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Random Functions and Turbulence

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by

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CONTENTS

PART III. ATMOSPHERIC TURBULENCE

FOREWORD TO THE ENGLISH EDITION

THE statistical theory of turbulence is a comparatively new branch of hydrodynamics and has only recently begun to be an independent academic discipline in the applied mathematics and engineering departments of the universities in various countries. For this reason, specialized text materials pertaining to this branch of science have not yet been well established.

In view of the fact that special mathematical tools are needed for the study of the statistical theory of turbulence, those who study this subject without adequate mathematical preparation generally find it difficult to pursue independent study. The only known book which can be used as a textbook in this field is *Turbulence* by J. O. Hinze (1959). A few good monographs and surveys on the subject are also available. None the less, the need for a comprehensive text on an introductory level, in our opinion, still exists.

This book is intended primarily as an introduction to the statistical theory of turbulence. Parts I and II offer a systematic description of the theory of random functions and its application in the investigation of turbulence. Part III deals with the application of the methods of statistical theory of turbulence to the solution of many practical problems of small- and large-scale atmospheric turbulence, particularly the problem of numerical weather analysis and prediction. This part, as well as Chapters 6 and 7, is monographic in character and can be used for further study of turbulence and its applications.

This book is written for applied mathematicians, physicists, as well hydrodynamists and meteorologists. It can be used as a textbook at the graduate level and as a reference book for scientists working in turbulence and related fields.

An early version of this book was first published in 1965 in Bulgarian and translated to Russian in 1967. The present edition is based on the Russian version of the book but it is completely revised with many supplements and new results. Chapter 6 has been completely rewritten.

Many people and institutions in the United States of America helped me in preparing the English Revised Edition:

Professor S. K. Kao, of the University of Utah, organized the translation from Russian to English which was carried out by Dr. Thomas F. Rogers from the Language Department of the same University. Professor Kao also read the entire translation, including my supplements, and has made many valuable corrections. He also made a generous contribution by writing a special section included in the book as appendix on the Lagrangian analysis of large-scale atmospheric turbulence, which represents a valuable supplement to the other questions investigated in Eulerian variables only. As a visiting scientist at the National Center for Atmospheric Research and in the Fluid Mechanics Program at Colorado State University, I was provided with time to revise the book. Dr. J. E. Cermak, Professor-in-

Charge, Fluid Mechanics Program, provided the opportunity to teach a course in turbulence using the book as a text to a group of graduate students of his program during the spring quarter of 1968. He also made support available under Grant AP-0091-06 of the Public Health Service for completion of the final draft of the English version. Mrs. Joella Matthews and Mrs. Mary Fox provided invaluable assistance by typing the manuscript. To all these people and institutions, to my students whose interest in the subject stimulated my work, I express my sincere gratitude.

I would like to thank again Professor L. Krastanov, Head of the Department of Meteorology, University of Sofia, and former President of the Bulgarian Academy of Sciences, for his encouragement and continued support, given to me always when needed, particularly during those years when I was working on the first edition of this book.

Finally, I express in advance my thanks to all English-speaking readers who will send me their remarks and comments concerning the book.

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INTRODUCTION

DURING the last three decades a new field has rapidly developed in physics and technology which has massive phenomena in the most general sense of the phrase as its object of investigation and which uses the theory of random functions as a mathematical method. The development of this new field of mathematics, which first appeared in the 1930's, is, as will be made clear by what follows, urgently required for the solution of a number of concrete practical problems.

Up to the present time, the theory of random functions has been so widely used that it would be difficult to enumerate the many occasions of its application. It will suffice to mention only the most important of these in order to clarify its great significance. Of these, first consideration must be accorded the investigation of fluctuation processes in radio receiver and electronic systems for the purpose of distinguishing a useful signal on a background of random noises as well as the transmission and processing of information in a number of respects.

The recent development of complicated calculating machines and automatic control systems, which not only greatly relieve man's physical labor but even perform certain functions of the human brain, has extended the range of application for the mathematical apparatus of random functions. Without this apparatus it would be impossible to account for various random perturbations in the performance of the aforementioned automatic systems. For these and a number of other tasks, the probability functions of one variable —time (random functions)—have found an application.

However, the latest developmental trends in physics and mechanics, particularly that of the theory of turbulence, have necessitated the study of random functions of more than one variable—so-called random fields (scalar and vector). In this area the most substantial contribution has been made by the Soviet school of probability theory. In addition to the present statistical theory of turbulent motion, various mathematical problems concerning the theory of random fields have also received elaboration. This latest field of application of random functions provides the particular object of discussion in the second and third parts of the present book.

Besides its independent scientific interest for the study of "pure" turbulence in hydrodynamics, the theory of random functions has further significance to the extent that its methods and results can be applied to various practical aspects of turbulence. This is especially true of atmospheric turbulence.

In the dynamics of the atmosphere turbulence is one of the most important factors related to a number of phenomena which are intrinsic to the life of man and to his work. It will suffice to mention that such important problems as the diffusion of solid impurities in the atmosphere and the control of air pollution above large industrial centers cannot be resolved without considering the turbulent condition of the atmosphere. Although thoroughly elaborated, existing semi-empirical methods relating to the theory of turbulence are inadequate for this purpose. Most recently, considerable attention has been given to the construction of a statistical theory of turbulence in Lagrangian variables; for this purpose, correlation and spectral theories of random functions have been very widely used.

Turbulent fluctuations of meteorological elements (wind velocities, temperatures, and air humidity) transform the atmosphere into a medium of random inhomogeneities. This phenomenon causes a unique scattering of light, sound, and radio waves, which diffuse in the atmosphere and appear to us in the familiar forms of noise in radio receiver systems, long-range transmissions in ultra short waves, the attenuation of sound, the scintillation of earthly and heavenly sources of light, etc. Some of these phenomena are useful, others are harmful. In order to avoid or minimize the influence of the latter, it is necessary for specialists in the areas of radio technology and radio physics, atmospheric acoustics, and optics to have a fundamental knowledge of atmospheric turbulence.

Airplanes equipped with automatic pilot systems are subjected during flight in the actual turbulent atmosphere to the activity of continuous random disturbances; an investigation of the stability of the work of control systems is impossible without a knowledge of the statistical structure of these disturbances.

A number of theoretical problems which concern the upper layers of the atmosphere and also pertain to astrophysics are related to turbulent motions in the ionosphere and also to the behavior of ionized interstellar gas. These problems find a possible explanation within the framework of the theory of turbulence. Thus, for example, the distribution of radio waves substantially depends upon the frequency of turbulent electron pulsations in the ionosphere. These can in turn be determined by the fluctuations of characteristics (phases and amplitudes) of radio waves.

Finally, a number of important problems concerning present methods for predicting weather—the interpolation and extrapolation of observations, their averaging, the choice of a network of reference points for the numerical (machine) solution of equations, etc.—can be solved by means of the theory of random functions and with a knowledge of the statistical structure of meteorological fields with synoptic scale motions (atmospheric macroturbulence). These problems have thus far and will yet attract the attention of an ever greater number of applied mathematicians and engineers, for whom an acquaintance with the meteorological aspect (physical basis) of the indicated problems has proved to be absolutely indispensable.

An immense number of investigations have appeared with respect to the theory of random functions and their application to the statistical theory of turbulence. The majority of these are survey articles, a few of them monographs. However, these investigations usually discuss individual concrete problems. Thus, they are highly specialized and extremely difficult for the beginner student. The purpose of this book is to overcome this deficiency with a systematic description of the theory of random functions of one or more variables within the bounds necessary for its application to the statistical theory of turbulence. Due to the many problems which it was necessary to outline in this book, we have endeavored to elaborate the main body of the book in a complete fashion in Chapters 2 through 8; elsewhere, as space has permitted, problems have been succinctly stated, while, for their more comprehensive formulation, the reader has been referred to original sources. A number of the author's contributions have also been included.

The book begins with a brief exposition of the classical theory of probability; a more detailed statement of the theory of random functions of one variable follows (random processes, Chapter 2). The inclusion of this material is dictated by the fact that, without the necessary minimum of knowledge in this area, the reader would experience great difficulty in understanding the specialized problems discussed in this book. The reading of Chapters 1 and 2 can serve as a basis for the further intensive study of the theory of random processes and its application to the various problems heretofore mentioned. In Chapter 3 the theory of random fields, which is at present only applicable to the theory of turbulent motions, is systematically discussed.

Chapters 4 through 7 discuss the current statistical theory of turbulent motion. The competent reader may decry the separate discussion of the various methods which apply to the theory. Although the indicated organization necessarily requires the consideration of one and the same problem on several ocasions and in different sections, we none the less consider such a means of discussion advisable, since it permits a clear presentation of the advantages and disadvantages as well as the potential possibilities of various methods.

In Chapters 8 through 10 the statistical theory of atmospheric turbulence is discussed in sufficient detail. The application of random functions to various problems of meteorology, climatology, and prognosis is also examined. The list of references appended at the end of the book does not include all the works which had appeared at the time this book was written but only those which, for one or another reason, are cited or which the author regards as necessary to include. For convenience they are given for each part disjointly. Besides, in preparing the Russian and the English edition of the book many new references have been used. They are appended as "additional references" and denoted in the text by a star; for example[(23*), (50*)], etc. In citing all references either the name of the author followed by the number of the corresponding reference or only this number have been used.

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PART I

Elements of the Theory of Random Functions

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CHAPTER 1

CERTAIN DATA ON THE THEORY OF PROBABILITY

§ 1. Random Variable and Distribution Functions

According to the theory of probability a variable is called random which, as a result of experimentation under defined conditions, can assume any one of a number of possible numerical values, which would be impossible to predict in advance. However, the range of possible numerical values for this variable can be known in advance. For example, when dice are thrown, it is known in advance that one of six ciphers will appear; when the temperature of air is measured, degrees of temperature will be recorded which vacillate around the mean temperature at the point of measurement, etc. In the first instance we are dealing with a discrete quantity, while in the second with a continuous random quantity. In order to comprehend the statistical laws of random quantities, such data are inadequate and it is also necessary to know the probability of the given values. In this connection the concept of the function of distribution has been introduced into the theory of probabilities. This concept serves as the mathematical expression of a law according to which to every interval of possible values of a continuous random variable (or to each of its possible values in the case of a discrete random variable) the probability of its occurrence during that interval (or to accept these values) is juxtaposed. Let x be an arbitrary real number and ξ be a random variable. Let us examine event $\xi < x$, consisting of the fact that random variable ξ will have a value less than x. The probability of this event is generally designated by $P(\xi < x)$. It is evidently a function of x. Let us introduce

$$F(x) = P(\xi < x). \tag{1.1}$$

Function $F(x)$ is called an integral distribution function. The random variable ξ is conered to be known if its distribution function $F(x)$ is given.

According to its definition $F(x)$ exhibits the following fundamental characteristics.

Above all, since $F(x)$ is a probability, its values are included in the bounds

$$0 \leqslant F(x) \leqslant 1. \tag{1.2}$$

The integral law of distribution $F(x)$ is a continuous nondecreasing function x, i.e. if $x_2 > x_1$, then $F(x_2) \geqslant F(x_1)$. In fact, let the two events $\xi < x_1$, and $x_1 \leqslant \xi < x_2$ occur, where $x_1 < x_2$. It is evident that these two events are mutually exclusive in the sense that they cannot exist simultaneously. In combination they are equivalent to a single event $\xi < x_2$. According to the familiar theorem of the combination of probabilities[7] the probability of the equivalent event $\xi < x_2$ will be

$$P(\xi < x_2) = P(\xi < x_1 \quad \text{or} \quad x_1 \leqslant \xi < x_2) = P(\xi < x_1) + P(x_1 \leqslant \xi < x_2),$$

from which follows

$$P(x_1 \leqslant \xi < x_2) = P(\xi < x_2) - P(\xi < x_1) = F(x_2) - F(x_1). \tag{1.3}$$

In other words, the probability of the occurrence of the random variable ξ in the semi-closed interval (x_1, x_2) is equal to the increase of the function $F(x)$ in the same interval. Because the left part (1.3) is always non-negative there is an inequality

$$F(x_2) \geqslant F(x_1). \tag{1.4}$$

In the theory of probability only such random quantities are usually considered whose possible values are finite. Accordingly, the probability of the inequality $\xi < x$ is arbitrarily near unity in the vicinity of a sufficiently large x and arbitrarily near zero in the vicinity of negative x with sufficiently large absolute value. On the basis of this and of the previously proven characteristic of monotonicity of $F(x)$, it is possible to write

$$F(-\infty) = \lim_{x \to -\infty} F(x) = 0, \quad F(\infty) = \lim_{x \to \infty} F(x) = 1. \tag{1.5}$$

For a discrete random variable, the distribution function $F(x)$ is a discrete function and changes for the values x_1 which are possible values of ξ, according to which, between two adjacent values of x, $F(x) = \text{const}$. Consequently diagram $F(x)$ is a stepped curve.

In the case of a continuous random quantity, $F(x)$ is a differentiable function. Its derivative

$$f(x) = F'(x) = \lim_{\Delta x \to 0} \frac{F(x+\Delta x) - F(x)}{\Delta x} \tag{1.6}$$

is called a differential function of distribution or (more often) a probability density. According to (1.3) one can also write function $f(x)$ as

$$f(x) = \lim_{\Delta x \to 0} \frac{P(x \leqslant \xi < x+\Delta x)}{\Delta x}, \tag{1.7}$$

according to which the meaning of the second term is clear.

From (1.7) it follows that, with accuracy to infinitesimals of a higher order relative to Δx, the product

$$f(x)\,\Delta x \approx P(x \leqslant \xi < x+\Delta x) \tag{1.8}$$

expresses the probability of the occurrence of the quantity ξ in the interval $(x, x+\Delta x)$. Bearing in mind (1.3), let us integrate (1.6) within the limits between a and b

$$\int_a^b f(x)\,dx = F(b) - F(a) = P(a \leqslant \xi < b). \tag{1.9}$$

It follows from the geometrical interpretation of a definite integral that the probability $P(a \leqslant \xi < b)$ can be represented on a graph as a curved trapezoid, formed by segment ab, ordinates at the ends of this segment and limited above by the curve $f(x)$. Inasmuch as $F(x)$ is a nondecreasing function,

$$f(x) \geqslant 0. \tag{1.10}$$

If in (1.9) we establish $a = -\infty$, $b = +\infty$ and use (1.3), then we easily arrive at

$$\int_{-\infty}^{\infty} f(x)\,dx = F(\infty) - F(-\infty) = 1. \tag{1.11}$$

The latter signifies that the area beneath the curve $f(x)$ is equal to unity. With $a = -\infty$, $b = x$ from (1.9) we still have

$$F(x) = P(\xi < x) = P(-\infty < \xi < x) = \int_{-\infty}^{x} f(x)\,dx. \tag{1.12}$$

EXAMPLE 1. *Uniform distribution.* This is the law of distribution according to which the probability density $f(x)$ in the interval (a, b) acquires a constant value, and outside of which it equals zero (Fig. 1). According to the definition, $f(x) = c$. Since, according to (1.11), the area of the rectangle constructed on the segment ab (Fig. 1) should equal unity, it therefore follows that

$$c = \begin{cases} \dfrac{1}{b-a} & a < x < b \\ 0 & x < a, \quad x > b. \end{cases} \tag{1.13}$$

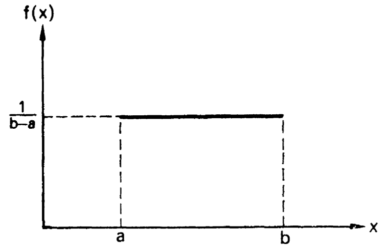

FIG. 1. Function $f(x)$ with uniform distribution.

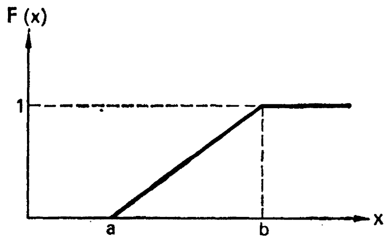

FIG. 2. Function $F(x)$ with uniform distribution.

According to (1.12),

$$F(x) = \begin{cases} \dfrac{x-a}{b-a} & a < x < b \\ 1 & x > b \\ 0 & x < a. \end{cases} \tag{1.14}$$

EXAMPLE 2. *The normal distribution.*

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-a)^2}{2\sigma^2}}, \tag{1.15}$$

which is also called Gaussian distribution. With $x = a$ function $f(x)$ has the maximum

$$f_{\max} = \frac{1}{\sigma\sqrt{2\pi}}. \tag{1.16}$$

Moreover, with $|x| \to \infty$ $f(x) \to 0$. The form of the curve substantially depends upon the value of the parameter σ (Fig. 3).

It is easy to verify that $f(x)$, defined by the expression (1.15), statisfies (1.11). In this case, according to (1.12), the integral distribution function will be

$$F(x) = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{(x-a)^2}{2\sigma^2}}\, dx = \frac{1}{2} + \Phi\left(\frac{x-a}{\sigma}\right), \tag{1.17}$$

where

$$\Phi(u) = \frac{1}{\sqrt{2\pi}} \int_{0}^{u} e^{-\frac{t^2}{2}}\, dt \tag{1.18}$$

is the familiar Gaussian function (errors integral).

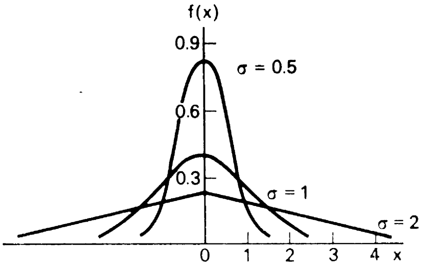

FIG. 3. A curve of normal distribution with various values for parameter σ.

In a number of cases the results of observations of probability phenomena constitute not one number but two, three or in general n numbers $\xi_1, \xi_2, \ldots, \xi_n$. We can consider these numbers as constituting a multidimensional random variable.

$$\xi = (\xi_1, \xi_2, \ldots, \xi_n). \tag{1.19}$$

It is convenient to give these quantities a geometrical interpretation. We will thus regard variables $\xi_1, \xi_2, \ldots, \xi_n$, despite their nature, as coordinates of a random point of an n-dimensional Euclidean space, or as components of the n-dimensional random vector ξ. This allows us to use a graphic geometrical representation. Let us here consider only two-dimensional random quantities. The random vector on a plane with components (ξ_1, ξ_2) can serve as an example of this variable. Let us assume that the two-dimensional random variable is given, if its two-dimensional function of distribution is known.

$$F(x_1, x_2) = P(\xi_1 < x_1, \xi_2 < x_2). \tag{1.20}$$

This function expresses the probability of a simultaneous occurrence of events $\xi_1 < x_1$ and $\xi_2 < x_2$ or, geometrically, the probability of occurrence of the end point of vector $\xi = (\xi_1, \xi_2)$ in the shaded area in Fig. 4.

Inasmuch as $\xi_1 < -\infty$ and $\xi_2 < -\infty$ are impossible events while $\xi_1 < \infty$ and $\xi_2 < \infty$ are, according to the terminology of the theory of probability, reliable events and, moreover,

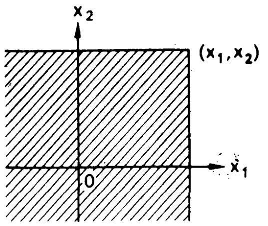

FIG. 4. A geometrical interpretation of a two-dimensional random quantity.

function $F(x_1, x_2)$ is continuous, we will therefore write, similarly to (1.5),

$$F(x_1, -\infty) = F(-\infty, x_2) = 0, \quad F(\infty, \infty) = 1. \tag{1.21}$$

It is evident, moreover, that

$$F(x_1, \infty) = P(\xi_1 < x_1 \text{ and } \xi_2 < \infty) = P(\xi_1 < x_1) = F(x_1), \quad F(\infty, x_2) = F(x_2). \tag{1.21'}$$

Let us construct the ratio

$$\frac{P(x_1 \leqslant \xi_1 < x_1 + \Delta x_1 \text{ and } x_2 \leqslant \xi_2 < x_2 + \Delta x_2)}{\Delta x_1 \, \Delta x_2}. \tag{1.22}$$

If with $\Delta x_1 \to 0$ and $\Delta x_2 \to 0$ there is a limit, then it is designated by $F(x_1, x_2)$ and is called a two-dimensional density of distribution of probability and, according to (1.20),

$$f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \, \partial x_2}. \tag{1.23}$$

By analogy to (1.8), $f(x_1, x_2)\, dx_1\, dx_2$ expresses the probability that point M is located within the limits of the elementary area $dx_1\, dx_2$ with the coordinates (x_1, x_2). Then the probability of point M to be located in the arbitrary area S will be equal to a double integral

$$P(M \in s) = \iint_S f(x_1, x_2)\, dx_1\, dx_2. \tag{1.24}$$

It is evident that $f(x_1, x_2) \geqslant 0$. The function $F(x_1, x_2)$ is èxpressed by means of $f(x_1, x_2)$ also by means of a double integral

$$F(x_1, x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(x_1', x_2')\, dx_1'\, dx_2'. \tag{1.25}$$

Hence in the case of $x_1 = x_2 = \infty$ with the use of (1.21) we arrive at

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1\, dx_2 = 1. \tag{1.26}$$

If the two-dimensional density of distribution $f(x_1, x_2)$ is known, then it is not difficult to determine the corresponding one-dimensional densities of the random variables ξ_1 and ξ_2. In fact, from (1.21) and (1.25) in the case of $x_2 = \infty$ we obtain

$$F(x_1) = \int_{-\infty}^{x_1} \int_{-\infty}^{\infty} f(x_1', x_2')\, dx_1'\, dx_2'.$$

Differentiating with respect to x_1 we find

$$f(x_1) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_2, \tag{1.27}$$

and by analogy

$$f(x_2) = \int_{-\infty}^{\infty} f(x_1, x_2)\, dx_1. \tag{1.28}$$

Random variables ξ_1 and ξ_2 are called independent, if events $\xi_1 < x_1$ and $\xi_2 < x_2$ are independent. According to the theorem of multiplication of probabilities, the probability of the joint occurrence of two independent events equals the product of the probabilities of these events. Thus from (1.20) it directly follows that, in the case of independent random quantities, the relationship

$$F(x_1, x_2) = F(x_1)F(x_2) \tag{1.29}$$

holds.

The reverse assertion is also true: if (1.29) is satisfied for $F(x_1, x_2)$, then the random quantities are independent. Differentiating (1.29) with the use of (1.6) and (1.23), we find

$$f(x_1, x_2) = f(x_1)\, f(x_2). \tag{1.30}$$

As an example let us examine the two-dimensional normal distribution

$$f(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{(1-r^2)}}\, e^{-A(x_1, x_2)},$$

$$A(x_1, x_2) = \frac{1}{2(1-r^2)}\left[\frac{(x_1-a_1)^2}{\sigma_1^2} - 2r\,\frac{(x_1-a_1)(x_2-a_2)}{\sigma_1\sigma_2} + \frac{(x_2-a_2)^2}{\sigma_2^2}\right]. \tag{1.31}$$

Hence also from (1.27) we obtain

$$f(x_1) = \frac{1}{\sigma_1\sqrt{2\pi}}\, e^{-\frac{(x_1-a_1)^2}{2\sigma_1^2}}$$

and the analogous expression $f(x_2)$. Consequently, if the two-dimensional random variable (ξ_1, ξ_2) has a normal distribution, then each of the variables ξ_1, ξ_2 also has a normal distribution. In the case of $r = 0$ the function $f(x_1, x_2)$, defined by the expression (1.31), satisfies the condition (1.30). Therefore parameter r characterizes the dependence of random quantities. We will explain the meaning of this parameter in greater detail in the following section.

§ 2. Numerical Characteristics of Random Variables

From the point of view of the theory of probabilities, to define a random variable, it is necessary, as demonstrated by the foregoing, to know the aggregate of its possible values (or the interval in which these values are included in the case of a continuous random variable) and the probabilities corresponding to them. In other words, it is necessary to know its distribution function. In practice, however, it is often difficult to determine the analytical form of this function and it is not always necessary. It is sufficient to know only certain mean