OPEN CHANNEL FLOW

Numerical Methods and Computer Applications

Roland Jeppson

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Preface

This book was developed over several years while teaching courses in open channel flow to graduate students. Initially, while on the quarter system, two open channel graduate-level courses were taught: the first dealing with steady-state flow was a four-credit course, including one credit for a laboratory; and the second dealing with unsteady flow, that is, numerical solutions of the St. Venant equations, was a three-credit course. When the university switched to the semester system, these two courses were both expanded into three-credit graduate-level semester courses, and the amount of material covered was essentially that in the current book. Before undertaking the second course, most of the students had also taken a course dealing with numerical methods in engineering. While the material was developed and intended to complement lectures in this subject area, it should also be useful to the practicing engineer. There are numerous example problems throughout the book that elucidate principles, formulate and set up problems, and/or apply techniques of problem solutions. Thus, the book is also intended for self-study for those who have taken courses in fluid mechanics and hydraulics.

The basic principles of conservation of mass, energy, and momentum are emphasized in the hope that this will help students master this important subject rather than just learn routine techniques in solving the large host of open channel applications. In so doing, students will enhance and enlarge their understanding of the fundamental principles of fluid mechanics and apply them in solving complex real problems. This emphasis is accomplished by devoting an entire chapter (Chapter 2) to the energy principle as it applies to open channel flow. Chapter 3 is devoted to the momentum principle, but since energy and conservation of mass have been covered previously, all three principles are used in setting up and solving problems. (Since the principle of conservation of mass is relatively easily implemented in solving open channel problems, a separate chapter is not devoted to it.) Many of these equations are nonlinear, and therefore numerical means for solving them are covered in addition to the open channel hydraulics. Real channels generally do not consist of a single channel of constant size, but rather a series of channels with different sizes and control structures, and/or parallel systems. Therefore, in dealing with these principles, they are applied repeatedly to link the equations together, which must be solved simultaneously to obtain depths, velocities, and flow rates throughout channel systems. Again, numerical means for accomplishing the solution for a system of nonlinear equations are covered, and the techniques for accomplishing such solutions are documented through computer codes and program listings. The progression from a single channel to a multichannel system is a distinguishing feature that sets this book apart from other books on this subject.

Seldom is the flow in real channels uniform, that is, the depth varies with position along the channel. Except near control structures, these variations in depth can be handled as gradually varied flow, that is, the flow is assumed to be one-dimensional, or the dependent variables are only a function of the position along the channel. Such gradually varied flows are described by an ordinary differential equation (ODE), for which closed-form solutions are only possible using very restrictive assumptions and, therefore, seldom apply in practice. The longest chapter in the book, Chapter 4, deals with gradually varied flows. It begins by deriving the general gradually varied flow equation, and documents numerical methods for solving this first-order ODE. Computer codes based on mathematical numerical methods rather than the traditional standard step method for solving a single ODE are provided, and these are then applied to solve a variety of problems associated with upstream and downstream controls, side weirs, etc. Again, as in previous chapters, after the student is thoroughly familiar with how gradually varied flow in single channels can be solved, he or she is shown how to set up and solve gradually varied flows in a system of channels.

This instruction involves numerical methods for solving systems of ordinary differential and nonlinear (with some linear) algebraic equations simultaneously. To assist in the instruction, computer codes and/or computer programs are provided. Following the line-by-line instructions for the computer to follow is in fact a most effective means of learning how such complex problems can be solved. The setting up of simultaneous algebraic equations in Chapters 2 and 3 provides the basis for setting up the equations that govern gradually varied flows in channel systems. The extension is that now, not only are there algebraic equations involved, but also ODEs. Commonly available software packages are not capable of solving combined systems of ODEs and algebraic equations without "add-in." Thus, much of the material in Chapter 4 is not available in other textbooks on this subject.

The variety of these complex problems is almost unlimited, but in an attempt to provide the student with the tools needed to set up and solve a particular problem that he or she may encounter, a large number of example problems are provided as an integral part of the text. The solutions to these example problems generally contain numerous computations and therefore require the use of a computer. Thus, many of the example problems contain computer programs. There are also a large number of homework problems at the end of the chapters. These problems provide the student not only with experience to solve problems somewhat similar to the example problems, but require him or her to also apply the principles to solve problems that expand upon the text material.

The material covered in Chapters 1 through 4 assumes that the channel's geometry is rectangular, trapezoidal, circular, or can be defined by simple parameters such as bottom width, side slope, diameter, etc. Chapter 5, entitled "Common techniques used in practice and controls," describes how the geometry of natural channels can be defined using a table of xy values for its bottom shape, and how quantities such as areas, perimeters, and top widths can be obtained from this data rather than just by solving an equation. It then covers water measurements in open channels, gates, and transitions, and concludes with a section dealing with total least cost design of channels.

The last two chapters, Chapters 6 and 7, deal with unsteady flow. Chapter 6 derives the various forms of unsteady flow equations for one-dimensional flow, that is, the St. Venant equations, and describes their characteristics. By assuming that the difference between the slopes of the energy line and channel bottom is the same, these unsteady flow equations can be solved along characteristic lines; this concept can then be used to obtain solutions to a variety of problems with upstream and downstream controls. Such simplified solutions provide the student with a good understanding of unsteady channel flow, and become almost indispensable in setting up the complete unsteady flow equations for a variety of problems, as described in Chapter 7. Initially, the material also included a follow-on chapter dealing with solutions of the two- and three-dimensional unsteady flow equations, but that has now been deleted. These two- and three-dimensional equations are now only derived.

I would like to express my heartfelt gratitude to Dr. Oulhaj Ahmed at the Institut Agronomique et Véterinaire Hassan II, Rabat, Morocco, who has translated this book into French and used it as course material for years now. He had translated material years ago, and recently updated that translation to include this book. The French version of this book is available at the Institut Agronomique et Véterinaire Hassan II, Rabat, Morocco.

An electronic "User's Manual" is also available on the CRC Press Web site (www.crcpress.com) that contains the solutions to the homework problems that are located at the end of each chapter. To obtain these solutions, please contact the publisher (Taylor & Francis Group).

I wish to thank the many students who have participated in this course and in other courses. The satisfaction and joy associated with teaching are truly enormous, and the enthusiasm of students adds much thereto. It is difficult to think of any profession that is as rewarding as teaching at a university. I sincerely hope that this book will contribute to the important subject of open channel flow and the use of numerical methods in engineering practice.

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GUIDELINES FOR STUDYING THIS BOOK

The details of learning and internalizing a new subject until it becomes an integral part of your working knowledge cannot be described here. Rather, as a university student, or a professional, by now you should know how you learn best. Unless you are genius, it is unlikely that you will learn the subject of open channel flow by just reading, as you would read a newspaper, a magazine such as *National Geographic*, or a novel. Rather you will have to acquire the ability to set up and solve problems. Therefore, in these guidelines, I will suggest objectives and goals that you should set, and establish a means for measuring whether these goals are being systematically met. In other words, you will not really understand the subject of open channel flow unless you are able to set up mathematical equations that properly describe an open channel flow system and then solve those equations. To help in this process, this book contains numerous example problems, and an even larger number of homework problems. Some of these problems are what one might call routine, for example, problems that simply deal with solving a specified equation for a specified unknown variable. Other problems require that you identify the basic principles that apply, and how the proper application of these principles produces the system of equations that needs to be solved simultaneously to get the numerical values that describe how the channel system will perform under given conditions. It is hoped that this course will help you appreciate how the mathematics and mathematical methods that were developed mainly for purely theoretical reasons in your previous courses on linear algebra, calculus, and differential equations, suddenly assume great importance when dealing with engineering problems.

In your previous math courses, practically all of the learning was associated with linear equations and differential equations for which “closed-form” solutions are possible. Real engineering problems are, most frequently, governed by nonlinear and differential equations that must be solved using numerical techniques. This is certainly true for open channel flow. In brief, engineering mathematics comes down, ultimately, to numerical results, for example, numbers that define a problem’s properties, such as depth, velocity, flow rate, force, etc., and the variations of these quantities in space and time. Consequently, much of what you will learn during this course deals with numerical methods. You will find that the material in the appendixes, especially Appendixes B and C, will need to be studied in detail, and fully mastered. Since numerical methods require a large amount of number crunching, beyond what can be practically accomplished by hand, it will be vital that you use a computer to solve many problems. Thus, plan on making your computer a heavily relied upon workhorse in studying this subject. It is for good reason that the title of this book contains not only open channel flow, but also numerical methods and computer applications.

Throughout the text there are listings of computer programs in Fortran, C (or C^{++}), and math applications software such as Mathcad and TK-Solver that are used to solve example problems and to illustrate concepts. A folder on the CD-ROM on the back cover of this book contains MATLAB® programs that accomplish the same tasks of many of the Fortran programs. You might not be thoroughly familiar with these languages and/or applications for obtaining the numerical solutions, but it is relatively easy to gain sufficient understanding of Fortran (or C^{++}) to follow the logic and computations needed to implement solutions. You will discover that it is often much easier to fully understand how principles and equations are used to solve problems by studying these programs than by just reading the text. The listings of these programs will be a very important tool for you to accomplish the goals (as listed below) that you should establish for yourself during the courses you take using this book.

Most of our bodies of knowledge in engineering, and its practice, are based on, relatively, a few fundamental principles and/or laws. In open channel flow, the three underlying principles are (1) conservation of mass, (2) conservation of energy, and (3) conservation of momentum. A thorough and comprehensive understanding of these principles is vital to properly understanding open channel flows and to solving complex channel systems. The proper application of these relatively simple concepts often entails considerable insight; therefore, devote time and effort in understanding these principles completely, and how they are used in setting up the equations that provide solutions to a wide variety of open channel situations. The problems solved in Chapters 1 and 2 use only the first two of these principles and gets you acquainted with equations such as Manning's equation and Chezy's equation, which describe how energy is dissipated due to fluid friction. Chapter 3 adds the momentum principle to your working tools. Chapter 4 and Appendix C devote much space to solving first-order ODEs, because the vast majority of steady-state open channel flows are mathematically described by such equations. The ability you acquire in numerically solving ODEs related to open channel flows will enhance your ability to cope with many other engineering problems, since ODEs are a most important body of knowledge in engineering. The first five chapters assume that flow conditions do not change with time, that is, they deal with the subject of steady-state open channel flow. The last two chapters are devoted to unsteady flows in open channels. Initial conditions, or what the flow consists of at time zero, require that steady-state solutions be obtained. Thus, you will need to have a good understanding of Chapters 1 through 5 before beginning Chapter 6. Since there is a semester's amount of study (or more) in Chapters 1 through 5, to study the material in this book will take at least two semesters of graduate-level course work. In fact, the last time I taught the last two chapters to PhD students in Fluid Mechanics Hydraulics Program at Utah State University, Logan, Utah, it took me two semesters, the second at the request of the students, so they could solve more general unsteady open channel problems.

Now, let us define goals you should set for yourselves. Below, only the goals for the first portion of the book will be outlined, that is, for steady-state flows. After studying this portion of the book, you should be able to set your goals associated with solving unsteady open channel flow problems. (You should repeatedly read these goals and assess your progress in completing them.)

Goal 1: Develop computer software that will solve for any of the variables associated with Manning's equation for (a) rectangular channels, (b) trapezoidal channels, (c) circular channels, and (d) natural, or irregularly shaped, channels. A vital part of this goal is to understand why the Newton method works and how it is implemented to solve for variables that cannot be placed on the left side of the equal sign by manipulating an equation, for example, solving implicit equations.

Goal 2: Develop a similar software that solves the combined Chezy and Chezy-C equations. The simultaneous solution of these two equations can be considered a more fundamentally sound approach to open channel flow than using the empirical Manning's equation. A similar comparison in pipe flow is use of the Darcy–Weisbach equation (with the friction factor therein being a function of the relative roughness of the pipe wall, and the Reynolds number associated with the flow) versus use of the empirical Hazen–Williams equation.

Goal 3: Develop computer software that will solve the energy equation (and energies equate at two different positions) for any of the variables associated therewith for (a) rectangular channels, (b) trapezoidal channels, (c) circular channels, and (d) natural, or irregularly shaped, channels.

Goal 4: Develop computer software that will solve the critical flow equation for any of the variables associated therewith for (a) rectangular channels, (b) trapezoidal channels, (c) circular channels, and (d) natural, or irregularly shaped, channels.

Goal 5: Develop computer software that will solve the momentum equation (and momentum functions equated at two different positions) for any of the variables associated therewith for (a) rectangular channels, (b) trapezoidal channels, (c) circular channels, and (d) natural, or irregularly shaped, channels.

Goal 6: Combine the solution capabilities of Goals 1 through 5 into a single software package that is easy to use by allowing you to select the type of problem you are solving, and what variable(s) is (are) to be solved.

Upon completing these first six goals, you will have developed a program similar to program CHANNEL that is available on the CD-ROM on the back cover of this book. You might wish to use your software to solve the problems given at the end of Chapter 3 "Problems to solve using program CHANNEL."

In addition to, or in conjunction with, the achievement of the above goals you should establish and complete the following goals:

Goal 7: Using the energy equations and/or critical flow equations associated with channel systems that consist of branched and parallel channels, write out the system of equations that describe the flow rates, velocities, and depths throughout the system.

Goal 8: Use linear algebra in combination with the expanded Newton method to solve the system of equations from Goal 7.

Goal 9: Use the momentum principle, in addition to the energy and critical flow equations, to define and solve problems involving branched and parallel channel systems that have controls such as gates that cause hydraulic jumps to occur.

The above goals deal with flow situations in which the depth does not vary with position along the channel, except in the immediate position of control structures, such as gates, intakes, etc., or, in other words, uniform flows occur. Similar goals need to be set to handle gradually varied flows (GVFs), or situations in which depths, velocities, and possibly flow rates vary with position along the channel. These latter types of flows are governed by ODEs, if steady state, or, if unsteady, by partial differential equations.

Goal 10: Become thoroughly familiar with the general ODE that defines GVFs, which allows for lateral inflow/outflow and changing channel size and shape, and how this equation simplifies depending upon the conditions.

Goal 11: When possible solve the GVF equation by numerical integration, otherwise learn and obtain numerical solutions of this equation using techniques designed to solve ODEs, and apply these numerical methods to solving GVF problems in single channels. Part of this goal should be to develop computer software that implements the solutions.

Goal 12: Be able to write out the system of equations that define channel systems that involve both algebraic equations and ODEs.

Goal 13: Solve these combined systems of algebraic equations and ODEs using the Newton method in combination with numerical solutions of ODEs. Again, part of this objective is to develop computer software to obtain the solutions. This later software (or computer program), because of the variety of equations involved, will not be a general program, but will need to be modified to handle different given situations.

Chapter 4 is devoted to helping you achieve Goals 10 through 13. Because the subject of GVF is more complex, and the type of problems more varied, Chapter 4 is longer than Chapters 1 through 3 combined.

Computer Programs with Listings of Code

1 Dimensions, Terminology, and Review of Basic Fluid Mechanics

1.1 INTRODUCTION

Open-channel flow is distinguished from closed-conduit flow by the presence of a free surface, or interface, between two different fluids of different densities. The two most common fluids involved are water and air. The presence of a free surface makes the subject of open-channel flow more complex, and more difficult to compute commonly needed information about the flow, than closed-conduit flow, or pipe flow. In pipe-flow problems, the cross-sectional area of the flow is known to equal the area of the pipe. In open-channel flow, the area depends upon the depth of flow, which is generally unknown, and must be determined as part of the solution processes. Coupling this added complexity with the fact that there are more open-channel flows around us than there are pipe flows, emphasizes the need for engineers, who plan work in water-related fields, to acquire proficiency in open-channel hydraulics. The wide use of computers in engineering practice reduces the need for graphical, table lookup, and other techniques learned by engineers who received their training a decade ago.

Technical fields apply very specific meanings to words. With a knowledge of these meanings it is possible for individuals educated in that field to communicate much more effectively and, with fewer words, convey clear and concise information. Often these words have a more general, less concise, meaning in their general use and, therefore, have a less concise meaning for the public at large. Other words are coined especially for a technical discipline. This chapter introduces the terminology used in open-channel flow. It is important that some terminology be fully mastered to effectively read and understand the remainder of this book, and to converse verbally, or in writing with open-channel hydraulic engineers.

1.2 ONE-, TWO-, AND THREE-DIMENSIONAL FLOWS

The dimensionality of a flow is defined as the number of independent space variables that are needed to describe the flow mathematically. If the variables of a flow change only in the direction of one space variable, e.g., in the direction along the channel x, then the flow is described as **one dimensional**. For such flows, variables such as depth Y and velocity V are only functions of x, i.e., Y(x) and V(x). If the variables of the flow change in two directions, such as the position along the channel x, and the position from the bottom of the channel y, or the position across the channel z, then the flow is described as **two dimensional**. For two-dimensional flows the mathematical notation of variables contains two arguments such as V(x,y) and Y(x,y), or V(x,z) and Y(x,z). If the variables of the flow change in three directions, such as the position along the flow, with the vertical position within the flow, and the horizontal position across the flow, then the flow is **three dimensional**. If a Cartesian coordinate system with axes x, y, and z (note here that lower case y is not the depth of flow, Y) is used, then three-dimensional flows are described mathematically by noting that the variables of the flow are a function of all three of these independent variables, or the velocity, for example, is denoted as V(x,y,z), to indicate that its magnitude varies with respect to x, y, and z in space, and since velocity is a vector, its direction also depends on x, y, and z. If the velocity also changes with time, this additional dependency will be denoted by V(x,y,z,t).

A two-dimensional flow that changes in time would have its velocity described as V(x,y,t) or V(x,z,t), and thus depends upon three independent variables and, from a mathematical point of view, is three dimensional. However, in fluid mechanics such flows are called **two dimensional, unsteady**. A flow with V(x,z) is called **two dimensional, steady**. The notation for the variables of a **one-dimensional, unsteady** flow consists of Y(x,t), V(x,t), etc.

Since the computations needed to solve a one-dimensional problem are much simpler than a two-dimensional steady problem, we wish to define the flow as one dimensional, if assuming this does not deviate too much from reality. The equations describing one dimensional, non-time-dependent or steady-state flows, are either just algebraic equations, or ordinary differential equations, whereas equations needed to describe two-dimensional flows are almost without exception partial differential equations. A one-dimensional flow that does change with respect to time must be described mathematically by partial differential equations also. Furthermore, because of the complex nature of the two- and three-dimensional flow equations that describe real flows, a very small number of closed form solutions to the mathematical boundary value problems governed by these partial differential equations are available for theoretically simple flows. Therefore, it is necessary to resort to approximate numerical methods to solve general two- and three-dimensional flows.

The term **hydraulics of open channel** flow is often used for one-dimensional free surface flow. The assumption made, that allows the flow in an open channel to be defined as one dimensional, is that the average velocity at a cross section can be used, and it is not necessary to be concerned with variations of the velocity with depth, or position across the flow. Based on this assumption the velocity, V(x) at any position along the channel equals the flow rate Q at this section divided by the cross-sectional area. Since the velocity varies from the bottom of a channel to the top at any position, and this velocity distribution may vary across the channel, correction factors are sometimes utilized to provide more accurate values of the kinematic energy per unit weight, and the momentum flux when the average velocity is used in the appropriate formula. However, the assumption is retained that the flow is one dimensional. These correction factors are discussed later in this chapter.

1.3 STEADY VERSUS UNSTEADY FLOW

A fluid flow is steady if none of the variables that can be used to describe the flow change with respect to time. Mathematically, steady flow is described by having partial derivatives of such variables as the depth of flow, the velocity, the cross-sectional area, etc., with respective to time all equal to zero, e.g., $\partial Y/\partial t = 0$, $\partial V/\partial t = 0$, $\partial A/\partial t = 0$.

A fluid flow is unsteady if any of the variables that describe the flow changes with respect to time. Thus a flow is unsteady if the depth at a given position in an open-channel flow changes with respect to time. Mathematically, a flow would be determined to be unsteady if any of the partial derivatives of any variable that describe the flow such as the depth, the velocity, the cross-sectional area, etc., with respect to time is different from zero. Generally, if one variable changes with respect to time all variables do.

Since for steady flows the variables that describe the flow are not a function of time, the independent variable t is not included in describing the flow, mathematically. If the flow is one dimensional in space, then the variables of the flow are only a function of that space variable. Thus the depth varies only as a function of the position variable x, or Y(x). If the flow is unsteady then the dependent variable Y varies with x and t, and this is denoted as Y(x,t). For a three-dimensional time-dependent flow, the depth is a function of the Cartesian coordinate system x,y, and z as well as time, and therefore the depth is denoted mathematically as Y(x,y,z,t).

1.4 UNIFORM VERSUS NONUNIFORM FLOW

A flow is **uniform** if none of the variables that describe the flow vary with respect to position, x along the channel. Therefore, the flow is uniform if the depth, velocity, and cross-sectional area are all constant. When dealing with one-dimensional open-channel hydraulics, uniform flows are not dependent on x. In fact, there is not such a thing as one-, two-, or three-dimensional **uniform flow** since the variables of the flow do not change with respect to x, y, or z. In theory, there could be a time-dependent uniform open-channel flow, but in practice such flows never actually exist. To have a uniform unsteady flow, the depth would have to increase (or decrease) at the same rate throughout the entire length of channel so that at all times neither the depth nor the velocity changes with respect to position, but is constantly changing with time. Mathematically, a flow is uniform if $\partial Y/\partial x = 0$, $\partial V/\partial x = 0$, $\partial A/\partial x = 0$, etc., throughout the flow. Setting partial derivatives with respect to the position x equal to zero, and having this zero occur throughout the flow is synonymous with stating that there is no dependency upon x.

A flow is **nonuniform** if any of the variables of the flow vary from position to position. Thus, in a one-dimensional channel flow, if the depth either increases or decreases from one position to another in a channel, the flow is nonuniform. A nonuniform flow may be one, two, or three dimensional, and may be either steady or unsteady. The flow over the crest of a dam's spillway is nonuniform but steady if the flow rate does not change. If, however, the flow rate is either increasing or decreasing with time then the flow is nonuniform and unsteady.

Nonuniform flows will be further subdivided into **gradually varied**, **rapidly varied**, and **spatially varied**. When the radius of curvature of the streamlines is large, e.g., the streamlines are nearly straight, such that the normal component of acceleration can be ignored, a nonuniform flow will be referred to as **gradually varied**. In a gradually varied open-channel flow the pressure will increase in the vertical direction just as it does in the same fluid for a uniform flow. This variation of pressure with depth is hydrostatic. A **rapidly varied** flow occurs when the change in depth is too rapid to ignore the normal acceleration component of the flow, and the pressure distribution is not hydrostatic.

Another way of looking at the difference between gradually varied and rapidly varied flow is that it is possible to use one-dimensional hydraulic equations for gradually varied flows, but rapidly varied flows are two dimensional, or three dimensional. Because of the complexities involved in solving two- and three-dimensional open-channel flows, often rapidly varied flows are handled by utilizing one-dimensional open-channel equations that are modified by experimental coefficient that account for the deficiencies in the one-dimensional assumption. This utilization of experimental coefficients distinguished **one-dimensional hydraulics** from pure fluid mechanics.

There is no parameter of measurement of the radius of curvature, or other characteristic of the flow with a threshold value that separates a gradually varied from a rapidly varied flow. Rather the distinction is subjective. Almost everyone would agree that the flow over a dam's spillway crest is rapidly varied, whereas the flow in the channel upstream from the dam is gradually varied. Likewise the flow immediately downstream from a sluice gate, where the flow is contracting rapidly from the gate height, is rapidly varied whereas the flow both upstream and further downstream from the gate is gradually varied until the position downstream from the gate where a hydraulic jump occurs, should this be the case. The flow through the hydraulic jump is rapidly varied, again. Whether a flow through a transition between two channels of different sizes is rapidly or gradually varied may be debatable. The classification will depend upon how rapidly the transition changes the channel's cross section, and how accurate it is necessary that the computed results correspond to the actual flow characteristics. An abrupt enlargement or an abrupt contraction will cause a small section of rapidly varied flow to occur. Whereas the solution of one-dimensional hydraulic equations for gradually varied flow may represent an accurate method for solving the variation of depth across a 50 ft long smoothly formed transition. If so, the flow is gradually varied.

Spatially varied flows are those portions of the main channel flow over which either lateral inflow or lateral outflow occurs. Therefore, in a spatially varied flow, the flow rate changes with position along the channel. Strictly speaking, the joining of two channels creates a spatially varied flow for a short distance. Generally, however, spatially varied flows occur where the lateral inflow or outflow is over some length of channel. A side weir that runs parallel to the direction of the channel over which discharge occurs creates a section of spatially varied flow. In this case spatially varied flow has distributed outflow from the channel, and the flow rate decreases in the direction of the main channel flow. Water accumulating from rainfall over a roadway surface and flowing into the gutters along the sides of the roadway causes a spatially varied open-channel flow in the gutters. In this case the channel flow in the gutter has a lateral inflow, and the flow rate increases in the direction of flow. As this gutter flow crosses the grates of a storm drain a spatially varied outflow occurs in the gutter, but in the storm drain that receives the flow from the grates, a spatially varied inflow occurs. If all the gutter flow can enter the storm drain then the channel flow in the gutter terminates at the end of the spatially varied flow.

1.5 PRISMATIC VERSUS NONPRISMATIC CHANNELS

Definitions that are closely associated with uniform and nonuniform flows, but apply to the channel rather than the flow in the channel, are **prismatic** and **nonprismatic** channels. A prismatic channel has the same geometry throughout its length. This may consist of a trapezoidal section, a rectangular section, a circular section, or any other fixed section. If the shape and/or size of the section changes with position along the channel, the channel is referred to as a **nonprismatic** channel. In theory, it is possible for a natural channel created by nature to be prismatic. However, in practice natural channels are nonprismatic.

1.6 SUBCRITICAL, CRITICAL, AND SUPERCRITICAL FLOWS

An open-channel flow is classified according to how the average velocity, V, of the flow compares with the speed, c, of a small amplitude gravity wave in that channel. If V is less in magnitude than c, then the flow is **subcritical**. If V is greater in magnitude than c, then the flow is **supercritical**, and if V = c, then the flow is **critical**. The speed of a small amplitude gravity wave is given by

$$c = \sqrt{\frac{gA}{T}} = \sqrt{gY_d} \tag{1.1}$$

where

- g is the acceleration of gravity
- A is the cross section of the flow
- T is the top width of the flow
- Y_d is the hydraulic depth, A/T

Subcritical flows behave differently from supercritical flow because in a subcritical flow the effect of downstream changes are noted by the fluid and it adjusts in anticipation of that downstream occurrence. Thus, if an obstruction exists in a subcritical flow the depth will gradually increase to the depth needed to pass by the obstruction. This signal that something exists downstream is propagated continuously to the upstream flowing fluid by gravity waves. These gravity waves can travel upstream because the velocity of flow is smaller than their speeds.

In supercritical flows the effect of changes cannot travel upstream because the velocity in the channel exceeds the propagation speed of gravity waves. Therefore, flow does not adjust itself for downstream conditions. For example, if a channel containing a supercritical flow ends abruptly, the depth at the end of the channel at the free overfall will be the same as if the channel had continued. If the flow were subcritical, the depth would decrease toward critical depth at the end of a free overfall, however.

As a consequence of whether gravity waves can move upstream or not, subcritical flows have their control "downstream," whereas supercritical flows are "upstream controlled." An example of both downstream and upstream control exists at a gate in a channel. Upstream from the gate the flow must be subcritical because the gate, which is downstream, controls the depth, velocity, area, etc., of the flow. Downstream from the gate the flow will be supercritical, and the gate determines the magnitude of the variables of the flow. If the gate is lowered, for example, it will decrease the downstream depth while increasing the downstream velocity, and has the opposite effect on the upstream flow.

The Froude number, F_r, is the ratio of the velocity in a channel divided by the speed of propagation, or celerity of a small amplitude gravity wave c, or

$$F_r = \frac{V}{c} = \frac{V}{\sqrt{gA/T}} = \sqrt{\frac{Q^2T}{gA^3}} \tag{1.2}$$

Therefore, the determination of whether a flow is subcritical, critical, or supercritical is commonly accomplished by computing the Froude number of the flow. If this value is less than unity then the flow is subcritical. If the Froude number is exactly equal to one, then the flow is critical, and if the Froude number is larger than unity, then the flow is supercritical. It turns out that the Froude number is also the ratio of inertia to gravity forces. A more in-depth treatment of subcritical and supercritical flows and their associated Froude numbers is given in subsequent chapters.

1.7 TURBULENT VERSUS LAMINAR FLOW

The Reynolds number, or the ratio of inertia to viscous forces, is used to distinguish whether a flow is laminar or turbulent. For open-channel flows the Reynolds number is defined by using the hydraulic radius, R_h, or the cross-sectional area A divided by the wetted perimeter P as the length variable. Thus the Reynolds number is defined by

$$R_e = \frac{VR_h}{\nu} = \frac{Q}{\nu P} = \frac{\rho VR_h}{\mu} \tag{1.3}$$

where

V is the average velocity of the flow
Q is the volumetric flow rate
μ is the absolute viscosity
ν is the kinematic viscosity of the fluid

If the Reynolds number is less than 500, the flow is laminar. Otherwise, the flow is turbulent. Often $4R_h$ is used as the length parameter in Reynolds number for channel flows because this is equivalent to the diameter of a pipe, e.g., the hydraulic radius of a pipe is $R_h = (\pi D^2/4)/(\pi D) = D/4$. When using this latter definition, the numerators on the left side of Equation 1.3 should be multiplied by 4.

Laminar flows are rare in open channels if the fluid is water. Examples of laminar flow might be the sheet flow over the surface of a watershed produced by precipitation, or the lateral flow over the crest of a roadway as it moves toward the side gutter. To have laminar water flow in an open channel, the depth generally has to be very small, in conjunction with a not too large velocity, since the kinematic viscosity of water is about 1.2×10^{-5} ft²/s.

1.8 REVIEW OF BASIC FLUID MECHANICS PRINCIPLES

The rest of this chapter provides basic theory upon which the book is based. The presentation assumes that you are familiar with fluid mechanics, and therefore the remainder of this chapter should be considered a review; but this review is slanted toward open-channel hydraulics. Books dealing with engineering fluid mechanics will contain a more thorough treatment of this subject material.

The application of fluid mechanics in solving engineering problems involves a thorough understanding of the following four related items: (1) Physical properties of fluids, e.g., density, specific weight, viscosity, surface tension, and how these cause pressures to change, resistance to motion, etc. (2) The conservation of mass, or the continuity principle. (3) The conservation of energy and its dissipation into non-recoverable forms. (4) Utilization of momentum fluxes as vector quantities to deal with external forces on fluid in motion. A section for each of these four important subjects follows as the rest of this chapter. The specific application of these subjects to the flow of water in open channels constitutes the remaining chapters of this book. The review in this chapter will introduce the symbols that will be used through the rest of the book, and subsequent chapters are written assuming that you are acquainted with these symbols and their meanings. The problems at the end of this chapter have been listed under four similar headings. Problems given under subsequent headings generally also require an understanding of the principles involved in the previous headings.

1.9 PHYSICAL PROPERTIES OF FLUIDS AND THEIR EFFECTS ON OPEN-CHANNEL FLOWS

In solid mechanics, since the object being dealt with generally stays together its mass or weight is used when dealing with the effects it has on its environment. With fluids, however, total mass or total weight, generally, have no significance since these totals are directly related to the length of time the flow has been occurring. Rather, mass per unit volume or weight per unit volume are used. These quantities are the density ρ and the specific weight γ, or the fluid respectively. In the SI (International System of Units) the density ρ is given in kilograms per cubic meter, e.g., ρ (kg/m^3). In ES units (English System of Units) the density is in slugs per cubic foot, ρ (Slug/ft^3).

Specific weight γ is related to density through Newton's second law of motion,

$$\text{Force} = \text{Mass} \times \text{Acceleration}.$$

Weight is a force due to resisting gravity, and therefore $\gamma = \rho g$ in which g is the acceleration of gravity and equals 32.2 fps^2 in ES units or equals 9.81 m/s^2 when using SI units. (With more digits of precision g = 32.174049 fps^2 in ES units, and g = 9.80685 m/s^2 in SI units.) In SI units the specific weight is generally given as kilonewtons per cubic meter, e.g., γ (kN/m^3) and in ES units γ is given in pounds per cubic foot, e.g., γ (lb/ft^3). The density and specific weight for water vary moderately with its temperature as shown in Table 1.1.

The weight of a fluid causes pressure to increase with depth. If z is taken as the vertical coordinate, positive upward against gravity, from a selected datum and there is no motion in the fluid, then the pressure varies according to the hydrostatic law,

$$\frac{dp}{dz} = -\gamma = -\rho g \tag{1.4}$$

and if the fluid is incompressible (which is another way of indicating that γ and ρ are constant values), then this equation integrates to

$$p_2 - p_1 = \gamma(z_2 - z_1) = \rho g(z_2 - z_1) \tag{1.4a}$$

When the fluid is a liquid with a free surface it is convenient to use a coordinate h (e.g., independent variable) that has its origin on this surface and is positive downward. h is related to z by dh = −dz, and, therefore, Equations 1.4 and 1.4a become as following with p considered a function of h:

TABLE 1.1
Properties of Water Related to Temperature

Temperature		Density, ρ (Mass/Vol.)		Specific Weight (Weight/Vol.)		Abs. Viscosity, μ		Kinematic Viscosity, ν		Vapor Pressure, p_v	
						$\mu \times 10^3$	$\mu \times 10^5$	$\nu \times 10^6$	$\nu \times 10^5$		
°C	°F	kg/m^3	Slug/ft^3	kN/m^3	lb/ft^3	(N·s/m^2)	(lb·s/ft^2)	(m^2/s)	(ft^2/s)	kN/m^2 (abs)	psia
0	32.0	999.8	1.940	9.805	62.42	1.785	3.728	1.785	1.922	0.61	0.09
5	41.0	1000.0	1.940	9.807	62.43	1.518	3.170	1.518	1.634	0.87	0.13
10	50.0	999.7	1.940	9.804	62.41	1.307	2.730	1.307	1.407	1.23	0.18
15	59.0	999.1	1.939	9.798	62.37	1.139	2.379	1.140	1.227	1.70	0.25
20	68.0	998.2	1.937	9.789	62.32	1.002	2.093	1.004	1.080	2.34	0.34
25	77.0	997.0	1.934	9.777	62.24	0.890	1.859	0.893	0.961	3.17	0.46
30	86.0	995.7	1.932	9.764	62.16	0.798	1.667	0.801	0.863	4.24	0.61
35	95.0	994.1	1.929	9.749	62.06	0.720	1.504	0.724	0.780	5.62	0.82
40	104.0	992.2	1.925	9.730	61.94	0.653	1.364	0.658	0.708	7.38	1.07
45	113.0	990.2	1.921	9.711	61.82	0.596	1.245	0.602	0.648	9.58	1.39
50	122.0	988.0	1.917	9.689	61.68	0.547	1.142	0.554	0.596	12.33	1.79
55	131.0	995.7	1.932	9.764	62.16	0.504	1.053	0.506	0.545	15.73	2.28
60	140.0	983.2	1.908	9.642	61.38	0.466	0.973	0.474	0.510	19.92	2.89
65	149.0	980.5	1.902	9.615	61.21	0.433	0.904	0.442	0.475	25.00	3.63
70	158.0	977.8	1.897	9.589	61.04	0.404	0.844	0.413	0.445	31.16	4.52
75	167.0	974.9	1.892	9.561	60.86	0.378	0.789	0.388	0.417	38.53	5.59
80	176.0	971.8	1.886	9.530	60.67	0.354	0.739	0.364	0.392	47.34	6.87
85	185.0	968.6	1.879	9.499	60.47	0.333	0.695	0.344	0.370	57.58	8.35
90	194.0	965.3	1.873	9.466	60.26	0.315	0.658	0.326	0.351	70.10	10.17
95	203.0	962.2	1.867	9.436	60.07	0.294	0.614	0.306	0.329	79.36	11.51
100	212.0	958.4	1.860	9.399	59.83	0.282	0.589	0.294	0.317	101.33	14.70

$$\frac{dp}{dh} = \gamma = \rho g \tag{1.4b}$$

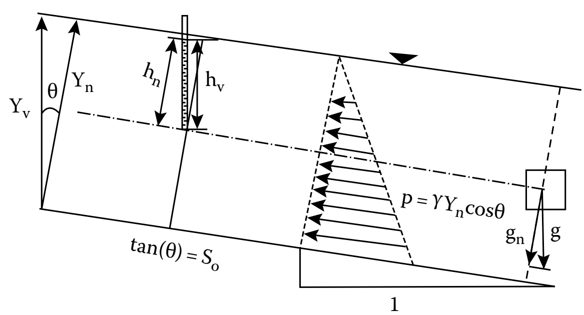
and

$$p = \gamma h = \rho g h \tag{1.4c}$$

In the last equation, the pressure p assumes that atmospheric pressure is the reference base pressure, and therefore p is gage pressure. Atmospheric pressure must be added to gage pressure to get absolute pressure.

A few example problems follow that illustrate the use of Equations 1.4; however, it is worthwhile to examine how the pressure varies with depth in a channel with a steep bottom slope that contains a flow, before leaving the subject of pressure within a fluid caused by gravity. If the angle that the channel bottom makes with the horizontal is denoted by θ (note that the $\tan\theta = S_o$ the slope of the channel bottom), then the normal depth Y_n is related to the vertical depth Y_v by the cosine of this angle, or $Y_n = Y_v\cos\theta$, as shown in the sketch below.

Since gravity acts on fluid elements in the vertical direction, the height water would rise in a piezometer with its opening pointing downward, and located at any position will equal $\gamma h_n\cos\theta = g\rho h_n\cos\theta = \gamma h_v\cos^2\theta$, where h_n is the normal distance from the water surface down to the point being considered, and h_v is the vertical distance from the water surface to the point as shown on the sketch. Thus, the pressure at the bottom of the channel equals $\gamma Y_n\cos\theta = \gamma Y_v\cos^2\theta$.



An alternative, to the above method for determining the pressure distribution, is to note that the acceleration of gravity g acts in the vertical direction on any fluid element. The component of the acceleration in the normal direction is $g_n = g\cos\theta$, and therefore $p = \rho g_n h_n = \rho g h_n\cos\theta = \gamma h_v\cos^2\theta$.

Should the bottom of the channel be curved instead of having a constant bottom slope, then the normal acceleration in the fluid due to the curvature of the channel bottom will affect the pressure distribution as illustrated in the sketches below. The fluid at the very bottom of the channel will have a normal acceleration equal to v^2/r and will add to the gravitational acceleration when the curvature is concave upward and subtract from g when the curvature is concave downward. Note that since the fluid is not free falling under gravity's acceleration, upward accelerations add to g and downward accelerations subtract from g. The component of acceleration on a fluid particle in the direction of the radius of curvature equals $g\cos\theta + v^2/r$, where θ now is the angle between the vertical and the radius of curvature, and therefore at any point in the fluid $dp/dr = \rho(g\cos\theta + v^2/r)$. Since the radii of curvature of streamlines other than the bottom streamline will not, in general, equal the radius of curvature of the channel, it is not possible to determine the pressure distribution without making assumptions related to the radius of curvature between the channel bottom and the free surface.

The bulk modulus $E_{\underline{v}}$ is a similar quantity in dealing with fluids, as is the modulus of elasticity when dealing with solids. It has the same units as pressure and is the reciprocal of the compressibility. The bulk modulus equals the change in pressure needed to cause a decrease in volume divided by that decreased volume divided by the volume of fluid involved, or

$$E_{\underline{v}} = -\frac{\Delta p}{\Delta \cancel{V} / \cancel{V}} = -\frac{dp}{dv/v} = \frac{dp}{d\rho/\rho} \tag{1.5}$$

The second and third parts of Equation 1.5 are obtained by taking one unit mass of a fluid so $\cancel{V}$ becomes the specific volume (volume/unit mass) v which is the reciprocal of the density ρ, and therefore $dv/v = -d\rho/\rho$. The bulk modulus of pure water equals 320,000 psi.

A small amount of free air entrained in the water can significantly reduce the bulk modulus of the mixture. If x is taken as the fraction of free air (not dissolved) mixed in water, then the density of the air–water mixture is given by $\rho_m = x\rho_a + (1 - x)\rho_w$ in which ρ_a and ρ_w are the densities of air and water, respectively. To obtain the bulk modulus of this mixture note from Equation 1.5 that $\Delta\cancel{V} = -\cancel{V}\,\Delta p/E_{\underline{v}}$. The change in volume of the mixture $\Delta\cancel{V}_m$ will be the sum of the changes in volume of the air and the water or, $\Delta\cancel{V}_m = \Delta\cancel{V}_a + \Delta\cancel{V}_w$. By taking an original volume of a unit amount ($\cancel{V} = 1$), this last expression for the change in volume becomes the following upon substituting for the $\Delta\cancel{V}$'s:

$$-\frac{\Delta p}{E_m} = -\frac{x\Delta p}{E_a} = -\frac{(1-x)\Delta p}{E_w} \quad \text{or} \quad E_m = \frac{E_a E_w}{(1-x)E_a + xE_w}$$

where

E_a is the bulk modulus of air

E_w denotes the bulk modulus of water

Since the air mass in a water–air mixture generally is an extremely small fraction of the water mass, the air's temperature will remain constant when it is compressed. The compression of a gas at constant temperature is referred to as an isothermal process. The bulk modulus of a gas undergoing an isothermal process equals its absolute pressure, or $E_a = p_{abs}$. Therefore, the bulk modulus for a small fraction x of air entrained in water becomes

$$E_m = \frac{p_{abs} E_v}{(1-x)P_{abs} + xE_v} \tag{1.6}$$

where the symbol E_v has been used for the bulk modulus of water again instead of E_w. The table below indicates how the bulk modulus is affected by a small fraction of air. Note that a small amount of air entrainment reduces the density of the mixture very modestly, but reduces the bulk modulus by orders of magnitude. These values assume air at atmospheric pressure of 14.7 psia, and at a temperature of 60°F, so its density equals 0.00237 slugs/ft^3.

Fraction of air x	0.0000	0.0001	0.0005	0.001	0.005	0.01	0.10
Density of mixture ρ_m	1.940	1.940	1.939	1.938	1.930	1.921	1.746
Bulk modulus of mixture $E_m \times 10^{-2}$ (psi)	3,200	1,007	26.927	7.115	2.913	1.463	0.147

EXAMPLE PROBLEM 1.1

A sensitive pressure transducer is used to record the pressure at the bottom of a river that carries a sediment load that causes the density to increase linearly from 1.945 slug/ft_3 on the surface at a rate of 0.012 slug/ft^3/ft of depth. Develop the equation that gives the depth of flow from this pressure reading. What is the error if the sediment load is ignored, and the density taken equal to 1.94 and a pressure of 3.5 psi is recorded?

Solution

The problem is solved by defining $\rho = 1.945 + 0.012h$, substituting this into Equation 1.4b, separating variables, and integrating. The result is

$$p = g(1.945h + 0.006h^2),$$

where

p is in pound per square foot
h is in feet

Applying the quadratic formula gives the depth h as a function of the pressure reading as

$$h = -162.083 + 83.333\left(3.783 + 0.024\frac{p}{g}\right)^{1/2}.$$

Substituting $p = 3.5 \times 144$ into the above equation gives $h = 7.856$ ft. If $\rho = 1.94$ (constant), then $p = 62.4h$, or $h = 8.077$ ft, or an error of +0.221 ft.

EXAMPLE PROBLEM 1.2

Water is being drawn from a well whose water level is 1500 ft below the ground surface. Determine the significance of the compressibility of water in determining the head the pumps must supply if the friction loss in the well pipe is 25 ft, and the pump must supply 80 psi of pressure at the ground surface. Assume the bulk modulus for water remains constant and equal to 320,000 psi.

Solution

First, it is necessary to determine the relationship between density and pressure since the effects of increasing density of the fluid are to be taken into account. The definition of a fluid's bulk modulus $E_{\underline{v}}$ provides this relationship since $E_{\underline{v}} = -\Delta p/(\Delta\underline{V}/\underline{V}) = \rho dp/d\rho$, in which $\underline{V}$ is fluid volume and $\Delta\underline{V}$ is the change in this volume due to the pressure increase Δp. Separating variables in this equation and integrating the density from ρ_o (the density at atmospheric pressure, which will be taken as 1.94 slugs/ft^3) to ρ and integrating the pressure from 0 (atmospheric) to p gives

$$\frac{\rho}{\rho_o} = \text{Exp}\left(\frac{p}{E_{\underline{v}}}\right)$$

(An alternative to integrating between the two limits is to just integrate and add a constant to the resulting equation. This constant can then be determined from a known condition. In this case the known condition is that the density equals $\rho_o = 1.94$ slugs/ft^3 when the pressure, $p = 0$.)

Substituting this expression into the hydrostatic Equation 1.4b gives

$$\frac{dp}{dh} = g\rho = g\rho_o\text{Exp}\left(\frac{p}{E_{\underline{v}}}\right)$$

Again separating variables and integrating gives

$$p = -E_{\underline{v}} \mathrm{Ln}\left(1 - \frac{g\rho_o h}{E_{\underline{v}}}\right)$$

The value of h to use in this equation is the sum 1500 ft, the pressure head needed at the surface, 80 × 144/(32.2 × 1.94) = 184.41 ft, and the frictional loss of 25 ft, or h = 1709.41 ft. Substituting this value for h in the above equation gives a pressure of 742.41 psi at the pump in the well. If the compressibility of the water is ignored, the pressure is obtained from p = gρh/144 = 741.55 psi. This small difference of about 1 psi points out that the compressibility of water is not very significant for most engineering applications.

An exception is whenever the speed of pressure waves are concerned, because for such applications a small amount of free air in water can dramatically change the speed of this wave as noted above.

EXAMPLE PROBLEM 1.3

Water is flowing at a rate of 800 cfs in a 6 ft wide rectangular channel at a constant depth of 4 ft. The bottom of the channel changes slope by means of a circular arc of radius 50 ft. The depth of flow through this arc remains constant at 4 ft. Assuming that the streamlines of this flow are concentric circles (have the same center of curvature), determine the pressure along a radial line at the beginning of the arc where it connects to the straight upstream channel that has a bottom slope $S_o = 0.15$, and also determine the pressure distribution along a vertical radial line.

Solution

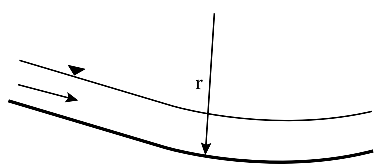
Using the upstream bottom slope θ = arctan(0.15) = 8.531°. The velocity in the channel equals 33.333 fps and therefore,

$$\frac{dp}{dr} = \rho\left\{g\cos(8.531°) + \frac{(33.333)^2}{r}\right\}$$

or after integrating p from 0 to p as r is integrated between 46 and r, this results in

$$p = \rho\left\{31.844r + 1111.11\,\mathrm{Ln}\left(\frac{r}{46}\right) - 1464.81\right\}$$

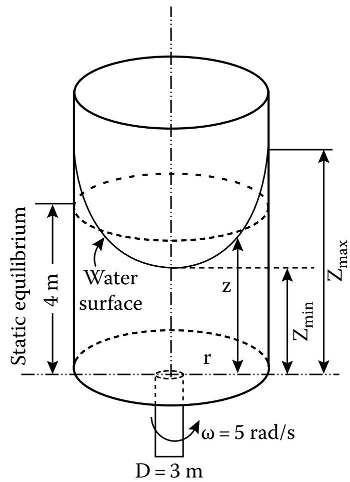
to give the pressure distribution as a function of r. On the bottom r = 50, and the pressure here is p = 426.75 psf = 2.964 psi, at the channel bottom, where r = 50 ft. Ignoring the added pressure due to the radius of curvature, the pressure at the bottom of the channel is p = 4γ/144 = 1.73 psi. This latter amount is 1.234 psi too small.



Flip-bucket spillways cause high-velocity water to have a considerable normal component of acceleration, as illustrated in this example. To find the added forces on such structures caused by changing the direction of the fluid can be handled easier through the use of the momentum principle that will be discussed in detail in Chapter 3.

EXAMPLE PROBLEM 1.4

A circular tank with a 3 m diameter (or radius $r_o = 1.5$ m) initially containing water to a depth of $h_o = 4$ m is rotated at an angular velocity ω = 5 rad/sec for a long time until the water is brought into solid body rotation. Determine the shape of the water surface in the tank, and its depth at the center and outside walls of the tank.



Solution

In this problem the fluid (water) is being accelerated. The equation for fluid statics $\partial p/\partial z = -\rho g$ (with $\partial p/\partial x = 0$ and $\partial p/\partial y = 0$) which accounts only for gravitational acceleration in the z-direction (vertical) can be generalized to the following: $\partial p/\partial z = -\rho(g + a_z)$, $\partial p/\partial x = -\rho a_x$, $\partial p/\partial y = -\rho a_y$. When applied to the rotating tank, with z as the vertical coordinate, and r the radial coordinate, these equations become $\partial p/\partial z = -\rho g$ and $\partial p/\partial r = -\rho a_r = \rho r\omega^2$. From the definition of a differential $dp = (\partial p/\partial r)dr + (\partial p/\partial z)dz$. Along any constant pressure surface, such as the free surface, $dp = 0$, and therefore the slope of the water surface at any radial position r is $dz/dr = -a_r/(g + a_z) = r\omega^2/g$. Separating variables and integrating gives the following parabolic relationship between z and r for the water surface:

$$z = \frac{r^2\omega^2}{2g} + C$$

The constant C can be evaluated by noting that the same amount of water exists in the tank after rotation as before, or

$$\int_0^{r_o} 2\pi rz\,dr = 2\pi\int_0^{r_o} r\left(\frac{\omega^2 r^2}{2g} + C\right)dr = 2\pi\left(\frac{\omega^2 r_o^4}{8g} + \frac{r_o^2}{2}C\right) = \pi h_o r_o^2$$

or solving for C

$$C = h_o - \frac{r_o^2\omega^2}{4g} = 4 - \frac{(1.5\times 5)^2}{4(9.81)} = 4 - 1.433 = 2.567\,\text{m}.$$

At the center axis of the tank the water depth will be $z_{in} = C = 2.567$ m, and at the outside $z_{max} = C + (r_o \times \omega)^2/(2g) = 2.567 + 2.867 = 5.433$ m. As a check on the computations, etc., we might note that if the volume of the cylinder formed by z_{max} as its height has the volume of the paraboloid subtracted from it, then the original volume of the cylinder with a height h_o should occur. The volume of a paraboloid equals 1/2 the area of the base time the height. Therefore, $Ah_o = A\{z_{max} - 0.5(z_{max} - C)\} = A(z_{max} + C)/2$, or $h_o = (z_{max} + C)/2 = (5.433 + 2.567) = 4$ m.

Before leaving this problem, it is worth noting that the Bernoulli equation cannot be applied across the streamlines in this rotation tank, e.g., from the inside radius to the outside. If this were done then the total head on the free surface would be the sum of the elevation and velocity heads (since $p = 0$) or

$$z + \frac{V^2}{2g} = z + \frac{(r\omega)^2}{2g} = H$$

or

$$z = H - \frac{r^2\omega^2}{2g},$$

but this equation has a constant minus the velocity head whereas the above equation indicated that z was equal to a constant plus the velocity head. The reason is that the Bernoulli equation is based on irrotational flow, whereas this is rotational flow. A free vortex represents irrotational flow, but its equation indicates that the transverse component of velocity v_t = Constant/r. The forced rotation in the tank gives $v_t = r\omega = r \times \text{Constant}$.

EXAMPLE PROBLEM 1.5

Water is flowing in a natural channel, and the flow rate Q is to be determined by measuring the velocity on the surface and the depth of flow at various positions x across the channel. These measurements have produced the values in the table below.

Position, x (m)	0.0	4	8	12	16	20	24	28	32	36
Depth, Y (m)	0.0	4.7	8.3	11.2	12.7	13.3	13.4	12.8	12.0	11.8
Velocity, Vs (m/s)	0.0	0.35	0.45	0.47	0.49	0.50	0.49	0.50	0.48	0.48
	40	44	48	52	56	60	64	68	72	75
	11.5	11.7	12.3	13.3	14.5	15.0	14.8	12.7	8.0	0.0
	0.48	0.48	0.48	0.49	0.50	0.50	0.45	0.35	0.20	0.0

It has also been determined that the velocity varies according to the same dimensionless profile from the bottom of the channel at any position x according to the data given in the table below. (In this table y′ = y/Y is the dimensionless depth, with y beginning at the bottom at this position x, and likewise v′ = v/Vs is the dimensionless velocity, in which Vs is the velocity on the surface at this position x.)

Dimensionless depth y′	0.0	0.04	0.08	0.12	0.20	0.28	0.40	0.52	0.64	0.72	0.80	0.90	1.0
Dimensionless velocity v′	0.0	0.22	0.375	0.500	0.675	0.815	0.955	1.040	1.070	1.068	1.06	1.03	1.0

Solution

There are a number of methods that could be used to solve this problem, including plotting the data from the above tables and determining area under the curves. However, since in this book we wish to emphasize the use of the computer to do numerical computations, the problem will be solved by writing a computer program. First, however, let us note that a dimensionless flow rate per unit width q′ can be obtained by integrating the dimensionless velocity profile given in the second table, or $q' = \int v'dy'$, with limit from 0 to 1. The flow rate per unit width q equals the dimensionless q′ multiplied by the depth Y and the surface velocity at any position, or $q = \int v\,dy = VsY\int v'dy' = VsYq'$. Thus, the flow rate Q, which is the integral of the unit flow rate q times dx, or $Q = \int q dx$, can be determined from $Q = q'\int VxY\,dx$, with the limits of this integration from 0 to the total width of the section, or 75 m. These integrations will be accomplished by using the subroutine SIMPR, which is described in Appendix B, and implements Simpson's rule to numerically evaluate integrals. To provide the integrand as a continuous function of the variable being integrated, a cubic spline function will be used, i.e., subroutine SPLINESU, also described in Appendix B is utilized. The program EXPRB1_5 (both in FORTRAN and C) is given below to provide the solution. The following are the key components of this program: (1) The first two READS store the data from the above two tables in arrays, with arrays YP and VP storing the dimensionless velocity profile data in the second table, and arrays X, Y, and VS storing the data in the first table. (2) The cubic spline function SPLINESU is called three times to provide (a) the second derivatives d^2v'/dy'^2, (b) the second derivatives d^2Y/dx^2, and (c) the second derivatives d^2Vs/dx^2 corresponding to the points in the tables. (3) The subroutine SIMPR is called next to integrate the dimensionless velocity profile. It calls on function subprogram VPROF to provide v' corresponding to any dimensionless depth y', and accomplishes this by using the d^2v'/dy'^2 supplied by the first call to SPLINESU. (4) The subroutine SIMPR is called again to provide the integral $\int(VsY)dx$, and subprogram Dq provides the arguments for this integration. (5) Finally by multiplying q' (the result from the first integration) by the result from the second integration the flow rate Q is printed out. (Note that the C program contains the function that supplies what is integrated as argument 1, as mentioned in Appendix B if the name is not equat used; thus allowing the two different arguments vprof (for v') and qp (for Y*Vs).)

Listing of program EXPRB1_5.FOR

```
      EXTERNAL VPROF,Dq
      REAL DUM(30)
      COMMON YP(20),VP(20),X(30),Y(30),VS(30),
     &D2VP(20),D2VS(30),D2Y(30),I1,I2,NP,NX
      I1=1
      I2=2
      READ(2,*) NP,(YP(I),VP(I),I=1,NP)
      READ(2,*) NX,(X(I),Y(I),VS(I),I=1,NX)
      CALL SPLINESU(NP,YP,VP,D2VP,DUM,0)
      CALL SPLINESU(NX,X,Y,D2Y,DUM,0)
      CALL SPLINESU(NX,X,VS,D2VS,DUM,0)
      CALL SIMPR(VPROF,0.,1.,qPRIM,1.E-6,20)
      WRITE(*,*)' Integral of dimensionless'/
     &' velocity profile=',qPRIM
      I1=1
      I2=2
      CALL SIMPR(Dq,0.,X(NX),Q,1.E-4,20)
      WRITE(*,*)' Flowrate, Q =',Q*qPRIM
      END
      FUNCTION Dq(XX)
      COMMON YP(20),VP(20),X(30),Y(30),VS(30),
     &D2VP(20),D2VS(30),D2Y(30),I1,I2,NP,NX
1     IF(XX.LT.X(I2) .OR. I2.EQ.NX) GO TO 2
      I1=I2
      I2=I2+1
      GO TO 1
2     IF(XX.GE.X(I1) .OR. I1.EQ.1) GO TO 3
      I2=I1
      I1=I1-1
      GO TO 2
      END
      FUNCTION VPROF(YY)
      COMMON YP(20),VP(20),X(30),Y(30),VS(30),
     &D2VP(20),D2VS(30),D2Y(30),I1,I2,NP,NX
1     IF(YY.LT.YP(I2) .OR. I2.EQ.NP) GO TO 2
      I1=I2
      I2=I2+1
      GO TO 1
2     IF(YY.GE.YP(I1) .OR. I1.EQ.1) GO TO 3
      I2=I1
      I1=I1-1
      GO TO 2
```

```
3     DYP=YP(I2)-YP(I1)
      A=(YP(I2)-YY)/DYP
      B=1.-A
      VPROF=A*VP(I1)+B*VP(I2)+((A*A-1.)*A*D2VP(I1)+
     &(B*B-1.)*B*D2VP(I2))*DYP**2/6.
      RETURN
3     DX=X(I2)-X(I1)
      A=(X(I2)-XX)/DX
      B=1.-A
      AA=A*(A*A-1.)*DX*DX/6.
      BB=B*(B*B-1.)*DX*DX/6.
      DEPTH=A*Y(I1)+B*Y(I2)+AA*D2Y(I1)+BB*D2Y(I2)
      VSURF=A*VS(I1)+B*VS(I2)+AA*D2VS(I1)+BB*
     &D2VS(I2)
      Dq=DEPTH*VSURF
      RETURN
      END
```

Listing of program EXPRB1_5.C

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
float yp[20],vp[20],x[30],y[30],vs[30],d2vp[20],d2y[30],d2vs[30];
int i1,i2,np,nx;
extern float simpr(function equat,float xb,float xe,\
  float err,int max);
extern void splinesu(int n,float *x,float *y, float *d2y,\
  float *d,int ity);
float vprof(float yy){float a,b,dyp;
 while((yy>=yp[i2])&&(i2<np-1)){i1=i2;i2++;}
 while((yy<yp[i1])&&(i1>1)){i2=i1;i1--;} dyp=yp[i2]-yp[i1];
  a=(yp[i2]-yy);b=1.-a;
 return a*vp[i1]+b*vp[i2]+((a*a-1.)*a*d2vp[i1]+(b*b-1.)*b*d2vp[i2])\
  *dyp*dyp/6.;
}  // End of vprof
float qp(float xx){float a,b,dx,aa,bb,depth;
 while((xx>=x[i2])&&(i1<nx-1)){i1=i2;i2++;}
 while((xx<x[i1])&&(i1>1)){i2=i1;i1--;}
 dx=x[i2]-x[i1];a=(x[i2]-xx)/dx;b=1.-a;
 aa=a*(a*a-1.)*dx*dx/6.;bb=b*(b*b-1.)*dx*dx/6.;
 depth=a*y[i1]+b*y[i2]+aa*d2y[i1]+bb*d2y[i2];
 return depth*(a*vs[i1]+b*vs[i2]+aa*d2vs[i1]+bb*d2vs[i2]);
}  // End of qp
void main(void){FILE *fili; char filnam[20];int i;
  float dum[30];qprim,q;
 i1=1;i2=2; printf("Give input file name\n");
 scanf("%s",filnam);
 if((fili=fopen(filnam,"r"))==NULL){
  printf("Cannot open file\n");exit(0);}
 fscanf(fili,"%d",&np);
 for(i=0;i<np;i++)fscanf(fili,"%f %f",&yp[i],&vp[i]);
 fscanf(fili,"%d",&nx);
 for(i=0;i<nx;i++)fscanf(fili,"%f %f %f",&x[i],&y[i],&vs[i]);
 splinesu(np,yp,vp,d2vp,dum,0);splinesu(nx,x,y,d2y,dum,0);
 splinesu(nx,x,vs,d2vs,dum,0);
 qprim=simpr(vprof,0.,1.,1.e-6,20); i1=1;i2=2;
```

```
 printf("Integral of dimensionless velocity profile =%f\n",qprim);
 printf("Flowrate, Q =%f\n",qprim*simpr(qp,0.,x[nx-1],1.e-4,20));
}
```

The following is the input file needed to solve the problem (EXPRB1_5.DAT):

```
13 0. 0. .04 .22 .08 .375 .12 .5 .2 .675 .28 .815 .4 .955 .52
  1.04 .64 1.07 .72 1.068 .8 1.06 .9 1.03 1 1
20 0 0 0 4 4.7 .25 8 8.3 .45 12 11.2 .47 16 12.7 .49 20 13.3 .5
  24 13.4 .49 28 12.8 .48 32 12. .48 36 11.8 .48 40 11.5 .48 44
  11.7 .48 48 12.3 .48 52 13.3 .49 56 14.5 .5 60 15. .5 64 14.8
  .45 68 12.7 .35 72 8. .2 75 0. 0.
```

The solution produced is

Integral of dimensionless velocity profile = 8.726063e−01

Flowrate, Q = 342.505700

Viscosity is the fluid property that defines its resistance to motion. By definition, the absolute viscosity μ of a fluid is defined as the coefficient that relates the internal shearing stress τ within the fluid to the velocity gradient at this point in the flowing fluid, or

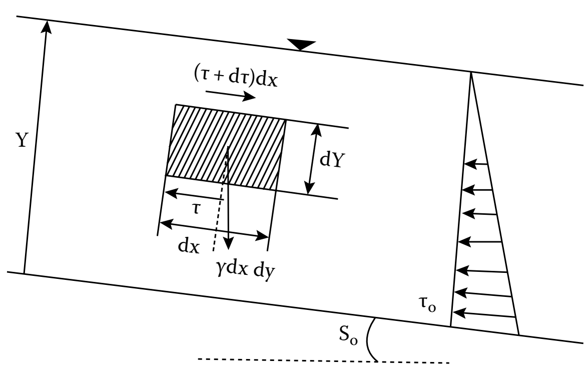
$$\tau = -\mu\left(\frac{\partial v}{\partial n}\right) \tag{1.7}$$

where n is a direction normal to velocity v. Fluids, such as water, whose absolute viscosities are not dependent upon the rate of shear, $\partial v/\partial n$, are referred to as **Newtonian fluids**. The viscosity of water does depend upon its temperature, as shown in Table 1.1. If the viscosity of a fluid does change with the rate of shear (μ is a function of $\partial v/\partial s$ or τ, i.e., $\mu(\partial v/\partial s)$ or $\mu(\tau)$) then the fluid is referred to as a **non-Newtonian** fluid. Silly putty is a non-Newtonian fluid that behaves similar to an elastic solid if sheared rapidly (i.e., bounces as a rubber ball), but flows as a very viscous fluid if given sufficient time to do this. Mud and debris flows, which are open-channel flows, of a fluid that is a mixture of water and soil (and possibly debris from the channel) that have mobilized are other examples involving non-Newtonian fluids.

The kinematic viscosity v equals the absolute viscosity divided by the fluid density, or

$$v = \frac{\mu}{\rho} \tag{1.8}$$

Effects that viscosity has on fluid motion can be examined by considering a small section of uniform, steady flow in a channel in which an element of fluid with a differential length dx, and a height dy is taken as shown in the sketch below.



The channel has a bottom slope equal to S_o and is wide enough that only the shearing stress on the bottom is important (the side shears can be neglected). Since the flow is steady and uniform, the summation of forces on the differential fluid element in the x-direction must equal zero. This summation produces the following:

$$-\tau dx + (\tau + d\tau)dx + S_o\gamma\, dx\, dy = 0$$

In this equation the slope of the channel bottom is small so that the $\sin\theta = \tan\theta = S_o$. Simplifying gives $d\tau = -S_o\gamma\, dy$, which can be integrated between the bottom of the channel to any depth y, or

$$\int_{\tau_o}^{\tau} d\tau = -\gamma S_o \int_0^Y dy$$

which gives

$$\tau = \tau_o - S_o\gamma y$$

An examination of the section of this flow from the bottom to the top with a length dx shows that $\tau_o = \gamma Y S_o$. For any arbitrary section in which the shear stress on the side is considered, it can be shown that

$$\tau_o = \frac{\gamma A S_o}{P} = \gamma R_h S_o \tag{1.9}$$

where
- A is the cross-sectional area
- P is wetted perimeter
- R_h is the hydraulic radius, A/P

which reduces to the above for the wide section being considered here. Substituting for τ_o gives the following equation that defines the shearing stress as a function of the position from the bottom y:

$$\tau = \gamma S_o(Y - y) \tag{1.10}$$

which shows the shearing stress varies linearly from $\gamma S_o Y$ at the bottom to zero at the top of the channel flow. Substituting Equation 1.10 into Equation 1.7 allows for the velocity distribution from the bottom to the top of the channel to be determined. Making this substitution gives

$$dv = \left(\frac{\gamma S_o}{\mu}\right)(Y - y)dy$$

which integrates to give the parabolic relationship,

$$v = \frac{\gamma S_o}{\mu(Yy - y^2/2)} \tag{1.11}$$

It turns out that Equation 1.11 is correct only for laminar flows in which the viscous forces dominate over the inertia forces. For a turbulent flow the fluid near the walls has its velocity increased, whereas the velocity near the top of the flow away from the solid boundaries has its velocity decreased from the velocity given by Equation 1.11 by mixing of the flows. In other words, the large velocity portion of the flow is continually being mixed with the small velocity portion of the flow tending to make the velocity distribution more nearly the same throughout the flow than that described by Equation 1.11. This mixing is referred to as momentum transfer within the flow, and results in the need to modify Equation 1.7 for turbulent flows. This momentum transfer causes the inertial forces to play a more important role in determining the distribution of the velocity with depth. For turbulent flows it is necessary to multiply the velocity gradient in Equation 1.7 by an additional eddy viscosity term ε to get the shearing stress, or μ is replaced by $(\mu + \varepsilon)$. This eddy viscosity varies with y in a manner that cannot be defined easily for different flow situations, and therefore has limited practical value.

Even though Equations 1.7 and 1.11 hold only for laminar flows, Equation 1.10 is valid for turbulent flows as well.

EXAMPLE PROBLEM 1.6

A trapezoidal channel with a bottom width b = 3 m, and a side slope m = 2 has a bottom slope $S_o = 0.0008$ and is 5000 m long. This channel contains a uniform flow of Q = 20 m^3/s at a depth Y = 1.6 m. Determine the total shear force on the sides of this channel caused by the flowing water.

Solution

The shear stresses on the boundaries of this channel are given by Equation 1.9. Computing the area and wetted perimeter gives A = (b + mY), Y = 9.920 m^2, P = 10.155 m. Substituting into Equation 1.9 gives

$$\tau_o = 9800\left(\frac{9.920}{10.115}\right)(0.0008) = 7.658 \text{ N/m}^2.$$

The area over which this shear stress acts equals the perimeter times the length of channel or 10.155 × 5000 = 50,775 m^2. Therefore, the total shear force equals

50,775 × 7.658 = 388,848 N.

Alternatively, this shear force equals the component of the weight in the channel in the direction of the flow, or weight XS_o. This computation gives 9800(9.92 × 5000)(0.0008) = 388,848 N. It should be noted in this problem that the shear force does not involve using the flow rate. In actuality, it is the flow rate and the channel roughness that determines the depth of flow.

EXAMPLE PROBLEM 1.7

Water at a rate of Q = 25 m^3/s flows around a circular bend in a rectangular channel with a width of 4 m. The inside radius of this bend is 4 m. If the depth upstream from the bend is Y_o = 3 m, determine the velocity distribution through a radial line through this bend under the following assumptions: (1) an average velocity through the depth of flow can be used so that the velocity can be considered only a function of r (distance from the center of the bend); (2) the shear stress is proportional to the velocity gradient dv/dr; and (3) the shear stress is zero along the circle midway between the two channel walls and varies linearly to τ_o at the two walls.

Solution

The solution starts by noting from the problem's description that $|\tau| = a(dv(r)/dr) = \tau_o(r - 6)/2$, in which a is a constant. Integrating this equation and evaluating the constant of integration gives the following equation for the velocity distribution:

$$v = C\left\{\left(\frac{6r - r^2}{2}\right) - 16\right\} \quad \text{(Note that v = 0 when r = 4 and r = 8)}$$

To evaluate the constant C, the integral of vYdr is equated to the flow rate Q, or

$$Q = \int vY\,dr = 25$$

If the depth is assumed constant at Y = 3 m across the circular portion of the channel then

$$25 = 3C\int\left[6r - \frac{r^2}{2} - 16\right]dr = 3C\left[\frac{6r^2}{2} - \frac{r^3}{6} - 16r\right]\Bigg|_4^8 = 3C[5.33], \quad \text{or} \quad C = 1.563$$

The assumption of constant depth is not good since the centrifugal forces of the flow around the bend will cause the depth at its outside to be larger than at the inside. The slope of the water surface will be normal to the total acceleration vector, or $dY/dr = v^2/(gr)$. Substituting for v and integrating gives

$$Y = \left(\frac{C^2}{g}\right)\left\{\frac{r^4}{16} - 2r^3 + 26r^2 - 192r + 256\,\text{Ln}(r)\right\} + C_1$$

Since the depth is assumed to equal 3 m when r = 6 m (the mid-radius of the bend), C_1 can be evaluated as

$$C_1 = 3 + \left(\frac{C^2}{g}\right)(108.31).$$

The above expressions for Y and v are again substituted in $25 = \int vY\,dr$ and this integrated from 4 to 8 with the result,

$$25 = 16C - 0.015982C^3$$

The three roots of this equation are 1.5663, –32.395, and 30.828. The first root is the desired one, i.e., C = 1.5663. The depths and velocities through the bend are given in the table below.

r	**4**	**4.5**	**5**	**5.5**	**6**	**6.5**	**7**	**7.5**	**8**
Y	2.800	2.808	2.847	2.917	3.000	3.077	3.131	5.157	3.162
V	0.000	1.371	2.350	2.937	3.133	2.937	2.350	1.371	0.000

1.10 CONSERVATION OF MASS, OR CONTINUITY EQUATIONS

A basic principle of the mechanics is that mass is conserved. When this principle is applied under steady-state flow conditions to one-dimensional fluid flows it is generally expressed that the mass flow rate or flux past a second section equals the mass flow rate past the first section plus the mass flow rate entering between the two sections, or

$$(\rho VA)_2 = (\rho VA)_1 + (\rho VA)_{1-2} = (\rho VA)_1 + \left(\int \rho vY\,dx = \int \rho q\,dx\right) \tag{1.12}$$

where

the Vs represent the average velocities at the designated positions

the As represent the corresponding cross-sectional areas normal to these velocities

When dealing with an incompressible fluid the density ρ is constant and can be divided out of Equation 1.12 to give a volumetric flow equation. The volumetric flow rate with dimensions of L^3/t will be given the symbol Q, so Equation 1.12 can be written in either of the following two forms:

$$Q_2 = Q_1 + \int q(x)dx$$

or

$$(VA)_2 = (VA)_1 + \int q(x)dx \tag{1.13}$$

where q(x) is the lateral inflow (negative for outflow) in units of flow rate per unit length (L^2/t) being equal to v_iY, and the integrations are over the length between the two sections 1 to 2. If the lateral inflow is constant, then the last term in Equation 1.13 can be simplified to qL_{1-2} where L_{1-2} is the distance between section 1 and 2.

The average velocity V in Equations 1.12 and 1.13 represents the point velocity integrated over the cross-sectional area of the flow divided by this area, or

$$V = \frac{\int v\,dA}{A} \tag{1.14}$$

When dealing with two- or three-dimensional flows, or an unsteady one-dimensional flow, the continuity equation becomes a partial differential equation rather than one of the above algebraic equations. The general equation that applies for three-dimensional unsteady flow of a compressible fluid is

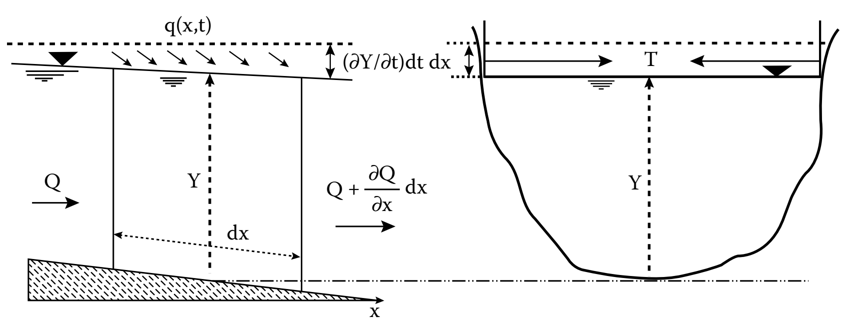
$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = -\frac{\partial(\rho)}{\partial t} \tag{1.15}$$

where u, v, and w are the velocity components in the x, y, and z coordinate directions, respectively. For either steady flow, or the flow of an incompressible fluid the right-hand side of this equation becomes zero. For an incompressible fluid the continuity equation reduces to

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \tag{1.16}$$

The continuity equation for an unsteady one-dimensional flow in an open channel needs to apply for the entire cross section of the channel rather than at a point as is the case with Equations 1.15 and 1.16. To develop this equation consider a differential length of a channel flow as shown in the sketch below whose depth, etc. changes with time, as well as the position x along the channel. The principle of conservation of mass requires that the following, in relationship to this control volume, CV, be true:

$$(\text{Vol. in CV in time } \Delta t) - (\text{Vol. leaving CV in time } \Delta t) = \text{Vol. change in } \Delta t,$$



or substituting the appropriate quantities into this equation gives

$$\left[Q - \left\{Q + \frac{\partial Q}{\partial x}dx\right\} + q(x,t)dx\right]dt = T\left(\frac{\partial Y}{\partial t}\right)dt\,dx$$

Simplifying and dividing by dxdt gives

$$\frac{\partial Q}{\partial x} - q(x,t) + T\frac{\partial Y}{\partial t} = 0 \tag{1.17}$$

This equation, coupled with a second partial differential equation of motion, will be the basis for solving unsteady channel flows in Chapter 6. It is worthwhile noting that if $\partial Q/\partial x$ is of the same positive magnitude as the magnitude of lateral inflow q(x,t) at all points x and at all times, then $\partial Y/\partial t = 0$, i.e., the flow is steady state, which means at any position x the depth and velocity are constant. However, it is important to note that q and $\partial Q/\partial x$ represent two different quantities. q(x,t) is the lateral inflow that can in general vary along the channel as well as vary with time. It is generally known and not part of what is solved for. Should lateral outflow occur, then q(x,t) is a negative quantity. The term $\partial Q/\partial x$ represents the change in flow rate in the channel direction x for a given instant in time, and in general varies from instant to instant. Sometimes an additional subscript is applied to partial derivatives to denote what is being held constant. In Equation 1.17 the two independent variables are x and t. Therefore, it is understood that t is constant when dealing with partial derivatives with respect to x and x is held constant when the partial derivative is with respect to t. Q is unknown and therefore its variation with x and t is a part of the solution.

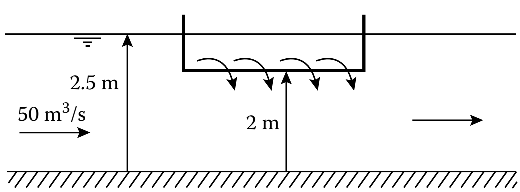
Since the volumetric flow rate Q equals the average velocity V times the cross-sectional area A, the first term in Equation 1.17 can be expanded to give the following alternative continuity equation for one-dimensional unsteady flow in a channel:

$$A\frac{\partial V}{\partial x} + V\frac{\partial A}{\partial x} - q(x,t) + T\frac{\partial Y}{\partial t} = 0 \tag{1.17a}$$

Equation 1.17 generally cannot be solved by itself because it involves two unknowns, or dependent variables, Q and Y. The principle that as many independent equations must exist as there are unknowns, dictates that another equation must be obtained. This second equation comes from Newton's second law of motion, and will be given later in Chapter 6. Equation 1.17 and this second equation are often referred to as the St-Vernant equations that govern one-dimensional unsteady open-channel hydraulic problems.

EXAMPLE PROBLEM 1.8

A 10 m wide rectangular channel has a 10 m long section of side weir that discharges a volumetric flow rate per unit length as given by the equation $q = 0.5/2gH^{3/2}$ in which H is the depth of water in the channel above the weir crest. The weir crest is 2 m above the channel bottom, and the depth in the channel at the beginning of the side weir is 2.5 m, and the flow rate here equals 50 m^3/s. If the depth of flow through the side weir is such that $V^2/2g + Y$ = constant, determine the flow rate in the channel over the length of the side weir.



Solution

Letting Q_o and Y_o be the flow rate and depth, respectively, at the beginning of the side weir the constant referred to can be determined from constant $= Q_o^2/(2gA_o^2) + Y_o = (50/25)^2/19.62 + 2.5 = 2.704$. Thus, the following implicit equation provides the depth Y:

$$Y = 2.704 - \frac{Q^2}{19.62(10Y)^2}$$

This equation will converge with Q given if the Y computed on the left of the equal sign is iteratively used for the Y on the right of the equal sign. After determining Y the lateral outflow can be determined by

$$q = 2.215(Y - 2)^{3/2}$$

and the flow rate determined from Equation 1.13, or

$$Q = 50 - \int q\,dx$$

Since this integration cannot be completed in closed form, it must be carried out using numerical techniques. The small FORTRAN program listed below does this under the assumption that if small enough steps are used for dx then the Q at the current position x can be used to compute the depth, etc.

Techniques designed to solve ordinary differential equations (ODEs) provide a better alternative for solving this problem than the crude numerical integration used in the above program. The ODE for this problem is $dQ/dx = 0.5/2g(Y - 2)^{3/2}$. Numerical methods for solving ODEs will be described in Chapter 4 and are extensively utilized in solving gradually varied flow problems.

Listing of FORTRAN program that numerically integrates the above equation in the solution for Example Problem 1.8

```
      Q=50.
      Y=2.5
      WRITE(6,100) Q,Y
100   FORMAT(' x',7X, 'Q',7X, 'Y',/,19('_'),
     &/,' 0',2F8.3)
      DO 30 I=1,1000
10    Y1=2.704-(Q/Y)**2/1962.
      IF(ABS(Y1-Y).LT. .000001) GO TO 20
      Y=Y1
      GO TO 10
20    Q=Q-.02215*(Y1-2.)**1.5
      IF(MOD(I,100).EQ.0) WRITE(6,110) I/100,Q,Y1
110   FORMAT(I3,2F8.3)
30    CONTINUE
      END
```

Solution from Progaram

X	Q(cfs)	Y(ft)
0	50.000	2.500
1	49.208	2.500
2	48.397	2.515
3	47.569	2.523
4	46.723	2.530
5	45.859	2.537
6	44.977	2.545
7	44.078	2.552
8	43.161	2.559
9	42.227	2.556
10	41.275	2.573

1.11 ENERGY PRINCIPLE

The equations of motion are of extreme importance to all mechanics, whether dealing with fluids or solids. The equations of motion are based on Newton's second law, that states that force = mass × acceleration. In this section the scalar application of this fundamental law of mechanics is discussed, as it applies to fluids in motion. The section is entitled "Energy Principle" because this scalar application gives equations similar to the energy equation in Thermodynamics. In the next section, the vector application of Newton's second law to fluid motion is discussed under the heading "Momentum Principle." The application of Newton's second law to a general three-dimensional unsteady fluid flow results in the **Navier Stokes** equations, whose development can be found in books dealing with fluid mechanics. Applying the same law to an ideal fluid that is inviscid (viscous shear stresses are ignored) produces the **Euler** equations of motion. Neither of these equations is presented herein. Rather simplified versions of these equations are given for one-dimensional and special flows, and these are most useful in handling problems of one-dimensional open-channel hydraulics.

It will be necessary to define acceleration in fluid motion, first. Acceleration is a vector that represents the time rate of change of the velocity vector, or **a** = d**V**/dt. Since velocity in a fluid is a function of the position as well as time in general, the full derivative d**V**/dt is obtained by the chain rule in calculus, or using Cartesian coordinates,

$$\frac{dV}{dt} = \frac{DV}{dt} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial V}{\partial y}$$

The quantity d**V**/dt is called the substantial derivative and is often denoted by D**V**/Dt to make it clear that the partial derivative with respect to time is not intended. Since dx/dt, dy/dt, and dz/dt equal the velocity components u, v, and w in the x-, y-, and z-directions, respectively, the above equation becomes

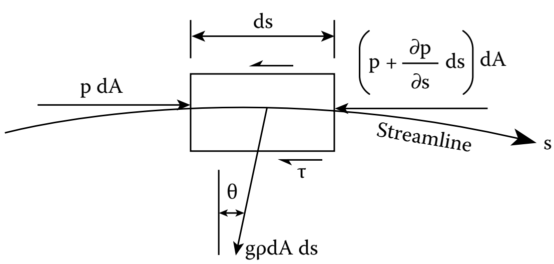
$$\frac{dV}{dt} = \frac{DV}{dt} = u\frac{\partial V}{\partial x} + v\frac{\partial V}{\partial y} + w\frac{\partial V}{\partial z} + \frac{\partial V}{\partial t}$$

The terms u∂**V**/∂x, v∂**V**/∂y, and w∂**V**/∂z are called the **convective** accelerations and can be visualized as the acceleration that occurs as streamlines within a flow converge together. The term ∂**V**/∂t is the **local** acceleration, and represents the variation in velocity that occurs at a point in the fluid with time. Also note that the above equations are vector equations since every term contains the velocity vector **V**.

If the flow is one dimensional, then the magnitude of the velocity is only a function of the coordinate in that direction and time, if the flow is unsteady, or V(s,t). s will be used for the space coordinate rather than x, because the one-dimensional flow will be allowed in the curved direction along a streamline, and not restricted to just the direction of the channel. For such one-dimensional flows the magnitude of the acceleration is given by

$$a = \frac{dV}{dt} = V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t}$$

Consider an element of fluid of length ds and cross-sectional area dA along a streamline in the direction s, as shown in the sketch below. Applying Newton's second law $\sum F_s = ma_s$ to this element results in (note that only magnitudes are used since only one direction s is involved)



$$p\,dA - \left(p + \frac{\partial p}{\partial s} ds\right) dA - \tau dP\,ds - \rho g\,dA\,ds \times \sin(\theta) = \rho\,dA\,ds\left(V\frac{\partial V}{\partial s} + \frac{\partial V}{\partial t}\right)$$

where P is the perimeter of the element. Noting that $\sin\theta = \partial z/\partial s$, dividing by $(-\rho\ dA\ ds)$ and simplifying results in

$$\frac{1}{\rho}\frac{\partial p}{\partial s}+\frac{\tau}{\rho}\frac{dP}{dA}+g\frac{\partial z}{\partial s}+V\frac{\partial V}{\partial s}+\frac{\partial V}{\partial t}=0$$

It is helpful in understanding equations to follow their dimensions. In the equation immediately above, the dimensions are force divided by mass, since every term started out as a force which was divided by the $\rho dAds$, or the mass of the differential element.

If flow is steady then $\partial V/\partial t = 0$, and the variables are only a function of s and, therefore, the above equation can be integrated along a streamline to give

$$\frac{p}{\rho}+\int\frac{\tau\,dp}{\rho\,dA}ds+gz+\frac{v^2}{2}=\text{constant}$$

Now the dimensions of the equation are force times length divided by mass. The product of force and length is work or energy. Thus, upon integrating along the streamline each term in the equation represents a form of fluid energy per unit mass. Dividing by g results in the Bernoulli Equation, that has units of energy per unit weight of fluid, or

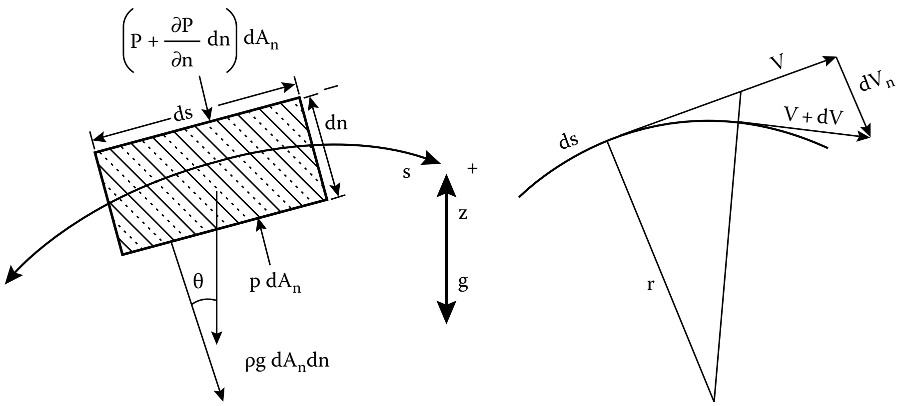
$$\frac{p}{\gamma}+z+\frac{v^2}{2g}+\int\frac{\tau\,dp}{\gamma\,dA}ds=H(\text{constant}) \tag{1.18}$$

The integral in Equation 1.18 cannot be evaluated in general since τ is not a known function of s, and therefore its value over the distance between a couple of points in the flow is designated as the headloss h_{L1-2} and Equation 1.18 is commonly applied between the two points in the following form:

$$\frac{p_1}{\gamma}+z_1+\frac{V_1^2}{2g}=\frac{p_2}{\gamma}+z_2+\frac{V_2^2}{2g}+h_{L_{1-2}} \tag{1.18a}$$

Each term in Equation 1.18 is considered a head because they all have the dimensions of length. In ES units this length is given in feet, and in SI units this length is in meters. It is important to note that terms are actually energy per unit weight of fluid. That is, when using ES units each term represents some form of energy in ft-lbs that each pound of fluid contains, and when using SI units each term is this energy in N-m that each Newton weight of fluid contains. p/γ is referred to as the pressure head, and represents the potential energy due to pressure per unit weight. z is the elevation head and is the potential energy due to the vertical position of the fluid per unit weight, and $V^2/2g$ is the kinetic energy per unit weight. The sum of p/γ and z is called the piezometric head.

To visualize these energies first consider the elevation head z. If a unit weight of fluid were a distance z above a datum then it would be capable of doing z times its weight of work in dropping under gravity to the datum. Since energy is the ability to do work, the potential energy that this unit weight of fluid has equals z. To visualize the potential energy due to the pressure of a fluid consider an infinitely large reservoir of liquid. At the bottom of this reservoir there is a cylinder that can extract work from the fluid due to its constant pressure here. This pressure acts over the area of the piston within the cylinder, and if the piston is permitted to move through a distance L, the amount of work done on it by the fluid equals pAL. The weight of fluid involved in doing this work equals γAL. Therefore, the energy per unit weight of this fluid is $pAL/(\gamma AL) = p/\gamma$. Note from the hydrostatic equations that $p/\gamma = h$ the height of the liquid above the cylinder. Thus, at the top of the reservoir the fluid contains elevation head, $z = h$, and at the bottom the liquid contains pressure head and at neither position does the liquid possess any velocity head since it is not moving, but these sum to a constant value. To understand that the velocity head $V^2/2g$ is the kinetic energy per unit weight, note that kinetic energy equals 1/2 the mass times the velocity squared, or $\rho(Vol)V^2/2$. Dividing this by the weight of this mass of fluid gives K.E./(unit weight) = $\{\rho(Vol)V^2/2\}/\{g\rho(Vol)\} = V^2/2g$.



Some applications in open-channel flow call for looking at what happens across streamlines. Flow over the crest of a dam is one such application. Therefore, consider applying Newton's second law normal to the streamlines, as shown in the sketch. If forces on this element due to viscous shear are ignored then the summation of forces in the direction of an outward normal equated to the mass times outward normal acceleration results in

$$pdA_n - \left(p + \frac{\partial p}{\partial n} dn\right) dA - \rho g \, dA_n \cos(\theta) = \rho dA_n \left(\frac{V^2}{r} + \frac{\partial V_n}{\partial t}\right)$$

The $\cos\theta = \partial z/\partial n$ and the last quantity within parenthesis is the acceleration in the normal direction. To see the latter, note that $V_n = f(s,t)$, and therefore by the chain rule,

$$a_n = \frac{\partial V_n}{\partial s} V + \frac{\partial V_n}{\partial t} \quad \text{where } V = \frac{\partial s}{\partial t}$$

From the small vector diagram in the sketch above it can be seen from similar triangles that

$$\frac{dV_n}{V} = \frac{ds}{r} \quad \text{and therefore } \frac{\partial V_n}{\partial s} = \frac{V}{r}$$

dividing the above force equation by $\rho dA_n dn$, and simplifying gives

$$\frac{1}{\rho}\frac{\partial p}{\partial n} + g\frac{\partial z}{\partial n} + \frac{V^2}{r} + \frac{\partial V_n}{\partial t} = 0$$

Special cases are worth examining. If the flow is steady and streamlines are straight then the last two terms in the last equation are zero and upon integrating in the n-direction, the following is obtained:

$$\frac{p}{\gamma} + z = \text{constant}$$

which indicates that the pressure distribution in the normal direction is hydrostatic in a moving flow provided the streamlines are straight. Since this is the assumption made to classify flows as one dimensional, pressure distributions are generally considered hydrostatic unless the channel is steep as discussed previously.

Before the above equation can be integrated with curved streamlines, it is necessary to have a relationship between V and r. Two special cases will be considered; irrotational flow, and solid body, or forced vortex, rotational flow. Real flows are between these two cases. Fluid viscosity must be absent for a flow to be irrotational, whereas the rotation of a forced vortex follows the solid body rotation law that the velocity increases with the radius of rotation, $V = r\omega$.

In general, a flow is irrotational if the vorticity or cross-product of the differential del operator and the velocity vector equal zero, or $\xi = \notin X\mathbf{V} = 0$ where (using the Cartesian coordinate system)

$$v = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

and i, j, and k are unit vectors in the x-, y-, and z-directions respectively.

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

and i, j, and k are unit vectors in the x, y, and z.

The component of vorticity for the plane of the curved flow above in the natural coordinate system is $\partial V_n/\partial s - \partial V/\partial n$ and if this flow is irrotational then this quantity equals zero. Since $\partial V_n/\partial s = V/r$, the term V^2/r in the above equation can be written as $V\partial V/\partial n$, permitting the equation to be integrated in the normal direction, giving

$$\frac{p}{\gamma} + z + \frac{v^2}{2g} = H\,(\text{constant}) \tag{1.18b}$$

or identically the same as Equation 1.18, without the viscous loss term; but since viscous forces were not included one would expect this. Therefore, if the flow is irrotational the Bernoulli equation applies across, as well as, along streamlines. The usual concept is that the Bernoulli equation is applied for the entire section of a one-dimensional flow. If the radius of curvature of the streamline is constant, it can be shown that for an irrotational flow the velocity times the radius is constant, or Vr = constant; thus, as r becomes zero V must become infinite.

For a forced vortex, the vorticity equals twice the actual rotation about an axis, or

$$\frac{\partial v_n}{\partial s} - \frac{\partial v}{\partial n} = 2\omega = \frac{2v}{r}$$

Therefore, $\partial V/\partial n = -V/r$ and integration of the above equation in the normal direction gives

$$\frac{p}{\gamma} + z - \frac{v^2}{2g} = H\,(\text{constant}) \tag{1.18bc}$$

(Note the minus sign in front of the velocity head.)

A centrifugal pump causes the flow within the housing of the pump to nearly follow this forced vortex law. The exception is that there must be a radial component of velocity for the fluid to pass onto and leave the pump impeller. This radial component will be governed by the usual Bernoulli equation and, therefore, the head produced by a pump will be approximated by (with $z_i = z_o$)

$$h_p = \frac{p_2 - p_1}{\gamma} + \left(\frac{V_i^2 - V_o^2}{2g}\right)_r + \left(\frac{V_o^2 - V_i^2}{2g}\right)_t$$

where subscripts i and o denote inlet and outlet, respectively, and subscripts r and t denote radial and tangential directions, respectively.

1.11.1 Kinetic Energy Correction Coefficient, α

The Bernoulli equation developed above is generally applied for an entire section of an open-channel flow, and the average velocity for the cross section is used to compute the velocity head, or kinetic energy per unit weight. Since the velocity is not constant throughout the entire section, the actual or exact kinetic energy per unit weight, K.E./Wt, can be defined by multiplying the velocity head by a correction coefficient α, or

$$\frac{\text{K.E.}}{\text{Wt}} = \alpha\left(\frac{V^2}{2g}\right)$$

The evaluation of α involves determining the exact kinetic energy and dividing this by the velocity head. The exact, or correct, kinetic energy K.E., passing a cross section is the integral over the area of the kinetic energy through a differential area dA, or

$$\text{K.E.} = \frac{\int \rho v^2 dQ}{2} = \frac{\int \rho v^3 dA}{2}$$

Dividing K.E. by $g\rho Q = g\rho VA$ gives K.E./Wt and, therefore,

$$\alpha = \frac{\text{K.E./Wt}}{v^2/2g} = \frac{\int v^2 dQ}{v^2 Q} = \frac{\int v^2 dA}{v^2 A} \tag{1.19}$$

For a natural channel, or when it is not possible to define v and A as functions of the depth, the integral sign is replaced by a summation, and Equation 1.19 becomes

$$\alpha = \frac{\text{K.E./Wt}}{v^2/(2g)} = \frac{\sum v^2 \Delta Q}{V^2 Q} = \frac{\sum v^2 \Delta A}{V^2 A} \tag{1.19a}$$

The value of α is only slightly larger than unity for typical turbulent open-channel flows. Furthermore, when the flow is subcritical, as it always is in natural rivers and streams, the velocity head will be smaller than the other terms in the Bernoulli equation, and therefore it is common practice to take α as 1.

EXAMPLE PROBLEM 1.9

Water flows over the crest of a small dam that is 4 m high at a rate of 48.2 m^3/s. The depth of water over the dam's crest is 6 m, and the channel is rectangular and 10 m wide. Determine the depth and velocity in the channel downstream from the dam. Ignore any frictional losses. Solve the problem twice; first, under the assumption that at both sections the kinetic energy correction coefficient is unity, and second, the αs are not unity, and that the following dimensionless velocity profiles are known.

Over the Dam's Crest								
y/Y	0.00	0.07143	0.2143	0.3571	0.5357	0.7143	0.8571	1.0000
v/V	0.00	0.3529	0.8235	1.0353	1.1764	1.2823	1.2235	1.0823
In the Channel at the Dam's Toe								
y/Y	0.00	0.0357	0.0714	0.1429	0.3571	0.7143	1.0000	
v/V	0.00	0.5743	0.8614	0.9763	1.0452	1.0567	1.0452	

Solution

For the first part with the α's = 1, the upstream velocity is determined from $V_1 = Q/A_1 = 48.2/60 = 0.8033$ m/s. Apply the Bernoulli equation,

$$Y_2 + \frac{Q^2}{2g(10Y_2)^2} = Y_1 + \frac{V_1^2}{2g} = 6.033 \text{ m}$$

This is a cubic equation that must be solved by trial, or an iterative technique such as the Newton Method described in Appendix B. The solution is $Y_2 = 0.461$ m. Another solution to this cubic equation is 6 m, but this is obviously not the wanted root since it is the upstream depth. The corresponding velocity is $V_2 = 48.2/4.61 = 10.46$ m/s. For the second part of the problem $q = 4.82 = \int v\, dy$, or $4.82V = \int (v/V)dy$, in which the limits of integration are 0–6. If dy is replaced by the dimensionless depth y/Y, then the upper limit changes to 1. There are many methods that can be used to evaluate this integral including plotting the profile and determining the area by counting squares. We will use a cubic spline as described in Appendix B to interpolate between the given values and Simpson's rule (described in Appendix C) to obtain a numerical integral. A FORTRAN program EPRB1_9.FOR listing to accomplish this task is given below.

```
      EXTERNAL EQUAT
      COMMON /TRA/Y(8),V(8),DQV(8),I1,IP,N
      I1=1
      IP=2
      WRITE(*,*)' Give no. and then this may pairs'
      READ(*,*) N,(Y(I),V(I),I=1,N)
      CALL SPLINESU(N,Y,V,DQV,D,0)
      CALL SIMPR(EQUAT,0.,Y(N),FLOW,1.E-5,30)
      WRITE(*,*) FLOW
      END
      FUNCTION EQUAT(YY)
      COMMON /TRA/Y(8),V(8),DQV(8),I1,IP,N
1     IF(YY.LT.Y(IP).OR.IP.EQ.N) GO TO 2
      I1=IP
      IP=IP+1
      GO TO 1
2     IF(YY.GT.Y(I1).OR.I1.EQ.1) GO TO 3
      IP=I1
      I1=I1-1
      GO TO 2
3     DY=Y(IP)-Y(I1)
      AY=(Y(IP)-YY)/DY
      BY=1.-AY
      EQUAT=AY*V(I1)+BY*V(IP)+((AY*AY-1.)*AY*DQV(I1)+(BY*BY-1.)*
     &BY*DQV(IP))*DY*DY/6.
      RETURN
      END
```

In the program, the subroutine SPLINESU is called to provide the values of the second derivatives at the points. The common block passes the arrays of dimensionless depths y/Y and corresponding dimensionless velocities, v/V, i.e., the values in the above table over the crest of the weir. The result of this numerical integration is 1.000 and (meaning that the above data truly give a proper dimensionless velocity profile), therefore, the average velocity V_1 is the same as above or $V_1 = 0.8033$ m/s. Next, the last two lines of the function subprogram EQUAT can be changed to the following:

```
 EQUAT=(AY*V(I1)+BY*V(IP)+((AY*AY-1.)*AY*DQV(I1)+(BY*BY-1.)*
&BY*DQV(IP))*DY*DY/6.)**3
```

Giving both the above tables of dimensionless values provides the solution for the kinetic energy correction coefficients: $\alpha_1 = 1.2557$, and $\alpha_2 = 1.0555$. Now the implicit equation that must be solved is

$$Y_2 + 1.0555\left\{\frac{4.82^2}{Y_2^2 19.62}\right\} = 6 + 1.2557\left(\frac{0.8033^2}{19.62}\right) = 6.0413$$

The solution is $Y_2 = 0.472$ m. The average velocity $V_2 = 4.82/0.472 = 10.21$ m/s.

EXAMPLE PROBLEM 1.10

A pump increases the pressure from its inlet to its outlet pipe by 52 psi. The inlet and outlet pipes are 10 and 12 in. in diameter, respectively, and the flow rate is Q = 6 cfs. Determine the amount of energy per unit weight that the pump supplies to the water. If the pump has an efficiency of 85%, determine the horsepower required to drive the pump.

Solution

In applying the energy equation to this problem the head of the pump needs to be added to the left side of the Bernoulli equation, or

$$\left(\frac{V^2}{2g} + \frac{p}{\gamma}\right)_1 + h_p = \left(\frac{V^2}{2g} + \frac{p}{\gamma}\right)_2$$

Upon substituting the known values into this equation gives $h_p = 120.97$ ft. To get energy from the head given by Bernoulli's equation multiply by the weight flow rate γQ, and therefore the horsepower is

$$HP = \frac{\gamma Q h_p}{550e} = 96.88\,\text{hp (550 is the conversion for hp from energy in ft-lb/s.)}$$

EXAMPLE PROBLEM 1.11

Assume that the water at the surface of a river flowing around a bend follows the free vortex law. The radius of the inside of the bend is 30 m, and at the outside of the bend the radius is 45 m. If the velocity at the inside of the bend is measured as 0.8 m/s, what is the velocity on the surface at the outside of the bend?

Solution

The free vortex law indicates that Vr = constant. This constant equals 0.8(30) and, therefore, the velocity at the outside of the bend is $V_o = (0.8)(30)/45 = 0.533$ m/s. Since at the bottom of the river the velocity will be smaller than at the surface, there is a net flow outward on the surface and an inflow at the bottom. This movement creates a secondary motion in a river that moves sediment toward the downstream inside of the bend. A pumping station here would bring in river sediments; therefore, such a pumping station is best located on the outside of the bend.

EXAMPLE PROBLEM 1.12

The velocity distribution in a wide open channel can be defined by the following logarithmic function between the channel bottom and a position y = 0.1 ft above the bottom: v = C/Ln(y), and a second-degree polynomial for the rest of the depth. When the depth of flow is 2.6 ft, velocities are measured at three positions with the following results:

Depth, ft	0.1	2.1	2.6
Velocity, fps	2.3	2.45	2.35

Determine the kinetic energy correction coefficient, α.

Solution

To obtain the equation for the velocity distribution over the bottom 0.1 ft substitute 2.3 in for v, and 0.1 for y in the given equation and solve for C. This gives C = 2.3 Ln(0.1) = −5.296. The equation for the rest of the profile is obtained by evaluating a,b, and c in $v = a + by + cy^2$ by Lagrange's interpolation equation described in Appendix B, or solving a, b, and c from the three equations obtained by substituting the y and v from above into this polynomial equation to give

$$v = 2.2694 + 0.317y - 0.110y^2$$

The flow rate q per unit width of channel is found first from $Q = \int v\, dA$ with the integration carried out in two parts, or

$$q = -5.296\int_0^{-0.1} \frac{dy}{Ln(y)} + \int_{0.1}^{2.4} (2.2694 + 0.317y - 0.11y^2)dy = 0.1715 + 6.0989 = 6.2705 \text{ cfs/ft.}$$

The first integral was evaluated numerically because of the complexity of a closed-form integration. Next, evaluate $\int v^3 dA$ as shown below.

$$\int v^3 dA = -5.296\int_0^{0.1} \frac{dy}{Ln(y)^3} + \int_{0.1}^{2.6} (2.2694 + 0.317y - 0.11y^2)^3 dy = 0.581 + 36.356 = 36.937$$

To evaluate α substitute into $\alpha = \int v^3 dA/(qV^2) = 36.937/[6.2705(2.4117)^2] = 1.013$.

1.12 MOMENTUM PRINCIPLE IN FLUID FLOW

An equally important skill for solving a fluid-flow problem to that associated with the use of the energy principle is the application of the momentum principle. The momentum principle must generally be utilized to solve problems dealing with external forces acting on the fluid. Its use occurs when dealing with the overall flow picture rather than a detailed examination of individual fluid particles, and their associated internal processes. The momentum principle produces vectors, and therefore is the natural choice when external forces are involved. It can be applied to one-, two- or three-dimensional flows, but when dealing with open-channel hydraulics the common use of the momentum principle will be for one- and two-dimensional flows.

The momentum principle in fluid mechanics is based on Newton's second law of motion, **F** = m**a**, where the force **F** and the acceleration **a** are vector quantities. Remember, to define a vector it is necessary that its direction as well as its magnitude are given, and that a change in direction (with the magnitude constant) is equally important to a magnitude change (with the direction remaining constant). Since acceleration is the time rate of change of velocity, Newton's second law can be written as

$$\mathbf{F} = m\frac{dV}{dt} = \frac{m}{dt}dV$$

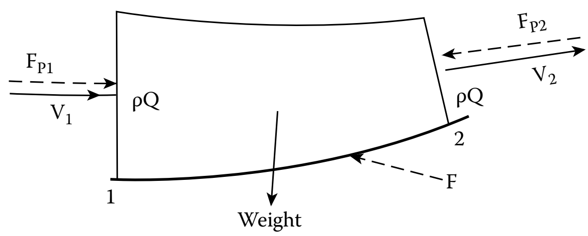
In writing the term after the last equal sign in the above equation the concept is that rather than dealing with a derivative we are dealing with a differential change in the velocity vector d**V** within the differential time dt. In solid mechanics dt is moved to the other side of the equal sign, associated with vector **F** and the product **F**Δt is called the linear impulse, and this is equated to the change in linear momentum that is defined as Δ(m**V**). In fluid mechanics, however, we deal with flow rates, and dm/dt will be denoted as m, the mass flow rate, which equals ρQ. The interpretation of m/dt is the change in mass per time or a mass flow rate ρQ. Therefore, for fluid applications the above equation can be written as

$$\mathbf{R} = \rho Q \Delta V$$

where **R** is the resultant force on the CV. When applied to a control volume, CV, of fluid between the two sections 1 and 2, as shown in the sketch below, it is written as

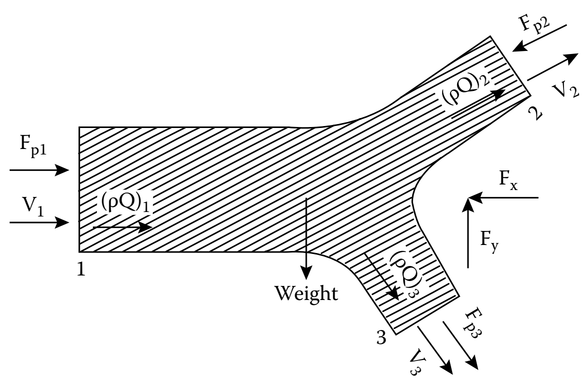
$$\mathbf{R} = \rho Q(V_2 - V_1) \tag{1.20}$$

The forces on the CV include the body forces, i.e., the weight of the CV, as well as external forces that may be applied by adjacent fluid and structures, etc., that act against the CV of fluid.



If the flow divides so that it leaves through sections 2 and 3 as shown below in the sketch, then the momentum equation would need to be written as

$$\mathbf{R} = (\rho QV)_2 + (\rho QV)_3 - (\rho QV)_1$$



The quantity (ρQ**V**) is called the momentum flux. A more generalized expression of the momentum principle states that the resultant external force acting on a CV of fluid equals the difference between the momentum flux vectors leaving the CV minus the momentum flux vectors entering the CV. Expressed as an equation the momentum principle becomes

$$\mathbf{R} = \sum (\rho QV)_{\text{leaving}} - \sum (\rho QV)_{\text{entering}} \tag{1.21}$$

Since Equation 1.21 is a vector equation, for two- and three-dimensional problems it can be written as two or three scalar equations, respectively, in coordinate directions of that space. For a one-dimensional problem the coordinate direction will coincide with the directions of the vectors on both sides of the equal sign in Equation 1.21 and the vectors become effectively scalar quantities. For a two-dimensional problem in Cartesian coordinates Equation 1.21 can be written as the two scalar equations,

$$\sum F_x = (\rho Q V_{x2} - \rho Q V_{x1}) \tag{1.21a}$$

and

$$\sum F_y = (\rho Q V_{y2} - \rho Q V_{y1}) \tag{1.21b}$$

where
the subscripts x and y represent the x and y coordinate directions, respectively
the 1 and 2 subscripts represent entering and leaving the control volume, respectively

The summation of forces in the x- and y-directions include all external and body forces acting on this CV.

The application of the momentum principle to most problems includes the following five steps:

1. Construct a control volume of the fluid. The upstream section(s) of this CV is (are) upstream from the occurrence of interest to a section of the flow where the velocity and other variables of the problem can be defined. Likewise, the downstream section(s) is (are) taken where fluid variables can be defined. Adjacent fluid and structures are removed from this CV.
2. Replace all removed fluid and structural components by the equivalent forces that they apply to the fluid in the CV.
3. Apply Equations 1.21a and 1.21b. In applying these equations it is vital to watch signs. Should the component of force or velocity be in the direction opposite to the positive direction taken for either x or y, then it is negative in these equations. Likewise, any vector not in either the x- or y-direction must be multiplied by the cos or sin of the appropriate angle to get the component in the coordinate direction.
4. Solve for the unknowns. Since Equations 1.21a and 1.21b are two equations, two unknowns can be obtained. For a one-dimensional problem only one unknown may exist, and for a three-dimensional problem three unknowns may exist. These unknowns may be the components of one unknown vector, such as the force.
5. If the unknown(s) solved for in step 4 consists of a vector, note that its sense will be opposite on the structure than on the fluid. The value obtained by the above procedure is the force on the CV of fluid.

In implementing step 2 above, the fluid adjacent to the CV of fluid is replaced by an equivalent force. In applying this to open-channel flows, this adjacent fluid will often have a hydrostatic pressure distribution, and the force will be a hydrostatic force. A hydrostatic, F_s, force equals the cross-sectional area times the pressure at the centroid of this area, or

$$F_s = p_c A = \gamma h_c A = \gamma A h_c \tag{1.22}$$

where h_c is the distance from the free surface to the centroid of the area. The last form given in Equation 1.22 contains the term Ah_c which is the first moment of area about the free surface. For rectangular and trapezoidal cross sections, this first moment of area is easily obtained. For a rectangle it equals the area times the distance from the surface to the center or $bY^2/2$. For a trapezoid it can be obtained by the method of composite areas, e.g., the rectangle and the two triangles and is $Ah_c = bY^2/2 + mY^3/3$. For other cross sections such as an arc of a circle, this first momentum of area must be obtained by integration. See Appendix A and Table A.1 for this first moment of area for a circular cross section.

EXAMPLE PROBLEM 1.13

A trapezoidal channel with a bottom width b_1 = 14 ft and a side slope m_1 = 1.5 divides into a rectangular channel with b_2 = 8 ft whose direction is 30° from the direction of the upstream channel, and a pipe with a 4 ft diameter whose direction is 45° from the direction of the upstream channel. The top of the pipe is below the water surface so it flows full. The depth in the downstream channel is 5.2 ft. The pipe drops in elevation by 110 ft over its length of 1500 ft, and delivers water at a pressure of 40 psi. The wall roughness for the pipe is e = 0.02 ft. Determine the resultant force on this structure for the design flow rate of Q_1 = 450 cfs.

Solution

First applying the Bernoulli equation between sections 1 and 2 and then sections 1 and 3 (or 2 and 3) along with the continuity equation and equations for pipe flow give the following five simultaneous equations to solve for the five unknowns Q_2, Q_3, f, Y_1, and p_t (pressure on top of pipe):

$$H = \frac{Q_1^2}{2gA_1^2} + Y_1 \frac{Q_2^2}{2gA_2^2} = \frac{Q_2^2}{64.4(38.4)^2} + 5.2 \tag{1}$$

$$H = \frac{Q_3^2}{2gA_3^2} + \frac{pt}{\gamma} + 4.0 \tag{2}$$

$$Q_2 + Q_3 = 450 \tag{3}$$

$$h_f = \frac{f(L/D)Q_3^2}{2gA_3^2} = 110 - \frac{40(144)}{\gamma} + H \text{ (the Darcy–Weisbach equation from pipe flow)} \tag{4}$$

and

$$\frac{1}{\sqrt{f}} = 1.14 - 2\log\left(\frac{e}{D} + \frac{9.35}{\mathrm{Re}\sqrt{f}}\right) \text{(the ColeBrook–White equation from pipe flow)} \tag{5}$$

The solution gives Q_2 = 322.40 cfs, Q_3 = 127.60 cfs, f = 0.03036, Y_1 = 5.965 ft and p_t = 33.17 psf. (To solve these five equations you need to use the techniques described in Appendix B as the Newton method, or Mathcad or MATLAB on your PC.) From these the forces on the CV become

$$F_1 = \gamma A_1 h_{c1} = \gamma\left(\frac{14Y_1^2}{2} + \frac{1.5Y_1^3}{3}\right) = 22{,}162 \text{ lbs}, \quad F_2 = \gamma A_2 h_{c2} = \gamma 41.6(2.6) = 6{,}749.2 \text{ lbs},$$

$$F_3 = \gamma A_3\left(\frac{p_t}{\gamma} + 2\right) = 1{,}985.1 \text{ lbs}.$$

Applying the momentum equation in the x- and y-directions, respectively, gives the following two equations:

x-direction

$$22{,}162 - 6{,}749.18\cos 30 - 1{,}985.1\cos 45 - R_x = 1.94[(322.4)7.75\cos 30$$
$$+ (127.6)10.15\cos 45 - 450(3.288)] = 3{,}105.1$$

and solving $R_x = 11{,}808$ lbs

y-direction

$$-6{,}749.2\sin 30 + 1{,}985.1\sin 45 + R_y = 1.94[(322.40)7.75\sin 30$$
$$-(127.6)10.154\sin 45] = 646.3$$

and solving $R_y = 2{,}617$ lbs

which results in R = 12,095 lbs at an angle of 12.5° from the horizontal with its direction downward to the right on the structure.

1.12.1 Momentum Flux Correction Coefficient, β

When the momentum principle is applied as if a three-dimensional problem is a two-dimensional problem, as in Example Problem 1.13, or as if a two-dimensional problem is a one-dimensional problem, i.e., when an average velocity is used for a section of flow, then $\rho \mathbf{Q}\mathbf{V}$ does not represent the exact momentum flux, EMF, passing the section. The EMF can be obtained by multiplying by a correction coefficient, β in a manner similar to the use of α as a correction for the velocity head. By definition this coefficient, β, is

$$\beta = \frac{\text{EMF}}{\rho QV} = \frac{\rho\int v\,dQ}{\rho QV} = \frac{\int v^2 dA}{V^2 A} \tag{1.23}$$

For a natural channel the integral is replaced by a summation and Equation 1.23 becomes

$$\beta = \frac{\text{EMF}}{\rho QV} = \frac{\rho\sum v\Delta Q}{\rho QV} = \frac{\sum v^2\Delta A}{V^2 A} \tag{1.23a}$$

and Equation 1.20 becomes

$$\mathbf{R} = (\rho\beta QV)_2 - (\rho\beta QV)_1 \tag{1.20}$$

EXAMPLE PROBLEM 1.14

The velocity distribution with depth in a wide rectangular channel is given by v = Ln(1 + 10y) where y is the distance from the channel bottom to the position where the velocity v is given. If the depth of flow Y = 5 ft, determine the momentum correction coefficient β.

Solution

Per unit width $\beta = \int v^2 dy/(qV)$ in which q is the volumetric flow rate per unit width and equal the depth time the average velocity V. To solve for β it is necessary to first solve $q = \int v dy = \int \text{Ln}(1 + 10y)dy$ with limits of 0 to 5. This can be integrated by letting x = 1 + 10y, and dy = dx/10; thus $q = \int \text{Ln}(x)dx/10$ with limits from 1 to 51, or q = [x Ln(x) − x]/10 evaluated between 51 and 1; giving q = 15.0523107. You will find it instructive to have your HP48 calculator obtain this answer by numerically integrating the above equation, and also have it carry out the symbolic integration. Since you will be using computer programs throughout this course, how about gaining some experience with this problem, but utilizing the Simpson's numerical integration described in Appendix B? The main program that calls SIMPR is listed below; the first column obtains q and also prints out the average velocity, and the second column integrates $\int v^2/dy$ between 0 and 5, and also prints out β.

```
EXTERNAL EQUAT
CALL SIMPR(EQUAT,0.,5.,VALUE)
WRITE(*,*) VALUE,VALUE/5.
END
FUNCTION EQUAT(Y)
EQUAT=ALOG(1.+10.*Y)
RETURN
END
```

```
EXTERNAL EQUAT
CALL SIMPR(EQUAT,0.,5.,VALUE)
WRITE(*,*) VALUE,VALUE/45.312708
END
FUNCTION EQUAT(Y)
EQUAT=ALOG(1.+10.*Y)**2
RETURN
END
```

The results are q = 15.05205 cfs/ft, V = 3.0104 fps from the first integration, and $\int v^2 dy = 48.73822$ and $\beta = 1.0756$.

EXAMPLE PROBLEM 1.15

Determine α and β for the natural channel in Example Problem 1.5.

Solution

The numerator in Equation 1.19 can be viewed as a double integral since the differential area dA = dy dx, or this numerator is

$$\iint v^3 dy\,dx = \int_0^{X_f}\left[V_s^3 Y \int_0^1 V'^3 dy'\right]dx = \left\{\int_0^1 v'^3 dy'\right\}\int_0^{X_f}\left(V_s^3 Y\right)dx$$

Note that the quantity within { } after the last equal sign is constant, since the dimensional velocity profile does not change from position to position and, therefore, it needs to be evaluated only once. Thus, the argument of numerical integration with respect to x (across the channel) involves the surface velocity Vs cubed, multiplied by the depth Y at this section, i.e., Vs^3Y, and both Vs and Y can be evaluated using the cubic spline interpolation of the data in the first table in Example Problem 1.5. The same applies for the numerator in Equation 1.23 to evaluate β; with the surface velocity squared rather than cubed. The program EXPR1_15 (in both FORTRAN and C) is given below to provide the numerator for both Equations 1.19 and 1.23. It also provides the area of the cross section, which can be done by evaluating the last integral in the above equation with the exponent of the velocity equal to zero, since $V_s^0 = 1$. Note that this program contains the additional variable NE that is used as the exponent of the surface velocity. When NE = 0, the area of the cross section is evaluated; when NE = 2, the numerator for β is evaluated, and when NE = 3, the numerator for α is evaluated.

Listing for Program EPR1_15.FOR

```
      EXTERNAL VPROF,Dq
      REAL DUM(30)
      COMMON YP(20),VP(20),X(30),Y(30),VS(30),
     &D2VP(20),D2VS(30),D2Y(30),I1,I2,NP,NX,NE
      READ(2,*) NP,(YP(I),VP(I),I=1,NP)
      READ(2,*) NX,(X(I),Y(I),VS(I),I=1,NX)
      CALL SPLINESU(NP,YP,VP,D2VP,DUM,0)
      CALL SPLINESU(NX,X,Y,D2Y,DUM,0)
      CALL SPLINESU(NX,X,VS,D2VS,DUM,0)
      NE=0
      CALL SIMPR(Dq,0.,X(NX),AREA,1.E-6,20)
      WRITE(*,*)' Area=',AREA
      NE=2
5     I1=1
```

```
      I2=2
      CALL SIMPR(VPROF,0.,1.,qPRIM,1.E-6,20)
      WRITE(*,*)' Integral of dimensionless',
     &velocity**',NE,' profile=',qPRIM
      I1=1
      I2=2
      CALL SIMPR(Dq,0.,X(NX),Q,1.E-4,20)
      WRITE(*,*)' Integral with respect to x='
     &,Q*qPRIM
      NE=NE+1
      IF(NE.LT.4) GO TO 5
      END
      FUNCTION VPROF(YY)
      COMMON YP(20),VP(20),X(30),Y(30),VS(30),
     &D2VP(20),D2VS(30),D2Y(30),I1,I2,NP,NX,NE
1     IF(YY.LT.YP(I2) .OR. I2.EQ.NP) GO TO 2
      I1=I2
      I2=I2+1
      GO TO 1
2     IF(YY.GE.YP(I1) .OR. I1.EQ.1) GO TO 3
      I2=I1
      I1=I1-1
      GO TO 2
3     DYP=YP(I2)-YP(I1)
      A=(YP(I2)-YY)/DYP
      B=1.-A
      VPROF=(A*VP(I1)+B*VP(I2)+((A*A-1.)*A*
     &D2VP(I1)+(B*B-1.)*B*D2VP(I2))*DYP**2/6.)**NE
      RETURN
      END
      FUNCTION Dq(XX)
      COMMON YP(20),VP(20),X(30),Y(30),VS(30),
     &D2VP(20),D2VS(30),D2Y(30),I1,I2,NP,NX,NE
1     IF(XX.LT.X(I2) .OR. I2.EQ.NX) GO TO 2
      I1=I2
      I2=I2+1
      GO TO 1
2     IF(XX.GE.X(I1) .OR. I1.EQ.1) GO TO 3
      I2=I1
      I1=I1-1
      GO TO 2
3     DX=X(I2)-X(I1)
      A=(X(I2)-XX)/DX
      B=1.-A
      AA=A*(A*A-1.)*DX**2/6.
      BB=B*(B*B-1.)*DX**2/6.
      DEPTH=A*Y(I1)+B*Y(I2)+AA*D2Y(I1)+BB*D2Y(I2)
      VSURF=A*VS(I1)+B*VS(I2)+AA*D2VS(I1)+
     &BB*D2VS(I2)
      Dq=DEPTH*VSURF**NE
      RETURN
      END
```

Listing for Program EXPR1_15.C

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
```

```
float yp[20],vp[20],x[30],y[30],vs[30],d2vp[20],d2y[30],d2vs[30];
int i1,i2,np,nx,ne;
extern float simpr(float (*equat)(float xx),float xb,float xe,\
  float err,int max);
extern void splinesu(int n,float *x,float *y, float *d2y,float *d,\
  int ity);
float vprof(float yy){float a,b,dyp;
 while((yy>=yp[i2])&&(i2<np-1)){i1=i2;i2++;}
 while((yy<yp[i1])&&(i1>0)){i2=i1;i1–;} dyp=yp[i2]-yp[i1];
 a=(yp[i2]-yy)/dyp;b=1.-a;
 return pow(a*vp[i1]+b*vp[i2]+((a*a-1.)*a*d2vp[i1]+(b*b-1.)\
   *b*d2vp[i2])*dyp*dyp/6.,ne);
} // End of vprof
float qp(float xx){float a,b,dx,aa,bb,depth;
 while((xx>=x[i2])&&(i1<nx-1)){i1=i2;i2++;}
 while((xx<x[i1])&&(i1>0)){i2=i1;i1--;}
 dx=x[i2]-x[i1];a=(x[i2]-xx)/dx;b=1.-a;
 aa=a*(a*a-1.)*dx*dx/6.;bb=b*(b*b-1.)*dx*dx/6.;
 depth=a*y[i1]+b*y[i2]+aa*d2y[i1]+bb*d2y[i2];
 return depth*pow(a*vs[i1]+b*vs[i2]+aa*d2vs[i1]+bb*d2vs[i2],ne);
} // End of qp
void main(void){FILE *fili; char filnam[20];int i;
 float dum[30],qprim,q;
 printf("Give input file name\n"); scanf("%s",filnam);
 if((fili=fopen(filnam, "r"))==NULL){printf("Cannot open file\n");
   exit(0);}
 fscanf(fili, "%d",&np);
 for(i=0;i<np;i++)fscanf(fili, "%f %f",&yp[i],&vp[i]);
 fscanf(fili, "%d",&nx);for(i=0;i<nx;i++)fscanf(fili, "%f %f %f",\
   &x[i],&y[i],&vs[i]);
 splinesu(np,yp,vp,d2vp,dum,0);splinesu(nx,x,y,d2y,dum,0);
 splinesu(nx,x,vs,d2vs,dum,0);
ne=0;printf("Area=%f\n",simpr(qp,0.,x[nx-1],1.e-6,20));ne=2;
L5:i1=0;i2=1; qprim=simpr(vprof,0.,1.,1.e-6,20); i1=0;i2=1;
printf("Integral of dimensionless velocity**%d\
  profile =%f\n",ne,qprim);
printf("Integral with respective to x =%f\n",\
  qprim*simpr(qp,0.,x[nx-1],1.e-4,20));
ne++; if(ne<4) goto L5;
}
```

Using as input the data in file EXPRB1_5.DAT, used in Example Problem 1.5, program EXPR1_15 produces the following as the solution (the units have been added):

AREA = 857.0909 m^2

Average velocity V = Q/A = 342.5057/857.0909 = 0.3991689 m/s
Integral of dimensionless velocity**2 profile = 0.8307201
Integral with respect to x = 153.15
Integral of dimensionless velocity**3 profile = 0.8180358
Integral with respect to x = 71.6953

From which V = Q/A = 342.5057/857.0909 = 0.399169 m/s

$\alpha = \mathrm{I}v^3 dA/(AV^3) = 71.6953/(857.0909\mathrm{x}.3991689^3) = 1.3108$
$\beta = \mathrm{I}v^2 dA/(AV^2) = 153.15/(857.0909\mathrm{x}.3991689^2) = 1.1189$

PROBLEMS

Terminology

1.1 Define the flows in the following situations according to (1) steady or unsteady; (2) uniform or nonuniform and if nonuniform, whether rapidly or gradually varied; (3) subcritical or supercritical; and (4) turbulent or laminar. If insufficient information is provided for a classification, indicate what other information is needed.

(a) A flow in a river that produces a storm hydrograph at the site of observation.

(b) The flow downstream from a gate supplying water to a prismatic channel from a constant water surface elevation reservoir.

(c) The same for as in (b) except the gate is slowly being closed.

(d) A constant flow rate entering a trapezoidal canal of constant cross section that supplies water to three turnouts (consider: (1) the flow upstream from the first turnout, (2) the flow between the turnouts, and (3) the section downstream from the last turnout). A depth of water is required for the canal to supply the last turnout.

(e) Water flow into a canal from a large lake with the control gate fully open. The canal has a constant cross section and a small bottom slope.

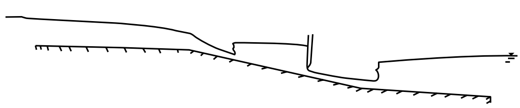
1.2 Water at a temperature of 55°F is flowing at a rate Q = 550 cfs in a trapezoidal channel with a bottom width of b = 12 ft, and a side slope m = 1.5. If the depth of flow is 4.5 ft, compute (a) the Froude number, (b) the Reynolds number, and (c) the speed of a small amplitude gravity wave in this flow. Classify the flow in this channel.

1.3 A cross section of a natural channel has the following transect data:

x (ft)	0	2	4	5	6	8	10	12	14	15	17
y (ft)	0	0.6	1.5	2.5	4.5	5.0	4.3	2.8	1.2	0.3	0

For a flow rate of Q = 280 cfs at a depth of 4.6 ft compute (a) the cross-sectional area, (b) the Froude number, and (c) the Reynolds number if the water temperature is T = 50°F.

1.4 The sketch below shows a long canal system whose bottom slope changes at several points, and is controlled by gates. Indicate the reaches where the flow is subcritical (downstream controlled) and where it is supercritical (under upstream control). Under uniform flow a steep channel will sustain a supercritical flow whereas a mild channel will sustain a subcritical flow.



1.5 Starting with the perfect gas law ρ = p/(RT) and the definition of bulk modulus, prove that a gas undergoing an isothermal compression or expansion has a bulk modulus equal to its absolute pressure.

1.6 The speed of sound propagates through a fluid with a velocity given by $c = \sqrt{E_v/\rho}$. Compare the speed of sound in pure water to that of water that contains 1% free air by volume.

1.7 Compute the pressure variation in the ocean to a depth of 4000 m if the compressibility is not ignored, and then compute it on the basis of an incompressible fluid. In this computation assume both g and E_v are constant. Take the density of ocean water as ρ = 1020 kg/m^3.

1.8 Water is flowing down a steep spillway with slope S_o = 0.30 at a depth of 0.8 m. Determine the pressure at the bottom of this channel.

1.9 A pressure transducer records 4.00 psi of pressure on the bottom of a steep channel with a bottom slope of two to one (horizontal-to-vertical distances). What are the normal and vertical depths of this flow? The bottom slope is constant.

1.10 At the base of a dam spillway the bottom of the channel suddenly changes from having a constant slope of S_o = 0.40 to a circular arc with a radius of 30 m. The normal depth of flow at this point of change is 0.75 m and the velocity V = 10 m/s. Assume the radius of curvature of the streamlines changes linearly from 30 to 20 m through this depth, and compute the pressure distribution through the depth of flow where the radius is 20 m, and also the pressure distribution along the bottom of the channel through the length where the radius of curvature varies. What is the force against a section of fluid per unit width of flow at the section where the circular arc first begins?

1.11 Water is being rotated in a cylindrical drum of 2 m radius and height 5 m at a rotational speed of (a) $\omega = 1200$ rpm and (b) $\omega = 40$ rpm. The axis of rotation is vertical. Compute the pressure distribution in the vertical direction at the following three radii: $r_1 = 0$, $r_2 = 1$ m, and $r_3 = 2$ m. When rotated, water is to the top of the tank.

1.12 Determine the average velocity, V, for a laminar flow that has a velocity profile given by Equation 1.11. How is the average velocity related to the velocity on the surface? How does this result relate to the area under a parabola?

1.13 Solve illustrative Example Problem 1.7 if the flow rate is $Q = 20\ m^3/s$ and Y still equals 3 m in the upstream channel.

Continuity

1.14 Prove whether the following two-dimensional flow fields are (1) continuous (satisfy the continuity equation), and (2) are rotational or irrotational.

(a) $u = U$ (constant) and $v = 0$
(b) $u = 10x/(x^2 + y^2)$ and $v = 10y/(x^2 + y^2)$. For this flow determine the tangential and radial components of velocity.
(c) $u = \sin(x)\cosh(y)$ and $v = \cos(2x)\sinh(y)$
(d) $u = 3x^2 - 2y^2$ and $v = 6xy$
(e) $u = \sinh(y)\cos(x)$ and $v = \cosh(y)\sin(x)$
(f) $u = 5(x^2 - y^2)$ and $v = 10xy$

1.15 The velocity and depth of water flowing in a 2 m diameter pipe are 3 m/s and 1.4 m at one section. At another section the depth equals 0.6 m. What is the velocity at this second section?

1.16 The flow from a trapezoidal channel with b = 10 m and m = 2 goes through a transition section and enters a circular culvert with an 8 m diameter. If the flow rate is 350 m^3/s and the depths in these two channels equal 4 m and 6.4 m respectively, what are the average velocities at the two sections.

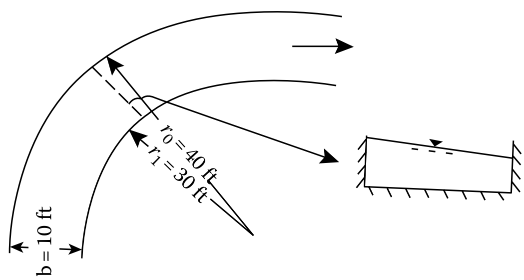
1.17 A river whose cross section is defined by the following x y data from the left bank (y is positive downward from the top of the left bank) has been gaged by a current meter giving the velocities shown at the 0.2 and 0.6 depth at five position across the river, when the river stage was 2 ft. Under the assumption that the average of these two velocities gives the average velocity for this incremental position across the river, determine the flow rate in the river.
Cross-section data for the river

x (ft)	0	2.0	4.0	6.0	8.0	10.0	12.0
y (ft)	0	0.8	1.1	2.3	2.1	1.0	0.1

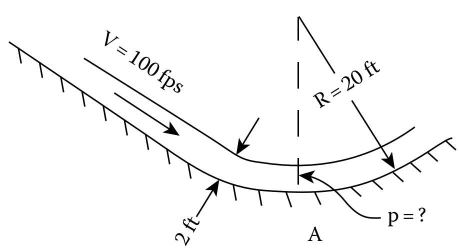
Current meter measurements

Position	1	2	3	4	5
x (ft)	1.2	3.6	6.0	8.4	10.8
$v_{0.2}$ (fps)	1.1	1.3	1.6	1.4	1.2
$v_{0.8}$ (fps)	1.3	1.5	1.9	1.5	1.3

1.18 Water in a rectangular channel flows around a 90° bend. The channel is 10 ft wide, the inside radius of the bend is $r_i = 30$ ft and the radius of the outside of the bend is $r_o = 40$ ft. A flow rate of Q = 400 cfs is in the channel when the depth of water at the inside of the bend is measured at $Y_i = 5$ ft. Make the following assumptions: (a) there is no secondary flow; (b) the angular momentum is constant, i.e., the flow around the bend follows the free vortex law; (c) the velocity does not change with position between the bottom of the channel to the water surface at any give radial distance. Determine the following: (1) The depth at the outside of the bend, (2) The velocity at the inside of the bend, V_i, and (3) the velocity at the outside of the bend. (To accomplish these tasks you should first develop the equation that gives the depth through the bend as a function of the radius r, and the free vortex constant, and then evaluate the free vortex constant by making sure that 400 cfs passes through the bend.)



1.19 Water comes off a spillway with a velocity of V = 100 fps at its toe. The spillway flip bucket at its end consists of a circular arc with a radius R = 20 ft. The depth of flow through the flip bucket is 2 ft. Assume that the velocity distribution through the depth of flow is constant. Derive the equation that gives the pressure distribution in the water on the spillway bucket through a vertical section passing through the center of the circle, i.e., the section vertically above point A on the sketch. What is the pressure at the bottom of the flow at this position?



Energy

1.20 Prove that for an irrotational flow that if the radius of curvature for a particle of fluid remains constant that its velocity is given by v = C/r (C = constant). Therefore, as $r\varpi 0$, then $v\varpi^{\infty}$.

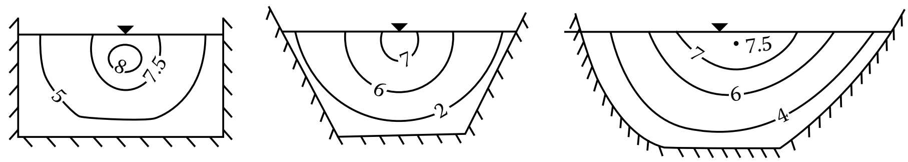
1.21 Obtain the equation y = f(x) that describes the top surface of water flowing over a sharp crested weir in a wide channel under the following assumptions: (a) the top surface remains horizontal for all x < 0 (where x has its origin at the weir crest), (b) there is no friction between this top streamline and the fluid below it, and (c) there is no pressure gradient in the normal direction at the top streamline. The average velocity and depth upstream from the weir are V_o and Y_o, respectively. What will cause the real flow to have a top surface that deviates from this mathematical description.

1.22 A weir at the end of a wide rectangular channel has its crest 1.2 m above the channel bottom, and the depth of water above the weir crest is 0.8 m. Under the assumption that the flow is inviscid, compute the thickness of the falling water at a point 2.5 m below the weir crest.

1.23 An irrotational flow occurs around a 90° circular bend in a rectangular channel. The channel is 10 ft wide, and the inside radius of the bend is 60 ft. If velocity at the inside of the bend is 8 fps, and the velocity does not vary in the vertical direction, determine the flow rate in this channel if the depth at the inside of the bend is 4.5 ft. What is the difference between the water surface elevation at the inside and outside of the bend under this ideal situation? How would the real flow around this bend deviate from this behavior?

1.24 Assume the velocity distribution is parabolic in a shallow depth of flow in a wide channel. Determine the value of the kinetic energy correction coefficient, α.

1.25 Lines of constant velocity are shown in cross sections of a flow in three channels. Determine the kinetic energy correction coefficient for each.



1.26 Lateral outflow at a rate of 2 cfs/ft is taking place over a 20 ft long side weir. There is zero flow downstream from this weir, and the width of the channel is 5 ft. Develop the equation that describes the depth of flow as a function of x across the length of the side weir. How would this solution be complicated if the discharge from the side weir depended upon the depth of water above its crest to the 3/2 power?

1.27 The velocity distribution in a wide rectangular channel is as given in illustrative Problem 1.13, v = Ln(1 + 10y). Determine the kinetic energy correction coefficient, α.

1.28 Assume the velocity distribution with depth in a trapezoidal channel with a bottom width b = 3 m and a side slope m = 1.2 is given by the equation v = 0.5 Ln(1 + 3y) (m/s). The depth of flow Y = 2 m. What is the volumetric flow rate? What is the kinetic energy correction coefficient, α? What is the average energy; per unit weight in this channel flow?

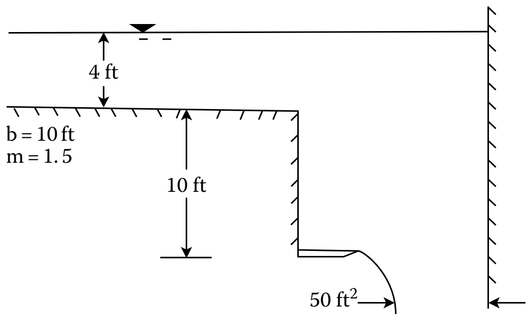
Momentum

1.29 Solve Example Problem 1.13 if the pressure delivered at the end of the pipe is 45 psi instead of 40 psi.

1.30 Compute the resultant force on a spillway bucket that turns a flow rate of q = 300 cfs/ft from its direction down the spillway at a slope of $S_o = 0.4$ to the horizontal direction. The depth of flow at the beginning of this bucket is 5 ft and at the end of the bucket is 4.6 ft.

1.31 If the spillway bucket of the previous problem turned through a total angle of 33° what would the force on the bucket be? How high would the jet of water rise above the bottom of the bucket and what would its velocity be at this highest point?

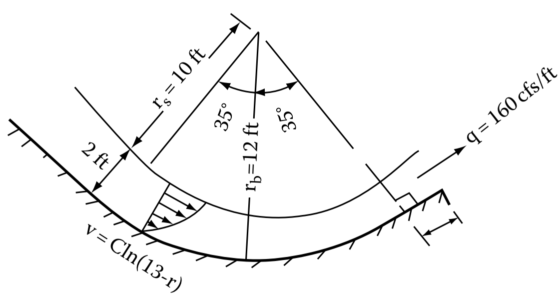
1.32 A structure takes water in the vertical direction from a position 10 ft below the bottom of a channel under atmospheric pressure. The area of this outlet is 50 ft^2. The upsteam channel is of trapezoidal shape with b = 10 ft and m = 1.5, and the depth in this channel just upstream from this structure is 4 feet. Compute the flow rate and the resultant force on this structure. Ignore frictional losses.



1.33 Determine the momentum flux correction coefficient, β for the velocity v = 0.5 Ln(1 + 3y) given for the flow in the trapezoidal channel of Problem 1.28 (b = 3 m, m = 1.2 and Y = 2 m).

1.34 A flip bucket at the toe of a spillway has a central portion consisting of a circle with radius of r_b = 12 ft. After a short smooth transition from the constant slope spillway face, the circular portion of the bucket begins at an angle of 35° to the left of the vertical and it ends at an angle of 35° to the right of the vertical as shown in the sketch. After the end of the circular portion a constant section 2 ft in length ends the bucket. A flow rate of q = 160 cfs/ft is coming down the spillway, and the depth normal to the bottom at the beginning of the circular arc is 2 ft. Assume that the velocity distribution can be defined by the following logarithmic function of the radius r. v = C ln(13 − r).

Do the following: (1) Determine the constant C (in integral tables you will find Iln(x)dx = x ln(x) − x). (2) Write the expression that provides the pressure gradient dp/dr through the circular portion of the spillway bucket. How would you obtain the pressure distribution at the end of the circular portion of the bucket at the plus 35° angle?



1.35 Repeat Example Problems 1.5 and 1.15, except use the following dimensionless velocity profile, rather than the one given in Example Problem 1.5.

Dimensionless depth, y′	0.00	0.03	0.05	0.10	0.15	0.20	0.60	0.80	0.90	1.00
Dimensionless velocity, V′	0.00	0.30	0.55	0.75	0.95	0.98	0.985	1.00	1.01	1.00

1.36 The program EXPRB1_5 that was written to solve Example Problem 1.5 integrated the dimensionless velocity profile to obtain a dimensionless unit flow rate q′. Modify this program so the actual velocity profile at each position x is integrated to get the actual unit flow rate q. (In doing this you can evaluate q for each unit of width of channel, and sum these q's to get the total flow rate Q.) What flow rate, channel area, and average velocity do you get with this process?

1.37 Using the approach used in the previous problem, determine the values of the energy correction coefficient α, and the momentum correction coefficient β for the natural channel in Example Problem 1.5.

2 Energy and Its Dissipation in Open Channels

2.1 INTRODUCTION

In applying the energy principle to the flow of liquids in open channels, it is necessary at the outset to be able to describe the resistance of fluid to motion. This resistance occurs because all real fluids have the property called viscosity, which causes internal shearing stresses to exist within the fluid as a consequence of a velocity gradient as it passes over a solid boundary. Viscosity is defined in Chapter 1. At the solid boundary, the velocity in the fluid must be zero, or agree with the velocity of the boundary, and increase therefrom for movement to take place. Thus it takes work, or energy, to cause motion of fluids relative to the boundaries that contain the fluids. This resistance and its effects are given several names such as "frictional resistance," "viscous shear," "friction factors," "friction losses," "friction energy dissipation," "frictional head loss," etc. Often, friction is omitted. In the case of liquids, the internal processes associated with fluid friction involve the conversion of useful fluid energy, which is originally in the form of a potential or kinetic energy, into a non recoverable energy, i.e., increase in temperature of the liquid. It is appropriate, therefore, to consider it a head loss, or dissipation of energy per unit weight of fluid. The increase in temperature is very small and of minor, if any, significance.

The first part of this chapter deals with how this head loss can be determined practically in computing depths of flow, and velocities that will occur if a given volumetric flow rate is to occur in a channel of a given size. These losses will be restricted to **uniform flows**. Nonuniform flows will be dealt with in Chapter 4. After being able to determine this loss, the second part of this chapter deals with the application of the energy principle to open channel flows. In this part, concepts associated with **specific energy** in open channel flow will be covered, as well as the differences between subcritical and supercritical flows.

2.2 APPROACHES TO FRICTIONAL RESISTANCE

The problem of frictional resistance in open channel flows is complex and depends on sound engineering judgment in selecting appropriate coefficients. The subject is made more complex by the fact that many channels are unlined, and therefore may have a **moveable** bed under some, or all flow conditions, that exist in that channel. The effects of bed movement are not considered in this chapter and are dealt with in articles and volumes dealing with sediment transport, its scour, and deposition and flow in alluvial channels. This chapter is restricted to flow resistance in **fixed bed channels**.

The different means for handling the resistance to motion in channels can be roughly classified as (1) the more fundamentally sound friction factor approach similar to that used for pipe flows when utilizing the Darcy–Weisbach equation and (2) based on tested and widely used empirical equations. In engineering practice, the use of empirical equations dominates in computing frictional losses in open channels. The reason for this is associated with the complexities involved with the friction factor approach. The same occurs in pipe flow computations in which the Hazen–Williams equation is used more extensively than is the Darcy–Weisbach equation. The use of empirical equations should be limited to ranges of situations for which they give good answers. A more fundamentally sound approach generally does not have these restrictions. As computers take over the task of doing the arithmetic, there will likely be a trend in engineering practice to switch to the friction factor approach. The friction factor approach will be described first and it will be followed by a section dealing with the use of Manning's equation.

Before beginning this discussion of fluid frictional losses in open channel flows, it should be pointed out that other complex phenomena may operate in dissipating fluid energy that are not covered by this theory, and using the results from this theory for these channel flows may produce erroneous answers. Such a flow, for example, exists in a steep mountain stream with very large bed elements that extend up to, or above, the water surface in many locations. Data taken from such a river/stream that flows through the Rocky Mountain Hydraulics Laboratory (Jarrett, 1991) indicate that energy dissipated is proportional to the velocity raised to the 8.3 power, whereas frictional theory for turbulent flows has the energy dissipation proportional to the velocity squared. Empirical equations, such as Manning's equation, are not appropriate for such flows. In fact, neither is Manning's equation appropriate to determine flow depths, etc. on steep spillways with slopes 0.2 or greater even though they are made of relatively smooth concrete, because it has not been developed to describe such large velocity flows that have large-scale turbulence associated with them. Neither is Manning's equation suited for extremely low-velocity flows through tall grass, for example.

2.2.1 Friction Factors in Open Channels

In 1768, a French engineer, Antoine Chezy, reasoned that the resistance to flow in an open channel would vary with the wetted perimeter and with the square of the velocity and that the force to balance this resistance would vary with the area of the cross section and with the slope of the channel. Therefore, he proposed that

$$\frac{V^2P}{(AS)} = \frac{V^2}{(R_hS)} = \text{Constant}$$

and would be the same for any similar channel. He used this in designing a canal for the Paris water supply. Years later, in 1897 his manuscript was published, and as his method became known and adopted by other engineers, the square root of the constant became know as the Chezy coefficient, and the equation

$$Q = CA(R_hS)^{1/2} \tag{2.1a}$$

or

$$V = C(R_hS)^{1/2} \tag{2.1b}$$

became known as Chezy's equation. While Chezy took C as a constant for a channel with a fixed wall roughness, we now know that C is a function of the Reynolds number ($R_e = 4VR_h/\nu = 4Q/(\nu P)$; also note as above that the hydraulic radius R_h is the area divided by the wetted perimeter, or $R_h = A/P$) of the flow as well as the relative roughness (e/R_h) of the channel wall, or $C = f(R_e, e/R_h)$. In other words, not only does C depend on the type of channel, but its value also depends on the flow conditions in that channel.

The processes associated with fluid resistance in open channel flows are similar to those that cause head losses in pipeline flows. Therefore, it is possible to get considerable insight into channel resistance by utilizing what is known from experimentation and theory, and has been adopted into practice in pipe fluid friction. The Darcy–Weisbach equation, $h_f = f(L/D)(V^2/2g)$ defines the frictional head loss in a pipe flow, in which f is a friction factor whose magnitude depends upon the relative roughness of the pipe wall, e/D, and the Reynolds number of the flow in the pipe. The Darcy–Weisbach equation is accepted as the fundamentally sound and best method for computing head losses or pressure drops in pipelines due to known flow rates. Therefore, this equation will be compared with Chezy's equation. The Darcy–Weisbach equation indicates that the slope of energy line, or the head loss, h_f, divided by the length, L, of pipe over which this loss occurs equals a friction factor, f, divided by the pipe diameter (which is a representative parameter for the pipe size with dimensions of length), multiplied by the velocity head, $V^2/2g$. If the hydraulic radius R_h multiplied by 4 is used in place of the pipe diameter, then the Darcy–Weisbach equation can be written as

$$S = \frac{h_f}{L} = \frac{fV^2}{4R_h(2g)} = \frac{fQ^2}{8R_h(gA^2)} \tag{2.2}$$

in which the friction factor, f, is a function of Reynolds number and the relative roughness, e/D, of the pipe wall (e is the roughness of the pipe wall and D is the diameter of the pipe). The substitution of $4R_h$ in place of D can be justified by noting that for a pipe, the hydraulic radius that equals the area divided by the perimeter is $R_h = (\pi D^2/4)/(\pi D) = D/4$.

It is worth noting that if f is dimensionless, then the terms separated by equal signs in Equation 2.2 are all dimensionless. Thus, Equation 2.2 can be obtained from dimensional analysis, and this analysis leads to the conclusion that f does depend on the two dimensionless parameters, R_e (Reynolds number) and e/D. The friction factor f will have the same value regardless of whether ES or SI units are used.

A comparison of Equation 2.1a and b with Equation 2.2 gives the following relationship between the Darcy–Weisbach friction factor, f, and Chezy's coefficient, C:

$$C = \sqrt{\frac{8g}{f}} \tag{2.3}$$

For pipe flow, it is known that the friction factor f can be defined well by a Moody diagram that is a plot of f as the ordinate against Reynolds number as the abscissa with different curves for different values of relative roughness e/D. This diagram is defined by the following equations for the following four different types of flow that occur:

1. Laminar flow

$$f = \frac{64}{R_e}$$

2. Hydraulically smooth flow (the wall roughness are well embedded within the laminar sublayer, less than 1/5 its size, and consequently have no influence on the magnitude of f)

$$\frac{1}{\sqrt{f}} = 2\mathrm{Log}_{10}(R_e\sqrt{f}) - 0.8$$

3. Transitional zone (in which both R_e and e/D determine f's magnitude; this is the Colebrook–White equation)

$$\frac{1}{\sqrt{f}} = -2\mathrm{Log}_{10}\left(\frac{e}{3.7D} + \frac{2.52}{R_e\sqrt{f}}\right) = 1.14 - 2\mathrm{Log}_{10}\left(\frac{e}{D} + \frac{9.35}{R_e\sqrt{f}}\right)$$

4. Wholly rough zone (in which R_e no longer effects f's magnitude)

$$\frac{1}{\sqrt{f}} = 1.14 - 2\text{Log}_{10}\left(\frac{e}{D}\right) = 2\text{Log}_{10}\left(\frac{3.7D}{e}\right)$$

Equations that define Chezy's C might be determined by substituting Equation 2.3 into the above four equations. When doing this for laminar flow, the following equation results:

$$C = \sqrt{\frac{gR_e}{8}} \tag{2.4}$$

Equation 2.4 applies for flows in which the Reynolds number is less than about 2100. Experimentation has shown that the value of 8 in Equation 2.4 does not hold constant for all channel, but no fully accepted range of values has been established. Since laminar flows are rare when dealing with water as the fluid, the value of 8 can probably be used, at least for a reasonable first approximation.

Substituting Equation 2.3 into the hydraulically smooth equation gives

$$C = \sqrt{32g}\,\text{Log}_{10}\left(\frac{gR_e}{0.887C}\right) \tag{2.5}$$

There is some question about whether the value 0.887 should be modified slightly to fit available experimental data. However, since very limited quality experimental data are available for hydraulically smooth flow, the derived value will be used in this book.

For flows that occur in the transitional zone in which both the relative roughness of the channel wall and the Reynolds number have an influence on the magnitude of the frictional resistance, the constants that are obtained from substituting Equation 2.3 into the above transitional equation might be modified slightly to give a better fit of some experimental data. (See ASCE Task Force, 1963.) With these modified coefficients the equation is

$$C = -\sqrt{32g}\,\text{Log}_{10}\left(\frac{e}{12R_h} + \frac{0.884C}{R_e\sqrt{g}}\right) \tag{2.6}$$

If the original coefficients that come from the Colebrook–White or other experimentally based equation are preferred, then Equation 2.6 can be written in the more general form

$$C = -c\text{Log}_{10}\left(\frac{e}{aR_h} + \frac{bC}{R_e\sqrt{g}}\right) \tag{2.6a}$$

in which the a, b, and c can be given slightly different values. For example, when using the Colebrook–White equation directly a = 12, b = 0.887, and c = $(32\,g)^{1/2}$.

For the wholly rough zone substitution of Equation 2.3 into the wholly rough equation that defines the Moody diagram gives

$$C = -\sqrt{(32g)}\,\text{Log}_{10}\left(\frac{e}{12R_h}\right) = \sqrt{(32g)}\,\text{Log}_{10}\left(\frac{12R_h}{e}\right) \tag{2.7}$$

These equations are summarized in Table 2.1, and Equations 2.5 through 2.7 are plotted on Figure 2.1 to give "Chezy C" diagram for turbulent flows. If Equation 2.4 were plotted on a left addition to Figure 2.1, it would result in a straight line with a slope of 1/2, since power equations such as Equation 2.4 plot as straight lines on log–log graph paper with the exponent in the equation giving the slope on this plot.

TABLE 2.1
Summary of Equations That Define Chezy's Coefficient, C

Type of Flow	Equation Giving C	Equation No.	Range of Application
Laminar	$C = (gR_e/8)$	2.4	$R_e < 2100$
Hydraulically smooth	$C = \sqrt{32g}\log_{10}\left(\dfrac{R_e\sqrt{g}}{0.887C}\right)$	2.5	$R_e < 2100$
Transition	$C = -\sqrt{32g}\log_{10}\left(\dfrac{e}{12R_h} + \dfrac{0.884C}{R_e\sqrt{g}}\right)$	2.6	$2100 < R_e < V_e/(C\nu) = 100$
Wholly rough	$C = -\sqrt{32g}\log_{10}\left(\dfrac{e}{12R_h}\right)$ or $C = \sqrt{32g}\log_{10}\left(\dfrac{12R_h}{e}\right)$	2.7	$R_e > V_e/(C\nu) = 100$

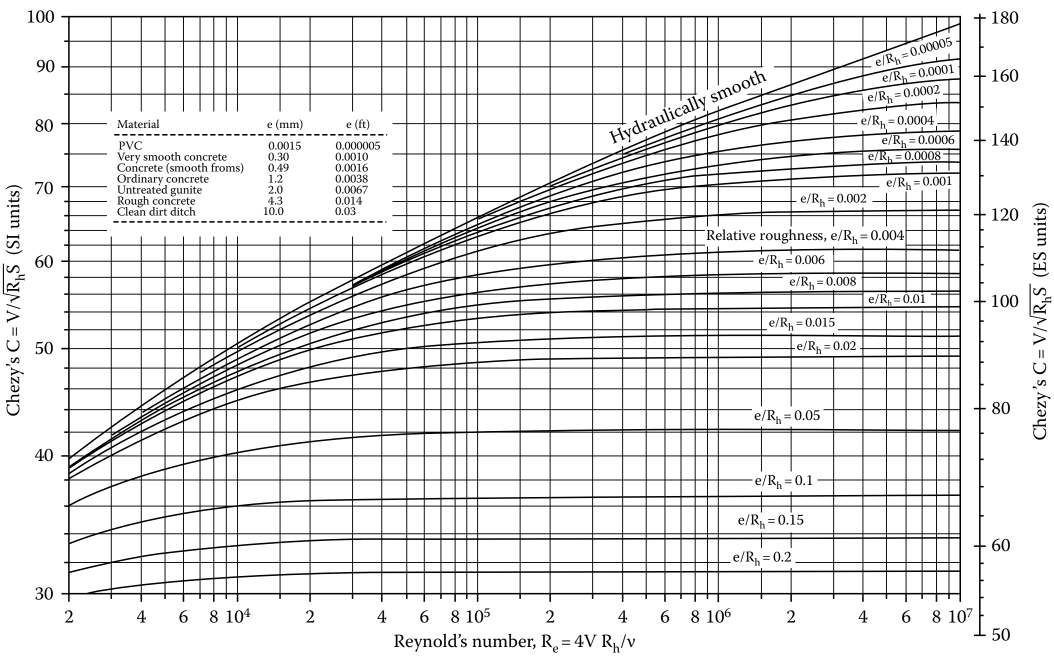


FIGURE 2.1 Diagram for Chezy's C for use in determining the flow rate, velocity and slope of the energy line, or head loss in open channels.

The friction factor f in the Darcy–Weisbach equation has a value that is independent of the units used. However, since C is not dimensionless, but has the dimensions of the square root of gravity, e.g., $L^{1/2}/T$, its value will be different when using SI units than when using ES units. In Figure 2.1, the values for C when using SI units are the left-side ordinate, whereas the right ordinate applies when using ES units. An alternative to having different values of C for different units would be to modify Chezy's equation to include g, or define the Chezy equation as

$$V = C_1\sqrt{gR_hS} \tag{2.1c}$$

However, historical developments have not done this, and therefore you must be sure that the appropriate value of C is used to the system of units that you are using.

Since many open channel flows do fall within the transitional type of flow, some discussion of the characteristics of Equation 2.6 are in order. It should be noted that C occurs on both sides of Equation 2.6 (the same is true of Equation 2.5). It is not possible to rearrange this equation so that C is on one side of the equation all by itself. Equations of this type are referred to as implicit equations, since an explicit solution of them is not possible. Use of general iterative techniques, such as the Newton method described in Appendix B can be used. However, because of the nature of Equations 2.5 and 2.6 they can be solved by a simple feedback iteration, called a Gauss–Seidel iteration, in which the C solved for on the left side of the equal sign is used for C on the right side of the equal sign for the next iteration. The value produced by the explicit Equation 2.7 can be used as the initial starting value for this iterative solution or easier, just start with a reasonable guess. The following few lines of FORTRAN and C code illustrate implementation of this iterative solution for a trapezoidal channel. For other types of cross sections, the two function statements at the top of this listing that define the area, A(B,FM,Y) and the wetted perimeter, P(B,FM,Y) need to be modified.

Program CHEZYC.FOR

```
      A(B,FM,Y)=(B+FM*Y)*Y
      P(B,FM,Y)=B+2*Y*SQRT(FM*FM+1.)
5     WRITE(6,*)' Give:B,FM,Y,G,VISC,E,Q'
      READ(5,*,ERR=30) B,FM,Y,G,VISC,E,Q
      AR=A(B,FM,Y)
      V=Q/AR
      G8=.884/SQRT(G)
      SQG=SQRT(32.*G)
      C1=SQG*ALOG10(12.*AR/P(B,FM,Y)/E)
10    RH=AR/P(B,FM,Y)
      RE=4.*V*RH/VISC
      C=-SQG*ALOG10(E/(12.*RH)+G8*C1/RE)
      IF(ABS(C-C1).LT. 1.E-8) GO TO 20
      C1=C
      GO TO 10
20    WRITE(6,*)' CHEZYS COEF=',C
      GO TO 5
30    STOP
      END
```

Program CHEZYC.C

```
#include <stdlib.h>
#include <stdio.h>
#include <math.h>
float a(float b,float m,float y){return (b+m*y)*y;}
float p(float b,float m,float y){return b+2.*y*sqrt(m*m+1.);}
void main(void){float b,m,y,g,visc,e,q,ar,v,g8,rh,re,c,c1,sqg;
 printf("Give: b,m,Y,g,Visc,e,Q\n");
 scanf("%f %f %f %f %f %f %f",&b,&m,&y,&g,&visc,&e,&q);
 ar=a(b,m,y); v=q/ar; g8=.884/sqrt(g);sqg=sqrt(32.*g);
 c=sqg*log10(12.*ar/p(b,m,y)/e);
 do {c1=c; rh=ar/p(b,m,y); re=4.*v*rh/visc;
   c=-sqg*log10(e/(12.*rh)+g8*c1/re);}while(fabs(c-c1)<1.e-8);
 printf("CHEZYS COEF =%f\n",c);}
```

A typical problem involves considerably more than solving one of the equations in Table 2.1 for Chezy's coefficient. The types of problems associated with steady flow can be categorized as follows:

1. The flow rate, or velocity is unknown, and all other variables are known.
2. The depth is unknown and all other variables are known. This type of problem typically asks a question like: With this given channel what will the depth of flow be if the flow rate equals Q.
3. One of the variables associated with the channel size is unknown, but the flow rate is known. This type of problem can be consider a design problem in which the size of channel needed to convey a specified flow rate is wanted. For a trapezoidal channel the unknown may be the bottom width, or the side slope, and for a circular section the diameter is the unknown.
4. All variables are known except the wall roughness.
5. The slope of the channel bottom is unknown, and all other variables are known. Of all problems this is the easiest, since its solution can be obtained most directly, by (a) solving the appropriate equation for Chezy's coefficient, and (b) solving Chezy's equation for S. (When the slope, S, refers to the channel bottom subscript o will be used and when the slope refers to the energy line subscript f will be used e.g., in most subsequent equations S_o or S_f will appear in Chezy's as well as Mannings equation.)

Problems in any of these categories might be view, as a mathematical problem of solving two nonlinear simultaneous equations; Chezy's equation and the appropriate equation from Table 2.1 that defines Chezy's coefficient. Appendix B describes the Newton method for solving systems of nonlinear equations. Alternately software packages, such a Mathcad, TK-Solver, or MATLAB can be used, or math packs for pocket calculators might be used. In addition to the procedure described above for case (5) problems under category (1) can be solved by cycling through the following steps: (a) solving for C (Equation 2.6), (b) using this C compute Q, or V from Equation 2.1a and b, and (c) based on this Q, or V update Reynolds number and repeat steps (a) through (c) until a small enough change occurs between consecutive values of Q or V that you are willing to accept the results. Convergence of this procedure will be rapid because changes in flow rate have a relatively small effect on the value of C. Should the flow be in the wholly rough zone then only one cycle of steps (a) and (b) above completes the solution since Chezy's coefficient is independent of the flow rate.

Typical values for wall roughnesses that are appropriate are given in Table 2.2.

EXAMPLE PROBLEM 2.1

What is the flow rate in a trapezoidal channel with b = 10 ft, m = 1.5, and a bottom slope of $S_o = 0.0005$ if the depth of flow is measured equal to 5 ft.

Solution

This problem falls in Category (1) above. The solution might begin by solving for C from Equation 2.7, which gives C = 127.49. Next the implicit Equation 2.6 is solved by the Gauss–Seidel iteration based on an assumed Reynolds number, i.e., 1.0E6 (or if one wishes the above C could be used in

TABLE 2.2
Value of Wall Roughness, e, for Different Channel Materials

Material	e (m)	e (ft)
PVC	0.0000015	0.000005
Very smooth concrete	0.00030	0.0010
Concrete (smooth forms)	0.00049	0.0016
Ordinary concrete	0.0012	0.0038
Untreated gunite	0.0020	0.0067
Rough concrete	0.0043	0.014
Clean dirt ditch	0.0100	0.03

Chezy's equation to compute the Reynolds number). The result is C = 125.15. Now using this C the velocity is V = 4.944 fps from Chezy's equation, and the associated Reynolds number equals 5,019,874 (assuming ν = 1.23E−5). To get the solution the steps of (a) solving C from Equation 2.6 based on the most recent Reynolds number, (b) solve V from the Chezy equation (Equation 2.1a and b) and updating $R_e = 4R_h/\nu$. The results are iteration # 2, C = 126.98, V = 5.017 fps, R_e = 5,093,390; iteration # 3, C = 126.99, which is close enough giving a flow rate Q = 439.0 cfs.

EXAMPLE PROBLEM 2.2

A trapezoidal channel with b = 8 ft, m = 1.2, and a bottom slope of 0.0006 is to convey a flow rate of Q = 300 cfs. The wall roughness of the channel is e = 0.004 ft. Determine the depth of flow in this channel.

Solution

An effective way to solve this problem is to use the Newton method (Appendix B) to solve Chezy equation and Equation 2.6 simultaneously for C and Y. For the Newton method these equation can be written as

$$F_1 = C + (32g)^{1/2}\,\mathrm{Log}_{10}\left(\frac{e}{12R_h} + \frac{0.884C}{\left(R_e\sqrt{g}\right)}\right) = 0 \qquad (1)$$

$$F_2 = Q - CA(R_hS)^{1/2} = 0 \qquad (2)$$

Using the Newton method the unknown vector, consisting of the two values C and Y, is updated by the following iterative equation:

$$\begin{Bmatrix} C \\ Y \end{Bmatrix}^{(m+1)} = \begin{Bmatrix} C \\ Y \end{Bmatrix}^{(m)} - \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$$

in which the vector z is the solution to the linear system of equations:

$$\begin{bmatrix} \dfrac{\partial F_1}{\partial C} & \dfrac{\partial F_1}{\partial Y} \\ \dfrac{\partial F_2}{\partial C} & \dfrac{\partial F_2}{\partial Y} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

Instead of actually taking the partial derivatives shown above a numerical approximation can be used. This numerical approximation evaluates the function (i.e., equation) twice with the second evaluation based on decreasing (or increasing) the variable that the derivative is taken with respect to by a small increment and then dividing the two values of the function by this increment.

The following FORTRAN program can be used to obtain this solution. It is designed to solve for C as well as any of the following variables: e, b, m, Y, Q, or S in a trapezoidal channel. Input to this program as well as the values to solve this problem consist of: 1 the number of the unknown variable which is 4 for Y for this problem, the list of known, including a guess for Y, in the following order: e = 0.004, b = 8, m = 1.2, Y = 4(guess), Q = 300, S = 0.0006, g = 32.2, and kinematic viscosity = 1.23E–5. The solution produced by the program is Y = 4.46 ft. The problem can be solved by using Figure 2.1 in connection with Equation 2.1a and b. You should at least verify Chezy's C from Figure 2.1. The value of C from the above procedure is C = 125.

Listing of FORTRAN program to solve problem using the Chezy equation (CH2PR2):

```
      REAL X(6),D(2,2),F(2)
      EQUIVALENCE (E,X(1)),(B,X(2)),(FM,X(3)),(Y,X(4)),(Q,X(5)),
     &(S,X(6))
      A(B,FM,Y)=(B+FM*Y)*Y
      RH(AR,B,FM,Y)=AR/(B+2.*Y*SQRT(FM*FM+1.))
      F1(C,SQG,RH1,RE,G8,E)=C+SQG*ALOG10(E/(12.*RH1)+G8*C/RE)
      F2(Q,C,AR,RH1,S)=Q-C*AR*SQRT(RH1*S)
```

```
1     WRITE(6,*)' Give: No. of UNK,1-e,2-b,3-m,4-Y,5-Q,6-S,g & Vis'
      READ(5,*) IUN,X,G,VISC
      IF(IUN.LT.1 .OR. IUN.GT.6) GO TO 1
      G8=.884/SQRT(G)
      SQG=SQRT(32.*G)
      C=SQG*ALOG10(12.*RH(A(B,FM,Y),B,FM,Y)/E)
20    AR=A(B,FM,Y)
      RH1=RH(AR,B,FM,Y)
      RE=4.*Q*RH1/(VISC*AR)
      F(1)=F1(C,SQG,RH1,RE,G8,E)
      F(2)=F2(Q,C,AR,RH1,S)
      C=.98*C
      D(1,1)=(F(1)-F1(C,SQG,RH1,RE,G8,E))/(.02040816*C)
      D(2,1)=(F(2)-F2(Q,C,AR,RH1,S))/(.02040816*C)
      C=C/.98
      X(IUN)=.95*X(IUN)
      AR=A(B,FM,Y)
      RH1=RH(AR,B,FM,Y)
      RE=4.*Q*RH1/(VISC*AR)
      D(1,2)=(F(1)-F1(C,SQG,RH1,RE,G8,E))/(.05263158*X(IUN))
      D(2,2)=(F(2)-F2(Q,C,AR,RH1,S))/(.05263158*X(IUN))
      X(IUN)=X(IUN)/.95
      FAC=D(2,1)/D(1,1)
      D(2,2)=D(2,2)-FAC*D(1,2)
      F(2)=F(2)-FAC*F(1)
      DIF=F(2)/D(2,2)
      X(IUN)=X(IUN)-DIF
      DIF1=(F(1)-DIF*D(1,2))/D(1,1)
      C=C-DIF1
      IF(ABS(DIF)+ABS(DIF1).GT. .001) GO TO 20
      WRITE(6,100) C,X(IUN),X
100   FORMAT(' C=',F8.2,' Unknown=',F10.3,/,' e=',F10.5,/,
     &' b=',F8.2,/,' m=',F8.2,/,' Y=',F8.3,/,' Q=',F10.2,/,
     &' S=',F8.5)
      WRITE(6,*)' Give 1 to solve another problem; otherwise 0'
      READ(5,*) IUN
      IF(IUN.EQ.1) GO TO 1
      STOP
      END
```

EXAMPLE PROBLEM 2.3

Determine the depth of flow in a circular channel with D = 10 ft, if its bottom slope equals 0.0005 and its wall roughness equals 0.004 ft and it is to convey a flow rate of Q = 150 cfs.

Solution

The above program can be modified by changing the function statement to obtain the area by a function subprogram, and function statement for the hydraulic radius to apply for a circular channel. These statements could consist of

```
      FUNCTION A(D,BETA,Y)
      BETA=ACOS(1.-2.*Y/D)
      A=D*D/4.*(BETA-COS(BETA)*SIN(BETA))
      RETURN
      END
```

and

```
      RH(AR,D,BETA,Y)=AR/(D*BETA)
```

The above listing might be altered with BETA replacing FM, and or the present FORTRAN names used with different meaning. The solution gives Y = 4.613 ft, and Chezy's C = 123.1.

EXAMPLE PROBLEM 2.4

You are to size a trapezoidal channel that is to carry a flow rate of Q = 50 m^3/s. The slope of the channel is 0.0012 and it is to be made of formed concrete. For stability of the channel sides their slopes are to be 1.5, and the depth is not to exceed 2 m. What should the bottom width be?

Solution

The computer program of Example Problem 2.2 will solve this problem. If this program is used the input consists of: 2,.00049,3,1.5,2,50,.0012,9.81,1.14E−6. The solution is b = 4.94 m. If Mathcad is available to you it would be a excellent experience to use it, or some other software package to get the same solution.

2.3 COMBINING THE CHEZY AND THE CHEZY C EQUATIONS

An alternative to solving Chezy's equation and the Chezy C equation for the transitional zone simultaneously for any of the variables is to eliminate C first by solving for it from Chezy's equation, and then substituting this in the equation that applies within the transitional zone where it occurs on both sides of the equal sign. By eliminating C between these two equations there is one equation and one of the variables can be solved as the unknown. The resulting equation is implicit for all variables except e, and so in general must be solved by an iterative technique. The HP48 calculator's SOLVE capability can be used; however, because A and P are functions of the depth and size variables the resulting equation becomes long and complex. However, this approach is readily implemented in a computer program, in TK-Solver, or Mathcad models where separate statements can be used to define A and P.

Solving Chezy's equation for C gives

$$C = \frac{Q\sqrt{P/(AS_0)}}{A} \quad \text{or} \quad C = V\sqrt{\frac{P}{AS_0}}$$

The transitional zone equation that gives C can be written as

$$F(\xi) = C + \sqrt{32g}\,\mathrm{Log}_{10}\left(\frac{eP}{12A} + \frac{0.221\nu CP}{Q\sqrt{g}}\right) = 0$$

If the first of the two equations above, that contains Q, is used that gives C using Q then the following equation results:

$$F(\xi) = \frac{Q\sqrt{P/(AS_0)}}{A} + \sqrt{32g}\,\mathrm{Log}_{10}\left(\frac{eP}{12A} + \frac{0.221\nu P^{\frac{3}{2}}}{A^{\frac{3}{2}}\sqrt{gS_0}}\right) = 0$$

and if the second of the above two equations that gives C using the velocity V then the following equation results:

$$F(\xi) = V\sqrt{\frac{P}{(AS_0)}} + \sqrt{32g}\,\mathrm{Log}_{10}\left(\frac{eP}{12A} + \frac{0.221\nu P^{\frac{3}{2}}}{A^{\frac{3}{2}}\sqrt{gS_0}}\right) = 0$$

After solving either of these two equations depending on whether the flow rate Q is given (or the unknown) or whether the velocity is given (or the unknown), then Chezy's equation is solve for the coefficient C if this value is desired. The program CHEZYCTC.FOR, which is given below, implements such a solution using the Newton method to solve for the selected unknown from the variables: m, b, S, Y, e, Q, or V if the channel is trapezoidal, and D, S, Y, e, Q, or V if the channel is circular. The technique used to accommodate both trapezoidal and circular channels is to not use X(1) (for m) and change V(2) to D (the Character string) if the channel is circular. When IC = 1, for a circular channel, then the equation that gives A and P for a circular channel are used; otherwise those that give these quantities for a trapezoidal channel are used. (See Statements starting with label 55.) The approach is very similar to that used in solving the DW and CW equations. This program does not contain the logic, however to generate a guess for the unknown, that is needed in the Newton method. Rather the user must supply this guess as well as the known values.

The variable and rule sheets from TK-Solver are listed below the program listing that handle first the equation that assumes that the flow rate Q is the flow variable, which is in the combined equation with the other channel property variables, and the second is for the equation that involves the V as the flow variable.

Listing of program CHEZYCTC.FOR for solving the Chezy equation for any of the variables for both a trapezoidal and a circular channel

```
      REAL X(7)
      CHARACTER*19 FMT/'(1X,A2,3H = ,F10.4)'/
      CHARACTER*1 V(7)/'m','b','S','Y','e','Q','V'/
1     WRITE(*,*)' Give 1=ES or 2=SI(or 0/=STOP),',
     &'0=trap or 1=cir. & Visc'
      READ(*,*) II,IC,VISC
      IF(II.LT.1) STOP
      G=32.2
      IF(II.GT.1) G=9.81
      VISC2=.221*VISC/SQRT(G)
      G32=SQRT(32.*G)
      IF(IC.GT.1) THEN
      I2=2
      V(2)='D'
      ELSE
      I2=1
      V(2)='b'
      ENDIF
      WRITE(*,100)(I,V(I),I=I2,7)
100   FORMAT(' Give No. of Unknown',/(I2,' - ',A2))
      READ(*,*) IU
      IF(IU.GT.5) GO TO 10
      WRITE(*,*)' Give 1 if Q will be given or 2',= if V is known'
      READ(*,*) IV
      GO TO 12
10    IV=IU-5
12    WRITE(*,*)' Give values to knowns &=,= GUESS for unknown'
      I3=7
      IF(IV.EQ.1) I3=6
      DO 20 I=I2,I3
```

```
      IF(IV.EQ.2 .AND. I.EQ.6) GO TO 20
      WRITE(*,"(A2,' = ',\)") V(I)
      READ(*,*) X(I)
20    CONTINUE
50    M=0
52    XX=X(IU)
      X(IU)=1.005*X(IU)
      DX=X(IU)-XX
55    IF(IC.EQ.1) THEN
      COSB=1.-2.*X(4)/X(2)
      BETA=ACOS(COSB)
      A=.25*X(2)**2*(BETA-COSB*SIN(BETA))
      P=X(2)*BETA
      ELSE
      A=(X(2)+X(1)*X(4))*X(4)
      P=X(2)+2.*X(4)*SQRT(X(1)**2+1.)
      ENDIF
      ADL=G32*ALOG10(X(5)*P/(12.*A)+VISC2*(P/A)**1.5/SQRT(X(3)))
      IF(IV.EQ.1) THEN
      F=X(6)*SQRT(P/(A*X(3)))/A+ADL
      ELSE
      F=X(7)*SQRT(P/(A*X(3)))+ADL
      ENDIF
      M=M+1
      IF(MOD(M,2).EQ.0) GO TO 60
      X(IU)=XX
      F1=F
      GO TO 55
60    DIF=DX*F/(F1-F)
      X(IU)=XX-DIF
      IF(ABS(DIF).GT. .00001 .AND. M.LT.30) GO TO 52
      IF(IV.EQ.1) THEN
      X(7)=X(6)/A
      ELSE
      X(6)=A*X(7)
      ENDIF
      DO 70 I=1,7
      FMT(18:18)='3'
      IF(I.EQ.3 .OR. I.EQ.5) FMT(18:18)='6'
70    WRITE(*,FMT) V(I),X(I)
      WRITE(*,FMT) 'C',X(7)*SQRT(P/(A*X(3)))
      GO TO 1
      END
```

```
=========== VARIABLE SHEET ============
St Input──── Name─── Output─── Unit────
              A       75
   10         b
   1          m
   5          Y
```

```
             P         24.142136
             Q         478.55564
   .001      S
   32.099844 g32
   .01       e
   .0000141  v
   32.2      g
             C         114.47967
=============== RULE SHEET =================
S Rule--------------------------------------
   A=(b+m*Y)*Y
   P=b+2*Y*sqrt(m^2+1)
   Q*sqrt(P/(A*S))/A+g32*Log(e*P/(12*A)+.221*v/
      sqrt(g*S)*(P/A)^1.5)=0
   C=Q*sqrt(P/(A*S))/A

=========== VARIABLE SHEET ============
St Input----- Name---- Output---- Unit------
             A         75
   10        b
   1         m
   5         Y
             P         24.142136
             V         6.3807419
   .001      S
   32.099844 g32
   .01       e
   .0000141  v
   32.2      g
             C         114.47967
============== RULE SHEET ================
S Rule------------------------------------
   A=(b+m*Y)*Y
   P=b+2*Y*sqrt(m^2+1)
   V*sqrt(P/(A*S))+g32*Log(e*P/(12*A)+.221*v/
      sqrt(g*S)*(P/A)^1.5)=0
   C=V*sqrt(P/(A*S))
```

For laminar flow or water to occur in open channels either the depth or the velocity must be very small. For example, assume water at 15.6°C (60°F) so its kinematic viscosity is $v = 1.123 \times 10^{-6}$ m²/s (1.217×10^{-5} ft²/s), and that the largest value of Reynolds number, VR_h/v allowed for laminar flow is 500, and that the channel is very wide so that $R_h = Y$, then the product of the velocity time the depth $VY \le 500(1.123 \times 10^{-6}) = 0.0005615$ for SI units or $VY \le 0.0061$ for ES unit. The tables below show these limiting values. The Froude number F_r in the third column of these tables is defined by $F_r = V/\sqrt{gY}$. Notice that as the depth becomes very small, in the order of 0.01 ft, or 0.003 m, and the velocity consequently larger that the flow may becomes supercritical (F_r greater than 1). To have conditions right to allow a flow to be simultaneously laminar and supercritical are not common. Furthermore, when the depth becomes small enough for Froude numbers to be larger than unity then the surface tension of the water becomes a significant factor. When water flows in very thin sheets on steep surfaces, it tends to form thread like streamlets. Also most channel surfaces, such as gutter, or road way beds, where very small depths of open channel water flows may be found in practice, are not smooth enough and the water will actually be seen to flow in the lower indentations. The sheet flow over watershed surfaces that occurs from rainfall, which does not infiltrate into the soil, tends to erode the smaller particles and by so doing forms a system of mini channel, which will grow in size in time especially if the rainfall is intense. The last columns in these tables give the bottom slope of the channel that would be required for the flow to take place as computed by the Chezy equation. Note that for super critical flows the bottom slopes must also be very large.

Limiting Depths, Velocities and Froude Numbers for Laminar Flow of Water in Wide Open Channels. T = 15.6°C (60°F) $\nu = 1.123 \times 10\,m^2/s$ ($1.217 \times 10\,ft^2/s$)

V (ft/s)	Y (ft)	F_r	S_o
0.020	0.304	0.006	0.000015
0.040	0.152	0.018	0.000117
0.0625	0.097	0.035	0.000447
0.125	0.049	0.100	0.003577
0.250	0.024	0.282	0.028619
0.500	0.012	0.799	0.228955
0.010	0.609	1.072	0.412688
0.025	0.243	0.271	0.026412
0.050	0.122	0.096	0.003302
0.100	0.061	0.034	0.000413
0.200	0.030	0.012	0.000052
0.300	0.020	0.007	0.000015

V (m/s)	Y (m)	F_r	S_o
0.0060	0.0936	0.0063	0.000008
0.0120	0.0468	0.0177	0.000062
0.0200	0.0281	0.0381	0.000288
0.0500	0.0112	0.1506	0.004495
0.1000	0.0056	0.4261	0.35962
0.1500	0.0037	0.7828	0.121372
0.0030	0.1872	1.0910	0.235792
0.0050	0.1123	0.5071	0.050931
0.0100	0.0561	0.1793	0.006366
0.0200	0.0281	0.0634	0.000796
0.0500	0.0112	0.0160	0.000051
0.1000	0.0056	0.0057	0.000006

2.4 EMPIRICAL FORMULA: USE OF MANNING'S EQUATION

Before the turn of the twentieth century, systematic research was underway to define better fluid resistance in open channel flow. It was recognized then that C in the Chezy equation was not constant under all flow conditions in a given channel. Bazin proposed the following formula that found use in the past to better define C:

$$C = \frac{157.6}{1 + m\sqrt{R_h}} \quad \text{(for ES units)}$$

in which m takes on a different value depending on the roughness of the channel wall. In 1868, Gauckler proposed that for flat slopes in open channel flow, C varies as the sixth root of the hydraulic radius. Others came to the same conclusion and the result has now been widely accepted throughout the world and is known in the United States as Manning's formula even though the name Manning is a misnomer and it has been proposed that the formula be called Gauckler–Manning's equation in part to undo the incorrect naming. In Europe the same formula is called the Strickler formula. Because of its wide use, this will be the formula used in this text book as an alternative to Chezy's formula. Manning's equation can be written as

$$V = \frac{C_u}{n} R_h^{2/3} \sqrt{S_o}$$

or

$$Q = \frac{C_u}{n} A R_h^{2/3} \sqrt{S_o} = \frac{C_u}{n} A \frac{A^{2/3}}{P^{2/3}} \sqrt{S_o} = \frac{C_u}{n} \frac{A^{5/3}}{P^{2/3}} \sqrt{S_o} \tag{2.8}$$

in which C_u equals 1 when using SI units, and C_u = 1.486 (the cube root of the number of feet per meter) when using ES units, and n is the roughness coefficient of the channel wall. In this equation the following symbols apply: A = the cross section of the flow in ft^2 when using ES units and in m^2 when using SI units; P is the wetted perimeter and is the length of contact between the water and the channel when viewed in a direction normal to the flow direction, and has units of ft in the ES system, and m in the SI system; R_h is the hydraulic radius, which is defined as the area A divided by the wetted perimeter P, with units of ft in ES units and m in SI units; and S_o is the slope of the channel bottom, which for uniform flow equal the slope of both the water surface in the channel as well as the energy line of the flow. The slope of the energy line equal the head loss divided by the length over which this loss occurs. Therefore S_o might be thought of as h_L/L for uniform flow. Since three digits of precision cannot be maintained in the selection of n, it is common to use C_u = 1.49 in ES units. Typical values for use in Manning's equation for different material that channel are constructed from are given in Table 2.3. If n value larger than 0.05, the largest value in Table 2.3, is needed to describe a given channels flow, other mechanisms than just fluid friction are likely involved and Manning's equation is probably inappropriate, e.g., a constant value of n will not describe depths, etc. over much of a range of flow rates. This condition exists for very small velocity flows that may be approaching flow through a porous media, or very large velocity flows.

It is worth noting that Manning's formula indicates that the head loss is proportional to the square of the velocity, or flow rate. On the Chezy C diagram this corresponds to the wholly rough zone. When the Reynolds number becomes small with a magnitude of 100,000 or less then according to the Chezy equation the head loss is proportional to the velocity to a power less than 2, but greater than 1. For laminar flow, Equation 2.4 indicates that the head loss is proportional to the velocity to the first power. Thus the use of Manning's equation will essentially duplicate the results obtained from Chezy's formula for very large Reynolds numbers when the flow lies in the wholly rough zone, provided corresponding roughness values are selected. On the other hand, the use of Manning's equation is questionable for low-velocity flows in small smooth channels.

A natural question is: What is the relationship between the values of Manning's n and the equivalent sand roughness e used in connection with Chezy's formula? The answer is that there is no direct

TABLE 2.3
Typical Values for Manning's n

Channel Material	**n**
Lined channels	
Smooth brass, glass, lucite, PVC	0.010
Cement plaster	0.011
Planed lumber, unpainted steel, trowelled concrete	0.012
Unplaned lumber, smooth asphalt, vitrified clay, brick in cement mortar, cast iron	0.013
Asphalt (rough), untreated gunite	0.016
Rough concrete	0.020
Corrugated metal	0.023
Natural channels	
Clean excavated earth	0.023
Earth (good condition), rock excavation, gravel (straight chan.)	0.025
Earth (straight with some grass)	0.026
Earth (winding, no grass), clean natural beds	0.030
Gravel beds (plus large boulders)	0.040
Earth (winding), weedy streams	0.050

relationship, or simple equation that can give n from e, or e from n. For any given e, the value of n is different depending on the channel type and size, and the flow rate, i.e., Reynolds number of this flow. To get an n that corresponds to an e for any given situation, one must solve both equations. For example, if one wanted to determine what n corresponds to e = 0.004 ft, it is first necessary to decide what channel and flow rate this correspondence is to apply for. To illustrate, assume a flow rate of 400 cfs occurs in a rectangular channel with a bottom width of 10 ft and bottom slope of 0.0005 has a known e = 0.004 ft. First the depth is solved from Chezy's equation. The depth is Y = 8.04 ft. Next Manning's equation is solved for n, giving n = 0.0141. Thus for this channel containing this flow rate the corresponding n = 0.0141 for an e = 0.004 ft. For this e, the value of n will be different for each different flow rate and each channel.

Table 2.4 illustrates this variation of n with several different variables that can be changed in a trapezoidal channel. Since a rectangular channel is a special trapezoidal channel with side slope m = 0, this table includes a couple of rectangular channels. All of the values for n given in this table correspond to a equivalent sand roughness of e = 0.004 ft in the Chezy equation. The first three columns in Table 2.4 are for the rectangular channel used in the above illustration with a bottom width of 10 ft and a bottom slope of $S_o = 0.0005$. The flow rate was varied from 8 to 400 cfs in this channel. The second columns gives the depths that are obtained by solving Chezy's formula, and the third column indicates the value of n that is solved by Manning's equation for this column 2. This third column of n values shows that n increases with Q and Y from 0.0131 to 0.0141, or a 7.6% change over this range of conditions.

The next three columns, i.e., columns 4, 5, and 6 in Table 2.4, were obtained similarly except for a trapezoidal channel with a side slope of 1.5 (with b = 10 ft and $S_o = 0.0005$ as for the rectangular channel). With the flow rate changing from 12 to 600 cfs, the variation in Manning's n is again 7.6%.

The next four groups of three columns each were obtained by holding the flow rate constant, letting the slope of the channel bottom vary, solving Chezy's equation for the depth, and based on this computed depth and other variables solving Manning's equation for n. The third column of each group of 3 shows the variation of n. These variations in n are from 3.9% to 5.1%.

The last three columns in Table 2.4 were obtained for a rectangular channel with a bottom slope of $S_o = 0.0005$ and the depth held constant at 2 ft. The first column of this group that gives the bottom width b was varied from 0.5 to 25.0 ft, and the flow rate in column 2 was obtained by solving Chezy's formula. The last column represents the solution of Manning's equation for this flow rate and channel. Note again a variation in the n value of 5.4%.

Table 2.5 represents the results from similar calculations to those used to get the values in Table 2.4 with the exception that they apply to circular cross sections, and the units are SI instead of ES. Approximately the same variations in n occur. Both of these tables represent common channel sizes, and the value of e = 0.004 ft is a typical value for man-made lined channel. The variation of 3%–8% shown in these tables is within the accuracy of selecting n for a given application. Therefore, we might conclude that for most practical problems it is satisfactory to use Manning's equation. Should the Reynolds number of the flow be less than 100,000, then it is probably best to use Chezy's formula even if it does entail a little more arithmetic. The accuracy of predicting flow rates and/or other variables in open channels cannot be expected to be better than the percentages shown in Tables 2.4 and 2.5.

If one assumes Manning's equation applies only in the wholly rough zone, then equating Chezy's and Manning's equations, $V = C(R_hS_o)^{1/2} = (C_u/n)R_h^{2/3}S_o^{1/2}$, with C defined by Equation 2.7 gives

$$\frac{C_u}{n}R_h^{1/6} = \sqrt{32g}\,\mathrm{Log}_{10}\left(\frac{12R_h}{e}\right)$$

If we take n as dimensionless, then this equation indicates C_u has dimensions of $L^{1/3}/t$. Making the assumption that n is dimensionless allows the same values of n to be used for both ES and SI units. This is what has occurred in practice, i.e., $C_u = 1$ for SI units, and for ES units C_u is the cubic root

TABLE 2.4
Variations of Manning's n with a Fixed Value of e = 0.004 ft ($\nu = 1.217 \times 10^{-5}$ ft²/s) and Other Variables Changed in a Trapezoidal Channel

Variables Held Constant

b = 10 ft, m = 0, S_o = 0.0005			b = 10 ft, m = 1.5, S_o = 0.0005			b = 10 ft, m = 0, Q = 300 cfs			b = 10 ft, m = 1.5, Q = 500 cfs			b = 2 ft, m = 0, Q = 5 cfs			b = 2 ft, m = 1.5, Q = 20 cfs			m = 0, Y = 2 ft, S_o = 0.0005		
Q	**Y**	**n**	**Q**	**Y**	**n**	**S_o**	**Y**	**n**	**S_o**	**Y**	**n**	**S_o**	**Y**	**n**	**S_o**	**Y**	**n**	**b**	**Q**	**n**
8	0.52	0.0131	12	0.64	0.0132	0.0001	12.39	0.0143	0.0001	8.05	0.0145	0.0001	2.77	0.0134	0.0001	2.58	0.0137	0.5	0.937	0.0130
80	2.39	0.0137	120	2.46	0.0137	0.0005	6.39	0.0141	0.0005	5.35	0.0142	0.0005	1.42	0.0132	0.0005	1.76	0.0134	1.0	2.758	0.0131
160	3.95	0.0139	240	3.62	0.0140	0.0010	4.88	0.0140	0.0010	4.46	0.0141	0.0010	1.08	0.0131	0.0010	1.49	0.0133	5.0	26.462	0.0135
240	5.37	0.0140	360	4.50	0.0141	0.0025	3.46	0.0138	0.0025	3.48	0.0139	0.0025	0.765	0.0130	0.0025	1.18	0.0132	10.0	61.932	0.0136
400	8.04	0.0141	600	5.88	0.0142	0.0050	2.69	0.0137	0.0050	2.87	0.0138	0.0050	0.594	0.0129	0.0050	0.988	0.0131	25.0	174.04	0.0137
Variation		7.6%			7.6%			4.4%			5.1%			3.9%			4.6%			5.4%

Note: Q is in cfs and Y is in ft, b is in ft.

TABLE 2.5
Variations of Manning's n with a Fixed Value of e = 0.001219 m ($\nu = 1.31 \times 10^{-6}$ m²/s) and Other Variables Changed in a Circular Channel

Variables Held Constant

D = 6 m, S_o = 0.0005			D = 2 m, S_o = 0.0005			D = 0.33333 m, S_o = 0.0005			D = 6 m, Q = 25 m³/s			D = 1 m, Q = 0.20 m³/s			D = 3 m, Y = 2 m			D = 0.5 m, Y = 0.3 m		
Q	**Y**	**n**	**Q**	**Y**	**n**	**Q**	**Y**	**n**	S_o	**Y**	**n**	S_o	**Y**	**n**	S_o	**Q**	**n**	S_o	**Q**	**n**
1.20	0.58	0.0135	0.073	0.204	0.0131	0.0007	0.0359	0.0134	0.0002	3.70	0.0147	0.0001	0.722	0.0136	0.0001	3.23	0.0142	0.0001	0.025	0.0134
6.00	1.29	0.0140	0.438	0.492	0.0134	0.0054	0.0987	0.0131	0.0004	2.97	0.0145	0.0003	0.498	0.0134	0.0005	7.26	0.0141	0.0005	0.056	0.0131
26.4	2.87	0.0145	1.02	0.770	0.0136	0.0122	0.1526	0.0131	0.0010	2.28	0.0144	0.0010	0.354	0.0132	0.0010	10.3	0.0141	0.0010	0.080	0.0131
40.0	3.77	0.0146	1.75	1.05	0.0137	0.0204	0.2087	0.0131	0.0020	1.57	0.0141	0.0020	0.294	0.0131	0.0020	14.6	0.0141	0.0020	0.113	0.0130
60.0	5.38	0.0147	3.43	1.83	0.0138	0.0306	0.3086	0.0131	0.0050	1.24	0.0139	0.0050	0.238	0.0130	0.0050	21.8	0.0140	0.0050	0.179	0.0130
Variation		8.8%			5.3%			2.3%			5.8%			4.6%			1.4%			3.1%

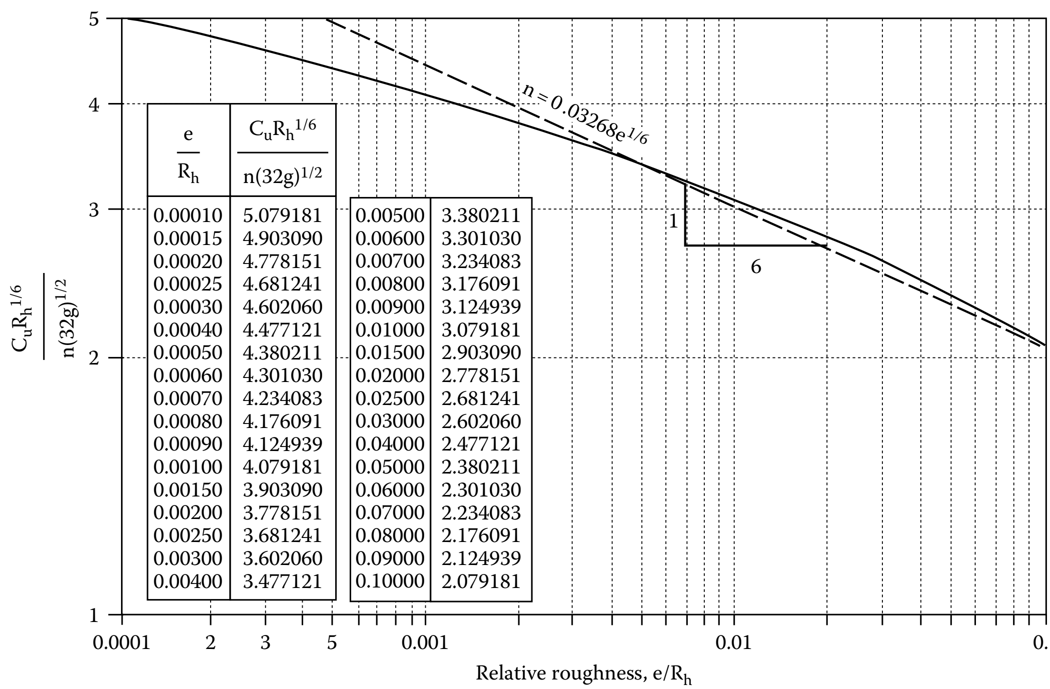
Note: Q is in m³/s and Y is in m, and D is in m.

of the number of feet per meter, or $C_u = (1/.3048)^{1/3} = 1.486$. However, it is generally accepted that n is not dimensionless. Often when working in one system of units (either ES or SI), the quantity on the right of the above equation, which is Chezy's C, as well as C_u are taken as dimensionless, and therefore the literature will often suggest that n has dimensions of $L^{1/6}$. A log–log plot of the above equation is given below. If n is assumed to have dimensions of $L^{1/6}$ then one would expect n to vary as the one-sixth root of the wall roughness size, e, or $n = Ke^{1/6}$ (where K is a constant). Such a relationship is shown on the graph as a dashed line, but any other line parallel to this line could be used depending on what n one selects to correspond to e for a given hydraulic radius. For the given dashed line R_h is taken equal to 3 ft and $C_u R_h^{1/6}/\{n(32g)^{1/2}\} = 2.079$ corresponding to $e/R_h = 0.1$ (Figure 2.2). Thus for e = 0.3 ft, n = 0.0267, or $n = 0.03268e^{1/6}$. This gives n = 0.0182 corresponding to e = 0.03 ft, and n = 0.0124 corresponding to e = 0.003 ft. Note from this graph that the dashed line would fit the curve closer if its slope were flatter; suggesting that the dimensions of n might be closer to $L^{1/7}$.

Let us now focus attention on solving Manning's equation, which can be solved directly if the flow rate Q, the roughness coefficient n, or the slope of the channel bottom S_o is the unknown. To solve for n, its place is interchanged in Equation 2.8 with Q. In solving for S_o Manning's equation becomes

$$S_o = \left\{\frac{nQ}{C_u}\frac{P^{2/3}}{A^{5/3}}\right\}^2 = \left\{\frac{nV}{C_u}R^{2/3}\right\}^2 = \left\{\frac{nQ}{C_u A}\left(\frac{P}{A}\right)^{2/3}\right\}^2 \tag{2.8a}$$

If one of the variables that goes into defining the cross-sectional area A, the wetted perimeter P, i.e., the hydraulic radius R_h is unknown, then Manning's equation becomes implicit in that variable, and must be solved by an iterative method such as the Newton's method, or by trial and error. Consider a trapezoidal section for example. For a trapezoid, the area and wetted perimeters are



e/R_h	$\frac{C_u R_h^{1/6}}{n(32g)^{1/2}}$	e/R_h	$\frac{C_u R_h^{1/6}}{n(32g)^{1/2}}$
0.00010	5.079181	0.00500	3.380211
0.00015	4.903090	0.00600	3.301030
0.00020	4.778151	0.00700	3.234083
0.00025	4.681241	0.00800	3.176091
0.00030	4.602060	0.00900	3.124939
0.00040	4.477121	0.01000	3.079181
0.00050	4.380211	0.01500	2.903090
0.00060	4.301030	0.02000	2.778151
0.00070	4.234083	0.02500	2.681241
0.00080	4.176091	0.03000	2.602060
0.00090	4.124939	0.04000	2.477121
0.00100	4.079181	0.05000	2.380211
0.00150	3.903090	0.06000	2.301030
0.00200	3.778151	0.07000	2.234083
0.00250	3.681241	0.08000	2.176091
0.00300	3.602060	0.09000	2.124939
0.00400	3.477121	0.10000	2.079181

FIGURE 2.2 Relationship between n parameter and relative roughness to establish relation of n to e/R_h.

defined respectively by A = (b + mY)Y, and $P = b + 2Y\sqrt{m^2+1}$, and Manning's equation written as a function of this unknown equal to zero for use in the Newton iterative Equation B.2 becomes

$$F(\xi) = nQ\left[b + 2Y\sqrt{m^2+1}\right]^{2/3} - [(b+mY)Y]^{5/3} C_u\sqrt{S_o} = 0 \tag{2.8b}$$

in which variable ξ represent b, Y, or m depending respectively whether the bottom width, the depth or the side slope is the unknown. The Newton method for solving implicit equations such as Equation 2.8b is discussed in Appendix B, and is illustrated by example problems below.

If the channel is circular, then it is best to introduce the additional angle $\beta = \cos^{-1}(1 - 2Y/D)$ as defined in Appendix A. Thus the depth Y is related to this angle (in radians) by $Y = D(1 - \cos\beta)/2$, and Manning's equation can be written as

$$F(\xi) = nQ(\beta D)^{2/3} - \left[\frac{D^2(\beta - \cos\beta\sin\beta)}{4}\right]^{5/3} C_u\sqrt{S_o} = 0 \tag{2.8c}$$

in which variable ξ now represents either β or D depending respectively whether the depth, or the diameter is unknown. Should the depth be unknown then β is first obtained by solving for it from Equation 2.8c, and thereafter the depth Y is computed by the equation above Equation 2.8c.

EXAMPLE PROBLEM 2.5

A flow rate of Q = 450 cfs is taking place in a trapezoidal channel with the following properties: b = 10 ft, m = 1, and $S_o = 0.0006$. Determine the uniform depth of flow in this channel if the appropriate value for Manning's n is, n = 0.013.

Solution

This problem can be solved using Equation 2.8b and the Newton method. The Newton method can easily be solved using a programmable pocket calculator using the following steps:

1. Assign the variables, Y, b, m, Q, S_o, and n to storage registers, and place the values for these variable in the assigned registers (this includes a guess for the unknown Y in this case).
2. Put the calculator in program mode and program Equation 2.8b into it with the value of the equation $F(\xi) = F(Y)$ being displayed upon completion of the equation.
3. Press the operate button on the calculator, and when it is complete store the value displayed in an unused register.
4. Retrieve the value from the register that hold Y, and increase it by a small amount such as 0.001.
5. Store this increased value in the same register for Y again.
6. Press the operate button again.
7. Recall the last value of the equation, subtract it from the current value and divide this difference by 0.001.
8. Recall Y, subtract the result from step 7, and also subtract 0.001 from it and store Y back in its register.
9. Repeat steps 4 through 8 until convergence has occurred. If you want you can also program these steps into your pocket calculator.

The implementation of these steps for this problem results in the following: Step 1

Register	1(Y)	2(b)	3(m)	4(Q)	5(S_o)	6(n)
Value	5	10	1	450	0.0005	0.013

from step 3 F(Y) = 2.902; from step 6 F(Y) = 2.738, after step 8 register # 1 contains 5.018 for Y. After iteration # 2 Y = 5.018, which represents no change to three digits beyond the decimal point, and the solution is terminated. If you have a calculator with the capability to solve implicit equations with a SOLVE key such as the HP 48xs, then all that is needed is to define the equation and give values to the variables.

EXAMPLE PROBLEM 2.6

A flow rate of Q = 30 cfs is to be carried by a pipe with a bottom slope of $S_o = 0.00028$, and a Manning's roughness coefficient of 0.013. If the depth is not to exceed 3/4 of the diameter, what size pipe should be used?

Solution

Since the depth is not to exceed 3/4 of the diameter, then $\cos\beta = 1 - 3/2 = -1/2$, and the equation that must be solved is

$$F(D) = nQ(\beta D)^{2/3} - \left\{\frac{D^2}{4}(\beta - \sin\beta\cos\beta)\right\}^{5/3} C_u\sqrt{S_o}$$

for the pipe diameter D. Using the Newton method, the solution to this equation is D = 4.49 ft. Taking the next standard pipe size would call for using a pipe with a 54 in. diameter.

EXAMPLE PROBLEM 2.7

A pipe of diameter 2 m and Manning's n = 0.013 has a bottom slope of $S_o = 0.00112$. Generate a table that gives the depths of flow that would be expected in this pipe under uniform flow conditions for flow rates of $Q = 0.5\,m^3/s$ to $Q = 4.5\,m^3/s$ in increments of $0.5\,m^3/s$.

Solution

The solution for each entry in this table requires that Equation 2.8c be solved by an iterative method such as the Newton method. The following BASIC program implements such a solution for the nine different flow rates giving the depth shown to the right of the BASIC listing.

BASIC program listing to solve Example Problem 2.7

```
10 INPUT "Give:n,So,D,Q1,DQ,est. for Y & No ",N,S,D,Q,DQ,Y,NO%
20 FOR I%=1 TO NO%
30 NCT%=0
40 ARG=1-2*Y/D
50 TANA=SQR(1-ARG*ARG)/ABS(ARG)
60 IF ARG>0 THEN BETA=ATN(TANA) ELSE BETA=3.14159265#-ATN
   (TANA)
70 A=.25*D*D*(BETA-ARG*SIN(BETA))
80 P=BETA*D
90 F=Q-A/N*(A/P)^.6666667*SQR(S)
100 IF NCT%>0 THEN GOTO 150
110 F1=F
120 Y=Y-.001
130 NCT%=1
140 GOTO 40
150 DY=.001*F1/(F1-F)
160 Y=Y-DY+.001
170 IF ABS(DY) > .00001 THEN GOTO 30
180 PRINT Q,Y
190 Q=Q+DQ
200 NEXT I
210 END
```

Q(m3/s)	Y(m)
0.5	0.42
1.0	0.60
1.5	0.74
2.0	0.87
2.5	0.99
3.0	1.10
3.5	1.22
4.0	1.33
4.5	1.46

EXAMPLE PROBLEM 2.8

A natural canal that has a bottom slope of 0.002, and a Manning's n = 0.018 has the cross section defined by the following transect data for a long distance. Determine the depth of flow in this canal if the flow rate is Q = 90 cfs.

x (ft)	0	2	4	7	9	11	14	18
y (ft)	0	0.8	1.5	3.0	3.3	2.5	1.2	0.0

Solution

From this data it is possible to interpolate along both sides of the canal to generate the following data giving the top width, the area, and perimeter for equal increments of the depth y.

Depth, Y	Top Width	Area	Perimeter
0.17	1.58	0.13	1.62
0.33	3.01	0.51	3.10
0.50	4.30	1.11	4.44
0.66	5.44	1.92	5.63
0.83	6.44	2.90	6.68
0.99	7.29	4.03	7.59
1.16	7.99	5.29	8.37
1.32	8.55	6.65	9.02
1.49	8.96	8.10	9.61
1.65	9.22	9.62	10.22
1.82	9.41	11.18	10.88
1.98	10.21	12.80	11.75
2.15	11.07	14.56	12.66
2.31	12.03	16.46	13.68
2.48	13.05	18.53	14.75
2.64	14.03	20.77	15.78
2.81	15.01	23.16	16.82
2.97	15.99	25.72	17.86
3.14	16.99	28.44	18.91
3.30	18.00	31.33	19.98

The solution might proceed by selecting a depth, and then by interpolation in the above table determine the corresponding area and perimeter, substitute these into Manning's equation, and compute the flow rate. Based on the difference between the computed flow rate and the wanted value of 90 cfs, adjust the depth and repeat the process. A more formal procedure would be to implement the Newton method in the above table interpolation to obtain a better estimate for the next depth to use. The answer is Y = 2.60 ft, with A = 20.21 ft^2 and P = 15.56 ft. See Appendix B.5 for details related to how areas and wetted perimeters can be defined from x y cross-sectional coordinates, etc.

Solving for the depth, or diameter, in circular sections by the Newton method requires that a good guess be supplied to start the iterative solution process, or else the method will fail. For circular sections, this guess must be much better than for trapezoidal channels. In the small BASIC program given under Problem 2.7 above it is left to the user to provide a satisfactory starting value. However, requiring the user to supply an initial guess is not always desirable. In most cases, an adequate initial guess can be easily generated within a computer program for solving for either the depth or diameter in a circular section, the two variables in Manning's equation for which an explicit solution is not possible.

If the area and wetted perimeter for a circular section are replaced in Manning's equation by functions of the auxiliary angle β that define these quantities, and terms rearranged, then Manning's equation can be written as

$$Q' = \frac{nQ}{C_u D^{8/3}\sqrt{S_o}} = \frac{(\beta - \cos\beta \sin\beta)^{5/3}}{10.079368\beta^{2/3}} = F(\beta) \qquad (2.9)$$

It should be noted that Q' is dimensionless if C_u has units of the cube root of length per time (with n taken as dimensionless). A close approximation of the above dependency of Q' on β is given by the following power equation:

$$Q' = 0.0313\beta^{4.0984} \qquad (2.10a)$$

or the inverse of this equation is,

$$\beta = 2.3286(Q')^{0.244} \qquad (2.10b)$$

Thus if the depth of flow is the unknown it is possible to compute $Q' = nQ/(C_u D^{8/3}\sqrt{S_o})$, and then from Equation 2.10b obtain a starting value for β. The reasonableness of this approximation is shown by the values in the small table below that gives the approximate β, the actual β and the difference corresponding to several values of Q'. Note that as the depth approaches the diameter, i.e., β approaches π the difference get larger.

Q'	β_{act}	β_{appr}	Difference
1.00E−06	0.0822	0.0800	0.0022
1.00E−05	0.1400	0.1403	−0.0003
1.00E−04	0.2388	0.2461	−0.0072
1.00E−03	0.4098	0.4316	−0.0218
1.00E−02	0.7160	0.7570	−0.0410
0.100	1.3478	1.3277	0.0201
0.200	1.7364	1.5723	0.1641
0.300	2.1842	1.7359	0.4484
0.330	2.4564	1.7767	0.6797
0.335	2.5955	1.7832	0.8123

Manning's equation will need to be solved frequently throughout the chapters of this book. It will be useful for you to develop a computer program, or program your pocket calculator so that it will be possible to readily obtain a solution to any variable that may be unknown in Manning's equation for either a trapezoidal or a circular section. Below is a listing of such a program in TURBO PASCAL, which utilizes the clear screen CLRSCR and the GOTOXY capabilities to give the program a little "user friendliness." You should study over the program carefully. The array element X[IU] contains the unknown. For a trapezoidal channel the correspondence between the X's and the variables are: X[1] = Q, X[2] = n, X[3] = S, X[4] = Y, X[5] = b, and X[6] = m. For a circular section the variables are the same through X[4], but then X[5] = D (the pipe diameter). A good way for you to understand how this program solves Manning's equation completely in either a circular or a trapezoidal section, and allows either ES or SI units to be used is for you to translate the program into FORTRAN, C or BASIC depending upon what you are most familiar with.

Listing of PASCAL program MANNING.PAS that completely solves Manning's equation in both trapezoidal and circular channels

```
Program Manning;
Const NX:array[1..7] of char=('Q','n','S','Y','b','m','D');
  VNAM:array[1..4] of string[12]=('flow rate','coefficient',
    'bottom slope','depth');
```

```
  JW:array[1..4] of integer=(2,4,6,3);
Var X:array[1..7] of real; F1,P,Beta,DF,DX,C,AA:real;
  ITY,I,IU,m:integer;
Function Expn(a,b:real):real;
  Begin if a<0 then Writeln('error in power',a,b)
  else Expn:=Exp(b*Ln(a)) End; {raises a to the power b}
Function A:real; Begin {computes area A & AA and perimeter P}
  If ITY=1 then begin P:=X[5]+2*X[4]*sqrt(sqr(X[6])+1);
    AA:=(X[5]+X[6]*X[4])*X[4];
  end else Begin P:=1-2*X[4]/X[5]; if P=0 then Beta:=Pi/2 else begin
  Beta:=sqrt(1-sqr(P))/abs(P); if P>0 then Beta:=arcTan(Beta) else
  Beta:=Pi-arcTan(Beta) end;AA:=sqr(X[5])/4*(Beta-P*sin(Beta));
    P:=Beta*X[5] End;
  A:=AA End;
Function F:real; Begin {Defines Manning's Equation for Newton Method}
  F:=X[2]*X[1]-C*A*Expn(AA/P,0.666667)*SQRT(X[3]) End;
Var Ch:Char;
Label L1;
BEGIN      {Start of program}
L1:ClrScr; GoToXY(1,10);Writeln('Do you want to use:');
  Writeln('1 - ES units, or'); Writeln('2 - SI units?');
  repeat Readln(Ch); until Ch in ['1','2','E','e','S','s'];
  If Ch in ['1','E','e'] then C:=1.486 else C:=1;
  ClrScr;GoToXY(1,10); Writeln('Is section:');
  Writeln('1 - Trapezoidal, or');Writeln('2 - Circular?');
  repeat Readln(Ch); until Ch in ['1','2','T','t','C','c'];
  if Ch in ['1','t','T'] then ITY:=1 else ITY:=2; ClrScr;
  Writeln('Give no. of unknown');
   For I:=1 to 4 do Writeln(I:2,' - ',NX[I]:1,' (',VNAM[I]);
   if ITY=2 then Writeln(' 5 - D (diameter)') else begin
     Writeln(' 5 - b (bottom width)');
     Writeln(' 6 - m (side slope)') end;
  repeat Readln(IU); until IU in [1..7]; ClrScr;
  Writeln('Give values for knowns');For I:=1 to 4 do if I<>IU then
    begin GoToXY(1,I+1);Write(NX[I]:1,' = '); Readln(X[I]) end;
  If (ITY=2) and (IU<>5) then begin GoToXY(1,6);Write('D = ');
    Readln(X[5]) end;
  If ITY=1 then Begin I:=5; if IU<>5 then begin I:=I+1;
    GoToXY(1,I); Write('b = ');Readln(X[5]) end;
    if IU<>6 then begin I:=I+1; GoToXY(1,I);
     Write('m = ');Readln(X[6]) end; End;
  Case IU of
    {1 thru 3 solve explicit eqs, 4,5 & 6 use Newton Method}
   1:X[1]:=C/X[2]*A*Expn(AA/P,0.6666667)*sqrt(X[3]);
   2:X[2]:=C/X[1]*A*Expn(AA/P,0.6666667)*sqrt(X[3]);
   3:X[3]:=sqr(X[1]*X[2]/(C*A*Expn(AA/P,0.6666667)));
   4,5,6:Begin If ITY=2 Then Begin If IU=4 then begin
     Beta:=2.3286*Expn(X[1]*X[2]/(C*Expn(X[5],
       2.6666667)*sqrt(X[3])),0.244);
      X[4]:=X[5]/2*(1-cos(Beta)) end else X[5]:=2*X[4] End Else
```

```
     case IU of 4:X[4]:=X[5]/2; 5:X[5]:=2*X[4]; 6:X[6]:=1;
      end; m:=0;
     repeat F1:=F; DX:=X[IU]/100; X[IU]:=X[IU]-DX;
      DF:=DX*F1/(F1-F); X[IU]:=X[IU]+DX-DF; m:=m+1;
        until (m>20) or (abs(DF)<0.0001);
     End; End; ClrScr;
  Writeln('Solution to unknown');
    If (ITY=2) and (IU=5) then Write('D = ') else
    Write(NX[IU]:1,' = '); Writeln(X[IU]:11:JW[IU]+1); Writeln;
  Writeln('Variables of Problem:');
   For I:=1 to 4 do Writeln(NX[I]:1,' = ',X[I]:10:JW[I]);
   If ITY=2 then Writeln('D = ',X[5]:10:3) else begin
    Writeln('b = ',X[5]:10:3); Writeln('m = ',X[6]:10:4) end;
  GoToXY(1,24);
    Write('Do You want to solve another problem?(Y or N) ');
   Readln(Ch); If (Ch='Y') or (Ch='y') then GoTo L1;
END.
```

Listing of FORTRAN program MANNING.FOR designed to solve Manning's equation in either trapzoidal or circular channels

```
      LOGICAL REPT
      REAL X(6),n,m
      CHARACTER*1 V(6)
      COMMON X,BETA
      EQUIVALENCE (Q,X(1)),(n,X(2)),(S,X(3)),(Y,X(4)),
     &(b,X(5)),(m,X(6))
      DATA V/'Q','n','S','Y','b','m'/
      WRITE(6,*)' Give 1.49 for ES units or 1. for SI'
     &,= units.'
      READ(5,*) C
      WRITE(6,*)' Give 1 for trap. sec., or 2 for cir.'
     &,= sec.'
      READ(5,*) ITYP
      IF(ITYP.EQ.1) THEN
      No=6
      ELSE
      No=5
      V(5)='D'
      ENDIF
1     WRITE(6,100) (I,V(I),I=1,No)
100   FORMAT(' Give No. of unknown:'/6(I2,'-',A1))
      READ(5,*) IUNK
      DO 10 I=1,No
      IF(I.LT.4.AND.I.EQ.IUNK) GO TO 10
      WRITE(6,101) V(I)
101   FORMAT(3X,A1,' = ',$)
      READ(5,*) X(I)
10    CONTINUE
      NCT=0
20    AA=A(ITYP)
```

```
      GO TO(21,22,23,24,24,24), IUNK
21    Q=C/n*AA*(AA/P(ITYP))**.6666667*SQRT(S)
      GO TO 40
22    N=C/Q*AA*(AA/P(ITYP))**.6666667*SQRT(S)
      GO TO 40
23    S=(n*Q/C*(P(ITYP)/AA)**.6666667/AA)**2
      GO TO 40
24    REPT=.TRUE.
25    F=n*Q-C*AA*(AA/P(ITYP))**.6666667*SQRT(S)
      IF(REPT) THEN
      REPT=.FALSE.
      F1=F
      X(IUNK)=X(IUNK)-.001
      AA=A(ITYP)
      GO TO 25
      ENDIF
      DIF=.001*F1/(F1-F)
      X(IUNK)=X(IUNK)+.001-DIF
      NCT=NCT+1
      IF(NCT.LT.20 .AND. ABS(DIF).GT..00001)
      * GO TO 20
      IF(NCT.EQ.20)WRITE(6,*)' FAILED TO CONVERGE' *,DIF
40    WRITE(6,105)(V(I),X(I),I=1,No)
105   FORMAT(' SOLUTION:',6(A2,' =',F12.6))
      WRITE(6,*)' Give 1 for another prob.; else 0'
      READ(5,*) NCT
      IF (NCT.EQ.1) GO TO 1
      STOP
      END
      FUNCTION A(ITYP)
      COMMON X(6),BETA
      IF(ITYP.EQ.1) THEN
      A=(X(5)+X(6)*X(4))*X(4)
      ELSE
      BETA=ACOS(1.-2.*X(4)/X(5))
      A=.25*X(5)*X(5)*(BETA-.5*SIN(2.*BETA))
      ENDIF
      RETURN
      END
      FUNCTION P(ITYP)
      COMMON X(6),BETA
      IF(ITYP.EQ.1) THEN
      P=X(5)+2.*X(4)*SQRT(X(6)**2+1.)
      ELSE
      P=X(5)*BETA
      ENDIF
      RETURN
      END
```

Listing of C program designed to solve Manning's equation in trapezoidal or circular channels

```
/*  Solves Manning's Equation in Trapezoidal & Circular Channels  */
#include <stdio.h>
#include <math.h>
int type; float beta;
float A(float b,float m,float Y) {
  if (type==1) return((b+m*Y)*Y); else
 {beta=acos(1.-2.*Y/b); return(0.25*b*b*(beta-0.5*sin(2.*beta)));}}
float P(float b,float m,float Y) {
 if(type==1) return(b+2.*Y*sqrt(m*m+1.));
 else return(b*sin(beta));}
main () {
float x[6],cc,AA,F,F1,DIF,dumm; int IUNK,nct,No,II,i; char u,V[7];
  dumm=sqrt(.04); /* drag floating pt lib. into the linker */
  strcpy(V,"QnSYbm\n");
 puts("Give 1.49 for ES units or 1. for SI units.");
   scanf("%f",&cc);
 puts("Give 1 for trapezoidal section, or 2 for circular section.");
   scanf("%d",&type); if (type==1) No=6; else {V[4]='D'; No=5;}
 a2:puts("Give No. of unknown:");
   for (i=0;printf(" %d - %c,",i,V[i]),i<No-1;i++);
 printf("\n"); scanf("%d",&IUNK);
 puts("Give:\n"); for (i=0;i<No;i++) if ((i!=IUNK) | (i>2)) {
  printf(" %c = ",V[i]); scanf("%f",&x[i]);}
 switch (IUNK) {
  case 0: x[0]=cc/x[1]*pow(A(x[4],x[5],x[3]),1.6666667)/pow(P(x[4],\
    x[5],x[3]),0.66667)*sqrt(x[2]); break;
  case 1: x[1]=cc/x[0]*pow(A(x[4],x[5],x[3]),1.6666667)/pow(P(x[4],\
    x[5],x[3]),0.66667)*sqrt(x[2]); break;
  case 2: AA=A(x[4],x[5],x[3]); DIF=x[1]*x[0]/cc*pow(P(x[4],\
    x[5],x[3])/AA,0.666667)/AA; x[2]=DIF*DIF; break;
  default: nct=0; do {II=0; a1:AA=A(x[4],x[5],x[3]);
    F=x[0]*x[1]-cc*AA*pow(AA/P(x[4],x[5],x[3]),0.66667)*sqrt(x[2]);
    if (II==0) {II=1; F1=F; x[IUNK]=x[IUNK]-0.001; goto a1;}
    DIF=0.001*F1/(F1-F); x[IUNK]=x[IUNK]+0.001-DIF;
    } while (abs(DIF)>0.00001 && ++nct<15);}
  for (i=0;i<No;i++) printf(" %c =%f\n",V[i],x[i]);
  puts(" Give 1 to solve another problem; else 0");
  scanf("%d",&II);
  if (II==1) goto a2;
}
```

The Newton method can be use to solve a linear equation. When the Newton method is used in solving a linear equation the solution will be obtained with the first iteration, even though this will generally not be known until the computations are done for the second iteration, and the function of the unknown equals zero. The logic of the above computer programs can, therefore, be simplified by omitting the separate equations that solve for those variable that can be explicitly obtained from Manning's equation. Below are listing of a FORTRAN program and a C program that only use the Newton method; even when solving for Q, n, and S.

Listing of FORTRAN program that uses the Newton method to solve all variables (MANNTC.FOR)

```
      PARAMETER (EX=.6666667,EX1=1.6666667)
      CHARACTER*1 VAR(6)/'Q','n','S','y','b','m'/,UNK
      CHARACTER*33 FMT/"(' Solution for ',A1,' = ',F10.3)"/
      REAL X(6),ns,neq,C/1.486/
C  X(1)=Q, X(2)=n, X(3)=S, X(4)=y, X(5)=b, X(6)=m
      LOGICAL R /.FALSE./
      WRITE(*,*)'Give 0 if ES; 1 if SI units'
      READ(*,*) IUNIT
      IF(IUNIT.EQ.1) C=1.
      WRITE(*,*)'Is the section: 1   trapzoidal, or 2   circular?'
      READ(*,*) ISECT
      IF(ISECT.EQ.2) THEN
      NVAR=5
      VAR(5)='D'
      ELSE
      NVAR=6
      ENDIF
      WRITE(*,*)'Give value to variables'
      DO 10 I=1,NVAR
      WRITE(*,"(2X,A1,' = '\)") VAR(I)
      IF(I.EQ.2 .AND. ISECT.EQ.1) THEN
      WRITE(*,"(' (1st for sides=) ')")
      READ(*,*) ns
      WRITE(*,"(' (now for bottom=) ')")
      ENDIF
10    READ(*,*) X(I)
15    WRITE(*,*)'Give symbol for unknown'
      READ(*,'(A1)') UNK
      IUNK=1
      DO 20 WHILE (VAR(IUNK).NE.UNK.AND.IUNK.LE.NVAR)
20    IUNK=IUNK+1
      IF(IUNK.GT.NVAR) GO TO 15
      DO 30 I=1,20
25    IF(ISECT.EQ.2) THEN
      COSB=1. 2.*X(4)/X(5)
      B=ACOS(COSB)
      A=.25*X(5)*X(5)*(B COSB*SIN(B))
      P=B*X(5)
      F=X(2)*X(1) C*SQRT(X(3))*A*(A/P)**EX
      ELSE
      P=X(5)+2.*X(4)*SQRT(X(6)**2+1.)
      neq=X(5)/P*X(2)+2.*X(4)*SQRT(X(6)**2+1.)/P*ns
      F=neq*X(1)*P**EX C*SQRT(X(3))*((X(5)+X(6)*X(4))*X(4))**EX1
      ENDIF
      IF(R) GO TO 28
      XX=X(IUNK)
      X(IUNK)=1.005*X(IUNK)
      F1=F
```

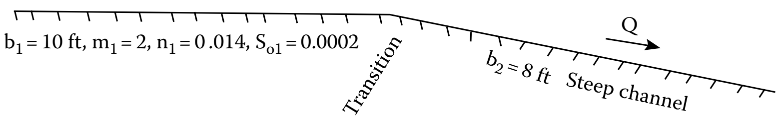
```
      R=.TRUE.
      GO TO 25
28    DIF=F1*(X(IUNK) XX)/(F F1)
      X(IUNK)=XX DIF
      IF(ABS(DIF).LT. .000001) GO TO 40
30    CONTINUE
40    IF(IUNK.EQ.2)FMT(32:32)='4'
      IF(IUNK.EQ.3)FMT(32:32)='7'
      WRITE(*,FMT) VAR(IUNK),X(IUNK)
      END
```

Listing of C program that uses the Newton method to solve all variables (MANNTC.C)

```
#include <stdio.h>
#include <math.h>
#include <stdlib.h>
main() {int units,sect,i,r,iunk,nvar=5,nct=0;
 float c,x[6],f,f1,xx,dif,cosb,a,b,p,ns,neq,ex=.6666667,\
   ex1=1.6666667;
 char var[7]="QnSybm\0",unk; clrscr();
 printf("Give 0 if ES; 1 if SI units ");scanf("%d",&units);
 if (units) c=1; else c=1.486;
 printf("Is the section: 1 - trapezoidal, or 2 - circular? ");
 scanf("%d",&sect);if(sect==2) {nvar=4;var[4]='D';}
 printf("Give value to variables\n");for(i=0;i<=nvar;i++){
  printf(" %c = ",var[i]); if((i==1)&&(sect==1)){
   printf(" (1st for sides =) ");scanf("%f",&ns);
   printf(" (now for bottom=) ");} scanf("%f",&x[i]);}
 do {printf("Give symbol for unknown ");scanf("%s",&unk);iunk=0;
 while (var[iunk]!=unk && iunk<=nvar) iunk++;} while(iunk>unk);
 do {r=1;
L1:if(nvar==4){cosb=1-2*x[3]/x[4];b=acos(cosb);
  a=.25*x[4]*x[4]*(b-cosb*sin(b)); p=b*x[4];
  f=x[1]*x[0]-c*sqrt(x[2])*a*pow(a/p,ex);} else {
  p=x[4]+2.*x[3]*sqrt(x[5]*x[5]+1.);
   neq=x[4]/p*x[1]+2.*x[3]*sqrt(x[5]*x[5]+1.)/p*ns;
  f=neq*x[0]*pow(p,ex)-c*sqrt(x[2])*pow((x[4]+x[5]*x[3])*x[3],\
   ex1);}
 if(r) {xx=x[iunk];x[iunk]=1.005*x[iunk];f1=f;r=0;goto L1;}
 dif=f1*(x[iunk]-xx)/(f-f1); x[iunk]=xx-dif; nct++;
 } while (nct<20 && fabs(dif)>.000001);
 printf("\nSolution for %c = %f",var[iunk],x[iunk]);}
```

EXAMPLE PROBLEM 2.9

Determine the maximum flow rate that can be accommodated in the channel shown below without causing the depth in the upstream channel to rise above its normal depth. The upstream channel has a bottom width $b_1 = 10\,ft$, a side slope $m_1 = 2$, a Manning's roughness coefficient, $n_1 = 0.014$, and a bottom slope, $S_{o1} = 0.0002$. The downstream channel is steep, i.e., under uniform flow conditions the depth will be less than critical depth and is rectangular in shape with a bottom width $b_2 = 8\,ft$. Investigate the relationship between the slope of bottom of the upstream channel on the flow rates and the depths that are possible in this channel under uniform flow conditions.



Solution

Since the downstream channel is steep, critical flow will occur at the end of the transition to 8 ft wide rectangular channel, the specific energy here is given by

$$E_c = 1.5Y_c = 1.5\left(\frac{q_2^2}{g}\right)^{1/3} = 1.5\left[\frac{(Q/b_2)}{g}\right]^{1/3} = 0.11787Q^{2/3}$$

The specific energy in the larger upstream channel must equal E_c or

$$Y_1 + \frac{(Q/A)_1^2}{(2g)} = E_c = 0.11787Q^{2/83}$$

If the depth upstream is to be uniform, then

$$Q = \frac{1.486}{n} A_1 \left(\frac{A}{P_1}\right)_1^{2/3} S_{o1}^{1/2}$$

These three equations allow for the variables E_c, Y_1, and Q to be solved, or if one wishes to eliminate the first equation then the latter two equations will solve for Y_1 and Q. Their solution is

$$Y = 4.146 \text{ ft}, \quad Q = 218.42 \text{ cfs}.$$

In general, a problem of this nature that is to determine what the maximum flow rate is that can occur in a channel with a transition from an upstream mild channel to a downstream steep channel requires the simultaneous solution of (1) the critical flow equation in the downstream channel; (2) the energy equation, which equates the specific energy in the upstream channel to the critical specific energy in the downstream channel; and (3) the uniform flow equation.

To investigate the relationship between the bottom slope of the upstream channel and the maximum uniform flow possible requires that the latter two above equations be solved for different values of S_{o1}. The table below shows the results from several such solutions:

Bottom slope S_{o1}	0.002	0.001	0.0009	0.0008	0.0007	0.0006	0.0005	0.0004
Flow rate Q (cfs)	0.457	6.703	9.466	13.59	19.88	29.66	45.37	71.72
Depth Y_1 (ft)	0.0616	0.376	0.476	0.610	0.792	1.094	1.406	1.922
	0.0003	0.0002	0.00018	0.00014	0.00012	0.0001	0.00008	
	119.4	218.4	251.4	344.1	412.1	504.7	638.6	
	2.734	4.146	4.566	5.662	6.404	7.352	8.627	

It is interesting to note that the flow rate decreases to extremely small values as the bottom slope of the upstream channel becomes larger. Thus it is very easy to "choke" the upstream flow, i.e., cause it to be above normal depth by reducing the size of a downstream channel that will have critical flow in it.

If you don't have a computer available, nor a programmable calculator, then it is possible to use graphical means for solving Manning's equation. To develop such a graphical solution let us define a dimensionless depth $Y' = Y/b$ for trapezoidal channels. (When defining a dimensionless depth for the specific energy later we will let $Y' = mY/b$ because this eliminates m from the resulting dimensionless equation.) Then after some algebraic manipulation the following equation can be obtained:

$$\frac{nQ}{C_u b^{\frac{8}{3}}\sqrt{S_0}} = \frac{(Y' + mY'^2)^{\frac{5}{3}}}{\left(1 + 2Y'\sqrt{1+m^2}\right)^{\frac{2}{3}}} = Q'$$

The parameter on the left of the equal sign might be taken as a dimensionless flow rate, denote as Q'. (Note that Q' is dimensionless if n is taken as dimensionless and C_u is assumed to have dimensions of $L^{1/3}/t$.) This equation shows that the flow rate parameter Q' is a function of the dimensionless depth $Y' = Y/b$, and the side slope m of the trapezoidal channel, and when $m = 0$ the channel is rectangular. This relationship is given in Figure 2.3 using several different curves for different m values. The main graph is a linear plot, and the insert gives the same graph except using log–log paper. This graph can be used to solve Manning's equation for several different unknowns (Figure 2.3). For example, to find the normal depth Y_o, one would first compute the flow rate parameter Q' from the known values; then enter this value on the ordinate of the graph, and read the corresponding dimensionless depth Y' on the abscissa; and finally compute Y_o as the product of Y' and the bottom width b.

Manning's equation can also be solved graphically for a circular section. To develop such a graphical solution Equation 2.9 can be used to define the relationship between the angle β and

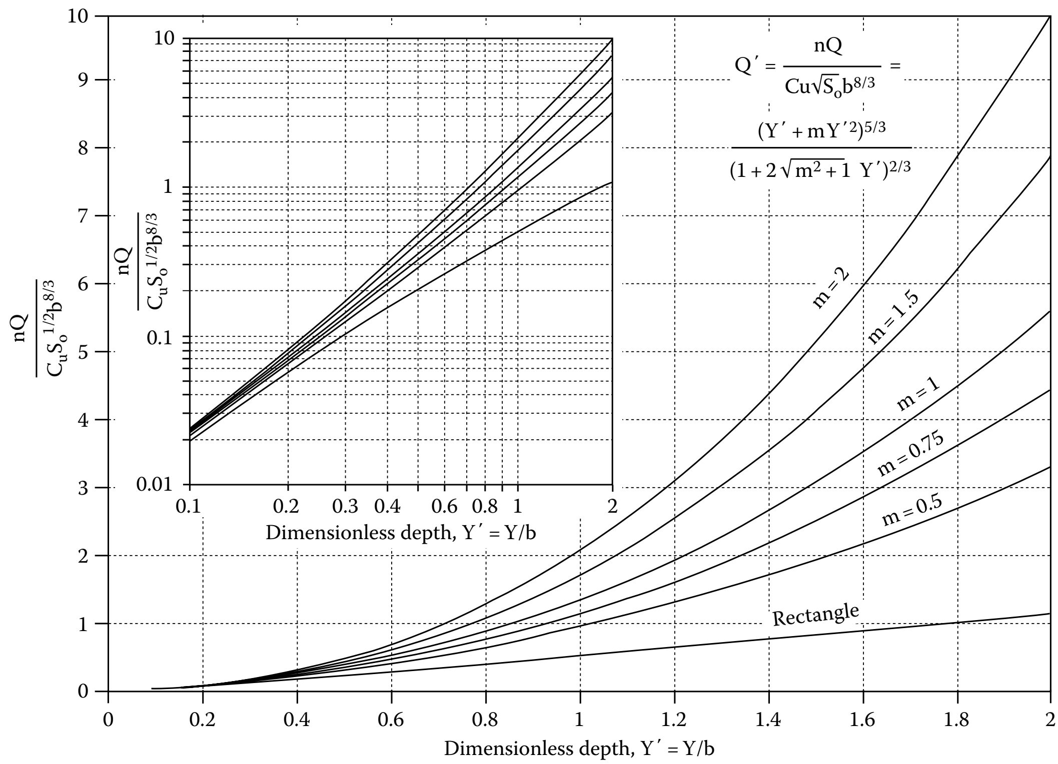


FIGURE 2.3 Plot of dimensionless Manning's equation in trapezoidal channels (including rectangular when m = 0).

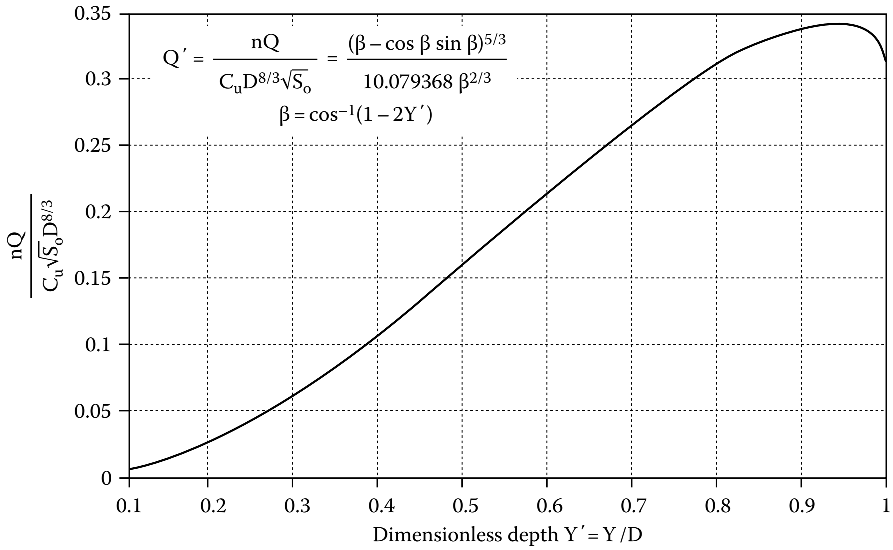


FIGURE 2.4 Plot of dimensionless Manning's equation in circular channels.

the flow rate parameter $Q' = (\beta - \cos\beta \sin\beta)^{5/3}/(10.0793684\beta^{2/3})$, and the angle β in turn can be obtained from the dimensionless depth $Y' = Y/D$ from $\beta = \cos^{-1}(1 - 2Y')$. Such a graphical solution of Manning's equations for circular channels is given in Figure 2.4. Table D.2 provides greater precision than can be obtained from the graph, and the example problems at the end of Table D.2 illustrate how the table can be used.

2.5 CHANNELS WITH VARYING WALL ROUGHNESS, BUT Q = CONSTANT

Uniform flow cannot exist in a channel in which the wall roughness varies in the direction of the channel unless the bottom slope varies in precisely the correct manner so the velocity and depth remain constant with x. This combination of n and S_o would create uniform flow only for one flow rate. However, uniform flow can occur in long channels with constant bottom slopes when the wall roughness varies along the position of the channel cross section. It is not uncommon to have a channel's sides with a different roughness than its bottom. An example is a laboratory flume whose sides are Plexiglas and its bottom is filled with gravel.

A modification to Manning's equation for channels with varying roughness coefficients in different portions of its cross section might be to compute an equivalent Manning's n that weights the individual n values according to the portion of the perimeter to which they apply. For a trapezoidal channel with a different Manning's n along the bottom, n_b, than that for the sides, n_s, the equivalent roughness coefficient would then be

$$n_{eq} = \frac{b}{P}n_b + \frac{2Y\sqrt{1+m^2}}{P}n_s$$

The last two program listings are designed to handle a trapezoidal channel that has a different roughness coefficient along the bottom than the sides of the channel using an equivalent roughness coefficient computed by this equation. The validity of using an equivalent roughness coefficient would need to be verified by field or laboratory measurements for a given channel. The need for verification is that Manning's equation is empirical and therefore it is not possible to use theory alone to derive an "equivalent" Manning's equation for channels with varying roughnesses along the cross section. The above formula will produce an n_{eq} that equals n_b and n_s when these are the same, whereas other methods will not. For example, one might be inclined to associate n with $P^{2/3}$ for which then n applies. Then $nP^{2/3}$ would be replaced by $\sum n_i P_i^{2/3}$. Using this approach for a trapezoidal channel with a different bottom roughness than side roughness would use one of the following Manning's equation:

$$Q = \frac{C_u A^{5/3}\sqrt{S_o}}{\left\{n_b b^{2/3} + \left[2Y\sqrt{m^2+1}\right]^{2/3} n_s\right\}}$$

or

$$Q = C_u A\sqrt{S_o}\,\frac{\left\{(Y/n_b)^{2/3} + \left[mY\Big/\left\{2n_g\sqrt{m^2+1}\right\}\right]^{2/3}\right\}}{\{n_b + n_s\}^{1/3}}$$

or

$$Q = C_u A\sqrt{S_o}\,\frac{\left\{Y/n_b + mY\Big/\left[2n_s\sqrt{m^2+1}\right]\right\}^{2/3}}{\left\{n_b + n_s\right\}^{1/3}}$$

The problem with any of these latter formulas is that they will not produce the same results as the original Manning's equation does when $n = n_b = n_s$.

2.6 SPECIFIC ENERGY, SUBCRITICAL AND SUPERCRITICAL FLOWS

When dealing with open channel flow it is convenient to reference the energy per unit weight from the channel bottom. Thus instead of having a horizontal datum from which the energy is referenced, a sloping data is used. The result is call the "specific energy," and it consists of the depth of flow in the channel plus the velocity head, or

$$E = Y + \alpha\frac{V^2}{2g} = Y + \alpha\frac{Q^2}{2gA^2} \tag{2.11}$$

in which α is the kinetic energy correction coefficient defined in Chapter 1. In Equation 2.11 Y is the depth of flow, and if the pressure distribution is hydrostatic, then this depth Y will equal the pressure head p/γ on the bottom of the channel. Therefore, regardless of the position within the channel flow the sum of the two terms in Equation 2.11 represent the distance between the channel bottom and the energy line. For a uniform flow the specific energy will be constant. If the channel is on a steep slope then it is necessary to adjust the depth as described in Chapter 1, that is, if Y represents the vertical distance through the fluid, then it needs to be multiplied by the cosine squared of the angle of the bottom slope. It is common in practice to ignore the kinetic energy correction coefficient α (i.e., assume $\alpha = 1$) and then the specific energy becomes

$$E = Y + \frac{V^2}{2g} = Y + \frac{Q^2}{2gA^2} \tag{2.11a}$$

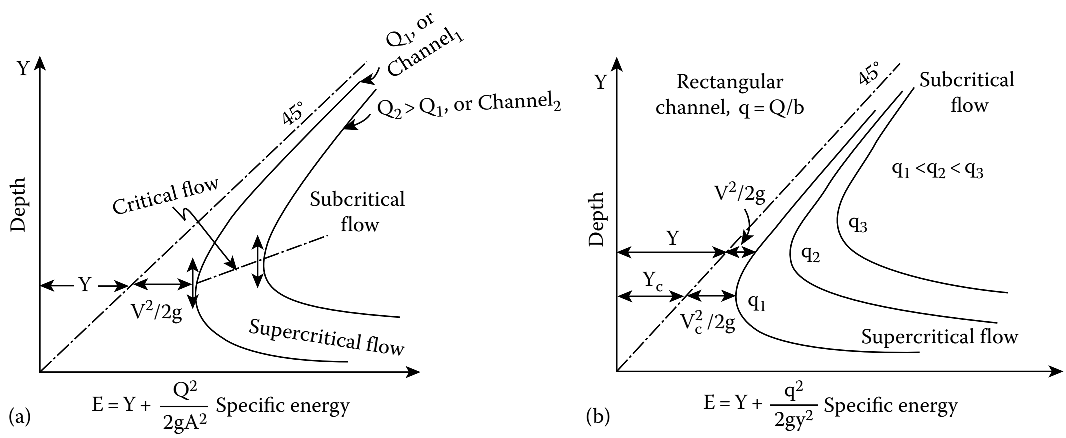


FIGURE 2.5 Sketches of specific energy diagrams in (a) a general channel and (b) in a rectangular channel in which the bottom width changes, but Q = constant.

For a rectangular channel, it is convenient to deal with the flow rate per unit width of channel, q = Q/b. For a rectangular channel the specific energy can be written as follows if α is assumed equal to 1, and the bottom slope of the channel is small enough so that the cos $\theta = 1$:

$$E = Y + \frac{V^2}{2g} = Y + \frac{q^2}{2gY^2} \tag{2.11b}$$

A plot of the depth Y as the ordinate, and the specific energy E as the abscissa is referred to as a specific energy diagram. In Figure 2.5 two sketches of specific energy diagrams are given. The first applies for any channel, and the second is specific for a rectangular channel. The following should be observed:

1. If the specific energy is held constant (i.e., a vertical lines is drawn on the specific energy diagram) then there are two depths. These depths are called alternative depths. The flow associated with the larger of these two depths produces **subcritical flow**, and the smaller depth is associated with **supercritical flow**.
2. As these two depths merge into a single depth, a minimum value for the specific energy occurs that can exist for any given flow rate. This depth is called **critical depth**, and the flow associated with it is called **critical flow** and will be denoted by Y_c. The specific energy associated with critical depth will be denoted by E_c.
3. As the flow rate in a given channel increases the specific energy curve is shifted to the right and upward, i.e., the critical depth is increased, and both the subcritical and supercritical depths are closer to the critical depth. In dealing with a rectangular channel the flow rate per unit width q increases (with Q constant) when the bottom width of the channel becomes less, i.e., a channel transition reduces the width of the channel.

That there are generally two depths associated with any given value of the specific energy can be understood best by examining Equation 2.11b. By multiplying this equation by Y^2 one notes that Equation 2.11b is a cubic equation. A cubic equation has three roots, generally, and if there are imaginary roots they occur in pairs. The third root of Equation 2.11b gives a negative value for Y, but since a negative depth is physically not possible, this root is ignored. From a mathematical view point, if the specific energy is reduced to a value less than the critical value, E_c, for a given flow rate in a given channel, then the alternative depths become imaginary, or complex roots of Equation

2.11b. For a general channel it is not obvious what type of equation (Equation 2.11b) represents, but two real positive roots of Y exist for values of $E > E_c$ and these are called alternative depths.

A simple explicit equation exists for computing the alternate depths in a **rectangular** channel. This equation can be used to obtain the depth upstream from a gate if the downstream depth is known or the downstream depth if the upstream depth is known, for example. To obtain this equation, equate E_1 to E_2 or

$$Y_1 + \frac{q^2}{2gY_1^2} = Y_2 + \frac{q^2}{2gY_2^2} \tag{2.11c}$$

Let the specific energy computed from the known depth be denoted by E_k and the other depth be given without a subscript. The Equation 2.11c becomes the following cubic equation:

$$Y^3 - E_k Y^2 + \frac{q^2}{2g} = 0 \tag{2.11d}$$

Since one depth Y_k is known this equation can be reduced to a quadratic equation by synthetic division or

$$\begin{array}{cccc|c} 1 & -E_k & 0 & q^2/2g & Y_k \\ & Y_k & Y_k(Y_k - E_k) & & \\ \hline 1 & Y_k - E_k & Y_k(Y_k - E_k) & 0 & \end{array}$$

giving

$$Y^2 + (Y_k - E_k)Y + (Y_k - E_k)Y_k = 0$$

Solving for Y gives

$$Y = \frac{1}{2}\left\{E_k - Y_k + \sqrt{(E_k - Y_k)^2 + 4Y_k(E_k - Y_k)}\right\} \tag{2.11e}$$

Since $E_k - Y_k$ is the velocity head $V_k^2/(2g) = V_h$ associated with the known depth, Y_k, Equation 2.11e can be written as

$$Y = \frac{1}{2}\left(V_h + \sqrt{V_h^2 + 4Y_k V_h}\right) \tag{2.11f}$$

Note that Equations 2.11e and f are valid only for situations in which $E_1 = E_2$ and the flow rate per unit width in the rectangular channel is the same at the two positions 1 and 2.

To find the minimum value of the specific energy, i.e., E_c and the corresponding critical depth that is associated with critical flow, the well known principle of calculus can be employed of setting the first derivative of E with respect to Y equal to zero. This principle will first be applied to Equation 2.11b, and later to 2.11a. Differentiation of Equation 2.11b with respect to Y and equating dE/dY to zero gives (with q held constant for a given specific energy curve)

$$\frac{dE}{dY} = 1 - \frac{q^2}{gY^2} = 0 \quad \text{or} \quad q^2 = gY^2 \quad \text{or} \quad q = \sqrt{gY}$$

which can be rewritten in several different ways as given by the following equations that define critical flow (the subscript c has been added to emphasize that these equations define critical flow conditions):

$$q_c = \sqrt{gY_c^3} \tag{2.12}$$

$$Y_c = \left(\frac{q_c^2}{g}\right)^{\frac{1}{3}} \tag{2.12a}$$

$$\frac{V_c^2}{2g} = \frac{Y_c}{2} \quad \text{(velocity head equal 1/2 the critical depth)} \tag{2.12b}$$

$$Y_c = \left(\frac{2}{3}\right)E_c \tag{2.12c}$$

$$E_c = 1.5Y_c \tag{2.12d}$$

It needs to be noted that Equations 2.12 apply only for rectangular channels.

Since the energy equation for a rectangular channel is a cubic equation (Equation 2.11d) in terms of depth Y, then if the specific energy E, and the flow rate per unit width q are known then the general solution of a cubic equation can be used (see CRC standard Math. Tables for methods to solve cubic equations), to solve it. This solution procedure consists of first substituting x + E/3 for Y in Equation 2.11d to eliminate the squared term. This substitution produces

$$x^3 - \frac{1}{3}E^2x + \frac{q^2}{2g} - \frac{2}{27}E^3 = 0$$

Let $a = -E^2/3$ and $b = q^2/(2g) - 2E^3/27$. Then the three roots for x are solved for next by the following three equations:

$$x_1 = A + B, \quad x_2 = 0.5(i\sqrt{3})(A - B) - 0.5x_1, \quad \text{and} \quad x_3 = 0.5(i\sqrt{3})(B - A) - 0.5x_1$$

in which

$$A = \left\{\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{1/2} - 0.5b\right\}^{1/3} \quad \text{and} \quad B = \left\{-\left(\frac{b^2}{4} + \frac{a^3}{27}\right)^{1/2} - 0.5b\right\}^{1/3}$$

If $b^2/4 + a^3/27 > 0$. there will be one real root and two conjugate imaginary roots, i.e., the specific energy $E < E_c$ and physically only a meaningless negative solution for Y exists.

If $b^2/4 + a^3/27 = 0$, there will be three real roots of which at least two are equal, i.e., critical flow occurs in which the two real roots gives the critical depth, and the other real root is a meaningless negative solution for Y. If $b^2/4 + a^3/27 > 0$ there will be three real and unequal roots, i.e., the two positive roots are the alternative depths and the third root is a meaningless negative depth Y.

After the solution for x values have been obtained the final step is to obtain the depths Y from

$$Y_1 = x_1 + \frac{1}{3}E; \quad Y_2 = x_2 + \frac{1}{3}E; \quad Y_3 = x_3 + \frac{1}{3}E$$

The program ALTDEP, listed below, implements this method for solving for the alternative depths. Generally to use the program one would first solve the specific energy equation to get E for the known depth, and then select the other depth from the output from the program as the alternative depth. However, the program lets us know what portion of the specific energy diagram the problem as been specified in. If two of the root are imaginary and the program gives the message "$E < E_c$ so negative real root and 2 imaginary roots" we know that physically the energy must be increased for the specified flow rate to be possible. An alternative to using the complex arithmetic involved in the above procedure for solving a general cubic equation is to use an implicit solution method, such as Newton's method, to solve for the depth desired by providing an appropriate guess, or software such as TK-Solver or Mathcad (or an HP calculator or a spreadsheet) that has the capability to solve implicit equations.

Listing of program ALTDEP.FOR

```
      COMPLEX AA,BB,C3,ARG,X1,X2,X3
      C3=.5*CSQRT(CMPLX(-3.,0.))
1     WRITE(*,*)' Give q, E & g or 0/=STOP'
      READ(*,*) Q,E,G
      IF(Q.LT.1.E-8) STOP
      A=-E*E/3.
      B=Q*Q/(2.*G)+2.*A*E/9.
      BH=.5*B
      BS4=B*B/4.
      YC=(Q**2/G)**.33333333
      ARG=CSQRT(CMPLX(BS4,0.)+CMPLX(A,0.)**3/27.)
      WRITE(*,*)' Yc =',YC,' Ec =',1.5*YC
      AARG=BS4+A**3/27.
      IF(AARG.GT.0.) THEN
      WRITE(*,*)' E<Ecsonegative real root and' &,'2',' imaginary
     &roots'
      ELSEIF(AARG.GT.-1.E-5 .AND. ARG.LT.1.E-5) THEN
      WRITE(*,*)' Critical condition'
      ELSE
      WRITE(*,*)'Alternative Depths&negative Y'
      ENDIF
      AA=(ARG-CMPLX(BH,0.))**.33333333
      BB=(-ARG-CMPLX(BH,0.))**.33333333
      X1=AA+BB
      X2=(AA-BB)*C3-.5*X1
      X3=(BB-AA)*C3-.5*X1
      WRITE(*,100) X1+E/3.,X2+E/3.,X3+E/3.
100   FORMAT(3(2F9.3,3X))
      GO TO 1
      END
```

Listing of Program ALTDEP.CPP

```
#include <iostream.h>
#include <stdlib.h>
#include <complex.h>
#include <iomanip.h>
void main(void){float a,b,q,e,g,bh,bs4,yc,aarg;
 complex aa,bb,c3,arg,x1,x2,x3;
 c3=.5*sqrt(3.)*complex(0.,1.);
L1:cout <<"Give q, E & g or 0 0 0 =STOP"<< endl;
 cin >>q>>e>>g; if(q<1.e-5) exit(0);
 a=-e*e/3.; b=q*q/(2.*g)+2.*a*e/9.; bh=.5*b;bs4=b*b/4.;
 yc=pow(q*q/g,.3333333);
 arg=sqrt(complex(bs4,0.)+pow(complex(a,0.),3.)/27.);
 cout <<"Yc ="<<yc<<" Ec ="<<1.5*yc<<"\n"; aarg=bs4+a*a*a/27.;
 if(aarg>0.)
   cout<<"E<Ec so negative real root and 2 imaginary roots"<<endl;
 else if((aarg>-1.e-5)&&(aarg<1.e-5))\
   cout <<"Critical condition"<< endl;
 else cout <<"Alternative Depths & negative Y"<< endl;
 aa=pow(arg-complex(bh,0.),.3333333);
   bb=pow(-arg-complex(bh,0.),.3333333);
 x1=aa+bb;x2=(aa-bb)*c3-.5*x1;x3=(bb-aa)*c3-.5*x1; cout.width(9);
 cout<<setprecision(3)<<x1+e/3.<<" "<<x2+e/3.<<" "<<x3+e/3.<< endl;
 goto L1;}
```

Example use of program

```
Input: 5 2 32.2
Output:
  Yc =   9.190971E-01  Ec =       1.378646
  Alternative depths and negative Y
    1.891     .000        .511     .000       -.402     .000
Input: 5 1.378646 32.2
Output:
  Yc =   9.190971E-01  Ec =       1.378646
  Critical condition
     .920     .000        .919     .000        -.460     .000
Input: 5 1 32.2
Output:
  Yc =   9.190971E-01  Ec =       1.378646
  E<Ec so negative real root and 2 imaginary roots
     .754     .444        .754    -.444        -.507     .000
Input: 0/
```

The above procedure can be simplified by using the arc cosine (and subsequently the cosine). To use this alternate method, first compute the angle θ from

$$\theta = \cos^{-1}\left\{\frac{\left[\left(6.75q^2/g - E^3\right)/27\right]}{(E/3)^3}\right\}$$

The three roots are then obtained from

$$Y_1 = \left(\frac{E}{3}\right)\left\{1 - 2\cos\left(\frac{\theta}{3}\right)\right\} \quad \text{(negative depth)}$$

$$Y_2 = \left(\frac{E}{3}\right)\left\{1 - 2\cos\left(\frac{[\theta + 2\pi]}{3}\right)\right\}$$

$$Y_3 = \left(\frac{E}{3}\right)\left\{1 - 2\cos\left(\frac{[\theta + 4\pi]}{3}\right)\right\}$$

Of course, if $E = E_c$ the two latter depths become equal, or become the critical depth Y_c associated with the given q. If the given E is less than the critical depth, then Y_2 and Y_3 are imaginary and the above procedure will not work because the argument for the arc cosine is not within the allowable range of −1 to +1. Thus if this alternative method is implemented in a computer program, as in ROOTSE listed below, the critical specific energy E_c, as described below, should be computed and if $E < E_c$ the computation of the roots should not be attempted.

Program ROOTSE.FOR

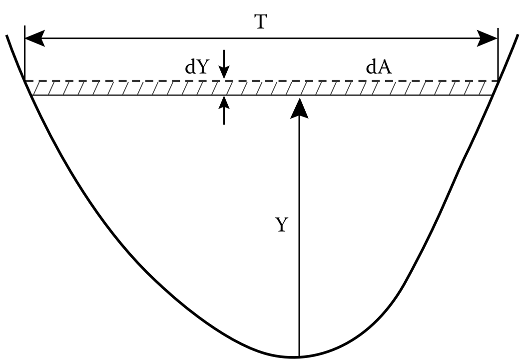
```
      PARAMETER (PI=3.14159265)
      REAL X(3)
      PI2=2.*PI
      PI4=4.*PI
1     WRITE(*,*)' Give: q,E,g'
      READ(*,*) q,E,g
      IF(q.LT.1.E-5) STOP
      Yc=(q*q/g)**.33333333
      Ec=1.5*Yc
      IF(E.LT.Ec) THEN
      WRITE(6,110) q,E,g,Yc,Ec
110   FORMAT(' q=',F8.3,' E=',F8.3,' g=',F8.2,/' Only 1 real
     &root(neg.)',' Yc=',F8.3,' Ec=',F8.3)
      GO TO 1
      ENDIF
      E3=E/3.
      THETA=ACOS(((6.75*q*q/g-E**3)/27.)/E3**3)
      X(1)=E3*(1.-2.*COS(THETA/3.))
      X(2)=E3*(1.-2.*COS((THETA+PI2)/3.))
      X(3)=E3*(1.-2.*COS((THETA+PI4)/3.))
      WRITE(6,100) q,E,g,Yc,Ec,X
100   FORMAT(' q=',F8.3,' E=',F8.3,' g=',F8.2,/,' Yc=',F8.3,'
     &Ec=',F8.3,' Roots are:',/,3F9.3)
      GO TO 1
      END
```

Program ROOTSE.C

```
#include <stdlib.h>
#include <stdio.h>
#include <math.h>
void main(void){float pi=3.14159265,q,g,e,yc,ec,theta,e3,y1,y2,y3;
L1:printf("Give: q,E,g\n"); scanf("%f %f %f",&q,&e,&g);
 if(q<1.e-5) exit(0);
 yc=pow(q*q/g,.33333333); ec=1.5*yc;
 if(e<ec) {printf("q=%8.3f E=%8.3f g=%8.2f \nOnly 1 real \
   root(beg.)\n Yc=%8.3f Ec=%8.3f\n",q,e,g,yc,ec); goto L1;}
 e3=e/3.; theta=acos(((6.75*q*q/g-pow(e,3.))/27.)/pow(e3,3.));
 y1=e3*(1.-2.*cos(theta/3.)); y2=e3*(1.-2.*cos((theta+2.*pi)/3.));
 y3=e3*(1.-2.*cos((theta+4.*pi)/3.));
 printf("q=%8.3f       E=%8.3f      g=%8.3f\nYc=%8.3f \
   Ec=%8.3f\nRoots      are:\n%9.3f      %8.3f \
   %8.3f\n",q,e,g,yc,ec,y1,y2,y3);
 goto L1;}
```

For any channel Equation 2.11a is differentiated, and dE/dY is set to zero to find the minimum value of E. This procedure leads to

$$\frac{Q^2}{gA^3}\frac{dA}{dY}=1$$



From the accompanying sketch it is clear that dA/dY equals the top width T for all channels and therefore the critical flow equation for all channels becomes

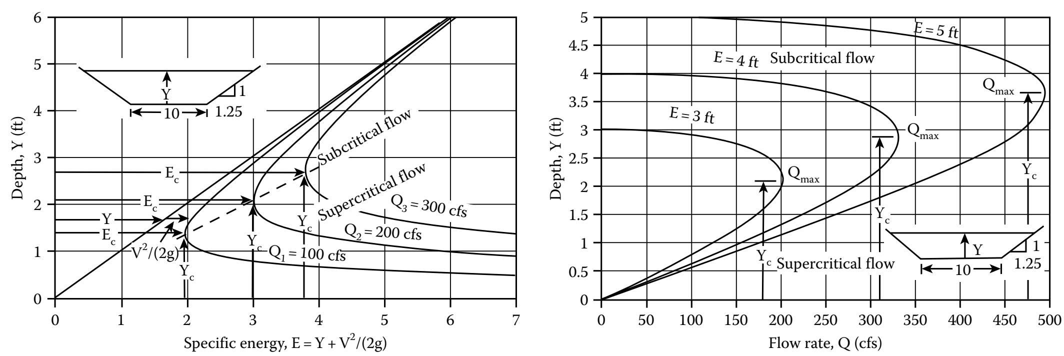
$$\frac{Q_c^2 T_c}{gA_c^3}=F_r^2=\frac{V^2}{c^2}=1 \tag{2.13}$$

in which F_r is the Froude number which is defined as the ratio of inertia to gravity forces. The speed at which a small amplitude gravity wave travel in an open channel is given by

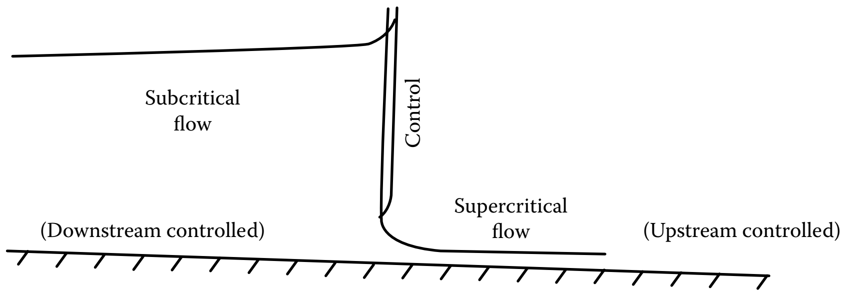
$$c=\sqrt{\frac{gA^3}{T}} \tag{2.14}$$

Proof of this equation is given in Chapter 3. From Equations 2.13 and 2.14, we see that the Froude number is also the ratio of the average velocity in the channel to the speed of a small amplitude gravity wave. From Equation 2.13, it can be concluded that critical flow occurs when the average velocity V of a channel flow exactly equals the speed c of a small amplitude gravity wave. For subcritical flows the Froude number F_r will be less than unity, since the denominator of Equation 2.13 increases with increasing depth, and for supercritical flows the Froude number will be larger than unity.

In a subcritical flow, the fact that the velocity in the channel is less than the speed of a small amplitude gravity wave allows these waves to move upstream against the flowing fluid. Thus in a sense the fluid can receive a signal that things are to change downstream, and it adjust gradually. This adjusting to downstream conditions is referred to as downstream control. Subcritical flows are always controlled by downstream conditions, i.e., an obstruction to the flow such as a partly closed gate will cause the depth upstream from the gate to increase, and the velocity to decrease. On the other hand if the flow is supercritical, the velocity in the channel exceeds the speed of a small amplitude gravity wave, and the flow does not receive a signal about downstream conditions. Therefore, if the flow is supercritical it will not change its depth or velocity to meet downstream conditions. For example, should the channel abruptly end, a supercritical flow would continue to the end of the channel at the same depth and velocity as it has upstream there from. Supercritical flows are, therefore, said to be upstream controlled. An example of a control that affects both the upstream flow as well as the downstream flow conditions is a sluice gate in a channel flow. Since subcritical flows are controlled downstream and supercritical flows are controlled by an upstream device we can conclude that the flow upstream from the gate must be subcritical, and the flow downstream from the gate must be supercritical.



Above a specific energy diagram and a depth-discharge diagram for a trapezoidal channel are given. The separate curves on the depth-discharge diagram show the relationship of the flow rate to changes in depth for a constant specific energy. The maximum point on these curves corresponds to critical conditions, i.e., the depth is critical and the Q_{max} is the maximum flow rate that could be obtained into this trapezoidal if it were supplied by a reservoir with a water surface elevation equal to the E for that curve. This assumes there is no entrance loss. You might note that the minimum point on the specific energy diagram corresponds to the maximum point on the depth-discharge diagram. For example, the curve for E = 3 ft on the depth–discharge diagram gives a maximum flow rate Q_{max} = 200 cfs, and the minimum specific energy on the specific energy diagram of the curve on the for Q = 200 cfs is E_c = 3 ft.



To further illustrate the meanings of these diagrams assume a reservoir with a head H = 5 ft supplies a steep trapezoidal channel with b = 10 ft, and m = 1.25, the flow rate that will enter the channel is Q = 494 cfs (the maximum flow rate possible on the depth-discharge diagram of the curve for E = 5 ft), and the depth at the entrance will be $Y_c = 3.62$ ft, the ordinate of the maximum point of this curve. These values are at the extreme, Q_{max} position of the curve for E = 5 ft, but these values could also be obtained by solving the critical flow equation $Q^2T/(gA^3) = 1$ and the specific energy equation $5 = Y + (Q/A)^2/(2\,g)$ simultaneously. Likewise if the reservoir level were 4 ft above the channel bottom $Q_c = Q_{max} = 331.5$ cfs and $Y_c = 2.87$ ft, or if H = 3 ft, $Q_c = Q_{max} = 201.2$ cfs and $Y_c = 2.12$ ft. It is not possible for the flow from a reservoir into a steep channel to be in the lower (the supercritical) portion of the depth-discharge diagram, because the flow rates would be less than the amount Q_{max} that can be supplied. However, if the channel is not steep, but mild, then the flow into the channel will be reduced by downstream conditions, i.e., the fluid frictional resistance. For example, for the trapezoidal channel for which the depth-discharge diagram is made if the reservoir supplied a specific energy of 5 ft, and the flow rate were 400 cfs, the depth would be 4.5 ft in the channel. If the channel has an n = 0.014, then Manning's equation could be solved to show that it would have to have a bottom slope of $S_o = 0.000701$ for this uniform flow to occur.

The need to determine critical depth, or critical flow conditions, occurs frequently in open channels. The illustrative problems, as well as the problems at the end of this chapter, point out a few situations where computation of critical depth is needed. Therefore, it is worth discussing how the above critical flow Equations 2.12 and 2.13 can be effectively solved. Should the channel be rectangular the solution of critical flow conditions is very easy since Equations 2.12 are all explicit. However, for a nonrectangular channel Equation 2.13 must be solved for the critical depth Y_c, and since this equation is implicit, the solution requires an iterative technique such as the Newton method. Alternatively graphical solution can be utilized as discussed below to get answers with adequate precision for most applications.

To solve Equation 2.13 by the Newton method, it can be rewritten as

$$F = Q_c^2 T_c - g A_c^2 = 0 \tag{2.15}$$

in which T_c and A_c are functions of the critical depth Y_c and the parameters that define the given channel.

EXAMPLE PROBLEM 2.10

Water enters a steep rectangular channel that has a bottom width of 8 ft from a reservoir whose water surface is 5 ft above the channel bottom. Determine the flow rate into the channel, and the depth of flow at the channel entrance. Neglect the entrance loss coefficient. A long distance downstream a gate exists that produces a depth $Y_2 = 1.5$ ft downstream from it. What is the depth immediately upstream from the gate?

Solution

Equations 2.12 apply to this problem because the channel is rectangular. Since the channel is steep, and a steep channel will contain a supercritical flow under uniform flow conditions, and the water in the reservoir (where V = 0) is subcritical, the flow must pass through critical depth at the entrance of the channel. Therefore, $Y_c = (2/3)\ E = (2/3)5 = 3.333$ ft, and the flow rate per unit width $q = (gY_c^3)^{1/2} = (32.3(3.333)^3)^{1/2} = 34.53$ cfs/ft, or $Q = bq = 8(34.53) = 276.3$ cfs. The second part of the problem requires that the specific energy downstream from the gate be solved, or $E_2 = Y_2 + q^2/(2gY_2^2) = 1.5 + (34.53/1.5)^2/64.4 = 9.729$ ft. Now the alternative depth to 1.5 is sought with $E = 9.729$ ft. One way is to extract the root 1.5 from the cubic equation, another is to solve the cubic equation with the Newton method starting with a "subcritical" guess for Y_1, or use the program ALTDEP. The solution for the depth upstream from the gate is $Y_1 = 9.525$ ft. A hydraulic jump (discussed in Chapter 3) will occur somewhere upstream from the gate changing the flow from super to subcritical.

EXAMPLE PROBLEM 2.11

Instead of the rectangular channel a steep pipe of 8 ft diameter is used. What is the flow rate and depth of flow at the entrance? A gate downstream causes a depth downstream from it of $Y_2 = 2.8$ ft. What is the depth immediately upstream from the gate?

Solution

Since this is a circular channel, the implicit Equation 2.13 applies. Since there are two unknowns, Y and Q, a second equation must be obtained. This second equation is the specific energy Equation 2.11a. If angle β is used initially as a substitute for Y, the two equations that must be solved simultaneously are

$$F_1 = Q^2(D\sin\beta) - g\left\{\frac{D^2}{4}(\beta - \cos\beta\sin\beta)\right\}^3$$

and

$$F_2 = \frac{E - D}{2(1-\cos\beta)} - \frac{\{1-K_L\}Q^2}{2g}\left\{\frac{D^2}{4}(\beta - \cos\beta\sin\beta)\right\}^2$$

The solution to these equations by the Newton method uses the following iterative equation

$$\begin{Bmatrix} Q \\ \beta \end{Bmatrix}^{(m+1)} = \begin{Bmatrix} Q \\ \beta \end{Bmatrix}^{(m)} - \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}$$

in which the Z values are obtained by solving the following linear system of two equations:

$$\begin{bmatrix} \dfrac{\partial F_1}{\partial Q} & \dfrac{\partial F_1}{\partial \beta} \\ \dfrac{\partial F_2}{\partial Q} & \dfrac{\partial F_1^2}{\partial \beta} \end{bmatrix} \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

in which F_1 and F_2 and its derivatives are evaluated using the current values, i.e., those with an m superscript. The solution gives Q = 207.2 cfs, and β = 1.474 rad, and from this angle $Y_c = D(1 - \cos\beta)/2 = 3.615$ ft. This solution assumed $K_L = 0$, e.g., ignores entrance losses. If the entrance loss coefficient is taken as 0.1, then the solution is Q = 197.2 cfs, and Y = 3.523 ft. To solve part two, $E_2 = Y_2 + (Q/A_2)^2/(2\,g) = 5.512$ ft. Now the energy equation must be solve using an implicit method because extracting a root or using the general solution to a cubic equation is not available. In fact since the variable β relates Y to A, it also needs to be solved. The solution is β = 1.787 rad, and $Y_1 = 4.859$ ft.

EXAMPLE PROBLEM 2.12

Instead of the rectangular channel (with b = 8 ft) of Example Problem 2.10 having a steep slope this problem deals with this channel on a "mild slope" with $S_o = 0.00075$. The channel has a Manning's n = 0.014, and is long. What are the depth and the flow rate? The entrance loss coefficient is $K_e = 0.1$. As a second part solve for the flow rate and the uniform depth if the channel is trapezoidal with b = 8 ft, and m = 1.2. (S_o is still 0.00075, n = 0.014 and H = 5 ft). Also solve these problems using Chezy's, rather than, Manning's equation, if e = 0.01 ft. As a fourth part of this problem assume the bottom width b of the trapezoidal channel is sought that will supply a flow rate Q = 400 cfs.

Solution

For these mild channels the two equations that govern are: (1) a uniform flow equation which is Manning's equation for the first part of the problem or

$$Q = \frac{C_u}{n} AR_h^{2/3} S_o^{1/2} \quad \text{or} \quad F_1 = nQP^{2/3} - C_u A^{5/3} S_o^{1/2} = 0$$

and (2) the energy equation

$$H = Y + \frac{1-K_L}{2g}\frac{Q^2}{\lambda^2} \quad \text{or} \quad F_2 = H - Y - \frac{1-K_L}{2g}\frac{Q^2}{\lambda^2} = 0$$

Using the Newton method, the solution to these two equations for the rectangular channel is: Q = 177.72 cfs, and Y = 4.602 ft, and for the trapezoidal channel is: Q = 336.03 cfs, and Y = 4.451 ft.

The program E_UN listed below is designed to solve Manning's and energy equations simultaneously using the Newton method. You should study either the FORTRAN or C versions of these programs to understand how this solution is accomplished. The subroutine FUN (void function in the C program) supplies the values to the two above equations whenever it is called. The main program numerically evaluates the four derivatives in the Jacobian matrix using $\partial F_i/\partial x = \{F_i(x + \Delta x) - F_i(x)\}/\Delta x$, in which Δx is obtained by multiplying the current value of the unknown x by 1.005 and then subtracting its current value or $\Delta x = 1.005x - x = 0.005x$. Carefully study the listing starting with the statement 30 SUM=0. to the statement IF(NCT.LT.30 .AND. SUM.GT.1.E-5) GO TO 30 (the do {} while; in the C-program) to see how the Newton method is implement in solving a system of simultaneous equation because this approach will be used repeatedly. The subroutine SOLVEQ is called on to solve the linear system of equation even though a 2 × 2 matrix such as occurs in this program could be solved with a few lines of code. Notice that the program is designed to solve for any two of the first 7 variables 1 = b, 2 = m, 3 = S_o, 4 = n, 5 = Q, 6 = H, 7 = Y, and 8 = K in the array X() and that the array ID() is given a value of 1 to identify these two unknown variables and a value of 0 if the variable is known. Also notice that the program accommodates either a trapezoidal (which includes a rectangular channel with m = 0) and a circular channel by giving ITYPE a 0 or a 1 respectively.

Program E_UN.FOR

```
C Solves Manning's (uniform flow) and Energy simultaneously
C for any 2 unknowns
C  See E_UN1 to solve Q & Y and method that can be used
C  with calculator
      CHARACTER*17 FMT/'(1X,A1,'' ='',F9.3)'/
      CHARACTER*1 CX(8)/'b','m','S','n','Q','H','Y','K'/
      CHARACTER*5 CH(0:1)/'value','guess'/
      READ(*,*) G,FKE,ITYPE
1     DO 10 I=1,7
      IF(ITYPE.EQ.1 .AND. I.EQ.2) GO TO 10
      WRITE(*,'(I2,2X,A1)') I,CX(I)
10    ID(I)=0
```

```
2     WRITE(*,*)' Give two numbers for 2 unknown variables'
      READ(*,*) I1,I2
      IF(I1.LT.1.OR.I1.GT.7.OR.I2.LT.1.OR.I2.GT.7) GO TO 2
      ID(I1)=1
      ID(I2)=1
      DO 20 I=1,7
      IF(ITYPE.EQ.1 .AND. I.EQ.2) GO TO 20
      WRITE(*,100) CH(ID(I)),CX(I)
100   FORMAT(' Give ',A5,' for ',A1,' = ',\)
      READ(*,*) X(I)
20    CONTINUE
      NCT=0
30    SUM=0.
      CALL FUN(F)
      I1=0
      DO 40 I=1,7
      IF(ID(I).EQ.0) GO TO 40
      XX=X(I)
      I1=I1+1
      X(I)=1.005*X(I)
      CALL FUN(F1)
      DO 35 J=1,2
      IF(ITYPE.EQ.1) THEN
      CX(1)='D'
      CX(2)=' '
      ENDIF
      IF(G.GT.15.) THEN
      Cu=1.486
      ELSE
      Cu=1.
      ENDIF
      X(8)=FKE
      FKE=(1.+FKE)/(2.*G)
      REAL F(2),F1(2),D(2,2)
      INTEGER*2 ID(8),INDX(2)
      COMMON X(8),G,FKE,Cu,ITYPE
      WRITE(*,*)' Give:g,entr. loss C. & 0=TRAP or 1=CIRLCE'
35    D(J,I1)=(F1(J)-F(J))/(X(I)-XX)
      X(I)=XX
40    CONTINUE
      CALL SOLVEQ(2,1,2,D,F,1,DD,INDX)
      I1=0
      DO 50 I=1,7
      IF(ID(I).EQ.0) GO TO 50
      I1=I1+1
      SUM=SUM+ABS(F(I1))
      X(I)=X(I)-F(I1)
50    CONTINUE
      NCT=NCT+1
      WRITE(*,*)' NCT=',NCT,' SUM=',SUM
      IF(NCT.LT.30 .AND. SUM.GT.1.E-5) GO TO 30
      WRITE(*,*)' Solution:'
      DO 60 I=1,8
      IF(ITYPE.EQ.1 .AND. I.EQ.2) GO TO 60
      IF(I.EQ.3) THEN
      FMT(16:16)='6'
```

```
      ELSEIF(I.EQ.4) THEN
      FMT(16:16)='3'
      ELSE
      FMT(16:16)='3'
      ENDIF
      WRITE(*,FMT) CX(I),X(I)
60    CONTINUE
      WRITE(*,*)' Give 1 to solve another prob. (0=STOP)'
      READ(*,*) I2
      IF(I2.EQ.1) GO TO 1
      END
      SUBROUTINE FUN(F)
      REAL F(2)
      COMMON X(8),G,FKE,Cu,ITYPE
      IF(ITYPE.EQ.1) THEN
      BETA=ACOS(1.-2.*X(7)/X(1))
      A=.25*X(1)**2*(BETA-COS(BETA)*SIN(BETA))
      P=X(1)*BETA
      ELSE
      A=(X(1)+X(2)*X(7))*X(7)
      P=X(1)+2.*X(7)*SQRT(X(2)**2+1.)
      ENDIF
      F(1)=X(4)*X(5)*P**.6666667-Cu*A**1.6666667*SQRT(X(3))
      F(2)=X(6)-X(7)-FKE*(X(5)/A)**2
      RETURN
      END
```

Program E_UN.C

```
// Solves Manning's (uniform flow) and Energy simultaneously for
// any 2 unknowns
// See E_UN1 to solve Q & Y and method that can be used with
// calculator
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
float x[8],g,fke,cu; int itype;
extern void solveq(int n,float **d,float *f,int itype,float *dd,\
  int *indx);
void fun(float *f){float beta,a,p;
 if(itype){beta=acos(1.-2.*x[6]/x[0]);
   a=.25*x[0]*x[0]*(beta-cos(beta)*sin(beta));p=x[0]*beta;}
 else {a=(x[0]+x[1]*x[6])*x[6];
   p=x[0]+2.*x[6]*sqrt(x[1]*x[1]+1.);}
 f[0]=x[3]*x[4]*pow(p,.6666667)-cu*pow(a,1.6666667)*sqrt(x[2]);
 f[1]=x[5]-x[6]-fke*pow(x[4]/a,2.);return;} //End of fun
void main(void){char *fmt=" %c =%9.3f\n",*ch[]={"value","guess"},\
  *cx="bmSnQHYK";
 float f[2],f1[2],xx,sum,**d,*dd; int id[8],indx[2],i,j,i1,i2,nct;
 d=(float**)malloc(2*sizeof(float*));
 for(i=0;i<2;i++)d[i]=(float*)malloc(2*sizeof(float));
 printf("Give: g,entrance loss C. & 0=TRAP or 1=CIRLCE\n");
 scanf("%f %f %d",&g,&fke,&itype);
 if(itype){stpcpy(cx[0],"D");stpcpy(cx[1]," ");} if(g>15.) cu=1.486;
 else cu=1.; x[7]=fke;fke=(1.+fke)/(2.*g);
L1: for(i=0;i<7;i++){if((!itype) || ((itype)&&(i != 1)))\
  printf("%2d %c\n",i+1,cx[i]);id[i]=0;}
```

```
do{printf(" Give two numbers for 2 unknown variables\n");
  scanf("%d %d",&i1,&i2);}while((i1<1)||(i1>7)||(i2<1)||(i2>7));
id[i1-1]=1;id[i2-1]=1;
for(i=0;i<7;i++){if((!itype) || ((itype)&&(i != 1)))
  {printf("Give %s for %c =",ch[id[i]],cx[i]);
scanf("%f",&x[i]);}} nct=0;
do{sum=0.; fun(f); i1=-1;
for(i=0;i<7;i++){if(id[i]){xx=x[i];i1++;x[i]*=1.005;fun(f1);
  for(j=0;j<2;j++) d[j][i1]=(f1[j]-f[j])/(x[i]-xx); x[i]=xx;}}
solveq(2,d,f,1,dd,indx); i1=-1;
for(i=0;i<7;i++){if(id[i]){sum+=fabs(f[++i1]);x[i]-=f[i1];}}
printf("nct= %d SUM=%f\n",nct,sum);}while((++nct<30)&&(sum>1.e-5));
printf("Solution:\n");for(i=0;i<8;i++){
  if((!itype) || ((itype)&&(i != 1))){
  if(i==2)stpcpy(fmt[8],"6"); else stpcpy(fmt[8],"3");
  printf(fmt,cx[i],x[i]);}}
printf("Give 1 to solve another prob. (0=STOP)\n");
scanf("%d",&i2); if(i2) goto L1;}
```

The above listing of the program will solve the fourth part of the problem in which a flow rate of Q = 400 cfs is given and b is wanted along with Y. The difference is for parts 1 and 2 variables 5 and 7 (Q and Y) are given as the unknowns and for part 3 variables 1 and 7 (b and Y) are identified as the unknowns. The solution for part 3 is b = 10.194 ft and Y = 4.416 ft.

If the flow rate Q and depth Y are always the two unknowns, then it is not necessary to solve the two equations simultaneously. Rather Manning's equation is substituted into the energy to eliminate the flow rate Q, or $F = H - Y - (1 + K_e)(C_u/n)^2(A/P)^{4/3}S_o/(2g) = 0$ and solve this equation for the depth Y. Thereafter solve for Q from either of the original equations. This is the approach you would use to solve parts 1 and 2 with a pocket calculator such as an HP with a SOLVE function. The program E_UN1, listed below use this latter approach to solve for Q and Y.

Program E_UN1.FOR

```
C Solves Manning's and Specific Energy for Y & Q by substituting
  Manning's equation
C into Energy and first solving for Y and thereafter Q. (This is
  the procedure
C to use with a pocket calculator) See program E_UN to solve any
  2 variables.
       COMMON B,FM,D,FN,SO,H,C,ITYPE
       WRITE(*,*)' Give: g,(0=trap or 1=cir)'
       READ(*,*) G,ITYPE
       IF(ITYPE.EQ.0) THEN
       WRITE(*,*)' Give: b,m,n,So,H,Ke'
       READ(*,*) B,FM,FN,SO,H,FKE
       ELSE
       WRITE(*,*)' Give: D,n,So,H,Ke'
       READ(*,*) D,FN,SO,H,FKE
       ENDIF
       Y=.8*H
       Cu=1.
       IF(G.GT.15.) Cu=1.486
       C=(Cu/FN)**2*SO*(1.+FKE)/(2.*G)
       NCT=0
10     F=FUN(Y)
       DIF=.05*F/(FUN(Y+.05)-F)
       Y=Y-DIF
       NCT=NCT+1
```

```
      WRITE(*,*) NCT,DIF
      IF(NCT.LT.30 .AND. ABS(DIF).GT. 1.E-5) GO TO 10
      CALL AREAP(Y,A,P)
      Q=CU/FN*A*(A/P)**.6666667*SQRT(SO)
      WRITE(*,100) Y,Q
100   FORMAT(' Solution: Y =',F8.3,' Q =',F10.2)
      END
      SUBROUTINE AREAP(Y,A,P)
      COMMON B,FM,D,FN,SO,H,C,ITYPE
      IF(ITYPE.EQ.1) THEN
      COSB=1.-2.*Y/D
      BETA=ACOS(COSB)
      P=D*BETA
      A=.25*D*D*(BETA-COSB*SIN(BETA))
      ELSE
      P=B+2.*Y*SQRT(FM**2+1.)
      A=(B+FM*Y)*Y
      ENDIF
      RETURN
      END
      FUNCTION FUN(Y)
      COMMON B,FM,D,FN,SO,H,C,ITYPE
      CALL AREAP(Y,A,P)
      FUN=H-Y-C*(A/P)**1.3333333
      RETURN
      END
```

Program E_UN1.C

```
/* Solves Manning's and Specific Energy for Y & Q by
  substituting Manning's Eq.into Energy and first solving for Y
  and thereafter Q. (This is the procedure to use with a pocket
  calculator) See program E_UN to solve any 2 variables. */
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
float b,m,d,n,so,h,c; int itype;
void areap(float y,float *a,float *p){float cosb,beta;
 if(itype){cosb=1.-2.*y/d;beta=acos(cosb);
   *p=d*beta;*a=.25*d*d*(beta-cosb*sin(beta));}
 else {*p=b+2.*y*sqrt(m*m+1.); *a=(b+m*y)*y;return;}}
float fun(float y){float *a,*p; areap(y,a,p);
  return(h-y-c*pow((*a)/(*p),1.333333));}
void main(void){float g,dif,ke,y,f,q,cu=1.,*a,*p; int nct=0;
 printf("Give: g(0=trap or 1=cir)\n"); scanf("%f %d",&g,&itype);
 if(itype){printf("Give: D,n,So,H,Ke\n");
   scanf("%f %f %f %f %f",&d,&n,&so,&h,&ke);}
 else {printf("Give: b,m,n,So,H,Ke\n");
    scanf("%f %f %f %f %f %f",&b,&m,&n,&so,&h,&ke);}
 y=.8*h; if(g>15.) cu=1.486; c=pow(cu/n,2.)*so*(1.+ke)/(2.*g);
 do{f=fun(y); dif=.05*f/(fun(y+.05)-f); y-=dif;
 printf("%d %f %f\n",++nct,dif,y);
 }while((nct<30) && (fabs(dif)>1.e-5));
 areap(y,a,p); q=cu/n*(*a)*pow((*a)/(*p),.6666667)*sqrt(so);
 printf("Solution: Y = %8.3f Q = %10.2f\n",y,q);
}
```

To solve the third part of the problem that requests that Chezy's equation be used instead of Manning equation, requires that 3 (rather than 2) equations be solved simultaneously for Q, Y, and C. These equations are as follows:

$$F_1 = Q - CA(R_hS_o)^{1/2} = 0 \quad \text{(Chezy's equation)}$$

$$F_2 = C + (32g)^{1/2}\text{Log}_{10}\left\{\frac{e}{(12R_h)} + \frac{0.884C}{(R_e g^{1/3})}\right\} = 0 \quad \text{(Chezy's C equation)}$$

$$F_3 = H - Y - \left\{\frac{(1+K_e)}{(2g)}\right\}\left\{\frac{Q}{A}\right\}^2 = 0 \quad \text{(Energy equation)}$$

Program UENCHEZ is designed to solve these three equations simultaneously. Notice that now the subroutine FUN provides values to 3 F values, and the part of the program that implements the Newton method has the DO loops changed from 2 to 3. The program UENCHEZ1 reduces the Newton solution to the depth by substituting Chezy's equation into the energy equation, as has been done previous when using Manning's equation as the uniform flow equation. Now however it is necessary to use a Gauss–Seidel type iteration to solve C associated with the current value of Q. This technique can be used with a calculator with a SOLVE function.

Program UENCHEZ.FOR

```
CHARACTER*17 FMT/'(1X,A1,'' ='',F9.3)'/
      CHARACTER*1 CX(9)/'b','m','S','e','Q','H','Y','C','K'/
      CHARACTER*5 CH(0:1)/'value','guess'/
      REAL F(3),F1(3),D(3,3)
      INTEGER*2 ID(9),INDX(3)
      COMMON X(9),G,FKE,G32,CG,ITYPE
      WRITE(*,*)' Give:g,k.vis.,entrance loss C.',
     &' & 0=trap,1=cir'
      READ(*,*) G,VIS,FKE,ITYPE
      IF(ITYPE.EQ.1) THEN
      CX(1)='D'
      CX(2)=' '
      ENDIF
      IF(G.GT.15.) THEN
      X(8)=100.
      ELSE
      X(8)=60.
      ENDIF
      X(9)=FKE
      FKE=(1.+FKE)/(2.*G)
      G32=SQRT(32.*G)
      CG=.884*VIS/(4.*SQRT(G))
1     DO 10 I=1,7
      IF(ITYPE.EQ.1 .AND. I.EQ.2) GO TO 10
      WRITE(*,'(I2,2X,A1)') I,CX(I)
10    ID(I)=0
2     WRITE(*,*)' Give 2 num. for 2 unknown var.',
      ' in addition to C'
      READ(*,*) I1,I2
      IF(I1.LT.1.OR.I1.GT.8.OR.I2.LT.1.OR.I2.GT.8) GO TO 2
      ID(I1)=1
      ID(I2)=1
```

```
      ID(8)=1
      DO 20 I=1,7
      IF(ITYPE.EQ.1 .AND. I.EQ.2) GO TO 20
      WRITE(*,100) CH(ID(I)),CX(I)
100   FORMAT(' Give ',A5,' for ',A1,' = ',\)
      READ(*,*) X(I)
20    CONTINUE
      NCT=0
30    SUM=0.
      CALL FUN(F)
      I1=0
      DO 40 I=1,8
      IF(ID(I).EQ.0) GO TO 40
      XX=X(I)
      I1=I1+1
      X(I)=1.005*X(I)
      CALL FUN(F1)
      DO 35 J=1,3
35    D(J,I1)=(F1(J)-F(J))/(X(I)-XX)
      X(I)=XX
40    CONTINUE
      CALL SOLVEQ(3,1,3,D,F,1,DD,INDX)
      I1=0
      DO 50 I=1,8
      IF(ID(I).EQ.0) GO TO 50
      I1=I1+1
      SUM=SUM+ABS(F(I1))
      X(I)=X(I)-F(I1)
50    CONTINUE
      NCT=NCT+1
      WRITE(*,*)' NCT=',NCT,' SUM=',SUM
      IF(NCT.LT.30 .AND. SUM.GT.1.E-5) GO TO 30
      WRITE(*,*)' Solution:'
      DO 60 I=1,9
      IF(ITYPE.EQ.1 .AND. I.EQ.2) GO TO 60
      IF(I.EQ.3 .OR. I.EQ.4) THEN
      FMT(16:16)='6'
      ELSE
      FMT(16:16)='3'
      ENDIF
      WRITE(*,FMT) CX(I),X(I)
60    CONTINUE
      WRITE(*,*)' Give 1 for another pb. (0=STOP)'
      READ(*,*) I2
      IF(I2.EQ.1) GO TO 1
      END
      SUBROUTINE FUN(F)
      REAL F(3)
      COMMON X(9),G,FKE,G32,CG,ITYPE
      IF(ITYPE.EQ.1) THEN
      BETA=ACOS(1.-2.*X(7)/X(1))
      A=.25*X(1)**2*(BETA-COS(BETA)*SIN(BETA))
      P=X(1)*BETA
      ELSE
      A=(X(1)+X(2)*X(7))*X(7)
      P=X(1)+2.*X(7)*SQRT(X(2)**2+1.)
```

```
      ENDIF
      Rh=A/P
      F(1)=X(5)-X(8)*A*SQRT(X(3)*RH)
      F(2)=X(8)+G32*ALOG10(X(4)/(12.*Rh)+CG*X(8)*P/X(5))
      F(3)=X(6)-X(7)-FKE*(X(5)/A)**2
      RETURN
      END
```

Program UENCHEZ.C

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
float x[9],g,fke,g32,cg; int itype;
extern void solveq(int n,float **d,float *f,int itype,\
  float *dd,int *indx);
void fun(float *f){float beta,a,p,rh;
 if(itype){beta=acos(1.-2.*x[6]/x[0]);
   a=.25*x[0]*x[0]*(beta-cos(beta)*sin(beta));p=x[0]*beta;}
 else {a=(x[0]+x[1]*x[6])*x[6];p=x[0]+2.*x[6]*sqrt(x[1]*x[1]+1.);}
   rh=a/p;
 f[0]=x[4]-x[7]*a*sqrt(x[2]*rh);
 f[1]=x[7]+g32*log10(x[3]/(12.*rh)+cg*x[7]*p/x[4]);
 f[2]=x[5]-x[6]-fke*pow(x[4]/a,2.);return;} //End of fun
void main(void){char *fmt=" %c =%9.3f\n",*ch[]={"value","guess"},\
  *cx="bmSeQHYCK";
 float f[3],f1[3],xx,sum,**d,*dd,vis;
 int id[9],indx[3],i,j,i1,i2,nct;
 d=(float**)malloc(3*sizeof(float*));
 for(i=0;i<3;i++)d[i]=(float*)malloc(3*sizeof(float));
 printf("Give: g,k.viscosity,entrance loss C. & 0=trap or 1=cir\n");
 scanf("%f %f %f %d",&g,&vis,&fke,&itype);
 if(itype){stpcpy(cx[0],"D");stpcpy(cx[1]," ");}
 if(g<15.) x[7]=100.; else x[7]=60.; x[8]=fke;
 fke=(1.+fke)/(2.*g);g32=sqrt(32.*g);cg=.884*vis/(4.*sqrt(g));
L1: for(i=0;i<7;i++){if((!itype) || ((itype)&&(i != 1)))
  printf("%2d %c\n",i+1,cx[i]);id[i]=0;}
 do{printf(" Give two numbers for 2 unknown variables\n");
   scanf("%d %d",&i1,&i2);
 }while((i1<1)||(i1>7)||(i2<1)||(i2>7)); id[i1-1]=1;id[i2-1]=1;
 for(i=0;i<7;i++){if((!itype) || ((itype)&&(i != 1)))
   {printf("Give %s for %c =",ch[id[i]],cx[i]);
   scanf("%f",&x[i]);}} nct=0;
do{sum=0.; fun(f); i1=-1;for(i=0;i<8;i++){if(id[i]){xx=x[i];i1++;
  x[i]*=1.005;fun(f1);
  for(j=0;j<3;j++) d[j][i1]=(f1[j]-f[j])/(x[i]-xx); x[i]=xx;}}
  solveq(3,d,f,1,dd,indx); i1=-1;
  for(i=0;i<8;i++){if(id[i]){sum+=fabs(f[++i1]);x[i]-=f[i1];}}
  printf("nct= %d SUM=%f\n",nct,sum);
  }while((++nct<30)&&(sum>1.e-5));
printf("Solution:\n");
for(i=0;i<9;i++){if((!itype) || ((itype)&&(i != 1))){
if((i==2)||(i==3))stpcpy(fmt[8],"6"); else stpcpy(fmt[8],"3");
printf(fmt,cx[i],x[i]);}}
printf("Give 1 to solve another prob. (0=STOP)\n");
scanf("%d",&i2); if(i2) goto L1;}
```