *all* solutions of equation (3).

Note. The relation (7) shows that the function $\psi(t)$ is the inverse of the function $G_0(x) + c$, which is the indefinite integral of the function $1/g(x)$. Thus all solutions $x = \psi(t)$ of equation (3) are described by the formula

$$\int \frac{dx}{g(x)} = t. \tag{8}$$

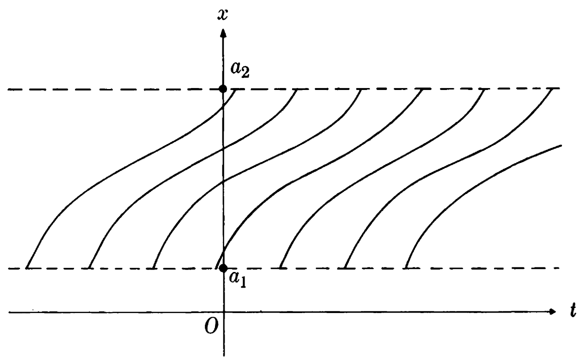

FIGURE 4

If the function $t = G_0(x) + c$ is taken as the unknown function, then we obtain for it the differential equation

$$\frac{dt}{dx} = \frac{1}{g(x)},$$

which is equivalent to equation (3). It is solved by the method presented in (A), which gives (8).

(C) We shall solve the equation

$$\dot{x} = f(t)g(x), \tag{9}$$

which is called an *equation with separable variables.* We shall assume that the function $f(t)$ is defined and continuous on the interval $r_1 < t < r_2$ and that the function $g(x)$ is defined and has a continuous derivative on the interval $a_1 < x < a_2$. Then Theorem 1 is applicable to equation (9), and the rectangle determined by the inequalities

$$r_1 < t < r_2, \qquad a_1 < x < a_2$$

serves as its domain Γ. For the sake of simplicity, we shall assume that $g(x)$ does not vanish on the interval $a_1 < x < a_2$. For the solution of (9) we form two auxiliary equations:

$$\frac{du}{dt} = f(t), \tag{10}$$

$$\frac{dx}{du} = g(x). \tag{11}$$

Equations (10) and (11) are solved by the rules given in (A) and (B). Let $u = \varphi_0(t)$ be some solution of equation (10) and $x = \psi_0(u)$ some solution