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PREFACE

The nature and theory of geometry, finite or infinite, relies on and springs from the 'interesting' examples.

It is, of course, well known that any projective or affine geometry of dimension larger than two corresponds to the lattice of vector subspaces of a vector space over a skewfield. In this context, a geometry of dimension two shall mean simply a projective or affine plane. That such geometries do not or may not correspond to such lattices is precisely what makes the study of projective planes intriguing. But, what are some examples, what makes them interesting and what is to be learned from them?

In the early 1960s, when I became interested in the general area of combinatorial geometry and, more particularly, in projective planes, there were very few examples of 'interesting' projective planes, particularly in the finite case. In fact, the then current hot research centered on the Lenz-Barlotti classification of projective planes by the extent of so-called point-line transitivities (see [5]) and the possible determination of examples in the various classes or the nonexistence of the same.

When I met Ted Ostrom in 1964, he was creating, to my mind, the most stunning, deep and, at the same time, simple examples of finite projective planes that had ever been seen.

In the early 1960s, T.G. Ostrom created the concept of the 'derivation of an affine plane'. We will get to the precise definition but, basically, this is a procedure by which one finite affine plane is transformed into another by a renaming of what are called 'lines'. Some of the lines of the new plane are subplanes of the old plane.

About this time, two striking results emerged. I might actually use the term 'shocking results' as this seemed to be the mood at the time.

The famous Hall planes (see [26]) had been constructed many years earlier by Marshall Hall, Jr., who distorted the multiplication of a finite field to create a coordinate structure for an affine plane which was remarkably different from the original plane coordinatized by the associated field. But it was A.A. Albert ([1]) who was first able to use the derivation procedure effectively and showed that the Hall planes may be realized from Desarguesian planes (field planes) by the Ostrom derivation process.

D.R. Hughes [27] had previously constructed a class of finite projective planes containing no point-line transitivities. These were marvelously interesting in their own right, but that they could be seen to be 'derivable' was an unbelievably propitious circumstance, since they 'derived' a class of finite projective planes admitting exactly one incident point-line transitivity—a completely 'new' class. Both Ostrom and G. Rosati were working on these planes independently, so these are called the Ostrom-Rosati planes (see [63], [69]).

I remember talking to Dan Hughes about this during a conference held at University of Illinois–Chicago Circle during the summer of 1967, where it seemed that the important open problem on derivation was to determine conditions for a plane to be derivable. But, Hughes mentioned that it was much more interesting and important to ask whether this technique was 'geometric' in any sense of the word. He kept insisting that he knew he was missing something; he just couldn't put his finger on it, so to speak.

I guess you could say that the motivation to write this book came from years of trying to put a metaphorical finger on the construction technique of derivation.

Perhaps the most productive study of derivable affine planes came from the simple idea of separating the derivable net from the affine plane which contained it. Now we have a net which is covered by affine Baer subplanes whose parallel classes coincide with the parallel classes of the net. Of course, from this point of view, there is nothing special about having Baer subplanes as opposed to simply having 'subplanes' whose parallel classes coincide with the parallel classes of the net. That is, we may consider 'subplane covered nets'.

In this monograph, we attempt to provide a completely self-contained account of the beautiful geometry that envelops the derivation process and the analysis of subplane covered nets.

In fact, the intuition of Hughes was correct; derivation is a geometric process and this book is an attempt to explain how this is so and how to generalize this explanation to understand completely the nature of subplane covered nets.

The ideas encountered in this monograph are amalgamations of ideas of T.G. Ostrom, A.A. Albert, D.R. Hughes, G. Rosati, J. Cofman, A. Barlotti, D.A. Foulser, R.C. Bose, R.H. Bruck, M. Hall, J.A. Thas, F. De Clerck, V. Jha, P.J. Cameron, N. Knarr, A. Bruen, J.C. Fisher, G. Lunardon, M. Walker, H. Lüneburg, M. Biliotti, and T. Grundhöfer, to mention a few who contributed to this area and related disciplines.

While I am enormously indebted to each of these mathematicians for their insights, these ideas have also been twisted and interwoven to fit into my scheme of doing things, so I accept full responsibility for any distortions.

I would like to thank Brian Treadway for programs that created all of the diagrams found in this book and for all the varied and many things he did in assembling the text.

I am most indebted to my wife who, after years of patiently listening to me 'spin' about geometries, still manages to provide continuous and unfailing support.

I dedicate this book to my wife, Bonnie L. Hemenover, with gratitude and love.

Norman L. Johnson

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Chapter 1

A BRIEF OVERVIEW

The reader unfamiliar with the terms used in this small chapter might like to skip this initially and go on to read the chapters on projective geometries and beginning derivation before returning to read this for a preview of what the book is all about.

Let π be a finite affine plane of over q^2. An affine subplane of order q is said to be a 'Baer subplane'. Let D denote a set of $q+1$ parallel classes such that for any two distinct affine points P and Q, such that the line $\ell_{P,Q}$ joining them is an element of one of the parallel classes of D, there exists a Baer subplane $\pi_{P,Q}$ that contains P and Q such that the set of $q+1$ parallel classes of $\pi_{P,Q}$ are those of D. In this situation, we call π a 'derivable' affine plane. Furthermore, D is called a 'derivation set'.

In the early $60's$, T.G. Ostrom ([62], [61]) realized that when such a set D exists, a potentially new affine plane $\pi(D)$ may be constructed from π in the following manner:

The 'points' of $\pi(D)$ are the points of π and the 'lines' of $\pi(D)$ are the lines of π which are not in a class of D together with the Baer subplanes $\pi_{P,Q}$.

A.A. Albert [1], showed that the derivation process applies to affine planes 'coordinatized' by finite fields K of order q^2. Considering the affine plane as an analogue to the real affine plane, points are elements (x, y) of $K \times K$ and lines are given by equations $y = xm + b$ and $x = c$ where juxtaposition denotes multiplication in K for all $m, b, c \in K$. The set D consists of the slopes (α) such that α is in the unique subfield of K of q

elements together with the parallel class (∞) containing the lines $x = c$ for $c \in K$.

Previously, in 1943, Marshall Hall Jr. had constructed an interesting and important class of affine planes by altering the multiplication in a field K and using this new coordinate structure to create the plane.

The planes derived from the field planes, which we call Desarguesian planes, turn out to be the planes of Hall.

There were a number of striking results concerning derivation proved in the early $'60's$ but, for us here, the main questions involve the nature of an affine plane which admits a derivation set and the nature of the substructure, called a 'derivable net', which contains the Baer subplanes that become 'new' lines of the 'new' plane.

What makes an affine plane derivable?

How can a derivation set be recognized?

Of course, we want to address the fundamental question:

Is derivation a geometric process?

Considering the last question, one wonders what it means to be geometric. Perhaps a reasonable definition might be that something is 'geometric' if it is closely connected to some property of an affine or projective geometry.

In fact, it turns out that, taking the above definition, derivation can be viewed geometrically.

In order for the reader to understand fully how to see this, we provide chapters on projective and affine geometries and a few results on projective and affine planes. Coordinatization is considered so as to better understand how to recognize a derivable affine plane and for use in later chapters on direct product nets and their reconstruction.

After focusing initially on the affine plane containing a derivable net, we consider the structure of the net itself. Hence, for such an analysis, we separate the net from the plane and study derivable nets themselves. At this point, we show how to associate a projective geometry with a derivable net. Furthermore, we discuss embedding and extension problems.

For example:

Is it possible to embed a derivable net into a projective space and what do embeddings imply about the structure of the net?

If a derivable net is structurally determined without the ambient affine plane containing it, must the net necessarily be ex-

tendible to an affine plane? Furthermore, if a net can be so extended, how many affine planes extend it?

Although we initially considered finite affine planes, there is really no good reason to do so apart from the intuition that is obtained. So, we consider general or arbitrary affine planes that could be derivable and arbitrary derivable nets and ask of their structure. For the arbitrary subplane situations, we require possibly infinite dimensional vector spaces over skewfields (structures that do not have necessarily commutative multiplication but otherwise have the properties of a field).

Thus, we study projective geometry, beginning derivation and ask what are some classes of derivable affine planes. To provide some examples and for reference to subsequent research discussions, we consider 'translation planes' and their duals. It will be seen that many dual translation planes automatically become derivable.

The ideas of Cofman [19] figure prominently into the relationship with derivable nets and affine geometry and these shall be fully developed.

The connection between derivable nets and projective geometry is the heart of the book although it is not the most general result that we shall prove. A derivable net is a net which is covered by subplanes of a certain type that we called 'Baer subplanes'. A natural generalization is the concept of a 'subplane covered net'.

Definition 1.1 *A 'subplane covered net' is a net with parallel class set $\mathcal{D}$ such that given any two distinct points P and Q that are incident with a line of the net, there is an affine subplane $\pi_{P,Q}$ containing P and Q whose parallel classes are exactly those of $\mathcal{D}$.*

All of the previous questions now are appropriate for subplane covered nets; **can they be embedded into a projective geometry, are there extensions to affine planes?**

The most important result in the book and the justification for its existence is the complete classification of subplane covered nets which establishes close connections with projective geometries and this shall be given in detail.

It is important to point out that herein lie various connections and applications to many types of finite geometries, although the results provided do make any assumptions about finiteness.

When Ostrom was working on finite derivable affine planes, Bruck [14], [15] had previously been giving the foundational studies for finite nets. In particular, extension problems were extremely important. Bruck was able to show that once a net has more than a certain 'critical' number of parallel classes, there is a unique extension to an affine plane provided there is an extension at all. The 'critical' number turns out to be exactly the number of parallel classes in a finite derivable affine plane minus the number of parallel classes in a derivation set; that is, if the order is q^2, the critical number is $q^2 - q$.

Ostrom [64] was able to show that any finite net of order q^2 with $q^2 - q$ parallel classes has at most two extensions to affine planes and, if two, they are related by the derivation process; one is the derived plane of the other.

The work of Bruck was influenced by the work of R.C. Bose [12] on partial geometries which was very graph theoretic. The ideas of extension came from the formulation of certain 'cliques' of the graph associated with a finite net.

In certain situations, it is convenient to study 'dual nets' as their point-line properties become phrased more closely to the point-line properties of affine or projective geometries.

J.A. Thas and F. De Clerck ([21], [75]) have provided basic and important contributions to both (finite) partial and semi-partial geometries concerning both embedding questions and characterization results. Certain finite dual nets satisfying a point-line property called the axiom of Pasch are, perhaps in somewhat disguised form, the duals of subplane covered nets.

Our treatment and characterization of subplane covered nets in the general case was influenced both by the work of Cofman on an associated affine geometry of three dimensions and the work of Thas and De Clerck on dual nets in the finite case.

The geometries associated with derivable nets turn out to be the projective geometries and the derivation process becomes a natural generalization of duality of a projective space. So, 'derivation' becomes a geometric process. It is one of the goals of this monograph to show how this is so.

The study of derivable nets which may be embedded in a three dimensional projective space leads to the analysis of subplane covered nets which also have a similar embedding into projective space although the dimension in this case can be infinite.

There are many affine planes which are covered by derivable nets or more generally covered by subplane covered nets in various ways. If an affine plane is covered by subplane covered nets, there is an associated projective space and hence an associated vector space and, when this occurs, there is probable cause to believe that the affine plane is a translation plane. This idea will be explored after the classification theorem for subplane covered nets has been given.

When an affine plane which is covered by subplane covered nets is, in fact, a translation plane, there are often other associated geometries. For example, translation planes which are covered by subplane covered nets defined by reguli in $PG(3,K)$ all of which share a common line correspond to flocks of quadratic cones in a three dimensional projective space. Furthermore, such translation planes are also associated with certain generalized quadrangles. Translation planes which are covered by derivable nets that share two common lines are connected with flocks of certain Minkowski planes and when the derivable nets are defined by reguli in $PG(3,K)$, the Minkowski plane is classical.

There are other connections with partial geometries and affine planes which are subplane covered. Note that the translation planes associated with the above mentioned geometries may be derived using one of the derivable nets involved in the covering. In this case, where the derivable nets initially share a common line, there is now a system of derivable nets sharing a Baer subplane of the derived plane so there is not a covering in the usual sense. However, translation planes admitting such a generalized cover correspond to what might be called 'partial flocks' of deficiency one in that such planes correspond to partial flocks 'missing' one conic.

We also consider the 'direct product' of two Desarguesian affine planes and show, under an assumption as to how the two sets of parallel classes line up, that such a direct product always produces a derivable net. This leads to the consideration of sets of such direct products which then turns to the study of 'parallelisms' in projective spaces. Recalling that a 'spread' in the three-dimensional projective space $PG(3,q)$ is a set of q^2+1 lines that covers the point set, a 'parallelism' is a set of $1+q+q^2$ spreads which covers the line set. In a sense which shall be made clear, a parallelism in $PG(3,q)$ leads to a spread in $PG(7,q)$ by the notion of direct products of affine planes. As we are emphasizing general theory concerning derivable nets, we, therefore, consider parallelisms in $PG(3,K)$ where K is a skewfield. There is a wonderful connecting theory where particular families of direct products of affine Desarguesian planes with spreads in $PG(3,K)$ are equivalent to certain spreads in $PG(7,K)$. However, since K may not be commutative, special care must be taken and, in fact, it is only true that 'left parallelisms' in $PG(3,K)$ produce 'right spreads' in $PG(7,K)$.

We have noted that there is a rich geometry associated with nets which are subplane covered. When there is at least one subplane of a net, one might consider what, if anything, can be determined as to the structure of the net. Since we are also interested in nets which can be embedded into translation planes and also derivable nets, we consider vector space nets which admit Baer subplanes. Here there is also a nice structure theory and when there are at least three Baer and Desarguesian subplanes, the net turns out to be a derivable net.

We have noted above connections to flocks of quadric sets with spreads which are unions of reguli. But, derivable nets, it is shown, do not always correspond to reguli but do, in fact, correspond to close cousins, the 'pseudo-regulus' nets. We generalize these ideas of flock spreads by the consideration of spreads which are unions of 'pseudo-reguli'. Furthermore, partial flocks, particularly of deficiency one, may be considered which entails an analysis of vector space nets admitting a Baer subplane that is pointwise fixed by particular groups (Baer groups).

So, we consider the structure and nature of vector space nets containing one or more Baer subplanes and their Baer groups. All of this leads to a complete theory of the spreads which correspond to partial flocks of quadric sets of deficiency one.

Thus, the major theme running through the text is how the existence of subplanes of a net or affine plane tend to determine the geometry and we formulate a reasonably complete theory in two major situations when the subplane(s) happens to be Baer or when there is a complete covering by subplanes.

Chapter 2

PROJECTIVE GEOMETRIES

Prerequisites and Background

This book is intended to be self-contained in the sense that only a knowledge of beginning group theory and linear algebra is required. However, an acquaintance with infinite dimensional vector spaces over skewfields is assumed.

There are, of course, a few theorems which we shall not prove.

At one point, we shall require the theorems of Artin-Zorn and the Bruck-Kleinfeld/Skornyakov/San Soucie theorem. A good reference for the first theorem is Hall [26] pp. 375-382.

Theorem 2.1 *(The Theorem of Artin-Zorn). A finite alternative division ring is a field.*

Theorem 2.2 *(Bruck-Kleinfeld/Skornyakov/San Soucie [16], [71], [70]).*

A projective plane which is a translation plane with respect to all lines incident with a given point is a translation plane with respect to all lines; the plane is a Moufang plane.

We shall be interested in affine planes which are covered by 'subplane covered nets' and shall require results of Gleason, Lüneburg and Kegel which we shall state without proof. We define both affine and projective Fano planes and planes which satisfy the little Reidemeister condition later in the book.

Theorem 2.3 *(Gleason [24]). A finite Fano plane is Desarguesian.*

We also require a technical result of Gleason which we shall give in the appropriate chapter.

Theorem 2.4 *(Gleason [24], Lüneburg [60], Kegel and Lüneburg [55]).*
A finite projective plane is Desarguesian if and only if it satisfies the little Reidemeister condition.

In the few occasions where a complete proof is not given, a sketch is provided from which the reader could easily fill-in the details. There are places in the text where a proof by the Klein quadric would provide an alternative proof which may be more elegant than the arguments given. However, we have chosen to provide proofs which are more directly related to derivable nets by coordinate and vector space methods and are thus able to avoid the use of the technical results and methods involved in the use of the Klein quadric.

Notation and conventions

We have adopted the convention of normally referring to 'points' by capital letters P, Q, R etc. and normally referring to 'lines' by small letters a, b, c etc. or by the symbol ℓ. Furthermore, ℓ_∞ shall be used, in particular, to denote the 'line at infinity' of an affine plane which is then equivalent to the 'set of parallel classes'. In addition, we shall use small Greek letters α, β etc. to denote parallel classes of affine planes with the exception that normally we reserve π to denote an affine plane and π^E the extension to the associated projective plane. In an affine plane, the unique line of the parallel class α incident with the (affine) point P is denoted by αP.

We shall normally use capital Greek letters Σ, Π to denote projective spaces.

Furthermore, we shall use the notation PQ to denote a line between two points P and Q of an affine or projective plane or a net when the points are collinear and use $a \cap b$ to denote the point of intersection of two lines a

and b of a projective plane or of an affine plane or a net when there is an intersection.

Normally, sets of points, lines etc., will be given calligraphic lettering $\mathcal{P}$, $\mathcal{L}$, etc. We shall employ the symbol '$\simeq$' to denote isomorphic structures.

We shall use '0' to denote both a zero vector or zero of a skewfield where context indicates the precise meaning. Occasionally, we shall also use the symbol 'O' to denote a zero vector.

Finally, we denote the end of a proof by the symbol $\square$.

In this chapter, we provide some of the fundamental background on projective and affine geometries required for the analysis of derivation and subplane covered nets and discuss briefly projective and affine planes.

2.1 Projective and Affine Geometries.

Definition 2.5 *Projective Geometry.*

Let V be a vector space over a skewfield K. We note that V need not be finite dimensional.

The 'projective geometry' $PG(V,K)$ is defined as the lattice of nonzero vector subspaces of V.

We shall use the terminology of 'points' for 1-dimensional vector subspaces, 'lines' for 2-dimensional vector subspaces, and 'planes' for 3-dimensional vector subspaces.

More generally, an 'i-dimensional projective subspace' shall mean a $i+1$-dimensional vector subspace.

In the case when V has finite dimension n, we shall use the notation $PG(n-1,K)$ interchangeably with $PG(V,K)$ and call $PG(V,K)$ the '$(n-1)$-dimensional projective geometry over K'.

If K is a finite field with q elements and V has finite dimension n, we shall use the notation $PG(n-1,q)$ to denote $PG(V,K)$ provided the specific field K is not important to the discussion.

We call $PG(2,K)$ the 'projective plane over K' and $PG(1,K)$ the 'projective line over K'. For historical reasons which we shall discuss in the chapter on Desarguesian Planes, we also refer to $PG(2,K)$ as the 'Desarguesian projective plane over the skewfield K'.

When it becomes important, we shall specify whether the vector space is to be a 'left' or a 'right' vector space. Although this is not important for *vector spaces over fields, it can be relevant when the vector space is defined over a skewfield.*

Definition 2.6 *Co-Dimension.*

Given any vector subspace W of V, it is possible to choose a basis B_W for W which extends to a basis B_V for V as is well known from linear algebra. The subspace generated by $B_V - B_W$ is called a 'complement' of W.

If a complement of a subspace has dimension k, we shall say that W has 'co-dimension' k.

Equivalently, W has co-dimension k if and only if the quotient space V/W has dimension k.

A subspace of co-dimension 1 is called a 'hyperplane' both in the projective and vector space senses.

Of course, if a vector space has finite dimension n and a subspace has dimension m then the co-dimension of the subspace is $n - m$.

The reason for the introduction of this terminology is that we will be considering possibly infinite dimensional vector spaces where the dimension of a subspace could be infinite whereas the co-dimension could be finite. In particular, we shall be considering subspaces of co-dimension two.

Remark 2.7 *If W is a vector subspace of V, we let $P(W)$ denote the corresponding projective subspace in $PG(V,K)$. For vector subspaces T and M, $T \oplus M$ shall denote the external direct sum.*

We now consider possible intersections.

Proposition 2.8 *(1) Two distinct projective hyperplanes $P(W)$, $P(S)$ intersect in a hyperplane $P(W \cap S)$ of $P(W)$ and/or $P(S)$.*

(2) Any two distinct projective subspaces of the same finite dimension n which are contained in a common projective subspace of dimension $n+1$ intersect in a projective subspace of dimension $n-1$.

(3) Any two distinct projective subspaces of the same finite co-dimension n which are contained in a common projective subspace of co-dimension $n-1$ intersect in a projective subspace of co-dimension $n+1$.

Proof: (1) W and S are vector subspaces of co-dimension 1. This means that the quotient spaces V/W and V/S have dimension 1. Since the subspaces are distinct, there exists a nonzero element $w \in W - S$ and a nonzero element $s \in S - W$. Let $S_1 \oplus \langle s \rangle = S$ and $W_1 \oplus \langle w \rangle = W$ where S_1 and W_1 are hyperplanes of S and W respectively.

Hence, $(W_1 \oplus \langle w \rangle) \oplus s = (S_1 \oplus \langle s \rangle) \oplus \langle w \rangle$ from which it follows that $V = (W \cap S) \oplus \langle w, s \rangle$.

Hence, $V/(W \cap S)$ has dimension 2 and $W \cap S$ has co-dimension 2 with respect to V and clearly co-dimension 1 with respect to either W or S. That is, $W \cap S$ is a hyperplane of W or S. This completes the proof of (1).

Assume the conditions of (2). Let W and S be vector subspaces of dimension n such that $\langle W, S \rangle = T$ has dimension $n + 1$. By the rank-nullity theorem, $\dim W + \dim S - \dim W \cap S = n + 1$ from which it follows that $\dim W \cap S = n - 1$ from which the projective analogue is immediately implied. This proves (2).

Assume the conditions of (3). Let W and S be vector subspaces of co-dimension n such that $\langle W, S \rangle = T$ has co-dimension $n - 1$.

So, $\langle W, S \rangle = W \oplus \langle s \rangle = S \oplus \langle w \rangle$ where $s \in S - W$ and $w \in W - S$. By (1), the co-dimension of $W \cap S$ in $\langle W, S \rangle$ is 2. So, the co-dimension of $W \cap S$ is $(n - 1) + 2$. This proves (3).□

Definition 2.9 *Affine Geometry.*

Let $PG(V, K)$ be a projective geometry. Let W be a vector subspace of V of co-dimension 1 and let $PG(W, K)$ denote the associated projective geometry.

The 'affine geometry' $AG(V, K)$ shall be defined as the deletion of $PG(W, K)$ from $PG(V, K)$ as follows:

The 'points' of $AG(V, K)$ are the points of $PG(V, K) - PG(W, K)$, the 'lines' of $AG(V, K)$ are the lines ℓ of $PG(V, K)$ which do not lie in W minus the point of intersection $\ell \cap W$.

(Note that a line is a 2-dimensional vector subspace and a hyperplane is a co-dimension-1 subspace corresponding to a co-dimension-1 vector subspace W. It follows that $\ell \cap W$ is a 1-dimensional vector subspace or rather a point of the projective space.)

Further, the 'planes' of $AG(V, K)$ are the planes of $PG(V, K)$ which do not lie within $PG(W, K)$ minus the intersection line and the 'subspaces' of

$AG(V,K)$ are the projective subspaces of $PG(V,K)$ minus the intersecting sets of points, lines, planes, etc.

We shall use the notation $AG(n-1,K)$ when $PG(V,K)$ is $PG(n-1,K)$ and $AG(n-1,q)$ when $PG(V,K)$ is $PG(n-1,q)$.

We furthermore call $AG(2,K)$ the 'affine plane over K' and/or the 'Desarguesian affine plane over the skewfield K'.

We refer to $AG(1,K)$ as the 'affine line over K'.

Definition 2.10 *Parallelism in $AG(V,K)$.*

Two hyperplanes of $AG(V,K)$ correspond to hyperplanes of $PG(V,K)$ minus the intersections on a given hyperplane W of $PG(V,K)$. We have noted previously that two distinct hyperplanes intersect in a hyperplane with respect to the subspaces themselves.

Hence, if the intersection relative hyperplane of the two projective subspaces lies within W then they are disjoint within $AG(V,K)$.

We define two hyperplanes of $AG(V,K)$ to be 'parallel' if and only if they are disjoint.

Two affine subspaces are said to be 'parallel' if and only if the projective versions intersect in the same subspace of the hyperplane which is deleted to form the space.

In particular, this means that any two lines are parallel if and only if they lie in a common $AG(2,K)$ and are disjoint.

Furthermore, two spaces are parallel if given any line of either space, there is a line of the remaining space which is parallel to it.

We also allow that any affine subspace is parallel to itself.

Definition 2.11 *Semi-linear and linear group.*

Let V be a left vector space over K. By this, we shall mean that scalar multiplication occurs on the left of the vector. Hence, αx shall denote the scalar multiplication of the vector x by the scalar $\alpha \in K$.

An additive mapping σ is said to be 'semi-linear' over K or 'K-semi-linear' if and only if $\sigma(\alpha x) = \alpha^{\rho}\sigma(x)$ for all $\alpha \in K$ and for all $x \in V$ where ρ is an automorphism of K.

We shall use the terminology that σ is 'K-linear' or simply 'linear' if and only if $\rho = 1$.

The group of all bijective semi-linear additive mappings over K is called the 'general semi-linear group over K' and is referred to by $\Gamma L(V,K)$.

The subgroup of $\Gamma L(V, K)$ consisting of linear mappings is called the 'general linear group over K' and is denoted by $GL(V, K)$.

We note that when V has finite dimension n over the skewfield K then $GL(V, K)$ is isomorphic to the set of non-singular $n \times n$ matrices with entries in K and furthermore $GL(V, K)$ is denoted by $GL(n, K)$. When K has finite cardinality q then $GL(V, K)$ is denoted by $GL(n, q)$.

We note that $GL(V, K)$ is a normal subgroup of $\Gamma L(V, K)$.

Definition 2.12 *When V has finite dimension n, and K is a field, the subgroup of $GL(n, K)$ consisting of the $n \times n$ matrices of determinant 1 is called the 'special linear group over K' and is denoted by $SL(n, K)$.*

It is easy to verify that $SL(V, K)$ is a characteristic subgroup of $GL(V, K)$ so that $SL(V, K)$ is also a normal subgroup of $\Gamma L(V, K)$.

Definition 2.13 *Projective semi-linear group.*

The 'projective semi-linear group' is the group induced on $PG(V, K)$ by $\Gamma L(V, K)$ and is denoted by $P\Gamma L(V, K)$.

Clearly, the projective general semi-linear group is $\Gamma L(V, K)$ modulo the subgroup of $\Gamma L(V, K)$ which leaves invariant each 1-dimensional vector subspace and hence leaves invariant each vector subspace.

Proposition 2.14 *The subgroup of $\Gamma L(V, K)$ which fixes each 1-dimensional subspace is the set of mappings of the form:*

$$g_\delta : x \longmapsto \delta x \, \forall \, \delta \in K - \{0\}.$$

Furthermore, $\mathcal{Z} = \{g_\delta \, ; \, \delta \in K\}$ is a normal subgroup of $\Gamma L(V, K)$.

Proof: Simply note that $\langle x \rangle = \langle \alpha x \rangle$ for any $\alpha \in K - \{0\}$. Hence, any mapping g which fixes each 1-dimensional subspace must map x onto $\delta_x x$ for some $\delta_x \in K$ which possibly depends on x. However, $x + y$ must map onto $\delta_{x+y}(x + y) = \delta_{x+y} x + \delta_{x+y} y$ which must be $\delta_x x + \delta_y y$ since g is additive, so it clearly follows that $\delta_x = \delta$ for all $x \in V$.

Since the indicated group is the kernel of the group induced on $PG(V, K)$, it follows that the subgroup is normal.□

Remark 2.15 *Note that* $\mathcal{Z} \cap GL(V,K) = \{g_\delta \ ; \ \delta \in Z(K)\}$ *where* $Z(K)$ *denotes the center of* K *(the set of elements which commute with all elements of* K*).*

Hence, we may make the following definitions.

Definition 2.16 *The 'projective general semi-linear group'* $P\Gamma L(V,K)$ *is the group* $\Gamma L(V,K)/\mathcal{Z}$*. The 'projective general linear group'* $PGL(V,K)$ *is* $GL(V,K)/(\mathcal{Z} \cap GL(V,K))$*. The 'projective special linear group'* $PSL(V,K)$ *is* $SL(V,K)/(\mathcal{Z} \cap SL(V,K))$*.*

2.2 Projective and Affine Planes.

In the previous section, we have defined projective and affine geometries and furthermore defined projective and affine planes over skewfields as $PG(2,K)$ and $AG(2,K)$ respectively where K is a skew-field.

Remark 2.17 *In* $PG(2,K)$*, every two distinct points are contained in a unique line and every two distinct lines intersect in a unique point.*

In $AG(2,K)$*, every two distinct points are contained in a unique line and every two distinct lines intersect in at most one point.*

Furthermore, two lines of $AG(2,K)$ *are 'parallel' if and only if they are disjoint or equal.*

However, point-line incidence structures satisfying the properties listed in the previous remark are not necessarily projective or affine geometries of the type $PG(2,K)$ or $AG(2,K)$.

Rather than attempt to deal with specific examples in this chapter, we close with two definitions and some fundamentals.

Definition 2.18 *Projective plane.*

A projective plane is a set of 'points' and a set of 'lines' with an incidence relation $\in$ *such that the following three properties are satisfied:*

(1) Any two distinct points are incident with a unique line,

(2) any two distinct lines are incident with a unique point, and

(3) there exist at least four points no three of which lie on a common line.

Definition 2.19 *Affine Plane.*

Let π be a projective plane and let ℓ_∞ denote a specific line of π.

An affine plane $A(\pi)$ is a set of 'points', 'lines' with an incidence relation and defined as follows:

The 'points' of the affine plane are the points of π which are not incident with the line ℓ_∞ and the 'lines' of the affine plane are the lines of π not equal to ℓ_∞ with the corresponding intersection points removed. Incidence in the affine plane is inherited from the incidence relation in π.

Furthermore, the points of ℓ_∞ are called the 'infinite points' and the line itself, the 'line at infinity' of $A(\pi)$.

If there are finitely many points, the projective plane is said to be a 'finite projective plane' and the corresponding affine planes, 'finite affine planes'.

Remark 2.20 *It is straightforward to show that any finite projective plane with $n+1$ points on a given line has $n+1$ points on any line. We say that the finite projective plane has 'order n' in this case.*

A projective plane of order n has then n^2+n+1 points and the same number of lines.

A corresponding affine plane, also said to be of order n, has n^2 points, n^2+n lines, $n+1$ parallel classes and n points per line.

Definition 2.21 *Let π_1 and π_2 denote two projective planes, two affine planes, two projective geometries or two affine geometries. The two structures are said to be 'isomorphic' if and only if there is a bijection σ from the points of π_1 onto the points of π_2 which induces a bijection from the lines of π_1 onto the lines of π_2 that preserves incidence.*

That is, P,Q are points on the line ℓ of π_1 if and only if $\sigma(P),\sigma(Q)$ are points on the line $\sigma(\ell)$ of π_2.

If $\pi_1=\pi_2$, the isomorphism is called a 'collineation'.

The set of all collineations under composition mapping forms a group called the 'collineation group'.

The following is basic to our discussions:

Theorem 2.22 *Let π_i for $i=1,2$ be projective planes and $A(\pi_i)$ for $i=1,2$ the corresponding affine planes obtained by the deletion of lines $\ell_{\infty,i}$ of π_i for $i=1,2$ respectively.*

Then $A(\pi_1) \simeq A(\pi_2)$ if and only if there exists an isomorphism σ from π_1 onto π_2 which maps $\ell_{\infty,1}$ onto $\ell_{\infty,2}$.

Proof: The reader is invited to complete the proof.□

Remark 2.23 *Given an incident point-line pair (P, ℓ), we shall adopt standard usage and say that the point P is 'on' the line ℓ and the line ℓ is 'thru' the point P.*

In the next chapter, we begin the discussion of derivation dealing first with certain derivable planes that fall into particular classes of a classification of projective planes by point-line transitivities due to Lenz and Barlotti (see Barlotti [5]).

Definition 2.24 *If G is a group acting as a permutation group on a set X and $x \in X$, then G_x is the subgroup which fixes x. G is said to be 'transitive' on X if and only if for any pair $x, y \in X$, there exists an element g of G such that $g(x) = y$.*

Definition 2.25 *By a '(P, ℓ)-transitivity' of a projective plane, we shall mean a collineation group that fixes P, all points on ℓ, all lines thru P and acts transitively on the non-fixed points on any line thru P.*

Chapter 3

BEGINNING DERIVATION

The concept of a finite derivable affine plane was conceived by T.G. Ostrom in the early 1960$'s$ (see [61] and [62]) and has been arguably the most important construction procedure of affine planes developed in the last thirty-five years. Certain finite affine planes may be 'derived' to produce other affine planes of the same order. For example, the Hall planes of order q^2 originally constructed by Marshall Hall Jr. [26] by coordinate methods were shown by Albert [1] to be constructible from any Desarguesian affine plane of order q^2 by the method of derivation. The Hughes planes [27] of order q^2 were shown to be derivable and the projective planes constructed were the first examples of finite projective planes of Lenz-Barlotti class II-1 (there is a single, incident, point-line transitivity). The planes obtained were independently discovered by T.G. Ostrom [63] and L.A. Rosati [69] and are called the 'Ostrom-Rosati planes'.

The description of a finite derivable affine plane is as follows:

Definition 3.1 *Let π denote a finite affine plane of order q^2 and let π^E denote the projective extension of π by the adjunction of the set ℓ_∞ of parallel classes as a line.*

Let $\mathcal{D}_\infty$ denote a subset of $q+1$ points of ℓ_∞. $\mathcal{D}_\infty$ is said to be a 'derivation set' if and only it satisfies the following property:

If A and B are any two distinct points of π whose join in π^E intersects ℓ_∞ in $\mathcal{D}_\infty$ then there is an affine subplane $\pi_{A,B}$ of order q containing A and B and whose $q+1$ infinite points are exactly those of $\mathcal{D}_\infty$.

Given any derivation set $\mathcal{D}_\infty$, there is a corresponding set $\mathcal{B}$ of $q^2(q+1)$ affine subplanes of order q each of which has $\mathcal{D}_\infty$ as its set of infinite points.

π is said to be 'derivable' if it contains a derivation set.

The main result of Ostrom is

Theorem 3.2 *(Ostrom [62]). Let π be a finite derivable affine plane of order q^2 with derivation set $\mathcal{D}_\infty$. Let $\mathcal{B}$ denote the associated set of $q^2(q+1)$ subplanes of order q each of which has $\mathcal{D}_\infty$ as its set of infinite points.*

Form the following incidence structure $\pi(\mathcal{D}_\infty)$: The 'points' of $\pi(\mathcal{D}_\infty)$ are the points of π and the 'lines' of $\pi(\mathcal{D}_\infty)$ are the lines of π which do not intersect $\mathcal{D}_\infty$ in the projective extension and the subplanes of $\mathcal{B}$.

Then $\pi(\mathcal{D}_\infty)$ is an affine plane of order q^2.

Proof: To prove this, we need only show that two distinct points are incident with a unique line and two lines either uniquely intersect or are disjoint.

Let P and Q be distinct points of π and let PQ denote the unique line of π containing P and Q. If PQ is not a line of $\mathcal{D}_\infty$ then PQ is also a line of $\pi(\mathcal{D}_\infty)$. If PQ is a line of $\mathcal{D}_\infty$, there is a subplane $\pi_{P,Q}$ containing P and Q whose parallel classes are exactly those of $\mathcal{D}_\infty$. Note that the infinite points together with P and Q generate a 'unique' subplane containing P and Q. Hence, given two distinct points of π, there is a unique line of $\pi(\mathcal{D}_\infty)$ containing the two points.

The lines of $\pi(\mathcal{D}_\infty)$ are of two types; lines of π and the subplanes of $\mathcal{B}$. Two distinct lines must either be disjoint or share a unique point. By the above remark on the generation of subplanes, we need only consider the situation where there is a line of each type. A subplane $\pi_{P,Q}$ and a line ℓ which does not lie in $\mathcal{D}_\infty$ must share at least one common point by a simple counting argument and cannot share more than two points since otherwise ℓ would not appear in the construction process.$\square$

Definition 3.3 *The affine plane $\pi(\mathcal{D}_\infty)$ is called the plane 'derived' from π by the derivation of $\mathcal{D}_\infty$ or merely the 'plane derived from π'.*

We shall now establish that any $AG(2, q^2)$ is a derivable affine plane.

We would like to determine the collineation group of the projective and affine geometries. It is now clear the subgroup which is induced from the