An Introduction to

MATHEMATICAL ANALYSIS

by

ROBERT A. RANKIN

Professor of Mathematics in the University of Glasgow

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TO

MY MOTHER

CONTENTS

VI. INTEGRATION

VII. CONVERGENCE AND UNIFORMITY

PREFACE

THIS book has grown out of lectures given by the author during the last sixteen years to students beginning honours courses in the Universities of Cambridge, Birmingham and Glasgow. It deals almost entirely with functions of a single real variable and includes the topics usually discussed in a first rigorous course on analysis, namely limits, continuity, differentiability, integration, convergence of infinite series, double series and infinite products.

The reader is expected to be familiar with, and have some facility in applying, elementary algebraic processes, including inequalities. No previous knowledge of the calculus or of limiting processes is assumed, although a certain amount of practice in the application of the elementary manipulative techniques of differentiation and integration will perhaps be advantageous to him, since it is not the primary purpose of the book to develop skill in these techniques.

In matters of logic a "naive" point of view is adopted, so that, for example, the notions of set and ordered pair are treated as being intuitively obvious, nor is there any mention of such underlying axioms as those of extension, specification and choice. The starting point is essentially Theorem 5.2.1, which states that a set of real numbers that is bounded above has a least upper bound. As mentioned in the Introduction (§ 1), the author believes that an earlier point of departure is unsuitable for the great majority of students beginning the study of analysis. From this point onwards the theory is developed logically and rigorously, theorems being proved in as general a form as is possible in their context. The exponential, logarithmic, trigonometric, hyperbolic and other special functions, as well as fractional powers, are established as applications of general theorems on, for example, infinite series and inverse functions, and their properties are found. These and other applications demonstrate the power and practical uses of the general theories developed and give point to them. In comparison, the inconvenience caused by the fact that the special functions are not available for illustrative purposes in the earlier parts of the book is small. This lack can, in any case, be turned to advantage by encouraging students to use their imagination to construct functions having various required properties; in this way, also, the erroneous preconception that a function must be definable by a formula can be eradicated.

A few features call, perhaps, for special mention:

(i) Care has been taken throughout to distinguish between a function f and its value $f(x)$ at a point x in its domain of definition. Where strict adherence to this might be inconvenient, as in the case of special functions defined explicitly by formulae, use has been made of braces $\{\ldots\}$ to indicate that the function, and not a particular value, is being considered; this, after all, is merely an extension of the existing notation used to distinguish a sequence $\{a_n\}$ from its nth term a_n.

(ii) The proof of the compactness of a closed interval (see § 11.3) is given in a form directly applicable to dissections (partitions), since it is in this form that applications are nearly always required.

(iii) The theories of functions of bounded variation, lengths of curves, and Riemann (or Riemann–Stieltjes) integration depend on certain elementary, if not particularly simple, properties of dissections. In each case these preliminary results are essentially the same. To make this clear and to avoid tedious repetition, this work has been separated off into a section on interval functions (§ 28), where certain general results are obtained that are immediately applicable to these three topics and free them from obscuring technical details. As a result, the Riemann–Stieltjes integral (at any rate for monotonic integrators) presents no more difficulty than does the Riemann integral, which is deduced from it as a particular case. The various applications of Riemann–Stieltjes integration made in the succeeding sections should convince the reader of its value as an analytical tool. Lebesgue integration is not discussed; although the methods of developing it have been much improved recently, there still does not seem to be any treatment simple enough for students beginning the study of analysis. Such students must have some integral at their disposal, and the Riemann integral, although unpopular in some quarters, fills this gap. Intuitively, its definition is acceptable to the student because of its connexion with area, despite its weakness as a tool in more advanced work.

(iv) The theory of curves, including integration and variation along a curve, is treated fairly fully (§ 31), a distinction being made between a curve and its carrier.

It is not expected that every reader and lecturer will wish to include everything in the book. While most of the material of the first five chapters may be regarded as essential, selections and omissions can be made more freely in the last two chapters, according to the interests of the student or teacher.

One of the main difficulties of analysis for the student arises from a lack of a sufficient number of examples. For this reason numerous exercises are given at the end of nearly every section. These exercises have a threefold object, to test understanding of the preceding theory, to provide practice in carrying out the techniques described, and to encourage imaginative thinking by requiring the student to provide his own examples and counter examples. They are of varying degrees of difficulty, the harder ones usually being taken from university examination papers. The author is indebted to the Syndics of the Cambridge University Press and the Universities of Glasgow and Birmingham for permission to include questions; their provenance is indicated in the text by a C, G or B, respectively, followed by the date.

It is a pleasure to thank my colleagues Dr Daniel Martin and Dr Alex. P. Robertson for many valuable criticisms and suggestions, and for their help in correcting the proofs; it is scarcely necessary to state that the imperfections and inaccuracies that remain are the sole responsibility of the author. Finally, I wish to express my gratitude to Miss Doris M. Caldwell for typing the greater part of the manuscript.

R. A. RANKIN

NOTE TO READER

All examples marked with an asterisk throughout the text have hints for solution which will be found on pages 583–599

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LIST OF SYMBOLS AND NOTATIONS

REFERENCES ARE TO SECTIONS AND SUBSECTIONS

$\Rightarrow$ 5.3 D (implies)

$\Leftarrow$ 5.3 D (is implied by)

$\Leftrightarrow$ 6.5 (implies and is implied by, is equivalent to, if and only if)

$\exists$ 6.5 (there exists (a) – existential quantifier)

$\forall$ 6.5 (for every, for all – universal quantifier)

$\ni$ 6.5 (such that)

$\neg$ 6.5 (not – negation sign)

$\in$ 5.1 (belongs to as a member)

$\notin$ 5.2 E (does not belong to)

$\subseteq$ 9.1 (is a subset of)

$\cup$ 9.1 (union symbol)

$\cap$ 9.1 (intersection symbol)

$\uparrow$ (increasing)

$\downarrow$ (decreasing)

$\to$ 7.1, 8.1, 10.1, 28.3, 28.4, 37.1 (tends to)

$\sim$ 31.2. (equivalence relation between curves)

$\sim$ 34.2 (asymptotic equality)

$[a, b]$, $[a, b[$, $]a, b]$, $]a, b[$ 9.1 (intervals)

$[\ldots]$ 8 (integral part)

$[\varepsilon]$ 7.1 B(i) (dependence on ε)

$[\ldots]_b^a$ 30.2 (difference of values)

$\{\ldots\}$ 6.1, 6.3 (functional notation)

$\circ$ 11.2 B (composition sign)

$^\circ$ 9.1 (interior sign).

cos	21.3 B, 24
$\cos^{-1}$	24.2
cosec	24.3
$\operatorname{cosec}^{-1}$	24.3 D
cosech	25.3
$\operatorname{cosech}^{-1}$	25.3 A
cosh	21.3 C, 25
$\cosh^{-1}$	25.2
cot	24.3
$\cot^{-1}$	24.3 D
coth	25.3
$\coth^{-1}$	25.3 A
exp	21.3 A
Im	16.1
inf	5.2
lim	7.1, 8.1, 10.1
$\lim\limits_{D}$	28.3
$\lim\limits_{\lvert D\rvert\to 0}$	28.4
$\overline{\lim}$	33.1
$\underline{\lim}$	33.1
Log	31.4
log	23.1, 26.2
$\log_{\mathscr{A}}$	26.2

max	3.2 E, 5.1	sech^{-1}	25.3 A
min	3.2 E, 5.1	sin	21.3 B, 24
osc	6.4 F, 16.4	$\sin^{-1}$	24.2
Ph	31.4	sinh	21.3 C, 25
ph	26.1	$\sinh^{-1}$	25.2
$\mathrm{ph}_{\mathscr{A}}$	26.1	sup	5.2
Re	16.1	tan	24.3
sec	24.3	$\tan^{-1}$	24.3
$\sec^{-1}$	24.3 D	tanh	25.3
sech	25.3	$\tanh^{-1}$	25.3

$\mathscr{A}^\circ$ 9.1 (interior of $\mathscr{A}$)

$\mathscr{A}^*(f)$ 33.1 (superior numbers)

$\mathscr{A}_*(f)$ 33.1 (inferior numbers)

$\mathscr{B}^*(a, b)$ 28.2 (subadditive interval functions dissectionally bounded above)

$\mathscr{B}_*(a, b)$ 28.2 (superadditive interval functions dissectionally bounded below)

$\mathscr{C}$ 6.1 (complex numbers)

$c+$, $c-$ 10.1

$\mathscr{D}(a, b)$ 28.2 (dissections of $[a, b]$)

$|dg|$ 29.5 E (dV_g)

e 23.1 F (2·71828 . . .)

i 16.1 ($\sqrt{(-1)}$)

I 6.2 B (identity function)

$\mathscr{I}$ 6.1 (integers)

$\mathscr{I}_X$, $\bar{\mathscr{I}}_X$ 6.1 (subsets of $\mathscr{I}$)

$J(f; a, b)$ 30.1 (8) (integral)

$J(f, g; a, b)$ 29.4 (3) (integral)

$L(a, b)$ 31.1 (5) (length of curve)

$L(f; D)$ 30.1 (3) (lower sum)

$L(f; a, b)$ 30.1 (5) (lower integral)

$L(f, g; D)$ 29.3 (14) (lower sum)

$L(f, g; a, b)$ 29.3 (18) (lower integral)

$\mathscr{L}(z)$ 26.2 (logarithm set)

$\mathscr{N}(\delta, c)$, $\mathscr{N}'(\delta, c)$, $\mathscr{N}^*(\delta, c)$ 9.1 (neighbourhoods)

n an integer throughout

O, o 34.2 (order symbols)

P 19.4, 36.1 (principal value)

$\mathscr{P}(z)$ 26.1 (phase set)

$\mathscr{R}$ 6.1 (real numbers)

$\mathscr{R}_X$, $\bar{\mathscr{R}}_X$, $\mathscr{R}'_X$, $\bar{\mathscr{R}}'_X$ 6.1 (subsets of $\mathscr{R}$)

$\mathscr{R}^n$ 31.1, 35.4 (n-dimensional space)

$\mathscr{R}_{M,N}$ 37.1 (2) (rectangle)

$\mathscr{R}(\mathscr{A})$ 30.2 (set of integrals)

$\mathscr{R}(a, b)$ 30.1 (set of integrals)

$\mathscr{R}(g; a, b)$ 29.4 (set of integrals)

$\mathscr{R}^*(g; a, b)$ 29.5 (set of integrals)

$S(f; D; \xi)$ 30.1 (10) (dissection sum)

$S(f, g; D; \xi)$ 29.4. (32) (dissection sum)

$\mathscr{S}(a,b)$, $\mathscr{S}_c(a,b)$ 28.1 (sets of subintervals)

$U(f; D)$ 30.1 (3) (upper sum)

$U(f; a, b)$ 30.1 (6) (upper integral)

$U(f, g; D)$ 29.3 (14) (upper sum)

$U(f, g; a, b)$ 29.3 (19) (upper integral)

V_g 28.5 (33) (total variation function)

$V(g, D)$ 28.5 (31) (variation)

$V(g; a, b)$ 28.5 (3) (total variation)

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CHAPTER 1

FUNDAMENTAL IDEAS AND ASSUMPTIONS

1. INTRODUCTION

IF ONE had to characterize in a single sentence the difference between algebra and analysis, one might say that algebra is concerned with finite sets of numbers, while analysis is concerned with infinite sets of numbers. This statement, like all attempts at concise summarization, is only partially accurate, but it does emphasize one of the causes of many of the difficulties that analysis presents to the average student. In the author's experience, the difficulties that the student encounters in the subject, and the errors that he makes, arise from three main causes, namely (i) failure of simple logic, such as the inability to distinguish between a necessary and a sufficient condition, (ii) lack of practice in manipulation of inequalities, and (iii) the unverified assumption that any process that can be carried out on a finite set of numbers can be extended to infinite sets of numbers.

Of these (i) can perhaps be overcome best by examples drawn from everyday life and by constantly drawing attention to the difference between implying and being implied by; (ii) can only be improved by frequent practice. As regards (iii), however, this can only be corrected by commencing the study of analysis at some suitable point as far back as possible and by proceeding thereafter as logically as possible.

There are several different opinions as to where this point of departure should be chosen. There is no doubt that the serious student of mathematics will wish, at some stage in his course, to start at the very beginning with a full discussion of the nature of number built up logically from certain simple axioms. However, at the stage when a knowledge of analysis is desirable, he will not normally possess the mathematical maturity that is necessary to appreciate the rather abstract, intangible and sometimes unexpected concepts, that occur in any serious discussion of the logical foundations of number and analysis.

For this reason it seems better to start further on and to make the assumption that the real and complex numbers exist and possess certain familiar properties. This then is our point of departure. In §§ 2–5 we set down our initial assumptions about numbers and their properties and draw from them various simple conclusions. From this foundation we proceed to discuss rigorously (i.e. logically and accurately) the mathematical subjects in the list of contents.

Before embarking on this programme, we remark that a logical proof of these initial assumptions cannot, of course, start from nothing, but must be based on certain initial axioms. It is usual to begin by setting down certain axioms for the natural numbers, i.e. the positive integers. From this foundation one constructs the other kinds of numbers, namely zero and the negative integers, the rational numbers, the real numbers and, finally, the complex numbers, and deduces their properties. There are various ways in which this can be done. For a recent treatment which seems particularly suitable for non-specialists see H. Thurston, *The number system* (Blackie, 1956). In it proofs of all our initial assumptions can be found.

2. ASSUMPTIONS RELATING TO THE FIELD OPERATIONS

2.1 Elementary operations. In this section and the three following we set down the various properties of real and complex numbers that we shall assume and use. As mentioned in § 1, all these properties could be deduced logically from certain simple initial axioms. The properties that we list here and in § 3 are not intended to form a logically complete set of postulates, nor are they chosen so as to be independent of each other. The main criterion for including a property is the frequency of its future use.

It is assumed that the integers (positive, zero and negative), the rational numbers, the real numbers and the complex numbers exist and that they obey the laws of algebra; that is, they can be combined by addition, subtraction, multiplication and division (except by zero) according to the familiar rules. In the language of abstract algebra this states that the real numbers, and also the complex numbers, form a field; this explains the title of the section. Examples of such rules are the following, in which letters stand for numbers, real or complex.

2.1 A. $a + b = b + a$ (commutative law for addition).

2.1 B. $a + (b + c) = (a + b) + c$ (associative law for addition).

2.1C. $ab = ba$ (commutative law for multiplication).

2.1D. $a(bc) = (ab)c$ (associative law for multiplication).

2.1E. $a(b + c) = ab + ac$ (distributive law).

2.1F. $a + 0 = a, \quad a + (-a) = 0, \quad b - a = b + (-a)$.

2.1G. $a0 = 0, \quad a1 = a$.

2.1H. If $a \neq 0$, then $aa^{-1} = 1$ and

$$\frac{b}{a} = b/a = ba^{-1}.$$

2.1I. If m and n are integers and a and b are non-zero real or complex numbers, then

$$a^m \,.\, a^n = a^{m+n}, \quad a^n b^n = (ab)^n, \quad (a^m)^n = a^{mn}.$$

These are known as the *index laws.*

2.1J. If $ab = 0$, then either $a = 0$ or $b = 0$ (integral domain property).

2.1K. If $a \neq 0$, then $a^0 = 1$; this can be deduced from the first index law by taking $m = 1$ and $n = 0$. It follows that $a(a^0 - 1) = 0$, and so $a^0 - 1 = 0$, by 2.1J. We make the convention that $0^0 = 1$, so that $a^0 = 1$ for all real or complex a. With this convention, the restriction in 2.1I that a and b are non-zero can be removed, provided that m and n are non-negative integers.

2.1L. $(-1)^2 = 1$.

2.2. Different kinds of numbers. It must always be borne in mind that, when we speak of numbers, we mean *finite* numbers. Whenever the word *infinity* and symbol ∞ are introduced*, they are always used with carefully defined meanings, and in no case are they regarded as numbers.

Further, the adjectives *real* and *complex* are not exclusive; a real number is also a complex number, but the converse may not hold. Similarly, the use of the word *or* in 2.1J is not exclusive; i.e., when we state that either $a = 0$ or $b = 0$, we do not preclude the possibility that both a and b may be zero. We occasionally use a more picturesque phraseology and say that a *vanishes* when we mean that $a = 0$.

---

* They occur for the first time in § 7.

The integers form a subclass of the rational numbers, and the rational numbers form a subclass of the real numbers. The letter n always denotes an integer wherever it appears in this book. A rational number a can be written as the quotient of two integers, i.e. as

$$a = \frac{p}{q},$$

where p and q are integers. Such a representation can be made unique by requiring q to be positive and p and q to possess no common factor other than unity. For example, the integer 0 can be represented in the unique form just mentioned by taking $p = 0$, $q = 1$. When we say that a positive integer b is a *factor* of, or *divides*, an integer c, we mean that c/b is an integer. An integer q is said to be *even* if 2 divides q; otherwise q is *odd*.

If a and b are rational numbers, so are $a + b$, $a - b$, ab and (when $b \neq 0$) a/b. A real number that is not rational is said to be *irrational* (see 3.2H and Ex. 3.1).

The statement of certain elementary properties of complex numbers is deferred to § 16.

3. ASSUMPTIONS RELATING TO THE ORDERING OF THE REAL NUMBERS

3.1. Elementary rules. In this section we are concerned exclusively with real numbers. It is assumed that the real numbers form an ordered system. By this we mean that, if we are given any two real numbers a and b, we can decide whether a is less than ($<$), equal to, or greater than ($>$) b, and only one of these three possibilities can occur; further $a < b$ is equivalent to $b > a$. We write $a \geqq b$, or $a \leqq b$, if a is not less than b, or if a is not greater than b, respectively; for example $2 \leqq 3$, and also $3 \leqq 3$. A real number a is positive, non-negative or negative according as $a > 0$, $a \geqq 0$ or $a < 0$, respectively.

The laws governing the use of inequality signs are assumed known; examples of these laws are the following:

3.1 A. If $a > b$ and $b > c$, then $a > c$.

3.1 B. If $a > b$ and $c \geqq d$, then $a + c > b + d$; in particular, $a + c > b + c$.

3.1 C. If $a > b$, then $-a < -b$.

3.1 D. If $a > b \geqq 0$ and $c \geqq d > 0$, then $ac > bd$. Here $a > b \geqq 0$ means that $a > b$ *and* $b \geqq 0$; two or more inequalities are frequently combined in this way. If the restrictions $b \geqq 0$, $d > 0$ are omitted, the result may be false, as is illustrated by taking $a = c = 1$, $b = d = -2$. If $c = d > 0$, we deduce, in particular, that $ac > bc$.

3.1 E. If $a > b > 0$, then

$$0 < \frac{1}{a} < \frac{1}{b}.$$

3.1 F. If a is any real number, an integer n exists such that $n > a$. This is the so-called *axiom of Archimedes*.

Similar results involving $\geqq$ and $\leqq$ hold, and will be referred to under the same reference numbers. For example, in 3.2 F below, we use 3.1 D in the form that, if $a \geqq b \geqq 0$ and $c \geqq d \geqq 0$, then $ac \geqq bd$.

The *modulus* or *absolute value* of a real number a is denoted by $|a|$, and is defined to be a, if $a \geqq 0$, and $-a$ if $a < 0$. Clearly $|a| \geqq 0$, and $|a| = 0$ if and only if $a = 0$; also $|-a| = |a|$.

3.2. Simple consequences

3.2 A. We note that, for any real numbers a and b, $|ab| = |a|\,|b|$. For each side is either ab or $-ab$, and both sides are non-negative and therefore equal.

3.2 B. The inequality $|a| \leqq \varrho$ (for $\varrho \geqq 0$) is equivalent to $-\varrho \leqq a \leqq \varrho$. For, if $|a| \leqq \varrho$, then $0 \leqq a \leqq \varrho$ if $a \geqq 0$, while if $a < 0$, then $-a \leqq \varrho$, i.e. $a \geqq -\varrho$; hence $-\varrho \leqq a \leqq \varrho$ in either case. The converse is proved similarly. Note that, in particular, $-|a| \leqq a \leqq |a|$. Similarly, $|a| < \varrho$ if and only if $-\varrho < a < \varrho$.

3.2 C. $|a + b| \leqq |a| + |b|$. By addition of the inequalities $-|a| \leqq a \leqq |a|$, $-|b| \leqq b \leqq |b|$, we obtain $-(|a| + |b|) \leqq a + b \leqq |a| + |b|$, which is the result stated. This result is known as the *triangle inequality* for a reason that will become apparent when we prove the corresponding result for complex numbers.

3.2 D. By taking $a = \alpha - \beta$, $b = \beta$ in 3.2 C we deduce the corollary:

$$|\alpha - \beta| \geqq |\alpha| - |\beta|.$$

Similarly, $|\alpha - \beta| = |\beta - \alpha| \geqq |\beta| - |\alpha|$, so that we have, by 3.2 B,

$$\big||\alpha| - |\beta|\big| \leqq |\alpha - \beta|.$$

3.2E. We define $\max(\alpha, \beta)$ to be α or β according as $\alpha \geqq \beta$ or $\alpha < \beta$. Similarly, $\min(\alpha, \beta)$ is β or α according as $\alpha \geqq \beta$ or $\alpha < \beta$. It is easily verified that

$$\max(\alpha, \beta) = \tfrac{1}{2}\{\alpha + \beta + |\alpha - \beta|\},$$
$$\min(\alpha, \beta) = \tfrac{1}{2}\{\alpha + \beta - |\alpha - \beta|\}.$$

We say that γ lies *between* two real numbers α and β if

$$\min(\alpha, \beta) \leqq \gamma \leqq \max(\alpha, \beta).$$

3.2F. If a is any real number, then $a^2 \geqq 0$. For, by 2.1L,

$$a^2 = (\pm |a|)^2 = |a|^2$$

and $|a|^2 \geqq 0$ by 3.1D (with $|a|$ in place of a and c, and $b = d = 0$). Clearly $a^2 > 0$ if $a \neq 0$.

3.2G. If $a \geqq 0$ and $b \geqq 0$, we have

$$(a - b)^2 \leqq |a^2 - b^2|.$$

To show this we clearly may assume that $a \geqq b \geqq 0$, since the inequality is unaltered when a and b are interchanged. We then have to prove that

$$(a - b)^2 \leqq a^2 - b^2,$$

which is equivalent to

$$2b(a - b) \geqq 0.$$

This is true since $2b \geqq 0$ and $a - b \geqq 0$.

3.2H. For the purpose of developing the theory it is not necessary to assume the existence of nth roots of real numbers, as their existence will be established, subject to certain conditions, as a consequence of Theorem 12.1.1. However, in exercises and for purposes of illustration, we shall assume that square-roots of non-negative numbers exist. That is, if a is any non-negative real number, there exists a non-negative number b with the property that $b^2 = a$, and we write $b = \sqrt{a}$. Clearly $(-b)^2 = a$ and it follows easily from 2.1J that $\sqrt{a}$ and $-\sqrt{a}$ are the only numbers x with the property that $x^2 = a$. Note that $\sqrt{\ }$ always denotes the non-negative square root, and that

$$\sqrt{(c^2)} = |c| \text{ for any real } c.$$

If n is a positive integer that is not the square of an integer, the real number $\sqrt{n}$ is irrational. In the general case, this is best proved by elementary arguments from the theory of numbers; in particular cases it can be proved by other methods (see Ex. 3.1, where $\sqrt{2}$ and $\sqrt{3}$ are considered). Many of the examples that we shall give will depend upon the existence of irrational numbers and, in particular, on the fact that an irrational number can be found between any two given real numbers (see Ex. 3.3). We shall rarely need to know that any particular number is irrational.

EXERCISES 3

*3.1. Show that $\sqrt{2}$ and $\sqrt{3}$ are irrational numbers.

*3.2. Prove that $\sqrt{3} - \sqrt{2}$ is irrational.

*3.3. If a and b are any two real numbers and $a < b$, show that there exist a rational number r and an irrational number s such that $a < r < b$ and $a < s < b$.

3.4. If r is rational and not zero and s is irrational prove that $r + s$ and rs are irrational.

*3.5. If (i) a, b, c and d are rational, (ii) $b > 0$, $d > 0$, (iii) $\sqrt{b}$ and $\sqrt{d}$ are irrational, and (iv) $a + \sqrt{b} = c + \sqrt{d}$, prove that $a = c$ and $b = d$.

3.6. If $p/q < r/s$, where p, q, r and s are real and $q > 0$, $s > 0$, show that

$$\frac{p}{q} < \frac{p+r}{q+s} < \frac{r}{s}.$$

*3.7. Suppose that $b > a$ and that $b > 0$. Without assuming the existence of square roots, prove that there exists a rational number r, such that $a < r^2 < b$.

*3.8. If the real number a is such that $a < 1 + \varepsilon$ for every positive ε, show that $a \leqq 1$.

*3.9. Find all real numbers x for which

$$\frac{1}{x-3} < \frac{1}{2-x} \quad (x \neq 2, \quad x \neq 3).$$

3.10. Show that, if p and q are real numbers, then

$$x^2 + 2px + q \geqq q - p^2$$

for all real x.

Prove that (i) if $q > p^2$, $x^2 + 2px + q > 0$ for all real x, (ii) if $q = p^2$, $x^2 + 2px + q \geqq 0$ for all real x, and (iii) if $q < p^2$,

$$x^2 + 2px + q > (\geqq)\, 0$$

if and only if either

$$x > (\geqq) - p + \sqrt{(p^2 - q)} \quad \text{or} \quad x < (\leqq) - p - \sqrt{(p^2 - q)}.$$

4. MATHEMATICAL INDUCTION

4.1. Statement of the principle of induction. A fundamental principle, which is used with great frequency to establish general results, is the *principle of induction*. No discussion of the logical

* Note: All asterisked examples throughout the text have hints for solution (see pages 583–599).

foundations of the natural numbers (i.e. the positive integers) can avoid including it in some form or other.

Let P(n) be some proposition involving the integer n. In applications. P(n) will usually be some equation or inequality, such as $2^n \geqq n + 5$, which may or may not be true for the value of n considered. Then the principle of induction may be formulated in two slightly different forms which we state as theorems.

THEOREM 4.1.1. *Suppose* (i) *that* P(N) *is true for some fixed* integer* N, *and* (ii) *that, for all* $N \geqq n$, P$(n+1)$ *is true whenever* P(n) *is true. Then* P(n) *is true for all integers* $n \geqq N$.

THEOREM 4.1.2. *Suppose* (i) *that* P(N) *is true for some fixed integer* N, *and* (ii) *that* P$(n + 1)$ *is true whenever* P(m) *is true for all integers* m *satisfying* $N \leqq m \leqq n$. *Then* P(n) *is true for all integers* $n \geqq N$.

In applying either of these theorems it can only be concluded that P(n) is true for all integers $n \geqq N$ if both of the conditions (i) and (ii) have been verified. It should also be noted that the words "P$(n + 1)$ is true" form part of the second condition, which has to be verified, and are not part of the conclusion of either theorem. Further, it cannot be too strongly emphasized that, in Theorem 4.1.1, P$(n + 1)$ must hold *whenever* P(n) holds, and that it is not sufficient, for example, merely to check this for particular values of n such as 1, 2 and 3.

Many mathematical functions (see later for general definition of function) and relations whose existence are generally accepted as obvious depend in reality upon the induction principle for their definition, existence and validity. This applies, in particular, to the definitions of expressions such as a^n, $\sum_{r=1}^{n} a_r$ and $\prod_{r=1}^{n} a_r$. It will suffice to consider two examples in detail.

4.1A. The *factorial function* $n!$ is defined inductively as follows:

$$0! = 1, \ (m + 1)! = (m + 1).\ m! \text{ for all integers } m \geqq 0. \quad (1)$$

* The word *fixed* as used here and elsewhere in the book is really superfluous, and for this reason no formal definition is given. The word is used in order to help the reader to remember which symbols, such as N, take the same value throughout the argument, and which symbols (sometimes called *dummy* symbols) can denote arbitrary members of sets (see § 5). Thus, in the theorem, n can be any integer greater than or equal to N. Similar remarks apply to the adjective *given*. In much the same way, although all numbers are finite by definition, the word *number* is occasionally qualified by the adjective *finite*, where it is thought that the reader might feel tempted to include infinity as a possible "value".

The existence of $n!$ for all positive integers n then follows from Theorem 4.1.1; for this purpose we take $N = 0$, and define $\mathrm{P}(n)$ to be the proposition that $n!$ is defined as a unique number by (1). Then condition (i) of Theorem 4.1.1 is satisfied, since $0! = 1$, by (1). So is condition (ii), since the definition of $(n+1)!$ follows from that of $n!$ by means of the relation $(n+1)! = (n+1).\,n!$. We conclude that $n!$ is defined for all $n \geqq 0$.

More generally, expressions such as

$$\sum_{r=N}^{n} a_r \quad \text{and} \quad \prod_{r=N}^{n} a_r \tag{2}$$

are defined inductively by

$$\sum_{r=N}^{N} a_r = \prod_{r=N}^{N} a_r = a_N,$$

and

$$\sum_{r=N}^{n+1} a_r = \left\{\sum_{r=N}^{n} a_r\right\} + a_{n+1}, \quad \prod_{r=N}^{n+1} a_r = \left\{\prod_{r=N}^{n} a_r\right\} a_{n+1},$$

for $n \geqq N$.

With the aid of 2.1A–2.1D it can be shown that the numbers denoted by the expressions (2) remain unaltered for different orderings and groupings of the $a_r\,(N \leqq r \leqq n)$, and we can prove, for example, that

$$b \sum_{r=N}^{n} a_r = \sum_{r=N}^{n} b\, a_r,$$

which extends 2.1E. Alternative notations for (2) are

$$a_N + a_{N+1} + \cdots + a_n \quad \text{and} \quad a_N\, a_{N+1} \cdots a_n,$$

respectively. The number of terms displayed need not be three in every instance. For example

$$n! = 1.2.3. \;\ldots\; .n = \prod_{r=1}^{n} r \quad (n \geqq 1).$$

In (2) the letter r can, of course, be replaced by any letter other than N, n or a.

4.1B. Let x and a be two different numbers, real or complex. Then

$$\frac{x^n - a^n}{x - a} = \sum_{r=0}^{n-1} x^{n-1-r} a^r = x^{n-1} + x^{n-2} a + \cdots + x\, a^{n-2} + a^{n-1}, \tag{3}$$

for every positive integer n. This is true for $n = 1$, since the right-hand side is to be interpreted as 1 (see 4.1A), and is also true for $n = 2$. If true for n, its truth for $n + 1$ follows, since

$$\frac{x^{n+1} - a^{n+1}}{x - a} = \frac{x(x^n - a^n) + a^n(x - a)}{x - a} = x\frac{x^n - a^n}{x - a} + a^n$$

$$= x\sum_{r=0}^{n-1} x^{n-1-r} a^r + a^n = \sum_{r=0}^{n-1} x^{n-r} a^r + a^n = \sum_{r=0}^{n} x^{n-r} a^r.$$

Hence (3) holds for all $n \geqq 1$.

4.2. Some further applications of the induction principle

4.2A. As a straightforward application of Theorem 4.1.1 we prove that $a^n > n$ for all positive integers n, when a is a given integer greater than unity. Since $a > 1$ the proposition is true for $n = 1$, and we take $N = 1$. We are now ready to test whether condition (ii) holds. Our assumption for this purpose is that $\mathrm{P}(n)$ is true, i.e. $a^n > n$, and we wish to deduce that $a^{n+1} > n + 1$. By 3.1D,

$$a^{n+1} = a \,.\, a^n > a\,n$$

and, since $a \geqq 2$ and $n \geqq 1$, $an \geqq 2n \geqq n + 1$, so that $a^{n+1} > n + 1$. Both conditions have now been verified and we draw the conclusion that $a^n > n$ for all positive integers n.

4.2B. As a slightly harder example we prove that $2^{n+1} > n^2$ for all positive integers n. Here the argument used in 4.2A will not apply when n is small, so that we check, first of all, that $2^{n+1} > n^2$ for $n = 1, 2, 3$ and then take $N = 3$ in Theorem 4.1.1. We observe first that, if $n \geqq 3$,

$$\left(1 + \frac{1}{n}\right)^2 \leqq \left(1 + \frac{1}{3}\right)^2 = \frac{16}{9},$$

by 3.1D and 3.1E. Now assume that $2^{n+1} > n^2$, where $n \geqq 3$. Then

$$(n+1)^2 = n^2\left(1 + \frac{1}{n}\right)^2 \leqq \frac{16}{9}n^2 < \frac{16}{9}2^{n+1} < 2^{n+2},$$

so that $\mathrm{P}(n + 1)$ holds when $\mathrm{P}(n)$ holds. Accordingly, $2^{n+1} > n^2$ for all $n \geqq 3$, and so for all $n \geqq 1$.

4.2C. The laws governing the use of inequalities can also be extended by induction to sums and products containing more than two terms. For example, take $a = 1$ in 4.1B and assume that $x > 1$.

Then, if $0 \leqq r \leqq n-1$, we have, by suitable extensions of 3.1D and 3.1B,

$$x^{n-1-r} a^r = x^{n-1-r} \geqq 1,$$

and

$$\frac{x^n - 1}{x - 1} = \sum_{r=0}^{n-1} x^{n-1-r} \geqq n.$$

From this we deduce that

$$x^n > n(x-1) \quad (x > 1,\ n \geqq 1), \tag{4}$$

an inequality of which we make considerable use.

4.2D. The binomial coefficient $\binom{a}{r}$ is defined for all a, real or complex, and integral $r \geqq 0$, by

$$\binom{a}{0} = 1, \quad \binom{a}{r} = \frac{a(a-1)(a-2)\cdots(a-r+1)}{r!} \quad (r \geqq 1).$$

In particular, when a is a positive integer n, we have

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} \quad (0 \leqq r \leqq n).$$

By expressing $(x+y)^{n+1}$ as $(x+y)(x+y)^n$ and using the relation

$$\binom{n}{r} + \binom{n}{r-1} = \binom{n+1}{r} \quad (1 \leqq r \leqq n).$$

we can easily apply Theorem 4.1.1 to prove the *Binomial Theorem* for a positive integral exponent n, namely

$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r = x^n + n\,x^{n-1} y + \frac{n(n-1)}{2!} x^{n-2} y^2$$

$$+ \frac{n(n-1)(n-2)}{3!} x^{n-3} y^3 + \cdots + y^n.$$

This is valid for all x and y, real or complex; when $x = 0$, x^0 is to be interpreted as 1, as mentioned in 2.1K.

4.2E. If $0 \leqq a < b$ and n is a positive integer, then $0 \leqq a^n < b^n$. This is easily proved by induction with the help of 3.1D. From this it follows conversely, that if $0 \leqq a^n < b^n$, where $a \geqq 0$, $b \geqq 0$ and n is a positive integer, then $a < b$. We deduce that if $a^n = b^n$, where $a \geqq 0$, $b \geqq 0$, then $a = b$.

4.2F. For any real a, $|a^n| = |a|^n$, where n is a positive integer. This again is easily proved by induction with the help of 3.2A. For it is true for $n = 1$, and if true for n, then

$$|a^{n+1}| = |a^n \cdot a| = |a^n|\,|a| = |a|^n\,|a| = |a|^{n+1}.$$

4.2G. *Cauchy's inequality.* Suppose that $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are $2n$ real numbers, where $n \geqq 1$. Then

$$\left(\sum_{r=1}^{n} a_r b_r\right)^2 \leqq \sum_{r=1}^{n} a_r^2 \sum_{s=1}^{n} b_s^2. \tag{5}$$

If $a_1 = a_2 = \cdots = a_n = 0$, the result is trivial; we therefore suppose that this is not the case. Then, by 3.2F, the numbers $a_1^2, a_2^2, \ldots, a_n^2$ are all non-negative, and at least one of them is positive. It follows that

$$A = \sum_{r=1}^{n} a_r^2 > 0. \tag{6}$$

Write

$$B = \sum_{r=1}^{n} b_r^2 \quad \text{and} \quad C = \sum_{r=1}^{n} a_r b_r.$$

By 3.2F,

$$(\lambda a_r + b_r)^2 \geqq 0$$

for every real number λ, and therefore

$$\sum_{r=1}^{n} (\lambda^2 a_r^2 + 2\lambda a_r b_r + b_r^2) = \sum_{r=1}^{n} (\lambda a_r + b_r)^2 \geqq 0;$$

i.e.

$$\lambda^2 A + 2\lambda C + B \geqq 0.$$

This is equivalent to the statement that

$$(\lambda A + C)^2 + A B \geqq C^2 \tag{7}$$

for all real λ. By (6), we may take $\lambda = -C/A$ in (7) and we obtain (5).

Further, when $a_1, a_2, \ldots, a_n$ are not all zero, equality can only hold in (5) when it holds in (7) with $\lambda = -C/A$, i.e. when

$$\sum_{r=1}^{n} (\lambda a_r + b_r)^2 = 0 \quad (\lambda = -C/A).$$

This implies that $\lambda a_r + b_r = 0$ for $r = 1, 2, \ldots, n$; i.e.

$$b_r = \frac{C}{A} a_r \quad (r = 1, 2, \ldots, n).$$

Consequently equality holds in (5) only when the numbers b_r are proportional to the numbers a_r $(1 \leqq r \leqq n)$.

EXERCISES 4

4.1. Show that, for $n \geqq 1$,

$$2^{n-1} \leqq n! \leqq n^n;$$

more generally, if N is a positive integer, show that

$$N!\,(N+1)^{n-N} \leqq n! \leqq N!\,n^{n-N}$$

for all $n \geqq N$.

4.2. Prove by induction that, for $n \geqq 1$,

$$\sum_{r=1}^{n} r = \tfrac{1}{2}n(n+1), \quad \sum_{r=1}^{n} r^2 = \tfrac{1}{6}n(n+1)(2n+1),$$

and

$$\sum_{r=1}^{n} r^3 = \tfrac{1}{4}n^2(n+1)^2.$$

4.3. Prove by induction that, for $n \geqq 1$,

$$\sqrt{\frac{\frac{5}{4}}{4n+1}} \leqq \frac{1.3.5.\ldots.(2n-1)}{2.4.6.\ldots.2n} \leqq \sqrt{\frac{\frac{3}{4}}{2n+1}}.$$

4.4. If $a_1, a_2, \ldots, a_n$ are real numbers $(n \geqq 1)$, prove that

$$\left|\sum_{r=1}^{n} a_r\right| \leqq \sum_{r=1}^{n} |a_r|.$$

*4.5. If n is a positive odd integer and a and b are real, prove that the inequality $a^n < b^n$ implies that $a < b$, and that $a^n = b^n$ implies that $a = b$.

*4.6. Let q be a positive integer and suppose that $a < b$, where b is positive if q is even. Without assuming the existence of qth roots, prove that there exists a rational number r such that $a < r^q < b$.

5. UPPER AND LOWER BOUNDS OF SETS OF REAL NUMBERS

5.1. Sets of real numbers. A collection of different objects of any kind is called a *set*, and any object a belonging to a set $\mathscr{A}$ is called a *member* or *element* of $\mathscr{A}$. We write this $a \in \mathscr{A}$, and use the symbol $\in$ in different grammatical constructions, so that, for example, it can mean "belonging to" or "contained in" as well as "belongs to" or "is contained in". We shall be concerned throughout with non-null sets, i.e. sets that contain at least one member. No two members of any set $\mathscr{A}$ are the same; i.e. we do not allow repetitions.

We say that a set $\mathscr{A}$ is *finite* if it contains only a finite number of elements. A non-null set which is not finite is called an *infinite* set. We do not require that a set shall be ordered; that is, we do not regard the members of a set as being arranged in any particular order.

For the remainder of the section we confine out attention to sets whose members are real numbers.* Our purpose is to consider whether a given set contains a maximum or a minimum member, and if it does not, to replace the concept of maximum or minimum member by something that does exist. For this purpose we consider the following special sets:

$\mathscr{A}_1$ consists of the five numbers 5, 7, 2, $-\frac{5}{2}$, $\sqrt{2}$.

$\mathscr{A}_2$ consists of all the positive integers 1, 2, 3, 4, ...

$\mathscr{A}_3$ consists of the numbers $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots$, the nth number in the list being $n/(n+1)$.

$\mathscr{A}_4$ consists of the number 1 and all the numbers in $\mathscr{A}_3$.

$\mathscr{A}_5$ consists of all real numbers a such that $0 < a < 1$.

$\mathscr{A}_6$ consists of all positive real numbers; i.e. all a such that $a > 0$.

$\mathscr{A}_7$ consists of all rational numbers, positive, negative or zero.

Of these seven sets only $\mathscr{A}_1$ is a finite set. It is clear that a finite set $\mathscr{A}$ always contains a greatest member, which we denote by

$$\max \mathscr{A} \quad \text{or} \quad \max_{a \in \mathscr{A}} a.$$

It also contains a least member denoted by

$$\min \mathscr{A} \quad \text{or} \quad \min_{a \in \mathscr{A}} a.$$

Thus $\max \mathscr{A}_1 = 7$ and $\min \mathscr{A}_1 = -\frac{5}{2}$. We can, of course, replace a by any other available letter.

An infinite set $\mathscr{A}$ need not contain a greatest member. Thus, of the six infinite sets just defined, only $\mathscr{A}_4$ contains a greatest member (namely 1) and only $\mathscr{A}_2$, $\mathscr{A}_3$ and $\mathscr{A}_4$ contain least members. This shows that, without examining a given infinite set $\mathscr{A}$, we cannot assert that it contains a greatest or least member, and for this reason it is best to avoid the notation $\max \mathscr{A}$ and $\min \mathscr{A}$ for infinite sets. For infinite sets the concepts of greatest and least member are replaced by those of supremum and infimum, as we now describe.

5.2. Suprema and infima of sets of numbers. Suppose that $\mathscr{A}$ is any given set of real numbers; $\mathscr{A}$ may be finite or infinite. We say

* The sets considered in this book will usually be sets of numbers. Nevertheless, we shall occasionally have to consider sets that are not of this kind. Thus in § 6.1 we consider sets whose members are ordered couples (x, y) of numbers x and y, in order to define a function; we shall also consider sets of functions.

that $\mathscr{A}$ is *bounded above* if there exists some number M, say, such that $a \leqq M$ for all $a \in \mathscr{A}$. We do not require M to be a member of $\mathscr{A}$. The number M is called *an upper bound* of, or for, $\mathscr{A}$; clearly any number greater than M is also an upper bound of $\mathscr{A}$. Similarly, if there exists some number m such that $a \geqq m$ for all $a \in \mathscr{A}$, then $\mathscr{A}$ is said to be *bounded below* and m is called a *lower bound* of $\mathscr{A}$. If $\mathscr{A}$ is both bounded above and bounded below we merely say that $\mathscr{A}$ is *bounded*. Every finite set is bounded, but this need not be so for infinite sets. Sets that are not bounded, or are not bounded above (below), are said to be *unbounded*, or *unbounded above* (*below*).

5.2A. For example, each of the numbers 63, $\frac{3}{2}$ and 1 is an upper bound for $\mathscr{A}_3$ and also for $\mathscr{A}_4$ and $\mathscr{A}_5$, so that these three sets are bounded above. Since we can always find a positive integer greater than any given real number M, the set $\mathscr{A}_2$ is not bounded above; neither are $\mathscr{A}_6$ and $\mathscr{A}_7$, for the same reason. Each of the sets $\mathscr{A}_2$, $\mathscr{A}_3$, $\mathscr{A}_4$, $\mathscr{A}_5$ and $\mathscr{A}_6$ is bounded below, since we can take $m = -1$, for example, in each case. The set $\mathscr{A}_7$ is not bounded below. Accordingly only $\mathscr{A}_1$, $\mathscr{A}_3$, $\mathscr{A}_4$ and $\mathscr{A}_5$ are bounded sets.

5.2B. We note that 1, which is an upper bound for $\mathscr{A}_3$ and $\mathscr{A}_4$, is also the least upper bound that can be found. This is obvious for $\mathscr{A}_4$, since $1 \in \mathscr{A}_4$. To prove it for $\mathscr{A}_3$ we observe that if b is any number less than 1, then by taking n to be any integer greater than

$$\frac{1}{1-b} - 1$$

we obtain

$$\frac{n}{n+1} = 1 - \frac{1}{n+1} > 1 - (1-b) = b,$$

so that b is not an upper bound for $\mathscr{A}_3$. We can prove similarly that 1 is the least upper bound that can be found for the set $\mathscr{A}_5$.

This suggests the following definition.

Definition 5.2.1. *A (finite) number σ is called the least upper bound or the supremum of a set $\mathscr{A}$ of real numbers if and only if the following two conditions are satisfied:*

(i) *$a \leqq \sigma$ for all $a \in \mathscr{A}$.*

(ii) *For every real σ' less than σ, there exists at least one $a \in \mathscr{A}$ such that $a > \sigma'$.*