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Detlef Lehmann

Mathematical Methods of Many-Body Quantum Field Theory

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CRC Press UK
Chapman & Hall/CRC Statistics and Mathematics
23 Blades Court
Deodar Road
London SW15 2NU
Tel: 020 8875 4370

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CHAPMAN & HALL/CRC
A CRC Press Company
Boca Raton London New York Washington, D.C.

Library of Congress Cataloging-in-Publication Data

Lehmann, Detlef.
Mathematical methods of many-body quantum field theory / Detlef Lehmann.
p. cm. -- (Chapman & Hall/CRC research notes in mathematics series ; 436)
ISBN 1-58488-490-8 (alk. paper)
1. Many-body problem. 2. Quantum field theory--Mathematical models. I. Title. II. Series.

QC174.17.P7L44 2004
530.14'3--dc22 2004056042

Direct all inquiries to CRC Press LLC, 2000 N.W. Corporate Blvd., Boca Raton, Florida 33431.

International Standard Book Number 1-58488-490-8
Library of Congress Card Number 2004056042
Printed in the United States of America 1 2 3 4 5 6 7 8 9 0
Printed on acid-free paper

Preface

In this book we develop the mathematical tools for the description of quantum many-body systems and apply them to the many-electron system. These are the formalism of second quantization, field theoretical perturbation theory, functional integral methods, bosonic and fermionic, and estimation and resummation techniques for Feynman diagrams. The physical effects discussed in this context are mainly BCS superconductivity, s-wave and higher l-wave, and we take a short look to the fractional quantum Hall effect. A central question of this book is, to what extent the approximations, which are done in the BCS theory of superconductivity, or more generally, in the theory of the weak coupling many-electron system, can be mathematically rigorously justified. Thus the style is mathematical in the sense of working with precise definitions and statements at all times, but, as we hope, close to the physics point of view in that we tried to emphasize actually how to compute things, not just proving that they exist and are well defined.

This book came into being as a combination of lecture notes, handed out to students attending the course Mathematical Physics III at TU Berlin, and the Habilitationsschrift of the author, entitled 'Perturbation Theory for the Many-Electron System with Short-Range Interaction and Its Resummation'. As such, we think that the text may be useful for the following groups of readers. First those who want to pursue the project of trying to mathematically rigorously explore the issue of exactly ximation schemes in this field actually do work, and when they can be shown to break down. Those researchers may find many results of this kind together with a great deal of worked-out detail which should also be useful for approaching similar problems. And second students, who are having trouble figuring out exactly what is going on in one or the other computation while reading tablished physics literature, may find some useful supplemental explanations.

I owe special thanks to my former supervisors, J. Feldman, H. Knoerrer and E. Trubowitz, who taught me field theoretical methods and to my former and current employers, ETH Zürich, IAS Princeton, UBC Vancouver and TU Berlin, for excellent working conditions and their, in part, very generous financial support over the last years.

Berlin, February 2004
Detlef Lehmann

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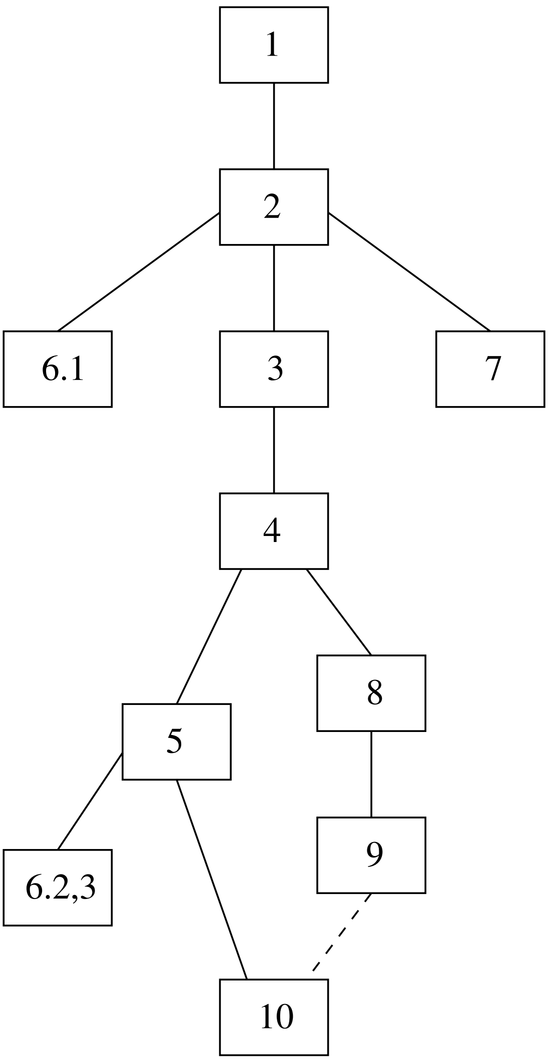

Chapter 1

Introduction

The computation of field theoretical correlation functions is a very difficult problem. These functions encode the physical properties of the model under consideration and therefore it is important to know how these functions behave. As it is the case for many mathematical objects which describe some not too idealized systems, also these functions, in most cases, cannot be computed explicitly. Thus the question arises how these functions can be controlled.

A quantum many-body system is given by a Hamiltonian $H(\lambda) = H_0 + \lambda H_{\rm int}$. Here, usually, the kinetic energy part H_0 is exactly diagonalizable and $H_{\rm int}$ describes the particle-particle, the many-body interaction. There are many situations where it makes sense to consider a small coupling λ. In such a situation it is reasonable to start with perturbation theory. That is, one writes down the Taylor series around $\lambda = 0$ which is the expansion into Feynman diagrams. Typically, some of these diagrams diverge if the cutoffs of the theory are removed. This does not mean that something is wrong with the model, but merely means first of all that the function which has been expanded is not analytic if the cutoffs are removed. The following example may be instructive. Let

$$G_\delta(\lambda) := \int_0^\infty dx \int_0^1 dk \, \tfrac{1}{2\sqrt{k+\lambda x+\delta}}\, e^{-x} \tag{1.1}$$

where $\delta > 0$ is some cutoff and the coupling λ is small and positive. One may think of $\delta = T$, the temperature, or $\delta = 1/L$, L^d being the volume of the system, and G_δ corresponds to some correlation function. By explicit computation

$$G_0(\lambda) = \lim_{\delta\to 0} G_\delta(\lambda) = \int_0^\infty dx\, (\sqrt{1+\lambda x} - \sqrt{\lambda x})\, e^{-x} = 1 + O(\lambda) - O(\sqrt{\lambda}) \tag{1.2}$$

Thus, the $\delta \to 0$ limit is well defined but it is not analytic. This fact has to show up in the Taylor expansion. It reads

$$G_\delta(\lambda) = \sum_{j=0}^{n} \binom{-\frac{1}{2}}{j} \int_0^\infty dx \int_0^1 dk\, \tfrac{x^j e^{-x}}{2(k+\delta)^{j+\frac{1}{2}}}\, \lambda^j \;+\; r_{n+1} \tag{UR}$$

Apparently, all integrals over k diverge for $j \geq 1$ in the limit $\delta \to 0$. Now, the whole problem in field theoretic perturbation theory is to find a rearrangement which reorders the expansion (UR) ('UR' for 'unrenormalized') into a new expansion

$$G_0(\lambda) = \sum_{\ell=0}^{n} \binom{\frac{1}{2}}{\ell} \int_0^\infty dx\, x^\ell\, e^{-x}\, \lambda^\ell \; - c\sqrt{\lambda} + R_{n+1} \tag{R}$$

('R' for 'renormalized') which, in this explicitly solvable example, can be obtained from (1.2) by expanding the $\sqrt{1+\lambda x}$ term. In (R), all coefficients are finite and, for small λ, the lowest order terms are a good approximation since $|R_{n+1}| \le n!\,\lambda^{n+1}$, although the whole series in (R), obtained by letting $n \to \infty$, still has radius of convergence zero. That is, the expansion (R) is asymptotic, the lowest order terms give us information about the behavior of the correlation function, but the expansion (UR) is not, its lowest order terms do not give us any information. The problem is of course that in a typical field theoretic situation we do not know the exact answer (1.2) and then it is not clear how to obtain (R) from (UR). Roughly speaking, this book is about the passage from (UR) to (R) for the many-electron system with short-range interaction which serves as a typical quantum many-body system. Thereby we will develop the standard perturbation theory formalism, derive the fermionic and bosonic functional integral representations, consider approximations like BCS theory, estimate Feynman diagrams and set up the renormalization group framework. In the last chapter we discuss a somewhat novel method which is devoted to the resummation of the nonanalytic parts of a field theoretical perturbation series.

In the first three chapters (2-4) we present the standard perturbation theory formalism, the expansion into Feynman diagrams. We start in chapter 2 with second quantization. In relativistic quantum mechanics this concept is important to describe the creation and destruction of particles. In nonrelativistic many-body theory this is simply a rewriting of the Hamiltonian, a very useful one of course. The perturbation expansion for $\exp\{-\beta(H_0 + \lambda V)\}$ is presented and Wick's theorem is proven. In chapter 4 we introduce anticommuting Grassmann variables and derive the Grassmann integral representations for the correlation functions. Grassmann integrals are a very suitable tool to handle the combinatorics and the rearrangement of fermionic perturbation series.

In the fifth chapter, we use these formulae to write down the bosonic functional integral representations for the correlation functions. These are typically of the form $\int F(\phi)\, e^{-V_{\rm eff}(\phi)} d\phi / \int e^{-V_{\rm eff}(\phi)} d\phi$. Here F depends on the particular correlation function under consideration but the effective potential $V_{\rm eff}$ is fixed once the model is fixed. Usually it is given by a quadratic part minus the logarithm of a functional determinant. In particular, we consider the case of an attractive delta-interaction and we give a rigorous proof that the global minimum of the full effective potential in that case is in fact given by the BCS configuration. This is obtained by estimating the functional determinant as a whole without any expansions and is thus a completely nonperturbative result.

In chapter 6, we discuss BCS theory, the Bardeen-Cooper-Schrieffer theory of superconductivity. Basically the BCS approximation consists of two steps. The interacting part of the full Hamiltonian, which is quartic in the annihilation and creation operators, comes, because of conservation of momentum, with three independent momentum sums. The first step of the approximation consists in putting the total momentum of two incoming electrons equal to zero. The result is a Hamiltonian, which is still quartic in the annihilation and creation operators, but which has only two independent momentum sums. Sometimes this model is called the 'reduced BCS model' but one may also call it the 'quartic BCS model'. The model, which has been solved by Bardeen, Cooper and Schrieffer in 1958 [6] is a quadratic model. It is obtained from the quartic BCS model by substituting the product of two annihilation or creation operators by a number, which is chosen to be the expectation value of these operators with respect to the quadratic Hamiltonian, to be determined selfconsistently. This mean field approximation is the second step of the BCS approximation.

In section 6.2 we show that the quartic BCS model is already explicitly solvable, it is not necessary to make the quadratic mean field approximation. This result follows from the observation that in going from three to two independent momentum sums one changes the volume dependence of the model in such a way that in the bosonic functional integral representation the integration variables are forced to take values at the global minimum of the effective potential in the infinite volume limit. That is, the saddle point approximation becomes exact. Even for the quartic BCS model the effective potential is a complicated function of many variables but with the results of chapter 5 we are able to determine the global minimum which results in explicit expressions for the correlation functions. For an s-wave interaction the results coincide with those of the quadratic mean field formalism, but for higher ℓ-wave interactions this is no longer necessarily the case.

Chapter 7 provides a nice application of the second quantization formalism to the fractional quantum Hall effect. We show that, in a certain long range limit, the interacting many-body Hamiltonian in the lowest Landau level can be exactly diagonalized. However, the long range approximation which is used there has to be considered as unphysical. Nevertheless we think it is worth discussing this approximate model since it has an, in finite volume, explicitly given eigenvalue spectrum which, in the infinite volume limit, most likely has a gap for rational fillings and no gap for irrational fillings. This is interesting since a similar behavior one would like to prove for the original model.

Chapters 8 and 9 are devoted to the rigorous control of perturbation theory in the weak coupling case. These are the most technical chapters. Chapter 8 contains bounds on individual Feynman diagrams whereas chapter 9 estimates sums of diagrams. First it is shown that the value of a diagram depends on its subgraph structure. This is basic for an understanding of renormalization. Then it is shown that, for the many-electron system with short range interaction, an n'th order diagram without two- and four-legged subgraphs allows a $const^n$ bound which is the best possible case. Roughly speaking, one can expect that a sum of diagrams, where each diagram allows a $const^n$ bound, is at least asymptotic. That is, the lowest order terms of such a series would be a good approximation in the weak coupling case and this is all one would like to have. Then it is shown that n'th order diagrams with four-legged subgraphs but without two-legged subgraphs are still finite but they produce $n!$'s. This is bad since, roughly speaking, a sum of such diagrams cannot expected to be asymptotic. That is, the computation of the lowest order terms of such an expansion does not give any information on the behavior of the whole sum. For that reason diagrams without two- and four-legged subgraphs are called 'convergent' diagrams but this does not refer to diagrams with four-legged but without two-legged subgraphs, although the latter ones are also finite. Finally diagrams with two-legged subdiagrams are in general infinite when cutoffs are removed (volume to infinity, temperature to zero).

In the ninth chapter we consider the sum of convergent diagrams. As already mentioned, such a sum can be expected to be asymptotic. More precisely, for a bosonic model one can expect an asymptotic series and for a fermionic model, one may even expect a series with a positive radius of convergence. In fact this is what we prove. We choose a fermionic model which has the same power counting as the many-electron system and show that the sum of convergent diagrams has a positive radius of convergence. The same result has been proven for the many-electron system in two dimensions and can be found in the research literature [18]. For those who wonder at this point how objects like the 'sum of all diagrams without two- and four-legged subgraphs' are treated technically we shortly remark that these sums are generated inductively by integrating out scales in a fermionic functional integral and then at each step Grassmann monomials with two and four ψ's are taken out by hand.

Diagrams with two-legged subdiagrams have to be renormalized. Conceptually, renormalization is nothing else than a rearrangement of the perturbation series. However, due to technical reasons, it may be implemented in different ways. One way of doing this is by the use of counterterms. In this approach one changes the model under consideration. Instead of a model with kinetic energy, say, $e_k = k^2/(2m) - \mu$, μ the chemical potential, one starts with a model with kinetic energy $e_k + \delta e$. The counterterm δe depends on the coupling and may also depend on k. Typically, for problems with an infrared singularity, like the many-electron system, where the singularity is on the Fermi surface $e_k = 0$, the counterterm is a finite quantity. It can be chosen in such a way, that the perturbation series for the altered model with kinetic energy $e(k) + \delta e$ does no longer contain any divergent diagrams. In fact, for the many-electron system with short-range interaction, it can be proven

[18, 20, 16] that, in two dimensions, the renormalized sum of all diagrams without four-legged subgraphs is analytic for sufficiently small coupling. This is true for the model with kinetic energy $e_k = k^2/(2m) - \mu$ which has a round Fermi surface $F = \{k \,|\, e_k = 0\}$ but also holds for models with a more general e_k which may have an anisotropic Fermi surface. Then, the last and the most complicated step in the perturbative analysis consists in adding in the four-legged diagrams. These diagrams determine the physics of the model.

At low temperatures the many-electron system may undergo a phase transition to the superconducting state by the formation of Cooper pairs. Two electrons, with opposite momenta k and $-k$, with an effective interaction which has an attractive part, may form a bound state. Since at small temperatures only those momenta close to the Fermi surface are relevant, the formation of Cooper pairs can be suppressed, if one substitutes (by hand) the energy momentum relation $e_k = k^2/(2m) - \mu$ by a more general expression with an anisotropic Fermi surface. That is, if momentum k is on the Fermi surface, then momentum $-k$ is not on F for almost all k. For such an e_k one can prove that four-legged subdiagrams no longer produce any factorials, an n'th order diagram without two-legged but not necessarily without four-legged subgraphs is bounded by $const^n$. As a result, Feldman, Knörrer and Trubowitz could prove that, in two space dimensions, the renormalized perturbation series for such a model has in fact a small positive radius of convergence and that the momentum distribution $\langle a^+_{k\sigma} a_{k\sigma} \rangle$ has a jump discontinuity across the Fermi surface of size $1 - \delta_\lambda$ where $\delta_\lambda > 0$ can be chosen arbitrarily small if the coupling λ is made small. Because of the latter property this theorem is referred to as the Fermi liquid theorem.

The complete rigorous proof of this fact is a larger technical enterprise [20]. It is distributed over a series of 10 papers with a total volume of 680 pages. J. Feldman has setup a webpage under www.math.ubc.ca/˜feldman/fl.html where all the relevant material can be found. The introductory paper '*A Two Dimensional Fermi Liquid, Part 1: Overview*' gives the precise statement of results and illustrates, in several model computations, the main ingredients and difficulties of the proof.

As FKT remark in that paper, this theorem is still not the complete story. Since two-legged subdiagrams have been renormalized by the addition of a counterterm, the model has been changed. Because e_k has been chosen anisotropic, also the counterterm δe_k is a nontrivial function of k, not just a constant. Thus, one is led to an invertability problem: For given e_k, is there a $\tilde{e}_k$ such that $\tilde{e}_k + \delta\tilde{e}_k = e_k$? If this question is addressed on a rigorous level, it also becomes very difficult. See [28, 55] for the current status. The articles of [28] and [20] add up to one thousand pages.

Another way to get rid of anomalously large or divergent diagrams is to resum them, if this is possible somehow. Typically this leads to integral equations for the correlation functions. The good thing in having integral equations is that the renormalization is done more or less automatically. The correlation functions are obtained from a system of integral equations whose solution can have all kinds of nonanalytic terms (which are responsible for the divergence of the coefficients in the naive perturbation expansion). If one works with counterterms one more or less has to know the answer in advance in order to choose the right counterterms. However, the bad thing with integral equations is that usually it is impossible to get a closed system of equations without making an uncontrolled approximation. If one tries to get an integral equation for a two-point function, one gets an expression with two- and four-point functions. Then, dealing with the four-point function, one obtains an expression with two-, four- and six-point functions and so on. Thus, in order to get a closed system of equations, at some place one is forced to approximate a, say, six-point function by a sum of products of two- and four-point functions.

In the last chapter we present a somewhat novel formalism which allows the resummation of two- and four-legged subdiagrams in a systematic and relatively elegant way which leads to integral equations for the correlation functions. Although this method too does not lead to a complete rigorous control of the correlation functions, we hope that the reader feels like the author who found it quite instructive to see renormalization from this point of view.

Chapter 2

Second Quantization

In this chapter we introduce the many-body Hamiltonian for the N-electron system and rewrite it in terms of annihilation and creation operators. This rewriting is called second quantization. We introduce the canonical and the grand canonical ensemble which is the framework in which quantum statistical mechanics has to be formulated. By considering the ideal Fermi gas, we try to motivate that the grand canonical ensemble may be more practical for computations than the canonical ensemble.

2.1 Coordinate and Momentum Space

Consider one electron in d dimensions in a finite box of size $[0, L]^d$. Its kinetic energy is given by

$$h_0 = \tfrac{\hbar^2}{2m}\Delta \tag{2.1}$$

and its Schrödinger equation $h_0\varphi = \varepsilon\varphi$ is solved by plane waves $\varphi(\mathbf{x}) = e^{i\mathbf{kx}}$. Since we are in a finite box, we have to impose some boundary conditions. Probably the most natural ones are Dirichlet boundary conditions $\varphi(\mathbf{x}) = 0$ on the boundary of $[0, L]^d$ but it is more convenient to choose periodic boundary conditions, $\varphi(\mathbf{x}) = \varphi(\mathbf{x}+L\mathbf{e}_j)$ for all $1 \leq j \leq d$. Hence e^{ik_jL} must be equal to 1 which gives $\mathbf{k} = (k_1, ..., k_d) = \frac{2\pi}{L}(m_1, ..., m_d)$ with $m_j \in \mathbb{Z}$. Thus, a continuous but bounded coordinate space gives a discrete but unbounded momentum space. Similarly, a discrete but unbounded coordinate space gives a continuous but bounded momentum space and a discrete and bounded coordinate space, with a finite number of points, gives a discrete and bounded momentum space with the same number of points.

To write down the Hamiltonian for the many-electron system in second quantized form, we will introduce annihilation and creation operators in coordinate space, $\psi(\mathbf{x})$ and $\psi^+(\mathbf{x})$. Strictly speaking, for a continuous coordinate space, these are operator-valued distributions. To keep the formalism simple, we found it convenient to introduce a small lattice spacing $1/M > 0$ in coordinate space which makes everything finite dimensional. We then derive suitable expressions for the correlation functions in the next chapters and at the very end, the limits lattice spacing to zero and volume to infinity are considered.

Thus, let coordinate space be

$$\begin{aligned}\Gamma &= \left\{\mathbf{x} = \tfrac{1}{M}(n_1, \cdots, n_d) \mid 0 \le n_i \le ML-1\right\} \\ &= \left(\tfrac{1}{M}\mathbb{Z}\right)^d / (L\mathbb{Z})^d \end{aligned} \tag{2.2}$$

Momentum space is given by

$$\begin{aligned}\mathcal{M} := \Gamma^\sharp &= \left\{\mathbf{k} = \tfrac{2\pi}{L}(m_1, \cdots, m_d) \mid 0 \le m_i \le ML-1\right\} \\ &= \left(\tfrac{2\pi}{L}\mathbb{Z}\right)^d / (2\pi M\mathbb{Z})^d \end{aligned} \tag{2.3}$$

such that $0 \le k_j \le 2\pi M$ or $-\pi M \le k_j \le \pi M$ since $-k_j = 2\pi M - k_j$. Removing the cutoffs, one gets

$$\frac{1}{L^d}\sum_{\mathbf{m}} = \frac{1}{(2\pi)^d}\left(\frac{2\pi}{L}\right)^d \sum_{\mathbf{m}} \stackrel{L\to\infty}{\to} \int_{[-\pi M, \pi M]^d} \frac{d^d\mathbf{k}}{(2\pi)^d}\,, \tag{2.4}$$

$$\frac{1}{M^d}\sum_{\mathbf{n}} \stackrel{M\to\infty}{\to} \int_{[-L/2, L/2]^d} d^d\mathbf{x} \tag{2.5}$$

A complete orthonormal system of $L^2(\Gamma) = \mathbb{C}^{N^d}$, $N = ML$, is given by the plane waves

$$\varphi_{\mathbf{k}}(\mathbf{x}) \equiv \varphi_{\mathbf{m}}(\mathbf{n}) = \frac{1}{(ML)^{\frac{d}{2}}}\, e^{i\frac{2\pi}{ML}\sum_{i=0}^d m_i n_i} = \frac{1}{N^{\frac{d}{2}}} e^{2\pi i \frac{\mathbf{mn}}{N}} \tag{2.6}$$

The unitary matrix of discrete Fourier transform is given by $F = (F_{\mathbf{mn}})$ where

$$F_{\mathbf{mn}} = \frac{1}{N^{\frac{d}{2}}} e^{-2\pi i \frac{\mathbf{mn}}{N}} \tag{2.7}$$

One has

$$F^* = F^{-1} = \bar{F} = \begin{pmatrix} & | & \\ \cdots & \varphi_{\mathbf{k}}(\mathbf{x}) & \cdots \\ & | & \end{pmatrix} \tag{2.8}$$

The discretized version of

$$\hat{f}(\mathbf{k}) = \int d^d\mathbf{x}\, e^{-i\mathbf{kx}} f(\mathbf{x})\,, \qquad f(\mathbf{x}) = \int \frac{d^d\mathbf{k}}{(2\pi)^d}\, e^{i\mathbf{kx}} \hat{f}(\mathbf{k}) \tag{2.9}$$

reads in terms of F

$$\begin{aligned}\hat{f}(\mathbf{k}) &= \frac{1}{M^d}\sum_{\mathbf{x}} e^{-i\mathbf{kx}} f(\mathbf{x}) = \left(\frac{L}{M}\right)^{\frac{d}{2}} \sum_{\mathbf{x}} F_{\mathbf{kx}} f(\mathbf{x}) = \left(\frac{L}{M}\right)^{\frac{d}{2}} (Ff)(\mathbf{k}) \\ f(\mathbf{x}) &= \frac{1}{L^d}\sum_{\mathbf{k}} e^{i\mathbf{kx}} \hat{f}(\mathbf{k}) = \left(\frac{M}{L}\right)^{\frac{d}{2}} \sum_{\mathbf{k}} F^*_{\mathbf{xk}} \hat{f}(\mathbf{k}) = \left(\frac{M}{L}\right)^{\frac{d}{2}} (F^*\hat{f})(\mathbf{x}) \end{aligned} \tag{2.10}$$

Derivatives are given by difference operators

$$\begin{aligned}\tfrac{\partial}{\partial x_i} f(\mathbf{x}) &= M\left(f(\tfrac{\mathbf{n}+\mathbf{e}_i}{M}) - f(\tfrac{\mathbf{n}}{M})\right) \\ &= \tfrac{1}{M^d}\sum_{\mathbf{y}} M M^d (\delta_{\mathbf{x}+\frac{\mathbf{e}_i}{M},\mathbf{y}} - \delta_{\mathbf{x},\mathbf{y}}) f(\mathbf{y})\end{aligned} \tag{2.11}$$

$$(\Delta f)(\mathbf{x}) = \tfrac{1}{M^d}\sum_{\mathbf{y}} M^2 M^d \sum_{i=1}^{d} (\delta_{\mathbf{x}+\frac{\mathbf{e}_i}{M},\mathbf{y}} + \delta_{\mathbf{x}-\frac{\mathbf{e}_i}{M},\mathbf{y}} - 2\delta_{\mathbf{x},\mathbf{y}}) f(\mathbf{y}) \tag{2.12}$$

which are diagonalized by F,

$$\tfrac{\partial}{\partial x_j}\varphi_{\mathbf{k}}(\mathbf{x}) = M(e^{2\pi i \frac{m_j}{N}} - 1)\varphi_{\mathbf{k}}(\mathbf{x}) \tag{2.13}$$

which gives

$$\begin{aligned}\left[F \tfrac{1}{i}\tfrac{\partial}{\partial x_j} F^*\right]_{\mathbf{m},\mathbf{m}'} &= M(e^{2\pi i \frac{m_j}{N}} - 1)\,\delta_{\mathbf{m},\mathbf{m}'} \\ &\overset{M\to\infty}{\to} \tfrac{2\pi}{L} m_j\,\delta_{\mathbf{m},\mathbf{m}'} = k_j\,\delta_{\mathbf{k},\mathbf{k}'}\end{aligned} \tag{2.14}$$

$$\begin{aligned}\left[F(-\Delta)F^*\right]_{\mathbf{m},\mathbf{m}'} &= \sum_{i=1}^{d} M^2\left(2 - 2\cos(\tfrac{2\pi m_i}{ML})\right) = \sum_{i=1}^{d} 4M^2 \sin^2\tfrac{\pi m_i}{ML}\,\delta_{\mathbf{m},\mathbf{m}'} \\ &\overset{M\to\infty}{\to} \left(\tfrac{2\pi}{L}\mathbf{m}\right)^2\,\delta_{\mathbf{m},\mathbf{m}'} = \mathbf{k}^2\,\delta_{\mathbf{k},\mathbf{k}'}\end{aligned} \tag{2.15}$$

In the following we will write k_j for the Fourier transform of $\frac{1}{i}\frac{\partial}{\partial x_j}$ instead of writing the exact discretized expressions.

2.2 The Many-Electron System

The N-particle Hamiltonian $H_N : \mathcal{F}_N \to \mathcal{F}_N$ is given by

$$H_N = -\tfrac{1}{2m}\sum_{i=1}^{N}\Delta_{\mathbf{x}_i} + \tfrac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}} V(\mathbf{x}_i - \mathbf{x}_j) \tag{2.16}$$

which acts on the antisymmetric N-particle Fock space

$$\begin{aligned}\mathcal{F}_N = \Big\{ F_N \in L^2\left[(\Gamma\times\{\uparrow,\downarrow\})^N\right] = (\mathbb{C}^{2|\Gamma|})^N \,\Big|\, \forall \pi \in S_n : \\ F_N(\mathbf{x}_{\pi 1}\sigma_{\pi 1}, \cdots, \mathbf{x}_{\pi N}\sigma_{\pi N}) = \mathrm{sign}\pi\, F_N(\mathbf{x}_1\sigma_1, \cdots, \mathbf{x}_N\sigma_N)\Big\}\end{aligned} \tag{2.17}$$

Since we assume a small but positive temperature $T = 1/\beta > 0$, we have to do quantum statistical mechanics. Conceptually, the most natural setting would be the

Canonical Ensemble: An observable has to be represented by some operator $A_N : \mathcal{F}_N \to \mathcal{F}_N$ and measurements correspond to the expectation values

$$\langle A_N\rangle_{\mathcal{F}_N} = \frac{Tr_{\mathcal{F}_N}\, A_N\, e^{-\beta H_N}}{Tr_{\mathcal{F}_N}\, e^{-\beta H_N}} \tag{2.18}$$

Example (The Ideal Fermi Gas): The ideal Fermi gas is given by

$$H_{0,N} = -\tfrac{1}{2m}\sum_{i=1}^{N}\Delta_{\mathbf{x}_i} \tag{2.19}$$

We compute the canonical partition function

$$Q_N := Tr_{\mathcal{F}_N}\, e^{-\beta H_{0,N}} \tag{2.20}$$

To this end introduce an orthonormal basis of $\mathcal{F}_1$ of eigenvectors of $-\Delta$ which is given by the plane waves

$$\Big\{\phi_{\mathbf{k}\sigma}(\mathbf{x}\tau) := \delta_{\sigma\tau}\,\tfrac{1}{L^{\frac{d}{2}}}e^{i\mathbf{k}\mathbf{x}}\;\big|\;(\mathbf{k},\sigma)\in\mathcal{M}\times\{\uparrow,\downarrow\}\Big\} \tag{2.21}$$

where the set of momenta $\mathcal{M}$ is given by (2.3). The scalar product is

$$(\phi_{\mathbf{k}\sigma},\phi_{\mathbf{k}',\sigma'},)_{\mathcal{F}_1} := \tfrac{1}{M^d}\sum_{\mathbf{x}\tau}\phi_{\mathbf{k}\sigma}(\mathbf{x}\tau)\bar{\phi}_{\mathbf{k}'\sigma'}(\mathbf{x}\tau) = \delta_{\sigma,\sigma'}\delta_{\mathbf{k},\mathbf{k}'} \tag{2.22}$$

and we have

$$-\Delta\phi_{\mathbf{k}\sigma} = \varepsilon(\mathbf{k})\phi_{\mathbf{k}\sigma}, \quad \varepsilon(\mathbf{k}) = \sum_{i=1}^{d} 2M^2(1-\cos[k_i/M]) \stackrel{M\to\infty}{\to} \mathbf{k}^2 \tag{2.23}$$

An orthogonal basis of $\mathcal{F}_n$ is given by wedge products or Slater determinants

$$\begin{aligned}\phi_{\mathbf{k}_1\sigma_1}\wedge\cdots\wedge\phi_{\mathbf{k}_n\sigma_n}&(\mathbf{x}_1\tau_1,\cdots,\mathbf{x}_n\tau_n)\\ &:= \tfrac{1}{n!}\sum_{\pi\in S_n}\mathrm{sign}\pi\,\phi_{\mathbf{k}_1\sigma_1}(\mathbf{x}_{\pi1}\tau_{\pi1})\cdots\phi_{\mathbf{k}_n\sigma_n}(\mathbf{x}_{\pi n}\tau_{\pi n})\\ &= \tfrac{1}{n!}\det\left[\phi_{\mathbf{k}_i\sigma_i}(\mathbf{x}_j\tau_j)\right]_{1\le i,j\le n}\end{aligned} \tag{2.24}$$

The orthogonality relation reads

$$\begin{aligned}
&\left(\ \phi_{\mathbf{k}_1\sigma_1}\wedge\cdots\wedge\phi_{\mathbf{k}_n\sigma_n}\ ,\ \phi_{\mathbf{k}'_1\sigma'_1}\wedge\cdots\wedge\phi_{\mathbf{k}'_n\sigma'_n}\ \right)_{\mathcal{F}_N}\\
&=\tfrac{1}{M^{nd}}\sum_{\mathbf{x}_1\tau_1\cdots\mathbf{x}_n\tau_n}\phi_{\mathbf{k}_1\sigma_1}\wedge\cdots\wedge\phi_{\mathbf{k}_n\sigma_n}(\mathbf{x}_1\tau_1,\cdots,\mathbf{x}_n\tau_n)\ \times\\
&\qquad\qquad\qquad\qquad\overline{\phi_{\mathbf{k}'_1\sigma'_1}\wedge\cdots\wedge\phi_{\mathbf{k}'_n\sigma'_n}}(\mathbf{x}_1\tau_1,\cdots,\mathbf{x}_n\tau_n)\\
&=\tfrac{1}{M^{nd}}\sum_{\mathbf{x}_1\tau_1\cdots\mathbf{x}_n\tau_n}\tfrac{1}{n!^2}\sum_{\pi\in S_n}\mathrm{sign}\pi\ \phi_{\mathbf{k}_1\sigma_1}(\mathbf{x}_{\pi1}\tau_{\pi1})\cdots\phi_{\mathbf{k}_n\sigma_n}(\mathbf{x}_{\pi n}\tau_{\pi n})\ \times\\
&\qquad\qquad\qquad\qquad\det\left[\phi_{\mathbf{k}'_i\sigma'_i}(\mathbf{x}_j\tau_j)\right]\\
&=\tfrac{1}{M^{nd}}\sum_{\mathbf{x}_1\tau_1\cdots\mathbf{x}_n\tau_n}\tfrac{1}{n!}\ \phi_{\mathbf{k}_1\sigma_1}(\mathbf{x}_1\tau_1)\cdots\phi_{\mathbf{k}_n\sigma_n}(\mathbf{x}_n\tau_n)\ \det\left[\phi_{\mathbf{k}'_i\sigma'_i}(\mathbf{x}_j\tau_j)\right]\\
&=\tfrac{1}{M^{nd}}\sum_{\mathbf{x}_1\tau_1\cdots\mathbf{x}_n\tau_n}\tfrac{1}{n!}\sum_{\pi\in S_n}\mathrm{sign}\pi\ \phi_{\mathbf{k}_1\sigma_1}(\mathbf{x}_1\tau_1)\cdots\phi_{\mathbf{k}_n\sigma_n}(\mathbf{x}_n\tau_n)\ \times\\
&\qquad\qquad\qquad\qquad\phi_{\mathbf{k}_{\pi1}\sigma_{\pi1}}(\mathbf{x}_1\tau_1)\cdots\phi_{\mathbf{k}_{\pi n}\sigma_{\pi n}}(\mathbf{x}_n\tau_n)\\
&=\tfrac{1}{n!}\ \det\left[(\phi_{\mathbf{k}_i\sigma_i},\phi_{\mathbf{k}'_j\sigma'_j})_{\mathcal{F}_1}\right]\\
&=\tfrac{1}{n!}\begin{cases}\pm1 \text{ if } \{\mathbf{k}_1\sigma_1,\cdots,\mathbf{k}_n\sigma_n\}=\{\mathbf{k}'_1\sigma'_1,\cdots,\mathbf{k}'_n\sigma'_n\}\\ 0\ \ \text{else}\end{cases}
\end{aligned}\tag{2.25}$$

Thus an orthonormal basis of $\mathcal{F}_N$ is given by

$$\left\{\sqrt{N!}\,\phi_{\mathbf{k}_1\sigma_1}\wedge\cdots\wedge\phi_{\mathbf{k}_N\sigma_N}\ \middle|\ \mathbf{k}_1\sigma_1\prec\cdots\prec\mathbf{k}_N\sigma_N,\ (\mathbf{k}_i,\sigma_i)\in\mathcal{M}\times\{\uparrow,\downarrow\}\right\}$$

where $\prec$ is any ordering on $\mathcal{M}\times\{\uparrow,\downarrow\}$. Another way of writing this is

$$\left\{\sqrt{N!}\ \bigwedge_{\mathbf{k}\sigma}(\phi_{\mathbf{k}\sigma})^{n_{\mathbf{k}\sigma}}\ \middle|\ n_{\mathbf{k}\sigma}\in\{0,1\},\ \sum_{\mathbf{k}\sigma}n_{\mathbf{k}\sigma}=N\right\}\tag{2.26}$$

Since

$$-\tfrac{1}{2m}\sum_{i=1}^{N}\Delta_{\mathbf{x}_i}\ \bigwedge_{\mathbf{k}\sigma}(\phi_{\mathbf{k}\sigma})^{n_{\mathbf{k}\sigma}}=\sum_{\mathbf{k}\sigma}n_{\mathbf{k}\sigma}\varepsilon(\mathbf{k})\ \bigwedge_{\mathbf{k}\sigma}(\phi_{\mathbf{k}\sigma})^{n_{\mathbf{k}\sigma}}\tag{2.27}$$

one ends up with

$$Q_N=Tr\,e^{-\beta H_{0,N}}=\sum_{\substack{\{n_{\mathbf{k}\sigma}\}\\ \sum n_{\mathbf{k}\sigma}=N}}e^{-\beta\sum_{\mathbf{k}\sigma}\varepsilon(\mathbf{k})n_{\mathbf{k}\sigma}}\tag{2.28}$$

for the canonical partition function of the ideal Fermi gas. ■

Because of the constraint $\sum n_{\mathbf{k}\sigma} = N$ formula (2.28) cannot be simplified any further. However, if we consider a generating function for the Q_N's which involves a sum over N, we arrive at a more compact expression:

$$
\begin{aligned}
Z(z) := \sum_{N=0}^{\infty} z^N Q_N &= \sum_{N=0}^{\infty} \sum_{\substack{\{n_{\mathbf{k}\sigma}\} \\ \sum n_{\mathbf{k}\sigma}=N}} \prod_{\mathbf{k}\sigma} \left(z\, e^{-\beta\varepsilon(\mathbf{k})}\right)^{n_{\mathbf{k}\sigma}} \\
&= \sum_{\{n_{\mathbf{k}\sigma}\}} \prod_{\mathbf{k}\sigma} \left(z\, e^{-\beta\varepsilon(\mathbf{k})}\right)^{n_{\mathbf{k}\sigma}} = \prod_{\mathbf{k}\sigma} \left[1 + z\, e^{-\beta\varepsilon(\mathbf{k})}\right]
\end{aligned} \tag{2.29}
$$

Thus, from a computational point of view, it is not too practical to have the constraint of a given number of particles, $\sum n_{\mathbf{k}\sigma} = N$. Therefore, instead of using the canonical ensemble one usually computes in the

Grand Canonical Ensemble: Let $\mathcal{F} = \oplus_{N=0}^{\infty} \mathcal{F}_N$, $H = \oplus_{N=0}^{\infty} H_N$. An observable has to be represented by some operator $A = \oplus_{N=0}^{\infty} A_N : \mathcal{F} \to \mathcal{F}$ and measurements correspond to the expectation values

$$
\langle A \rangle_{\mathcal{F}} = \frac{Tr_{\mathcal{F}}\, A\, e^{-\beta(H-\mu\mathbf{N})}}{Tr_{\mathcal{F}}\, e^{-\beta(H-\mu\mathbf{N})}} \tag{2.30}
$$

where the chemical potential μ has to be determined by the condition $\langle \mathbf{N} \rangle = N$, N being the given number of particles and $\mathbf{N}$ the number operator, $\mathbf{N}(1, F_1, F_2, \cdots) := (0, F_1, 2F_2, \cdots)$.

Example (The Ideal Fermi Gas): We compute the chemical potential $\mu = \mu(\beta, N, L) = \mu(\beta, \rho)$ for the ideal Fermi gas with density $\rho = N/L^d$ and we calculate the energy for the ideal Fermi gas.

To this end introduce the 'fugacity' z which is related to μ according to $z = e^{\beta\mu}$. One has

$$
\begin{aligned}
N = \langle \mathbf{N} \rangle_{\mathcal{F}} &= \frac{\sum_{N=0}^{\infty} N z^N Q_N}{\sum_{N=0}^{\infty} z^N Q_N} = \frac{z \frac{d}{dz} Z(z)}{Z(z)} = z \tfrac{d}{dz} \log Z(z) \\
&= z \tfrac{d}{dz} \sum_{\mathbf{k}\sigma} \log \left[1 + z\, e^{-\beta\varepsilon(\mathbf{k})}\right] = 2 \sum_{\mathbf{k}} \frac{z\, e^{-\beta\varepsilon(\mathbf{k})}}{1 + z\, e^{-\beta\varepsilon(\mathbf{k})}} \\
&\approx 2L^d \int \tfrac{d^d\mathbf{k}}{(2\pi)^d} \frac{z\, e^{-\beta\varepsilon(\mathbf{k})}}{1 + z\, e^{-\beta\varepsilon(\mathbf{k})}}
\end{aligned} \tag{2.31}
$$

where we have used (2.4) in the last line and the integral goes over $[-\pi M, \pi M]^d$. Recalling that $z = e^{\beta\mu}$ and introducing

$$
e_{\mathbf{k}} := \varepsilon(\mathbf{k}) - \mu \tag{2.32}
$$

we obtain in the zero temperature limit

$$
N = 2L^d \int \tfrac{d^d\mathbf{k}}{(2\pi)^d} \frac{e^{-\beta e_{\mathbf{k}}}}{1 + e^{-\beta e_{\mathbf{k}}}} \overset{\beta\to\infty}{\to} 2L^d \int \tfrac{d^d\mathbf{k}}{(2\pi)^d}\, \chi(e_{\mathbf{k}} < 0) \tag{2.33}
$$

which determines μ as a function of the density $\rho = N/L^d$. The expectation value of the energy is obtained from $Z(z)$ according to

$$\begin{aligned}\langle H_0\rangle_{\mathcal{F}} &= -\tfrac{d}{d\beta}\log Z + \mu N = -\tfrac{d}{d\beta}2\sum_{\mathbf{k}}\log\left[1+e^{-\beta e_{\mathbf{k}}}\right] + \mu 2\sum_{\mathbf{k}}\frac{e^{-\beta e_{\mathbf{k}}}}{1+e^{-\beta e_{\mathbf{k}}}} \\ &= 2\sum_{\mathbf{k}}\varepsilon(\mathbf{k})\,\frac{e^{-\beta e_{\mathbf{k}}}}{1+e^{-\beta e_{\mathbf{k}}}} \approx 2L^d\int\tfrac{d^d\mathbf{k}}{(2\pi)^d}\,\varepsilon(\mathbf{k})\,\chi\big(\varepsilon(\mathbf{k})<\mu\big) \end{aligned}\tag{2.34}$$

and the last approximation holds for large volume and small temperature. ■

2.3 Annihilation and Creation Operators

2.3.1 Coordinate Space

Let $\alpha \in \{\uparrow,\downarrow\}$ be a spin index and let

$$\delta_{\mathbf{x}\alpha}(\mathbf{x}'\alpha') := \delta_{\alpha,\alpha'}\,M^d\delta_{\mathbf{x},\mathbf{x}'} \tag{2.35}$$

For $F_N \in \mathcal{F}_N$ the wedge product $\delta_{\mathbf{x}\alpha}\wedge F_N \in \mathcal{F}_{N+1}$ is defined by

$$\begin{aligned}&(\delta_{\mathbf{x}\alpha}\wedge F_N)(\mathbf{x}_1\alpha_1,\cdots,\mathbf{x}_{N+1}\alpha_{N+1}) := \\ &\tfrac{1}{N+1}\sum_{i=1}^{N+1}(-1)^{i-1}\delta_{\mathbf{x}\alpha}(\mathbf{x}_i\alpha_i)\,F_N(\mathbf{x}_1\alpha_1,\cdots,\widehat{\mathbf{x}_i\alpha_i},\cdots,\mathbf{x}_{N+1}\alpha_{N+1})\end{aligned}\tag{2.36}$$

Then the creation operator at $\mathbf{x}$ is defined by $\psi^+(\mathbf{x}\alpha) : \mathcal{F}_N \to \mathcal{F}_{N+1}$,

$$\psi^+(\mathbf{x}\alpha)F_N := \sqrt{N+1}\,\delta_{\mathbf{x}\alpha}\wedge F_N \tag{2.37}$$

and the annihilation operator $\psi(\mathbf{x}\alpha) : \mathcal{F}_{N+1} \to \mathcal{F}_N$ is defined by the adjoint of ψ^+, $\psi(\mathbf{x}\alpha) = [\psi^+(\mathbf{x}\alpha)]^*$.

Lemma 2.3.1 **(i)** *The adjoint operator $\psi(\mathbf{x}\alpha) : \mathcal{F}_{N+1} \to \mathcal{F}_N$ is given by*

$$\left(\psi(\mathbf{x}\alpha)F_{N+1}\right)(\mathbf{x}_1\alpha_1,\cdots,\mathbf{x}_N\alpha_N) = \sqrt{N+1}\,F_{N+1}(\mathbf{x}\alpha,\mathbf{x}_1\alpha_1,\cdots,\mathbf{x}_N\alpha_N) \tag{2.38}$$

(ii) *The following canonical anticommutation relations hold:*

$$\begin{aligned}\{\psi(\mathbf{x}\alpha),\psi(\mathbf{y}\beta)\} = \{\psi^+(\mathbf{x}\alpha),\psi^+(\mathbf{y}\beta)\} = 0\,, \\ \{\psi(\mathbf{x}\alpha),\psi^+(\mathbf{y}\beta)\} = \delta_{\alpha\beta}\,M^d\delta_{\mathbf{x}\mathbf{y}}\,.\end{aligned}\tag{2.39}$$

Proof: (i) We abbreviate $\xi = (\mathbf{x}\alpha)$ and $\eta = (\mathbf{y}\beta)$. One has

$$\begin{aligned}
&\big(F_{N+1}, \psi^+(\xi)G_N\big)_{\mathcal{F}_{N+1}} = \\
&\quad \sum_{\xi_1\cdots\xi_{N+1}} \bar{F}_{N+1}(\xi_1, \cdots, \xi_{N+1})\sqrt{N+1}\,(\delta_\xi \wedge G_N)(\xi_1, \cdots, \xi_{N+1}) \\
&= \tfrac{1}{\sqrt{N+1}} \sum_{\xi_1\cdots\xi_{N+1}} \bar{F}_{N+1}(\xi_1, \cdots, \xi_{N+1}) \times \\
&\qquad\qquad \sum_{i=1}^{N+1} (-1)^{i-1} \delta_\xi(\xi_i) G_N(\xi_1, \cdots, \hat{\xi}_i, \cdots \xi_{N+1}) \\
&= \tfrac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} (-1)^{i-1} \sum_{\xi_1\cdots,\hat{\xi}_i,\cdots\xi_{N+1}} \bar{F}_{N+1}(\xi_1, \cdots, \xi, \cdots, \xi_{N+1}) \times \\
&\qquad\qquad G_N(\xi_1, \cdots, \hat{\xi}_i, \cdots \xi_{N+1}) \\
&= \tfrac{1}{\sqrt{N+1}} \sum_{i=1}^{N+1} \sum_{\eta_1\cdots\eta_N} \bar{F}_{N+1}(\xi, \eta_1, \cdots, \eta_N)\, G_N(\eta_1, \cdots, \eta_N) \\
&= \sqrt{N+1}\,\big(F_{N+1}(\xi, \cdot), G_N(\cdot)\big)_{\mathcal{F}_N} = \big(\psi(\xi)F_{N+1}, G_N\big)_{\mathcal{F}_N}\,. \qquad (2.40)
\end{aligned}$$

(ii) We compute $\{\psi, \psi^+\}$. One has

$$\begin{aligned}
&\big(\psi(\xi)\psi^+(\eta)F_N\big)(\xi_1, \cdots, \xi_N) \;=\; \sqrt{N+1}\;(\psi(\xi)\,\delta_\eta \wedge F_N)(\xi_1, \cdots, \xi_N) \\
&= (N+1)\;(\delta_\eta \wedge F_N)(\xi, \xi_1, \cdots, \xi_N) \qquad (2.41) \\
&= \delta_\eta(\xi)F_N(\xi_1, \cdots, \xi_N) + \sum_{i=1}^{N} (-1)^{(i+1)-1} \delta_\eta(\xi_i) F_N(\xi, \xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_N)
\end{aligned}$$

Since

$$\begin{aligned}
&\big(\psi^+(\eta)\psi(\xi)F_N\big)(\xi_1, \cdots, \xi_N) \;=\; \sqrt{N}\;\big(\psi^+(\eta)F_N(\xi, \cdot)\big)(\xi_1, \cdots, \xi_N) \\
&= N\tfrac{1}{N} \sum_{i=1}^{N} (-1)^{i-1} \delta_\eta(\xi_i) F_N(\xi, \xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_N) \qquad (2.42)
\end{aligned}$$

the lemma follows. ■

In the following theorem we show that the familiar expressions for the Hamiltonian in terms of annihilation and creation operators is just another representation for an N-particle Hamiltonian of quantum mechanics. So although these representations are sometimes referred to as 'second quantization', there is conceptually nothing new. We use the notation $\Gamma_s = \Gamma \times \{\uparrow, \downarrow\}$ ('s' for 'spin') and write $L^2(\Gamma_s) = \mathbb{C}^{|\Gamma_s|}$.

Theorem 2.3.2 **(i)** *Let* $h = (h_{\mathbf{x}\alpha,\mathbf{y}\beta}) : L^2(\Gamma_s) \to L^2(\Gamma_s)$ *(one particle Hamiltonian) and let*

$$H_{0,N} = \sum_{i=1}^{N} h_i \ : \ \mathcal{F}_N \to \mathcal{F}_N \tag{2.43}$$

where

$$\begin{aligned}(h_i F_N)(\mathbf{x}_1\alpha_1, \cdots, \mathbf{x}_N\alpha_N) &= \\ &\tfrac{1}{M^d} \sum_{\mathbf{y}\beta} h(\mathbf{x}_i\alpha_i, \mathbf{y}\beta)\, F_N(\mathbf{x}_1\alpha_1, \cdots, \mathbf{y}\beta, \cdots, \mathbf{x}_N\alpha_N)\end{aligned} \tag{2.44}$$

Then

$$\tfrac{1}{M^{2d}} \sum_{\substack{\mathbf{xy}\\ \alpha\beta}} \psi^+(\mathbf{x}\alpha) h(\mathbf{x}\alpha, \mathbf{y}\beta) \psi(\mathbf{y}\beta)\Big|_{\mathcal{F}_N} = H_{0,N} \tag{2.45}$$

(ii) *Let* $v : \Gamma \to \mathbb{R}$ *and let* $V_N : \mathcal{F}_N \to \mathcal{F}_N$ *be the multiplication operator*

$$(V_N F_N)(\mathbf{x}_1\alpha_1, \cdots, \mathbf{x}_N\alpha_N) = \tfrac{1}{2} \sum_{\substack{i,j=1\\ i\neq j}}^{N} v(\mathbf{x}_i - \mathbf{x}_j) F_N(\mathbf{x}_1\alpha_1, \cdots, \mathbf{x}_N\alpha_N) \tag{2.46}$$

Then

$$\tfrac{1}{2}\tfrac{1}{M^{2d}} \sum_{\substack{\mathbf{xy}\\ \alpha\beta}} \psi^+(\mathbf{x}\alpha)\psi^+(\mathbf{y}\beta)\, v(\mathbf{x}-\mathbf{y})\, \psi(\mathbf{y}\beta)\psi(\mathbf{x}\alpha)\Big|_{\mathcal{F}_N} = V_N \tag{2.47}$$

Proof: We abbreviate again $\xi = (\mathbf{x}\alpha)$ and $\eta = (\mathbf{y}\beta)$. One has

$$\begin{aligned}&\Big(\psi^+(\xi) h(\xi,\eta)\psi(\eta) F_N\Big)(\xi_1, \cdots, \xi_N) \ = \\ &= h(\xi,\eta) \tfrac{1}{\sqrt{N}} \sum_{i=1}^{N} (-1)^{i-1} \delta_\xi(\xi_i)\, (\psi(\eta)F_N)\, (\xi_1, \cdots \hat{\xi}_i \cdots, \xi_N) \\ &= h(\xi,\eta) \sum_{i=1}^{N} \delta_\xi(\xi_i) F_N(\xi_1, \cdots, \eta, \cdots, \xi_N)\end{aligned} \tag{2.48}$$

which gives

$$\begin{aligned}&\tfrac{1}{M^{2d}} \sum_{\xi,\eta} \Big(\psi^+(\xi) h(\xi,\eta)\psi(\eta) F_N\Big)(\xi_1, \cdots, \xi_N) \\ &= \sum_{i=1}^{N} \tfrac{1}{M^{2d}} \sum_{\xi,\eta} \delta_\xi(\xi_i) h(\xi,\eta) F_N(\xi_1, \cdots, \eta, \cdots, \xi_N) \\ &= \sum_{i=1}^{N} \tfrac{1}{M^{d}} \sum_{\eta} h(\xi_i,\eta) F_N(\xi_1, \cdots, \eta, \cdots, \xi_N)\end{aligned}$$

$$= \sum_{i=1}^{N} (h_i F_N)(\xi_1, \cdots, \xi_N) \tag{2.49}$$

This proves part (i). To obtain (ii) observe that because of (2.48) one has

$$\left(\psi^+(\xi)\psi(\xi)F_N\right)(\xi_1, \cdots, \xi_N) = \sum_{i=1}^{N} \delta_\xi(\xi_i)\, F_N(\xi_1, \cdots, \xi_N) \tag{2.50}$$

Since

$$\begin{aligned}
&\psi^{\pm}(\mathbf{x}\alpha)\psi^+(\mathbf{y}\beta)\, v(\mathbf{x}-\mathbf{y})\, \psi(\mathbf{y}\beta)\psi(\mathbf{x}\alpha) \\
&\quad = \psi^+(\mathbf{x}\alpha)\psi(\mathbf{x}\alpha)\, v(\mathbf{x}-\mathbf{y})\, \psi^+(\mathbf{y}\beta)\psi(\mathbf{y}\beta) \\
&\quad\quad - \delta_{\mathbf{x}\alpha}(\mathbf{y}\beta)\, \psi^+(\mathbf{x}\alpha) v(\mathbf{x}-\mathbf{y})\psi(\mathbf{y}\beta)
\end{aligned}$$

one gets, using (2.48) again,

$$\begin{aligned}
&\Big(\psi^+(\xi)\psi^+(\eta)v(\mathbf{x}-\mathbf{y})\, \psi(\eta)\psi(\xi)F_N\Big)(\xi_1, \cdots, \xi_N) \\
&= \sum_{i,j=1}^{N} \delta_\xi(\xi_i)\, \delta_\eta(\xi_j)\, v(\mathbf{x}-\mathbf{y})\, F_N(\xi_1, \cdots, \xi_N) \\
&\qquad - \sum_{i=1}^{N} \delta_\xi(\eta)\delta_\xi(\xi_i)\, v(\mathbf{x}-\mathbf{y})\, F_N(\xi_1, \cdots, \xi_N) \\
&= \sum_{\substack{i,j=1 \\ i\neq j}}^{N} \delta_\xi(\xi_i)\, \delta_\eta(\xi_j)\, v(\mathbf{x}-\mathbf{y})\, F_N(\xi_1, \cdots, \xi_N) \qquad (2.51) \\
&\qquad + \sum_{i=1}^{N} \left[\delta_\xi(\xi_i)\delta_\eta(\xi_i) - \delta_\xi(\eta)\delta_\xi(\xi_i)\right]\, v(\mathbf{x}-\mathbf{y})\, F_N(\xi_1, \cdots, \xi_N)
\end{aligned}$$

Since the terms in the last line cancel part (ii) is proven. ■

2.3.2 Momentum Space

Recall that the plane waves $\phi_{\mathbf{k}\sigma}(\mathbf{x}\tau) = \delta_{\sigma,\tau} L^{-\frac{d}{2}} e^{i\mathbf{kx}}$ are an orthonormal basis of $\mathcal{F}_1$. We define

$$a_{\mathbf{k}\sigma} = \tfrac{1}{M^d} \sum_{\mathbf{x}\tau} L^{\frac{d}{2}} \bar{\phi}_{\mathbf{k}\sigma}(\mathbf{x}\tau)\psi_{\mathbf{x}\tau} = \tfrac{1}{M^d} \sum_{\mathbf{x}} e^{-i\mathbf{kx}} \psi_{\mathbf{x}\sigma} \tag{2.52}$$

$$\Rightarrow\ a^+_{\mathbf{k}\sigma} = \tfrac{1}{M^d} \sum_{\mathbf{x}\tau} L^{\frac{d}{2}} \phi_{\mathbf{k}\sigma}(\mathbf{x}\tau)\psi^+_{\mathbf{x}\tau} = \tfrac{1}{M^d} \sum_{\mathbf{x}} e^{i\mathbf{kx}} \psi^+_{\mathbf{x}\sigma} \tag{2.53}$$

The following corollary follows immediately from the properties of ψ and ψ^+.