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Preface

This work gives a comprehensive description of various mathematical aspects of the problems originating in numerical solution of hyperbolic systems of partial differential equations. The material is presented in close relation with the important mechanical applications of such systems. They include both the Euler equations of gas dynamics and comparatively new fields such as magnetohydrodynamics (MHD), shallow water, and mechanics of solids. When considering the equations of gas dynamics, we mainly dwell on their applications to media with a complicated wide-range equation of state. Historically, high-resolution numerical schemes for hyperbolic conservation laws were first applied to purely gas dynamic problems. This can be explained by the fact that, due to the "convexity" of the Euler system, the Riemann problem for it generally has a unique solution. This is not true for more complicated MHD and solid dynamics equations. Although the solution of the MHD Riemann problem exists, it is too complicated to be used in regular calculations. In this book we give a collection of recipes for application of high-order nonoscillatory shock-capturing schemes to MHD modelling of complicated physical phenomena. Of great importance is also the problem of physical admissibility of solutions that are nonevolutionary if we solve the ideal MHD system. We discuss the current state of this problem and state our views on the stability of nonevolutionary shock waves.

The book deals with a number of new original problems called *nonclassical*. Among them are such problems as shock wave propagation in elastic rods and composite materials, ionization fronts in plasma, electromagnetic shocks in magnets, etc. We show that if a small-scale higher-order mathematical model results in oscillations of the discontinuity structure, the variety of admissible discontinuities can exhibit disperse behaviour. This variety includes discontinuities with additional boundary conditions not following from the hyperbolic conservation laws. Nonclassical problems are accompanied by a multiple nonuniqueness of solutions. An example of the selection rule is given that in certain cases permits one to easily make a correct, physically realizable choice.

The book is divided into seven chapters. For the reader's convenience, in Chapter 1 we introduce the main notions and definitions that make the book self-sufficient. The definitions are followed by mechanical examples that illustrate the essence of the subjects to be considered in the subsequent chapters. General properties of solutions are discussed. In Chapter 2 we formulate the approaches to numerical solution of quasilinear hyperbolic systems both in the conservative and nonconservative forms. The methods are subdivided into two classes: shock-fitting and shock-capturing schemes. Among shock-capturing schemes we choose only those belonging to the Godunov type, that is, which are based on the solution of the Riemann problem to determine fluxes through the computational cell interfaces. Since exact solutions are frequently not available, we describe also the methods based on approximate solutions and solutions of the linearized problem which always exists. The methods of increasing the order of accuracy are given, which include both the application of the generalized Riemann problem and various reconstruction procedures. Chapter 3 is devoted to the equations of gas dynamics. We present the exact solution of the Riemann problem for gases possessing complicated wide-range equations of state. The Courant–Isaacson–Rees, Roe, and Osher–Solomon methods are described. The cases are emphasized of fairly arbitrary equations of state. Genuine shock-fitting and floating shock-fitting techniques are discussed, including the self-adjusting grid approach. The applications include the problem of the chemically reacting airflow around blunt body at high angle of attack, modelling of shock-induced phenomena, jet-like structures in laser plasma, etc. Separately, both in Chapter 3 and Chapter 5, the problem is considered of the solar wind interaction with the magnetized interstellar medium. While investigating this problem, different aspects of the application of the introduced methods are comprehensively discussed. In addition to nonstationary hyperbolic systems, in Chapters 3 and 4 we dwell on the application of high-resolution methods to stationary, supersonic or supercritical, gas dynamic and shallow water equations. Chapter 4 also describes different Godunov-type methods for hyperbolic shallow water equations and is accompanied by a number of examples. Chapter 5 deals with MHD equations. For the reader's convenience we outline the assumptions adopted in ideal MHD and give the classification of discontinuities. The evolutionary property of MHD shock is discussed, emphasizing the degenerate (parallel and perpendicular) and singular (switch-on and switch-off) cases. The approaches to solving MHD system by nearly all modern high-resolution numerical methods are summarized. The problem of physical admissibility of nonevolutionary solutions is investigated, as well as its interaction with the application of shock-capturing numerical methods whose numerical dissipation can substantially exceed the physical dissipation existing in space plasma. Chapter 6 represents an attempt to outline the problems of solid dynamics that are governed by hyperbolic systems. For these problems, Courant–Isaacson–Rees type methods are formulated with the application to spallation phenomena, dynamics of Timoshenko-type shells, etc. Chapter 7 introduces the notion of nonclassical discontinuity and formulates evolutionary conditions for them. The correlation between the evolutionary conditions for discontinuities and the existence of their structure is investigated. The behavior of classical discontinuities near Jouget points on the shock adiabatic curves is explained. Further on, various examples are presented illustrating the application of the introduced theoretical basis to important physical phenomena.

The choice of the material naturally reflects the scientific interests of the authors and several important aspects arising in hyperbolic problems were discussed only briefly or were not discussed at all. We believe, however, that this book will substantially supplement existing literature devoted to this subject, since it concerns new areas of application and formulates new notions in order to clear out unexpected difficulties that may be encountered when one deals with nonconvex hyperbolic systems.

The book can be useful to graduate and postgraduate students majoring in the field of numerical, engineering, and applied mathematics and mechanics of continuous media. It is also aimed at the attention of specialists in pure and applied mathematics, and various fields of physics and mechanics where hyperbolic systems of partial differential equations find their application.

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Chapter 1
Hyperbolic Systems of Partial Differential Equations

1.1 Quasilinear systems

Let us define the system of first-order partial differential equations for the unknown vector-function $\mathbf{u}$ of the independent variables $\mathbf{x}$ and t as a system of relations

$$F_i\left(\mathbf{x}, t, \mathbf{u}, \frac{\partial \mathbf{u}}{\partial x_1}, \dots, \frac{\partial \mathbf{u}}{\partial x_m}, \frac{\partial \mathbf{u}}{\partial t}\right) = 0, \qquad i = 1, \dots, N. \tag{1.1.1}$$

Here

$$\mathbf{u} = [u_1, \dots, u_n]^{\mathrm{T}}, \quad \mathbf{x} = [x_1, \dots, x_m]^{\mathrm{T}}, \quad \frac{\partial \mathbf{u}}{\partial x_j} = \left[\frac{\partial u_1}{\partial x_j}, \dots, \frac{\partial u_n}{\partial x_j}\right]^{\mathrm{T}}$$

are vector-columns, $j = 1, \dots, m$.

This system is called determined if $N = n$. Later we shall consider only determined systems.

The system of (1.1.1) is called a system of quasilinear equations if the functions F_i are linear with respect to the derivatives of $\mathbf{u}$ occurring in (1.1.1) as arguments. If F_i are also linear with respect to $\mathbf{u}$, the system is called linear.

The system of the first-order quasilinear equations can be represented in the form

$$\tilde{A}\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{m} \tilde{B}_j \frac{\partial \mathbf{u}}{\partial x_j} = \tilde{\mathbf{c}}, \tag{1.1.2}$$

where the coefficient matrices $\tilde{A}$ and $\tilde{B}_j$ and the source term vector $\tilde{\mathbf{c}}$ depend on t, $\mathbf{x}$, and $\mathbf{u}$.

The vector-row $\mathbf{l} = [l_1, \dots, l_n]$ and the number λ are called a left eigenvector and an eigenvalue of a matrix A, respectively, if

$$\mathbf{l}A = \lambda \mathbf{l}, \quad \|\mathbf{l}\| \neq 0. \tag{1.1.3}$$

As a norm of a vector $\mathbf{a}$ we adopt the quantity $\|\mathbf{a}\| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{\sum_{k=1}^{n} a_k^2}$. The dot sign here stands for the scalar product of two vectors.

Similarly, the vector-column $\mathbf{r} = [r_1, \dots, r_n]^{\mathrm{T}}$ is called a right eigenvector of a matrix A if

$$A\mathbf{r} = \lambda \mathbf{r}, \quad \|\mathbf{r}\| \neq 0. \tag{1.1.4}$$

Owing to Eqs. (1.1.3)–(1.1.4), the eigenvalue λ of the matrix A is the root of the characteristic equation

$$\det(A - \lambda I) = 0, \tag{1.1.5}$$

where $I = \mathrm{diag}[1, \ldots, 1]$ is the $n \times n$ identity matrix.

Suppose all the eigenvalues λ of the matrix A are real. Let us enumerate them in increasing order, that is,

$$\lambda_1 \leq \ldots \leq \lambda_k \leq \ldots \leq \lambda_n. \tag{1.1.6}$$

The equality signs in Eq. (1.1.6) correspond to the case of the eigenvalue multiplicity.

If for any eigenvalue λ of multiplicity α the rank of the matrix $A - \lambda I$ is equal to $n - \alpha$, then both the right and left eigenvectors corresponding to all eigenvalues form a basis in the Euclidean space $\mathbf{E}^n(\mathbf{u})$.

It is easy to check that if $\lambda_k \neq \lambda_p$, the vectors $\mathbf{l}^k$ and $\mathbf{r}^p$ are mutually orthogonal. Indeed,

$$\begin{aligned} \mathbf{l}^k A = \lambda_k \mathbf{l}^k \quad &\Longrightarrow \quad \mathbf{l}^k A \cdot \mathbf{r}^p = \lambda_k \mathbf{l}^k \cdot \mathbf{r}^p, \\ A\mathbf{r}^p = \lambda_p \mathbf{r}^p \quad &\Longrightarrow \quad \mathbf{l}^k \cdot A\mathbf{r}^p = \lambda_p \mathbf{l}^k \cdot \mathbf{r}^p. \end{aligned}$$

Hence, $(\lambda_k - \lambda_p)\mathbf{l}^k \cdot \mathbf{r}^p = 0$.

Unfortunately, in a variety of mechanical applications of quasilinear systems (Euler gas dynamic equations, magnetohydrodynamic equations (MHD), solid dynamics equations, etc.) the multiplicity of eigenvalues can be greater than one and the choice of the set of independent nondegenerate eigenvectors requires additional analysis.

A matrix A is called nonsingular if $\lambda = 0$ is not its eigenvalue. Since in the applications to be studied in what follows the matrix $\tilde{A}$ in Eq. (1.1.2) is nonsingular, we can resolve the system for $\partial \mathbf{u}/\partial t$

$$\frac{\partial \mathbf{u}}{\partial t} + \sum_{j=1}^{m} A_j \frac{\partial \mathbf{u}}{\partial x_j} = \mathbf{b}, \tag{1.1.7}$$

where $A_j = \tilde{A}^{-1}\tilde{B}_j$ and $\mathbf{b} = \tilde{A}^{-1}\tilde{\mathbf{c}}$.

1.2 Hyperbolic systems of quasilinear differential equations

1.2.1 Definitions. Let us define a matrix P associated with Eq. (1.1.7) by the formula

$$P = P(\boldsymbol{\alpha}) = \sum_{j=1}^{m} \alpha_j A_j, \quad -\infty < \alpha_j < \infty. \tag{1.2.1}$$

The quasilinear system of (1.1.7) is called hyperbolic at the point $(\mathbf{x}, t, \mathbf{u})$ if there exists a nonsingular matrix $\Omega(\boldsymbol{\alpha})$ diagonalizing P,

$$\Omega^{-1} P \Omega = \Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_n], \tag{1.2.2}$$

where all eigenvalues λ_k of the matrix P are real and the norms of Ω and Ω^{-1} are uniformly bounded in $\boldsymbol{\alpha} = [\alpha_1, \ldots, \alpha_m]^{\mathrm{T}}$. If all the eigenvalues are distinct, the system is called strictly hyperbolic. The matrix Λ is here a diagonal matrix whose entries are the eigenvalues of the matrix P.

The hyperbolicity condition implies that there exists an independent basis $\{\mathbf{l}^1, \ldots, \mathbf{l}^n\}$ composed of the left eigenvectors of the matrix P. It is clear that there also exists a basis composed of the right eigenvectors. Note that the following relations hold:

$$\Omega^{-1} P = \Lambda \Omega^{-1}, \quad P\Omega = \Omega\Lambda.$$

Thus, the kth row of Ω^{-1} consists of the components of the kth left eigenvector $\mathbf{l}^k$ corresponding to the eigenvalue λ_k, see Eq. (1.1.3). In addition, the kth column of Ω consists of the corresponding components of the kth right eigenvector $\mathbf{r}^k$, see Eq. (1.1.4). In what follows we also use the notation $\Omega_L = \Omega^{-1}$ and $\Omega_R = \Omega$, where $\Omega_L\Omega_R = I$.

Thus, relations (1.2.2) can be rewritten as

$$\Omega_L P \Omega_R = \Lambda, \quad P = \Omega_R \Lambda \Omega_L.$$

As a norm of a matrix A we adopt the square root of the spectral radius of the matrix AA^T, where the spectral radius is equal to the largest eigenvalue of this matrix.

Since numerical solutions of the systems of hyperbolic quasilinear equations occurring in mechanical applications are frequently constructed on the basis of the system with two independent variables, we shall consider the essence of hyperbolicity for the system

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = \mathbf{b}. \tag{1.2.3}$$

Multiplying this system by the left eigenvector $\mathbf{l}^k$, we can transform it into the form

$$\mathbf{l}^k \cdot \left(\frac{\partial \mathbf{u}}{\partial t} + \lambda_k\frac{\partial \mathbf{u}}{\partial x}\right) = f_k, \qquad k = 1, \ldots, n; \tag{1.2.4}$$

where $f_k = \mathbf{l}^k \cdot \mathbf{b}$.

The system of (1.2.4) can be rewritten as

$$\mathbf{l}^k \cdot \left(\frac{d\mathbf{u}}{dt}\right)_k = f_k, \qquad k = 1, \ldots, n; \tag{1.2.5}$$

where $(d\mathbf{u}/dt)_k$ is the derivative of $\mathbf{u}$ with respect to t in the direction $dx/dt = \lambda_k$. This direction is called a characteristic direction and Eq. (1.2.5) is called a characteristic form of Eq. (1.2.3). If the system of (1.2.3) is linear and its coefficient matrix is constant, then the eigenvalues λ_k, which are also called characteristic velocities, are constant and the characteristic lines in the x–t plane become straight lines,

$$x = \lambda_k t + \text{const}. \tag{1.2.6}$$

In certain cases the system of (1.2.4), or (1.2.5), can be further simplified. If A does not depend on $\mathbf{u}$, Eq. (1.2.4) can be rewritten for Riemann invariants (Riemann 1860) w_k as

$$\left(\frac{dw_k}{dt}\right)_k = g_k, \quad k = 1, \ldots, n; \tag{1.2.7}$$

where

$$w_k = \mathbf{l}^k \cdot \mathbf{u}, \quad g_k = f_k + \sum_{j=1}^{m} u_j \left(\frac{dl_j^k}{dt}\right)_k = f_k + \sum_{j=1}^{m} u_j \left[\frac{\partial l_j^k}{\partial t} + \lambda_k \frac{\partial l_j^k}{\partial x}\right].$$

Choosing w_k as components of a new unknown vector, we can rewrite the system in the form such that each equation involves the derivatives of only one function of x, t, and $\mathbf{u}$, that is,

$$\frac{\partial w_k}{\partial t} + \lambda_k \frac{\partial w_k}{\partial x} = g_k(x, t, \mathbf{w}), \qquad k = 1, \ldots, n. \tag{1.2.8}$$

Although a quasilinear hyperbolic system generally cannot be written out for the Riemann invariants, these invariants play an important role in the construction of numerical solutions to these systems.

Note that the system of hyperbolic equations with two independent variables can be extended in a peculiar way. If we introduce the notation

$$\frac{\partial \mathbf{u}}{\partial x} = \mathbf{p}, \quad \frac{\partial \mathbf{u}}{\partial t} = \mathbf{q}, \tag{1.2.9}$$

then the system of (1.2.4) acquires the form

$$\mathbf{l}^k \cdot (\mathbf{q} + \lambda_k \mathbf{p}) = f_k, \qquad k = 1, \ldots, n. \tag{1.2.10}$$

By differentiating each equation of (1.2.10) with respect to t and x and taking into account the integrability condition $\partial \mathbf{q}/\partial x = \partial \mathbf{p}/\partial t$, we obtain

$$\mathbf{l}^k \cdot \left(\frac{\partial \mathbf{q}}{\partial t} + \lambda_k \frac{\partial \mathbf{q}}{\partial x}\right) = E_k, \quad \mathbf{l}^k \cdot \left(\frac{\partial \mathbf{p}}{\partial t} + \lambda_k \frac{\partial \mathbf{p}}{\partial x}\right) = G_k, \tag{1.2.11}$$

where E_k and G_k are the function x, t, $\mathbf{u}$, $\mathbf{p}$, and $\mathbf{q}$.

Thus, since $\mathbf{l}^k$ do not depend on $\mathbf{q}$ and $\mathbf{p}$, the extended system (1.2.11) can always be written out for the Riemann invariants (Courant and Lax 1949).

1.2.2 Systems of conservation laws.

Let us consider a system

$$\frac{\partial \mathbf{U}(\mathbf{x}, t, \mathbf{u})}{\partial t} + \sum_{j=1}^{m} \frac{\partial \mathbf{F}_j(\mathbf{x}, t, \mathbf{u})}{\partial x_j} = \mathbf{c}(\mathbf{x}, t, \mathbf{u}) \tag{1.2.12}$$

that is the consequence of the quasilinear system (1.1.7) for any of its solutions. The systems describing a number of mechanical problems (gas dynamics, MHD, shallow water equations, thermoelasticity and elastoviscoplasticity equations, etc.) can be written in the form that reflects the conservation of such fundamental physical properties as mass, momentum, energy, etc. and expressed by Eq. (1.2.12). If the number of such fundamental conservation laws is equal to the number of equations in (1.1.7), we say that this system is written in the conservation-law, or conservative, form. Most of hyperbolic systems are solved numerically on the basis of their conservative rather than quasilinear form. Moreover, in a Cartesian coordinate system they acquire the simplest form

$$\frac{\partial \mathbf{U}(\mathbf{u})}{\partial t} + \sum_{j=1}^{m} \frac{\partial \mathbf{F}_j(\mathbf{u})}{\partial x_j} = \mathbf{c}(\mathbf{x}, t, \mathbf{u}). \tag{1.2.13}$$

In this case $\mathbf{U}$ and $\mathbf{F}$ do not depend on the independent variables t and $\mathbf{x}$.

If $\mathbf{c} \equiv \mathbf{0}$, the system is said to be written in a strictly conservative, or divergent, form. The source term $\mathbf{c}$ can be both of physical (volume production of mass, momentum, and energy) and geometrical origin. This issue will be discussed in subsequent chapters. We shall assume $\mathbf{c} \equiv \mathbf{0}$ in the general discussion of hyperbolic systems, although the source term may appear in applications.

If the system is written in the conservative form, the quantities $\mathbf{U}$ and $\mathbf{F}_j$ are called a vector of conservative variables and a flux vector, respectively. The sense of these notions is readily understood if we consider a bounded region $V \subset \mathbf{E}^m(\mathbf{x})$ and let $\mathbf{n} = [n_1, \ldots, n_m]$ be the outward unit normal to the boundary ∂V of V. Then it follows from (1.2.13) that

$$\frac{d}{dt}\int_V \mathbf{U}\, dV + \sum_{j=1}^{m} \int_{\partial V} n_j \mathbf{F}_j(\mathbf{U})\, dS = \mathbf{0}, \tag{1.2.14}$$

that is, the time variation of the quantity $\mathbf{U}$ in the volume V is equal to its losses through the boundary.

1.3 Mechanical examples

Let us consider several examples of hyperbolic systems of partial differential equations that are frequently encountered in mechanical applications. The formulas to be presented below are useful for the implementation of modern high-resolution numerical methods.

1.3.1 Nonstationary equations of gas dynamics. The system of nonstationary Euler equations for primitive variables in Cartesian coordinates can be written as follows:

$$\frac{\partial \mathbf{u}}{\partial t} + A_i \frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{0}, \qquad \mathbf{u} = [\rho, u, v, w, p]^{\mathrm{T}}. \tag{1.3.1}$$

Here and further on in this section $i, j = 1, 2, 3$ and summation is adopted over repeated indices. If we introduce the Kronecker delta δ_{ij}, then the coefficient matrices can be written out as

$$A_i = \begin{bmatrix} v_i & \delta_{i1}\rho & \delta_{i2}\rho & \delta_{i3}\rho & 0 \\ 0 & v_i & 0 & 0 & \delta_{i1}/\rho \\ 0 & 0 & v_i & 0 & \delta_{i2}/\rho \\ 0 & 0 & 0 & v_i & \delta_{i3}/\rho \\ 0 & \delta_{i1}\rho c^2 & \delta_{i2}\rho c^2 & \delta_{i3}\rho c^2 & v_i \end{bmatrix}. \tag{1.3.2}$$

The entries of the coefficient matrices are constituted by the density ρ, the pressure p, the components $v_1 = u$, $v_2 = v$, $v_3 = w$ of the velocity vector $\mathbf{v}$, and the speed of sound $c = c(p, \rho)$.

It follows from the definition of hyperbolicity that the system remains hyperbolic if we perform an arbitrary differentiable self-invertible transformation of independent variables $\xi = \xi(x, y, z)$, $\eta = \eta(x, y, z)$, $\zeta = \zeta(x, y, z)$. In this case the system of (1.1.7) acquires the form

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial \xi} + B\frac{\partial \mathbf{u}}{\partial \eta} + C\frac{\partial \mathbf{u}}{\partial \zeta} = \mathbf{0}, \tag{1.3.3}$$

where $A = \xi_{x_i} A_i$, $B = \eta_{x_i} A_i$, and $C = \zeta_{x_i} A_i$. Here $\xi_{x_i} = \partial\xi/\partial x_i$, $\eta_{x_i} = \partial\eta/\partial x_i$, and $\zeta_{x_i} = \partial\zeta/\partial x_i$.

Let us introduce the vectors

$$\boldsymbol{\alpha} = [\xi_x, \xi_y, \xi_z], \quad \boldsymbol{\beta} = [\eta_x, \eta_y, \eta_z], \quad \boldsymbol{\gamma} = [\zeta_x, \zeta_y, \zeta_z]. \tag{1.3.4}$$

If the system of (1.3.3) is t-hyperbolic at the point $(\mathbf{x}, t, \mathbf{u})$, there must exist nonsingular matrices $\Omega_R^A(\boldsymbol{\alpha})$, $\Omega_R^B(\boldsymbol{\beta})$, and $\Omega_R^C(\boldsymbol{\gamma})$ such that

$$\Omega_L^A A\, \Omega_R^A = \mathrm{diag}[\lambda_k^A], \quad \Omega_L^B B\, \Omega_R^B = \mathrm{diag}[\lambda_k^B], \quad \Omega_L^C C\, \Omega_R^C = \mathrm{diag}[\lambda_k^C], \tag{1.3.5}$$

where λ_k^A, λ_k^B, and λ_k^C are all real, $k = 1, \ldots, 5$. The norms of $\Omega_{L,R}^A$, $\Omega_{L,R}^B$, and $\Omega_{L,R}^C$ must be uniformly bounded in $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$, respectively.

Without loss of generality we can consider only the diagonalization of the matrix

$$A = \begin{bmatrix} U & \alpha_1\rho & \alpha_2\rho & \alpha_3\rho & 0 \\ 0 & U & 0 & 0 & \alpha_1/\rho \\ 0 & 0 & U & 0 & \alpha_2/\rho \\ 0 & 0 & 0 & U & \alpha_3/\rho \\ 0 & \alpha_1\rho c^2 & \alpha_2\rho c^2 & \alpha_3\rho c^2 & U \end{bmatrix}. \tag{1.3.6}$$

In this formula $U = \alpha_i v_i$ is the contravariant component of the vector $\mathbf{v}$ along the curvilinear coordinate ξ.

The eigenvalues of the matrix A can be easily determined as

$$\lambda_1 = \lambda_2 = \lambda_3 = U, \quad \lambda_{4,5} = U \pm c\,(\alpha_i \times \alpha_i)^{1/2}. \tag{1.3.7}$$

We see that they are obviously real.

Although the matrix A has the eigenvalue of multiplicity three, it has a complete set of linearly independent eigenvectors. They constitute a matrix

$$\Omega_R^A = \begin{bmatrix} \tilde\alpha_1 & \tilde\alpha_2 & \tilde\alpha_3 & \rho/(c\sqrt{2}) & \rho/(c\sqrt{2}) \\ 0 & -\tilde\alpha_3 & \tilde\alpha_2 & \tilde\alpha_1/\sqrt{2} & -\tilde\alpha_1/\sqrt{2} \\ \tilde\alpha_3 & 0 & -\tilde\alpha_1 & \tilde\alpha_2/\sqrt{2} & -\tilde\alpha_2/\sqrt{2} \\ -\tilde\alpha_2 & \tilde\alpha_1 & 0 & \tilde\alpha_3/\sqrt{2} & -\tilde\alpha_3/\sqrt{2} \\ 0 & 0 & 0 & \rho c/\sqrt{2} & \rho c/\sqrt{2} \end{bmatrix}, \tag{1.3.8}$$

where $\tilde\alpha_j = \alpha_j/(\alpha_i \times \alpha_i)^{1/2}$. The kth column of Ω_R^A is the right eigenvector corresponding to the eigenvalue λ_k. The inverse of Ω_R^A can be constructed from the left eigenvectors of A as

$$\Omega_L^A = \begin{bmatrix} \tilde\alpha_1 & 0 & \tilde\alpha_3 & -\tilde\alpha_2 & -\tilde\alpha_1/c^2 \\ \tilde\alpha_2 & -\tilde\alpha_3 & 0 & \tilde\alpha_1 & -\tilde\alpha_2/c^2 \\ \tilde\alpha_3 & \tilde\alpha_2 & -\tilde\alpha_1 & 0 & -\tilde\alpha_3/c^2 \\ 0 & \tilde\alpha_1/\sqrt{2} & \tilde\alpha_2/\sqrt{2} & \tilde\alpha_3/\sqrt{2} & (\rho c\sqrt{2})^{-1} \\ 0 & -\tilde\alpha_1/\sqrt{2} & -\tilde\alpha_2/\sqrt{2} & -\tilde\alpha_3/\sqrt{2} & (\rho c\sqrt{2})^{-1} \end{bmatrix}. \tag{1.3.9}$$

The determinants of Ω_R^A and Ω_L^A are

$$\det \Omega_R^A = (\det \Omega_L^A)^{-1} = \rho c.$$

The spectral norm of the matrix Ω_R^A can be calculated from the formula

$$||\Omega_R^A|| = \sqrt{r\left(\Omega_R^A\,(\Omega_R^A)^{\mathrm{T}}\right)},$$

where r is the spectral radius of a corresponding matrix. The eigenvalues σ_k of $\Omega_{\mathrm{R}}^A(\Omega_{\mathrm{R}}^A)^{\mathrm{T}}$ are

$$\sigma_1 = \sigma_2 = \sigma_3 = 1, \quad \sigma_{4,5} = \frac{\phi \pm \sqrt{\phi^2 - 4\rho^2 c^6}}{2c^2},$$

where $\phi = \rho^2 + c^2 + \rho^2 c^4$, and consequently

$$\|\Omega_{\mathrm{R}}^A\|^2 = \max \sigma_k = \frac{\phi + \sqrt{\phi^2 - 4\rho^2 c^6}}{2c^2} > 1.$$

Since the norm of Ω_{R}^A and, hence, the norm of Ω_{L}^A are independent of the real parameters α_i, they are uniformly bounded in $\boldsymbol{\alpha}$. This is required for hyperbolicity of the considered system.

As shown by Warming, Beam, and Hyett (1975), the similarity transformation based on the matrix Ω_{R}^A not only diagonalizes A but also symmetrizes the matrices A_i and, hence, the matrices B and C. Note that such a presentation cannot be obtained for an arbitrary linear combination of noncommuting matrices.

Suppose we are solving the system in the conservation-law form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_j}{\partial x_j} = \mathbf{0}, \tag{1.3.10}$$

where

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho v_1 \\ \rho v_2 \\ \rho v_3 \\ e \end{bmatrix}, \quad \mathbf{F}_j = \begin{bmatrix} \rho v_j \\ \rho v_1 v_j + p\delta_{1j} \\ \rho v_2 v_j + p\delta_{2j} \\ \rho v_3 v_j + p\delta_{3j} \\ (e+p)v_j \end{bmatrix},$$

$$e = \frac{p}{\gamma - 1} + \frac{\rho \mathbf{v}^2}{2}, \quad \mathbf{v}^2 = v_1^2 + v_2^2 + v_3^2.$$

Here e is the total gas energy per unit volume and γ is the specific heat ratio, or adiabatic index.

Equation (1.3.1) can be expressed in the form

$$\frac{\partial \mathbf{U}}{\partial t} + \tilde{A}_j \frac{\partial \mathbf{U}}{\partial x_j} = \mathbf{0}, \tag{1.3.11}$$

where $\tilde{A}_j$ are the Jacobian matrices $\partial \mathbf{F}_j / \partial \mathbf{U}$.

The matrices A_j of the nonconservative form (1.3.1) and the matrices $\tilde{A}_j$ of the conservative form (1.3.11) are related via the similarity transformation $A_j = M^{-1}\tilde{A}_j M$, where M is the Jacobian matrix $\partial \mathbf{U}/\partial \mathbf{u}$. It is easy to find that

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ v_1 & \rho & 0 & 0 & 0 \\ v_2 & 0 & \rho & 0 & 0 \\ v_3 & 0 & 0 & \rho & 0 \\ \frac{1}{2}\mathbf{v}\cdot\mathbf{v} & \rho v_1 & \rho v_2 & \rho v_3 & \beta^{-1} \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -v_1/\rho & 1/\rho & 0 & 0 & 0 \\ -v_2/\rho & 0 & 1/\rho & 0 & 0 \\ -v_3/\rho & 0 & 0 & 1/\rho & 0 \\ \frac{1}{2}\beta(\mathbf{v}\cdot\mathbf{v}) & -\beta v_1 & -\beta v_2 & -\beta v_3 & \beta \end{bmatrix}, \tag{1.3.12}$$

where $\beta = \gamma - 1$.

Thus, the eigenvectors $\tilde{\mathbf{r}}^k$ and $\tilde{\mathbf{l}}^k$ of the matrix $\tilde{A}$ can be calculated by the formulas

$$\tilde{\mathbf{r}}^k = M\mathbf{r}^k, \quad \tilde{\mathbf{l}}^k = \mathbf{l}^k M^{-1}.$$

1.3.2 Stationary Euler equations. Let now the system of gas dynamic equations (1.3.3) be stationary and assume that the matrix A is nonsingular. Then we obtain the system

$$\frac{\partial \mathbf{u}}{\partial \xi} + \mathcal{B}\frac{\partial \mathbf{u}}{\partial \eta} + \mathcal{C}\frac{\partial \mathbf{u}}{\partial \zeta} = \mathbf{0}, \tag{1.3.13}$$

where $\mathcal{B} = A^{-1}(\eta_{x_i} A_i)$ and $\mathcal{C} = A^{-1}(\zeta_{x_i} A_i)$.

If this system is ξ-hyperbolic at the point $(\mathbf{x}, \mathbf{u})$, there must exist nonsingular matrices $\Omega_{\mathrm{R}}^{\mathcal{B}}(\boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\Omega_{\mathrm{R}}^{\mathcal{C}}(\boldsymbol{\alpha}, \boldsymbol{\gamma})$ such that

$$\Omega_{\mathrm{R}}^{\mathcal{B}}\, \mathcal{B}\, \Omega_{\mathrm{L}}^{\mathcal{B}} = \mathrm{diag}[\lambda_k^{\mathcal{B}}], \quad \Omega_{\mathrm{R}}^{\mathcal{C}}\, \mathcal{C}\, \Omega_{\mathrm{L}}^{\mathcal{C}} = \mathrm{diag}[\lambda_k^{\mathcal{C}}]$$

and $\lambda_k^{\mathcal{B}}$ and $\lambda_k^{\mathcal{C}}$ are all real, $k = 1, \ldots, 5$. The norms of the matrices $\Omega_{\mathrm{R}}^{\mathcal{B}}$ and $\Omega_{\mathrm{R}}^{\mathcal{C}}$ must be uniformly bounded in $\mathbf{x}$, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\gamma}$.

Without loss of generality we can consider (Pogorelov 1987) only the diagonalization of the matrix

$$\mathcal{B} = \frac{1}{Q^2 U}\begin{bmatrix} VQ^2 & -\rho d_1 U & -\rho d_2 U & -\rho d_3 U & \alpha_1 d_1 + \alpha_2 d_2 + \alpha_3 d_3 \\ 0 & Q^2 V + \alpha_1 c^2 d_1 & \alpha_1 c^2 d_2 & \alpha_1 c^2 d_3 & (-U d_1 + r_1 c^2)/\rho \\ 0 & \alpha_2 c^2 d_1 & Q^2 V + \alpha_2 c^2 d_2 & \alpha_2 c^2 d_3 & (-U d_2 + r_2 c^2)/\rho \\ 0 & \alpha_3 c^2 d_1 & \alpha_3 c^2 d_2 & Q^2 V + \alpha_3 c^2 d_3 & (-U d_3 + r_3 c^2)/\rho \\ 0 & -\rho c^2 d_1 U & -\rho c^2 d_2 U & -\rho c^2 d_3 U & (UV - qc^2)U \end{bmatrix}.$$

Here $U = \alpha_i v_i$ and $V = \beta_i v_i$ are the contravariant components of the vector $\mathbf{v}$ along the curvilinear coordinates ξ and η, respectively. Besides,

$$\begin{aligned} Q^2 &= U^2 - (sc)^2, \quad s^2 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2, \\ q &= \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3, \quad d_i = \alpha_i V - \beta_i U, \\ r_1 &= (\alpha_1\beta_2 - \alpha_2\beta_1)\alpha_2 + (\alpha_1\beta_3 - \alpha_3\beta_1)\alpha_3, \\ r_2 &= (\alpha_2\beta_3 - \alpha_3\beta_2)\alpha_3 + (\alpha_2\beta_1 - \alpha_1\beta_2)\alpha_1, \\ r_3 &= (\alpha_3\beta_1 - \alpha_1\beta_3)\alpha_1 + (\alpha_3\beta_2 - \alpha_2\beta_3)\alpha_2. \end{aligned}$$

The eigenvalues of the matrix $\mathcal{B}$ are the following:

$$\lambda_1^{\mathcal{B}} = \lambda_2^{\mathcal{B}} = \lambda_3^{\mathcal{B}} = \frac{V}{U}, \quad \lambda_{4,5}^{\mathcal{B}} = \frac{UV - qc^2 \mp cT}{Q^2}, \tag{1.3.14}$$

where

$$T = \left[D^2 - (s^2\bar{s}^2 - q^2)c^2\right]^{1/2}, \quad \bar{s}^2 = \beta_1^2 + \beta_2^2 + \beta_3^2, \quad D^2 = d_1^2 + d_2^2 + d_3^2.$$

It is easy to check that they are real for $U^2 > (sc)^2$.

The matrices $\Omega_{\mathrm{R}}^{\mathcal{B}}$ and $\Omega_{\mathrm{L}}^{\mathcal{B}}$ are constructed using the right and the left eigenvectors of the matrix $\mathcal{B}$ as follows:

$$\Omega_{\mathrm{R}}^{\mathcal{B}} = \begin{bmatrix} -d_3/D & d_2/D & d_1/D & 1/(2c^2\delta T) & -1/(2c^2\varepsilon T) \\ -d_2/D & -d_3/D & 0 & (d_1T + cb_1)/(2\delta\rho c D^2 T) & (d_1T - cb_1)/(2\varepsilon\rho c D^2 T) \\ d_1/D & 0 & d_3/D & (d_2T + cb_2)/(2\delta\rho c D^2 T) & (d_2T - cb_2)/(2\varepsilon\rho c D^2 T) \\ 0 & d_1/D & -d_2/D & (d_3T + cb_3)/(2\delta\rho c D^2 T) & (d_3T - cb_3)/(2\varepsilon\rho c D^2 T) \\ 0 & 0 & 0 & 1/(2\delta T) & -1/(2\varepsilon T) \end{bmatrix}$$

and

$$\Omega_{\mathrm{L}}^{\mathcal{B}} = \begin{bmatrix} -d_3/D & -d_2/D & d_1/D & 0 & [(\alpha_1\beta_2 - \alpha_2\beta_1)c^2 + \rho d_3]/(\rho c^2 D) \\ d_2/D & -d_3/D & 0 & d_1/D & [(\alpha_1\beta_3 - \alpha_3\beta_1)c^2 - \rho d_2]/(\rho c^2 D) \\ d_1/D & 0 & d_3/D & -d_2/D & [(\alpha_3\beta_2 - \alpha_2\beta_3)c^2 - \rho d_1]/(\rho c^2 D) \\ 0 & \delta\rho c d_1 & \delta\rho c d_2 & \delta\rho c d_3 & \delta T \\ 0 & \varepsilon\rho c d_1 & \varepsilon\rho c d_2 & \varepsilon\rho c d_3 & -\varepsilon T \end{bmatrix},$$

where

$$\begin{aligned} b_1 &= (\alpha_1\beta_2 - \alpha_2\beta_1)d_2 + (\alpha_1\beta_3 - \alpha_3\beta_1)d_3, \\ b_2 &= (\alpha_2\beta_3 - \alpha_3\beta_2)d_3 + (\alpha_2\beta_1 - \alpha_1\beta_2)d_1, \\ b_3 &= (\alpha_3\beta_1 - \alpha_1\beta_3)d_1 + (\alpha_3\beta_2 - \alpha_2\beta_3)d_2. \end{aligned}$$

The quantities δ and ε in these formulas are arbitrary nonzero real constants.

After proper normalization of the columns and rows of $\Omega_{\mathrm{R}}^{\mathcal{B}}$ and $\Omega_{\mathrm{L}}^{\mathcal{B}}$, respectively, these matrices will not only diagonalize the matrix $\mathcal{B}$ but also symmetrize all the matrices X_i of the linear combination forming $\mathcal{B}$ and $\mathcal{C}$

$$X_i = A^{-1}A_i, \quad \mathcal{B} = \eta_{x_i}X_i, \quad \mathcal{C} = \zeta_{x_i}X_i.$$

For this purpose we must choose

$$\delta^2 = \frac{cf + UT}{2\rho^2c^2D^2UT}, \quad \varepsilon^2 = \frac{UT - cf}{2\rho^2c^2D^2UT}, \quad f = s^2V - qU.$$

Note that from $U^2 > (sc)^2$ it follows that $|U|T > c|f|$.

In this case we obtain

$$\Omega_{\mathrm{L}}^{\mathcal{B}} X_i \Omega_{\mathrm{R}}^{\mathcal{B}} = \begin{bmatrix} \mathcal{D}_i & \mathcal{B}_i \\ \mathcal{B}_i^{\mathrm{T}} & \mathcal{C}_i \end{bmatrix},$$

where $\mathcal{D}_i$ and $\mathcal{C}_i$ are the diagonal and the symmetric matrix, respectively equal to

$$\mathcal{D}_i = \begin{bmatrix} v_i/U & 0 & 0 \\ 0 & v_i/U & 0 \\ 0 & 0 & v_i/U \end{bmatrix},$$

$$\mathcal{C}_i = \begin{bmatrix} \dfrac{cd_iQ^2 + (\nu_iU - \alpha_ic^2)(UT - cf)}{UQ^2T} & c_i \\ c_i & \dfrac{-cd_iQ^2 + (\nu_iU - \alpha_ic^2)(UT + cf)}{UQ^2T} \end{bmatrix},$$

$$c_1 = \frac{(\alpha_3\beta_2 - \alpha_2\beta_3)\tilde{c}}{QDT}, \quad c_2 = \frac{(\alpha_1\beta_3 - \alpha_3\beta_1)\tilde{c}}{QDT}, \quad c_3 = \frac{(\alpha_2\beta_1 - \alpha_1\beta_2)\tilde{c}}{QDT},$$

$$\tilde{c} = [(\alpha_2\beta_3 - \alpha_3\beta_2)u + (\alpha_3\beta_1 - \alpha_1\beta_3)v + (\alpha_1\beta_2 - \alpha_2\beta_1)w]\, c^2,$$

$$\mathcal{B}_1 = \frac{\alpha_3\beta_2 - \alpha_2\beta_3}{2\rho UDT}\mathcal{B}_c, \quad \mathcal{B}_2 = \frac{\alpha_1\beta_3 - \alpha_3\beta_1}{2\rho UDT}\mathcal{B}_c, \quad \mathcal{B}_3 = \frac{\alpha_2\beta_1 - \alpha_1\beta_2}{2\rho UDT}\mathcal{B}_c,$$

$$\mathcal{B}_c = \begin{bmatrix} w/\delta & -w/\varepsilon \\ -v/\delta & v/\varepsilon \\ -u/\delta & u/\varepsilon \end{bmatrix}.$$

Besides, such normalization results in a very sparse form of the matrix

$$(\Omega_{\mathrm{L}}^{\mathcal{B}})^{\mathrm{T}}\,\Omega_{\mathrm{L}}^{\mathcal{B}} = \begin{bmatrix} 1 & 0 & 0 & 0 & -c^{-2} \\ 0 & 1 & 0 & 0 & \alpha_1(\rho U)^{-1} \\ 0 & 0 & 1 & 0 & \alpha_2(\rho U)^{-1} \\ 0 & 0 & 0 & 1 & \alpha_3(\rho U)^{-1} \\ -c^{-2} & \alpha_1(\rho U)^{-1} & \alpha_2(\rho U)^{-1} & \alpha_3(\rho U)^{-1} & (c^2+\rho^2)\rho^{-2}c^{-4} \end{bmatrix},$$

whose eigenvalues σ_k are

$$\sigma_1 = \sigma_2 = \sigma_3 = 1, \quad \sigma_{4,5} = \frac{U\varphi \pm \sqrt{(U\varphi)^2 - 4[U^2 - (sc)^2]\rho^2c^6}}{2\rho^2c^4U},$$

where $\varphi = \rho^2c^4 + c^2 + \rho^2$.

The determinants of the matrices $\Omega_{\mathrm{R}}^{\mathcal{B}}$ and $\Omega_{\mathrm{L}}^{\mathcal{B}}$ are the following:

$$\det\Omega_{\mathrm{R}}^{\mathcal{B}} = (\det\Omega_{\mathrm{L}}^{\mathcal{B}})^{-1} = -\frac{\rho cU}{Q}.$$

The spectral norm of $\Omega_{\mathrm{L}}^{\mathcal{B}}$ is equal to $\sqrt{r}$, where $r = \max|\sigma_k|$. It is easy to see that for $U^2 > (sc)^2$ it is uniformly bounded in $\mathbf{x}, \boldsymbol{\alpha}$, and $\boldsymbol{\beta}$. The same is true for $\Omega_{\mathrm{R}}^{\mathcal{B}}$. Thus, the system of (1.3.13) is hyperbolic if the flow is supersonic with respect to ξ.

If we pass to the conservative variables $\mathbf{F}$, the system of (1.3.13) acquires the form

$$\frac{\partial\mathbf{F}}{\partial\xi} + \tilde{\mathcal{B}}\frac{\partial\mathbf{F}}{\partial\eta} + \tilde{\mathcal{C}}\frac{\partial\mathbf{F}}{\partial\zeta} = \mathbf{0}, \tag{1.3.15}$$

where

$$\tilde{A}_i = MA_iM^{-1}, \quad \tilde{A} = \xi_{x_i}\tilde{A}_i, \quad \tilde{\mathcal{B}} = \tilde{A}^{-1}(\eta_{x_i}\tilde{A}_i), \quad \tilde{\mathcal{C}} = \tilde{A}^{-1}(\zeta_{x_i}\tilde{A}_i)$$

and M is the Jacobian matrix $\partial\mathbf{F}/\partial\mathbf{U}$.

It is clear that $\tilde{\mathcal{B}}$ and $\tilde{\mathcal{C}}$ are diagonalized with the use of $\tilde{\Omega}_{\mathrm{R}}^{\mathcal{B}}$ and $\tilde{\Omega}_{\mathrm{R}}^{\mathcal{C}}$ such that

$$\tilde{\Omega}_{\mathrm{L}}^{\mathcal{B}}\,\tilde{\mathcal{B}}\,\tilde{\Omega}_{\mathrm{R}}^{\mathcal{B}} = \Omega_{\mathrm{L}}^{\mathcal{B}}\,M^{-1}\,\tilde{\mathcal{B}}\,M\,\Omega_{\mathrm{R}}^{\mathcal{B}} = \Omega_{\mathrm{L}}^{\mathcal{B}}\,\mathcal{B}\,\Omega_{\mathrm{R}}^{\mathcal{B}},$$
$$\tilde{\Omega}_{\mathrm{L}}^{\mathcal{C}}\,\tilde{\mathcal{C}}\,\tilde{\Omega}_{\mathrm{R}}^{\mathcal{C}} = \Omega_{\mathrm{R}}^{\mathcal{C}}\,M^{-1}\,\tilde{\mathcal{C}}\,M\,\Omega_{\mathrm{R}}^{\mathcal{C}} = \Omega_{\mathrm{L}}^{\mathcal{C}}\,\mathcal{C}\,\Omega_{\mathrm{R}}^{\mathcal{C}}.$$

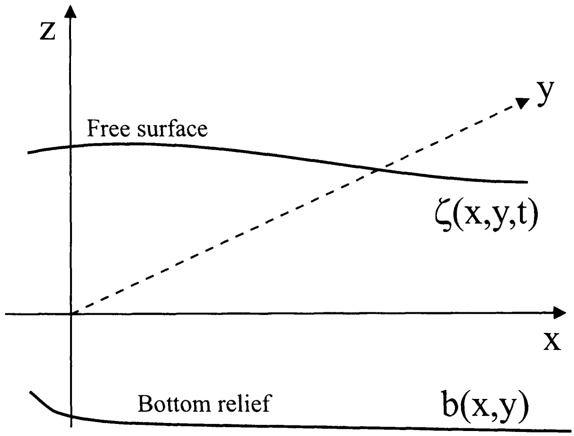

Figure 1.1 Shallow water layer.

1.3.3 Shallow water equations. If we consider the flow of an incompressible fluid, the mass conservation equation in Eq. (1.3.1) reduces to div $\mathbf{v} = 0$. Let the gravitation force act along the z-axis of the Cartesian coordinate system shown in Fig. 1.1. In this case the condition of the divergence-free velocity field must be accompanied by the momentum equation in the form

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} + \mathbf{F}_g, \tag{1.3.16}$$

where $\mathbf{F}_g = [0, 0, -g]^{\mathrm{T}}$ and g is the free-fall acceleration.

In the shallow water approximation the substantive time derivative along the z-axis is assumed negligibly small and we obtain (Stoker 1957)

$$p = \rho g(\zeta - z). \tag{1.3.17}$$

Here we assumed that pressure is equal to zero at the free-surface level $z = \zeta(x, y, t)$. Substituting expression (1.3.17) into Eq. (1.3.16) and adding the mass conservation equation, we arrive at the system governing the shallow water behavior in the form

$$\begin{aligned}
&\frac{\partial h}{\partial t} + \frac{\partial uh}{\partial x} + \frac{\partial vh}{\partial y} = 0, \\
&\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g\frac{\partial \zeta}{\partial x}, \\
&\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g\frac{\partial \zeta}{\partial y}.
\end{aligned} \tag{1.3.18}$$

Here $h(x, y, t) = \zeta - b(x, y)$, where $b(x, y)$ describes the shape of the bottom relief.

This system can be rewritten in the conservation-law form

$$\mathbf{U}_t + \mathbf{E}(\mathbf{U})_x + \mathbf{G}(\mathbf{U})_y = \mathbf{S}(\mathbf{U}) \tag{1.3.19}$$

with

$$\mathbf{U} = \begin{bmatrix} h \\ hu \\ hv \end{bmatrix}, \quad \mathbf{E} = \begin{bmatrix} hu \\ hu^2 + \frac{1}{2}gh^2 \\ huv \end{bmatrix}, \quad \mathbf{G} = \begin{bmatrix} hv \\ huv \\ hv^2 + \frac{1}{2}gh^2 \end{bmatrix}, \quad \mathbf{S} = \begin{bmatrix} 0 \\ -ghb_x \\ -ghb_y \end{bmatrix}.$$

Note that if we formally put $\rho = h$ and $p = \frac{1}{2}gh^2$, for $b(x, y) = \text{const}$ we obtain the gas dynamic system for barotropic gas with the specific heat ratio equal to 2. This system preserves the hyperbolic properties of the Euler equations. The methods of its solution will be discussed in Chapter 4.

1.3.4 Equations of ideal magnetohydrodynamics. The equations of ideal magnetohydrodynamics generalize the Euler gas dynamic equations in the presence of an electromagnetic field. In this case appropriate terms must be added to the momentum and energy equations of the Euler system and it must be supplemented by the Maxwell equations (see Landau and Lifshitz 1984; Jeffrey and Taniuti 1964; Kulikovskii and Lyubimov 1965; Akhiezer et al. 1975),

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho \mathbf{v} = 0, \tag{1.3.20}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{\nabla p}{\rho} - \frac{\mathbf{B} \times \operatorname{curl} \mathbf{B}}{4\pi\rho}, \tag{1.3.21}$$

$$\frac{dp}{dt} = c_e^2 \frac{d\rho}{dt}, \tag{1.3.22}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \operatorname{curl}(\mathbf{v} \times \mathbf{B}). \tag{1.3.23}$$

Here $\mathbf{B}$ is the magnetic field strength vector and c_e is the acoustic speed of sound. For the sake of simplicity, we substituted the energy equation, which will be written out in Chapter 5, by the equation expressing the definition of the speed of sound $(\partial p/\partial \rho)_S = c_e^2$, where S is entropy. In the corresponding equation (1.3.22) we also introduced the total, or substantive, time derivative

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + (\mathbf{v} \cdot \nabla) f.$$

If we take into account the vector analysis formula

$$\operatorname{curl}(\mathbf{v} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{B} + \mathbf{v} \operatorname{div} \mathbf{B} - \mathbf{B} \operatorname{div} \mathbf{v},$$

the quasilinear form of the MHD system acquires the form

$$\frac{\partial \mathbf{u}}{\partial t} + A_1 \frac{\partial \mathbf{u}}{\partial x} + A_2 \frac{\partial \mathbf{u}}{\partial y} + A_3 \frac{\partial \mathbf{u}}{\partial z} = \mathbf{0}, \tag{1.3.24}$$

where

$$\mathbf{u} = [\rho, u, v, w, p, B_x, B_y, B_z]^{\mathrm{T}},$$

$$A_1 = \begin{bmatrix} u & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u & 0 & 0 & \frac{1}{\rho} & 0 & \frac{B_y}{4\pi\rho} & \frac{B_z}{4\pi\rho} \\ 0 & 0 & u & 0 & 0 & 0 & -\frac{B_x}{4\pi\rho} & 0 \\ 0 & 0 & 0 & u & 0 & 0 & 0 & -\frac{B_x}{4\pi\rho} \\ 0 & \rho c_e^2 & 0 & 0 & u & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & u & 0 & 0 \\ 0 & B_y & -B_x & 0 & 0 & 0 & u & 0 \\ 0 & B_z & 0 & -B_x & 0 & 0 & 0 & u \end{bmatrix}, \quad A_2 = \begin{bmatrix} v & 0 & \rho & 0 & 0 & 0 & 0 & 0 \\ 0 & v & 0 & 0 & 0 & -\frac{B_y}{4\pi\rho} & 0 & 0 \\ 0 & 0 & v & 0 & \frac{1}{\rho} & \frac{B_x}{4\pi\rho} & 0 & \frac{B_z}{4\pi\rho} \\ 0 & 0 & 0 & v & 0 & 0 & 0 & -\frac{B_y}{4\pi\rho} \\ 0 & 0 & \rho c_e^2 & 0 & v & 0 & 0 & 0 \\ 0 & -B_y & B_x & 0 & 0 & v & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & v & 0 \\ 0 & 0 & B_z & -B_y & 0 & 0 & 0 & v \end{bmatrix}$$

$$A_3 = \begin{bmatrix} w & 0 & 0 & \rho & 0 & 0 & 0 & 0 \\ 0 & w & 0 & 0 & 0 & -\frac{B_z}{4\pi\rho} & 0 & 0 \\ 0 & 0 & w & 0 & 0 & 0 & -\frac{B_z}{4\pi\rho} & 0 \\ 0 & 0 & 0 & w & \frac{1}{\rho} & \frac{B_x}{4\pi\rho} & \frac{B_y}{4\pi\rho} & 0 \\ 0 & 0 & 0 & \rho c_e^2 & w & 0 & 0 & 0 \\ 0 & -B_z & 0 & B_x & 0 & w & 0 & 0 \\ 0 & 0 & -B_z & B_y & 0 & 0 & w & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & w \end{bmatrix}$$

The system of (1.3.20)–(1.3.23) must be supplemented by the equation

$$\operatorname{div} \mathbf{B} = 0 \tag{1.3.25}$$

expressing the absence of magnetic charge. It is not important for our current consideration, since if $\operatorname{div} \mathbf{B} = 0$ initially at $t = 0$, then the Faraday equation (1.3.23) will preserve the absence of magnetic charge later in time. This can be seen if we apply the divergence operator to the both parts of Eq. (1.3.23),

$$\frac{\partial \operatorname{div} \mathbf{B}}{\partial t} = \operatorname{div} \operatorname{curl} (\mathbf{v} \times \mathbf{B}) \equiv 0.$$

Note that we did not use Eq. (1.3.25) when deriving the system of (1.3.24).

For simplicity, we shall consider in this section only the one-dimensional system postponing the detailed description of the MHD equations to Chapter 5. Solution of the characteristic equation

$$\det(A_1 - \lambda I) = 0$$

gives the following eigenvalues:

$$\lambda_{1,2} = u, \quad \lambda_{3,4} = u \pm a_{\mathrm{A}}, \quad \lambda_{5,6} = u \pm a_{\mathrm{f}}, \quad \lambda_{7,8} = u \pm a_{\mathrm{s}}$$

with

$$a_{\mathrm{A}} = \frac{|B_x|}{\sqrt{4\pi\rho}}, \quad a_{f,s} = \frac{1}{2}\left[\left(c_e^2 + \frac{\mathbf{B}^2}{4\pi\rho} + \frac{|B_x| c_e}{\sqrt{\pi\rho}}\right)^{1/2} \pm \left(c_e^2 + \frac{\mathbf{B}^2}{4\pi\rho} - \frac{|B_x| c_e}{\sqrt{\pi\rho}}\right)^{1/2}\right].$$

The velocity a_A is called the Alfvén, or rotational, velocity; a_f and a_s are called the fast and slow magnetosonic velocities, respectively. All eigenvalues are clearly real.

Note that the equation for B_x in the one-dimensional treatment reduces to

$$\frac{\partial B_x}{\partial t} + u\frac{\partial B_x}{\partial x} = 0, \tag{1.3.26}$$

that is, to the one-dimensional convection equation for B_x. Of course, in the genuinely one-dimensional problem, with all quantities depending only on the space variable x, one can simply put $B_x \equiv \text{const}$ and omit the corresponding equation. By omitting Eq. (1.3.26), we reduce the system dimension to 7×7 (seven equations for seven dependent variables). Both the extended 8×8 system and the reduced one have real eigenvalues and a nondegenerate set of eigenvectors. It is apparent that the eigenvalues of the truly one-dimensional system are the same as those of the extended system. Only the eigenvalue $\lambda = u$ has the multiplicity 1.

Below we present a complete set of right and left eigenvectors for the extended system:

$$\begin{aligned}
\mathbf{r}^1 &= [1,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0,\ 0]^{\mathrm{T}},\\
\mathbf{r}^2 &= [0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0]^{\mathrm{T}},\\
\mathbf{r}^3 &= \left[0,\ 0,\ -\frac{b_z}{\sqrt{2}},\ \frac{b_y}{\sqrt{2}},\ 0,\ 0,\ b_z\sqrt{2\pi\rho}\,\operatorname{sgn}B_x,\ -b_y\sqrt{2\pi\rho}\,\operatorname{sgn}B_x\right]^{\mathrm{T}},\\
\mathbf{r}^4 &= \left[0,\ 0,\ -\frac{b_z}{\sqrt{2}},\ \frac{b_y}{\sqrt{2}},\ 0,\ 0,\ -b_z\sqrt{2\pi\rho}\,\operatorname{sgn}B_x,\ b_y\sqrt{2\pi\rho}\,\operatorname{sgn}B_x\right]^{\mathrm{T}},\\
\mathbf{r}^5 &= [\rho\alpha_f,\ \alpha_f a_f,\ -\alpha_s a_s b_y \operatorname{sgn}B_x,\ -\alpha_s a_s b_z \operatorname{sgn}B_x,\ \alpha_f \rho c_e^2,\ 0,\\
&\qquad\qquad 2\alpha_s\sqrt{\pi\rho}c_e b_y,\ 2\alpha_s\sqrt{\pi\rho}c_e b_z]^{\mathrm{T}},\\
\mathbf{r}^6 &= [\rho\alpha_f,\ -\alpha_f a_f,\ \alpha_s a_s b_y \operatorname{sgn}B_x,\ \alpha_s a_s b_z \operatorname{sgn}B_x,\ \alpha_f \rho c_e^2,\ 0,\ 2\alpha_s\sqrt{\pi\rho}c_e b_y,\ 2\alpha_s\sqrt{\pi\rho}c_e b_z]^{\mathrm{T}},\\
\mathbf{r}^7 &= [\rho\alpha_s,\ \alpha_s a_s,\ \alpha_f a_f b_y \operatorname{sgn}B_x,\ \alpha_f a_f b_z \operatorname{sgn}B_x,\ \alpha_s \rho c_e^2,\ 0,\\
&\qquad\qquad -2\alpha_f\sqrt{\pi\rho}c_e b_y,\ -2\alpha_f\sqrt{\pi\rho}c_e b_z]^{\mathrm{T}},\\
\mathbf{r}^8 &= [\rho\alpha_s,\ -\alpha_s a_s,\ -\alpha_f a_f b_y \operatorname{sgn}B_x,\ -\alpha_f a_f b_z \operatorname{sgn}B_x,\ \alpha_s \rho c_e^2,\ 0,\\
&\qquad\qquad -2\alpha_f\sqrt{\pi\rho}c_e b_y,\ -2\alpha_f\sqrt{\pi\rho}c_e b_z]^{\mathrm{T}},
\end{aligned}$$

$$\begin{aligned}
\mathbf{l}^1 &= \left[1,\ 0,\ 0,\ 0,\ -\frac{1}{c_e^2},\ 0,\ 0,\ 0\right],\\
\mathbf{l}^2 &= [0,\ 0,\ 0,\ 0,\ 0,\ 1,\ 0,\ 0],\\
\mathbf{l}^3 &= \left[0,\ 0,\ -\frac{b_z}{\sqrt{2}},\ \frac{b_y}{\sqrt{2}},\ 0,\ 0,\ \frac{b_z \operatorname{sgn}B_x}{2\sqrt{2\pi\rho}},\ -\frac{b_y \operatorname{sgn}B_x}{2\sqrt{2\pi\rho}}\right],\\
\mathbf{l}^4 &= \left[0,\ 0,\ -\frac{b_z}{\sqrt{2}},\ \frac{b_y}{\sqrt{2}},\ 0,\ 0,\ -\frac{b_z \operatorname{sgn}B_x}{2\sqrt{2\pi\rho}},\ \frac{b_y \operatorname{sgn}B_x}{2\sqrt{2\pi\rho}}\right],\\
\mathbf{l}^5 &= \left[0,\ \frac{\alpha_f a_f}{2c_e^2},\ -\frac{\alpha_s a_s}{2c_e^2}b_y \operatorname{sgn}B_x,\ -\frac{\alpha_s a_s}{2c_e^2}b_z \operatorname{sgn}B_x,\ \frac{\alpha_f}{2\rho c_e^2},\ 0,\ \frac{\alpha_s b_y}{4c_e\sqrt{\pi\rho}},\ \frac{\alpha_s b_z}{4c_e\sqrt{\pi\rho}}\right],\\
\mathbf{l}^6 &= \left[0,\ -\frac{\alpha_f a_f}{2c_e^2},\ \frac{\alpha_s a_s}{2c_e^2}b_y \operatorname{sgn}B_x,\ \frac{\alpha_s a_s}{2c_e^2}b_z \operatorname{sgn}B_x,\ \frac{\alpha_f}{2\rho c_e^2},\ 0,\ \frac{\alpha_s b_y}{4c_e\sqrt{\pi\rho}},\ \frac{\alpha_s b_z}{4c_e\sqrt{\pi\rho}}\right],\\
\mathbf{l}^7 &= \left[0,\ \frac{\alpha_s a_s}{2c_e^2},\ \frac{\alpha_f a_f}{2c_e^2}b_y \operatorname{sgn}B_x,\ \frac{\alpha_f a_f}{2c_e^2}b_z \operatorname{sgn}B_x,\ \frac{\alpha_s}{2\rho c_e^2},\ 0,\ -\frac{\alpha_f b_y}{4c_e\sqrt{\pi\rho}},\ -\frac{\alpha_f b_z}{4c_e\sqrt{\pi\rho}}\right],
\end{aligned}$$

$$\mathbf{l}^8 = \left[0,\ -\frac{\alpha_s a_s}{2c_e^2},\ -\frac{\alpha_f a_f}{2c_e^2} b_y \operatorname{sgn} B_x,\ -\frac{\alpha_f a_f}{2c_e^2} b_z \operatorname{sgn} B_x,\ \frac{\alpha_s}{2\rho c_e^2},\ 0,\ -\frac{\alpha_f b_y}{4c_e\sqrt{\pi\rho}},\ -\frac{\alpha_f b_z}{4c_e\sqrt{\pi\rho}}\right].$$

In these equations

$$\alpha_f = \sqrt{\frac{c_e^2 - a_s^2}{a_f^2 - a_s^2}}, \quad \alpha_s = \sqrt{\frac{a_f^2 - c_e^2}{a_f^2 - a_s^2}}.$$

Also, by definition, we put $\operatorname{sgn} 0 \equiv 1$.

We also normalized the tangential components of the magnetic field vector

$$b_y = \frac{B_y}{\sqrt{B_y^2 + B_z^2}}, \quad b_z = \frac{B_z}{\sqrt{B_y^2 + B_z^2}}.$$

The quantities b_y and b_z clearly degenerate in the absence of the tangential magnetic field. If one notices, however, that the only important relationship to be preserved is $b_y^2 + b_z^2 = 1$, the following regularization can be chosen:

$$b_y = \sin\psi, \quad b_z = \cos\psi.$$

The value of ψ is arbitrary. The compact form of the MHD eigenvector normalization presented here is based on the paper by Brio and Wu (1988) (see also Roe and Balsara 1996). Besides, they used $\psi = \pi/4$, though this does not seem to be important, and $\psi = 0$ or $\pi/2$ can also be a good choice. It is easy to check that $\det \Omega_R = 16\pi\rho^2 c^5$.

The formulas for the eigenvectors obviously become degenerate if $a_f = a_s$. This occurs for $B_y^2 + B_z^2 = 0$ and $c_e = a_A$. If this rare occasion happens, we can simply put $B_x = B_x(1+\varepsilon)$, where ε is a small constant.

1.3.5 Elasticity equations.

Let us consider as an example a linear equation describing the vibration of a uniform rod in the Timoshenko theory (Grigolyuk and Selezov 1973). A Timoshenko-type equation has the form

$$\frac{\partial^2 w}{\partial t^2} + a_1 \frac{\partial^4 w}{\partial x^4} - a_2 \frac{\partial^4 w}{\partial t^2 \partial x^2} + a_3 \frac{\partial^4 w}{\partial t^4} = 0, \tag{1.3.27}$$

where a_1, a_2, and a_3 are positive constants depending on the rod density, its geometrical parameters and elastic coefficients, and w is a deflection. In particular,

$$a_1 = \frac{EI}{\mu F l^2}, \quad a_2 = \frac{I}{Fl^2} + \frac{EI}{k\mu F l^2}, \quad a_3 = \frac{I}{kFl^2},$$

where F is the area of the cross-section of the rod, l is its length, μ is the shear modulus, I is the moment of inertia of the cross-section about the axis passing through the center of mass, EI is the bending rigidity, E is Young's modulus, and k is the shear coefficient. The variables x and w are normalized by l, and t is normalized by l/c, where $c = \sqrt{\mu/\rho}$, ρ is the density of material.

If we introduce

$$M = \frac{\partial^2 w}{\partial t^2}, \quad N = \frac{\partial^2 w}{\partial x^2},$$

Eq. (1.3.27) can be written as

$$a_3\frac{\partial^2 M}{\partial t^2} - a_2\frac{\partial^2 M}{\partial x^2} + a_1\frac{\partial^2 N}{\partial x^2} + M = 0, \tag{1.3.28}$$

$$\frac{\partial^2 N}{\partial t^2} - \frac{\partial^2 M}{\partial x^2} = 0. \tag{1.3.29}$$

If we also introduce

$$P = \frac{\partial M}{\partial t}, \quad Q = \frac{\partial N}{\partial t}, \quad R = \frac{\partial M}{\partial x}, \quad S = \frac{\partial N}{\partial x},$$

Eqs. (1.3.28)–(1.3.29) acquire the form

$$\begin{aligned}
&\frac{\partial P}{\partial t} - \frac{a_2}{a_3}\frac{\partial R}{\partial x} + \frac{a_1}{a_3}\frac{\partial S}{\partial x} = -\frac{1}{a_3}M,\\
&\frac{\partial Q}{\partial t} - \frac{\partial R}{\partial x} = 0, \quad \frac{\partial R}{\partial t} - \frac{\partial P}{\partial x} = 0, \quad \frac{\partial S}{\partial t} - \frac{\partial Q}{\partial x} = 0,\\
&\frac{\partial M}{\partial t} = P, \quad \frac{\partial N}{\partial t} = Q.
\end{aligned}$$

This system of the first-order equations can be rewritten in the form

$$\frac{\partial \mathbf{U}}{\partial t} + A\frac{\partial \mathbf{U}}{\partial x} = \mathbf{f},$$

where

$$A = \begin{bmatrix} 0 & 0 & -\alpha_2 & \alpha_1 & 0 & 0\\ 0 & 0 & -1 & 0 & 0 & 0\\ -1 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \alpha_1 = \frac{a_1}{a_3}, \quad \alpha_2 = \frac{a_2}{a_3};$$

$$\mathbf{U} = [P, Q, R, S, N, M]^{\mathrm{T}}, \quad \mathbf{f} = \left[-\frac{M}{a_3}, 0, 0, 0, P, Q\right]^{\mathrm{T}}.$$

The matrix A can be diagonalized,

$$A = \Omega_{\mathrm{R}}\Lambda\Omega_{\mathrm{L}},$$

where

$$\Lambda = \operatorname{diag}[\beta_1, -\beta_1, \beta_2, -\beta_2, 0, 0],$$

$$\beta_1 = \frac{1}{\sqrt{2}}\sqrt{\alpha_2 + \sqrt{\alpha_2^2 - 4\alpha_1}} = \sqrt{\frac{E}{\mu}}, \quad \beta_2 = \frac{1}{\sqrt{2}}\sqrt{\alpha_2 - \sqrt{\alpha_2^2 - 4\alpha_1}} = \sqrt{k};$$

and Ω_{R}, Ω_{L} have the block form

$$\Omega_{\mathrm{R}} = \begin{bmatrix} B & 0\\ 0 & I \end{bmatrix}, \quad \Omega_{\mathrm{L}} = \begin{bmatrix} B^{-1} & 0\\ 0 & I \end{bmatrix},$$

where $I = \mathrm{diag}[1, 1]$ is the 2×2 identity matrix, and the 4×4 matrices B and B^{-1} have the following structure:

$$B = \begin{bmatrix} -\beta_1^3 & \beta_1^3 & -\beta_2^3 & \beta_2^3 \\ -\beta_1 & \beta_1 & -\beta_2 & \beta_2 \\ \beta_1^2 & \beta_1^2 & \beta_2^2 & \beta_2^2 \\ 1 & 1 & 1 & 1 \end{bmatrix}, \quad B^{-1} = \frac{1}{2(\beta_2^2 - \beta_1^2)\beta_1\beta_2} \begin{bmatrix} \beta_2 & -\beta_2^3 & -\beta_2\beta_1 & \beta_1\beta_2^3 \\ -\beta_2 & \beta_2^3 & -\beta_2\beta_1 & \beta_1\beta_2^3 \\ -\beta_1 & \beta_1^3 & \beta_2\beta_1 & -\beta_1^3\beta_2 \\ \beta_1 & -\beta_1^3 & \beta_2\beta_1 & -\beta_1^3\beta_2 \end{bmatrix},$$

$$\det B = -4(\beta_2 - \beta_1)^2(\beta_2 + \beta_1)^2\beta_1\beta_2.$$

1.4 Properties of solutions

The purpose of this section is to provide a brief discussion of several mathematical properties of solutions of quasilinear hyperbolic equations. We restrict ourselves to those properties that are essential for further presentation. Detailed information can be found in a number of monographs and textbooks. The incomplete list includes those by Jeffrey and Taniuti (1964), Lax (1972), Rozhdestvenskii and Yanenko (1983), Jeffrey (1976), Le Veque (1992), Kulikovskii and Sveshnikova (1995), Godlewski and Raviart (1996), Serre (1996), Kröner (1997), and Toro (1997).

1.4.1 Classical solutions. We assume for simplicity that the system is homogeneous with the flux vector, depending only on the unknown vector itself but not on the independent variables

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{0}. \tag{1.4.1}$$

As shown in Section 1.3, this form of the system is frequently encountered in mechanical applications.

As is apparent from Eq. (1.1.7), the unknown vector must at least be differentiable. To pass from the integral form (1.2.14) to the corresponding differential form (1.2.13), we must also assume a definite smoothness of $\mathbf{U}$. If a sufficiently smooth solution does not exist, one must use the integral form.

Let us consider a Cauchy problem by specifying

$$\mathbf{U}(x, 0) = \mathbf{U}_0(x) \tag{1.4.2}$$

for Eq. (1.4.1).

The vector function $\mathbf{U}$ is called a classical solution of the Cauchy problem (1.4.1)–(1.4.2) if $\mathbf{U}$ is a continuously differentiable function that satisfies these equations pointwise.

Existence of smooth solutions

Before giving definition to generalized solutions of a hyperbolic system, we shall show that a classical solution, in fact, can sometimes exist only within a finite time interval even for a smooth initial distribution given by the function $\mathbf{U}_0$. For this purpose we shall consider the simplest case of the one-dimensional scalar equation

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0 \tag{1.4.3}$$

with the initial condition

$$U(x,0) = U_0(x), \quad -\infty < x < \infty. \tag{1.4.4}$$

Note that Eq. (1.4.3) can also be written in a quasilinear form

$$\frac{\partial U}{\partial t} + a(U)\frac{\partial U}{\partial x} = 0 \tag{1.4.5}$$

by putting $a(U) = \partial F(U)/\partial U$. Hence, the characteristic line (1.2.6) passing through the point $(x_0, 0)$ in the x–t plane becomes

$$x = x_0 + ta(U_0(x_0)). \tag{1.4.6}$$

It is apparent from Eq. (1.4.5) that

$$\frac{dU}{dt} = 0$$

along the characteristic line. This allows us to find a smooth solution using a so-called *method of characteristics* by specifying

$$U(x,t) = U_0(x_0),$$

where x_0 should be found from formula (1.4.6) describing the characteristic line.

For linear systems, when matrix A is independent of $\mathbf{U}$, characteristic lines $x = x_0 + at$ are straight lines that never intersect each other. This means that we can find the exact solution in the whole half-plane $\{-\infty < x < \infty,\ t > 0\}$. This solution has the form of a travelling wave

$$U(x,t) = U_0(x - at). \tag{1.4.7}$$

On the contrary, if $a(U) \neq \text{const}$ and for a certain $x_1 < x_2$,

$$\frac{1}{a(U_0(x_1))} < \frac{1}{a(U_0(x_2))},$$

then the characteristics that issue from $x = x_1$ and $x = x_2$ inevitably intersect each other. The classical solution no longer exists beyond the intersection point, since it must become discontinuous at it. This feature also outlines the limits of the method of characteristics.

An important notice must be given at this stage. Though we shall mainly deal with nonlinear systems in this monograph, linear equations are frequently encountered in numerical methods when a linearized system is used in the computational procedure to advance the solution within a small time interval.

If we linearize the system

$$\frac{\partial \mathbf{u}}{\partial t} + A(\mathbf{u})\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0} \tag{1.4.8}$$

in a small vicinity of some constant value $\mathbf{u}_0$, that is, assume $\mathbf{u} = \mathbf{u}_0 + \tilde{\mathbf{u}}(x,t)$, then $\tilde{\mathbf{u}}$ must satisfy the linear system

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + A(\mathbf{u}_0)\frac{\partial \tilde{\mathbf{u}}}{\partial x} = \mathbf{0} \tag{1.4.9}$$

to the second order of accuracy. The characteristic velocities λ_k and the eigenvectors $\mathbf{r}^k$ of the coefficient matrix A are constant in this case. Since A is diagonalizable, that is, $A = \Omega_R \Lambda \Omega_L$, on multiplying the system of (1.4.9) by Ω_L and introducing the Riemann variables $\mathbf{w} = \Omega_L \tilde{\mathbf{u}}$ we obtain

$$\frac{\partial \mathbf{w}}{\partial t} + \Lambda \frac{\partial \mathbf{w}}{\partial x} = \mathbf{0}, \tag{1.4.10}$$

where $\Lambda = \mathrm{diag}[\lambda_1, \ldots, \lambda_n]$.

Thus, the system splits into separate equations whose solutions w_k are travelling waves

$$w_k = w_k(x - \lambda_k t), \qquad k = 1, \ldots, n. \tag{1.4.11}$$

In this case the functions w_k are called Riemann invariants of the system (1.4.9) with constant coefficients a_{ij} constituting the matrix A. Travelling waves (1.4.11) propagate at constant velocities λ_k preserving their shape.

The general solution of the system (1.4.9) is the sum of n travelling waves propagating at the corresponding characteristic velocities,

$$\tilde{\mathbf{u}} = \sum_{k=1}^{n} \mathbf{r}^k w_k(x - \lambda_k t). \tag{1.4.12}$$

If the eigenvectors $\mathbf{r}^k$ are normalized so that $|\mathbf{r}^k| = 1$, the quantities w_k can be considered as the amplitudes of the corresponding linear waves.

Riemann waves

We now proceed to the properties of the nonlinear system (1.4.1). Let us consider an important special class of solutions to the Cauchy problem (1.4.1)–(1.4.2) that are called Riemann, or simple, waves. The latter notion seems to be more general, since it also includes such steady two-dimensional solution of the gas dynamic system (1.3.13) as the Prandtl–Meyer waves and similar waves in MHD and elasticity.

It is assumed that in simple waves the unknown vector $\mathbf{U}$ depends on a certain combination $\theta(x, t)$ of the independent variables, that is, $\mathbf{U} = \mathbf{U}(\theta(x, t))$. In this case from Eq. (1.4.1) we obtain

$$(A - \lambda I)\frac{d\mathbf{U}}{d\theta} = \mathbf{0} \tag{1.4.13}$$

with

$$\lambda = -\frac{\partial \theta}{\partial t} \bigg/ \frac{\partial \theta}{\partial x}. \tag{1.4.14}$$

To obtain nontrivial solutions, we must demand

$$\det(A - \lambda I) = 0, \tag{1.4.15}$$

that is, λ must be the eigenvalue of the matrix A, thus coinciding with one of the characteristic velocities $\lambda_k(\mathbf{U})$ of the system (1.4.1). The increments dU_k coincide in this case with the corresponding increments of small disturbances propagating through the uniform distribution U_k. It is clear that the derivative $d\mathbf{U}/d\theta$ is parallel in this case to the corresponding right eigenvector of the matrix A

$$\frac{d\mathbf{U}}{d\theta} = \alpha \mathbf{r}, \tag{1.4.16}$$

where the eigenvector number is omitted.

Equation (1.4.16) defines a family of integral curves tangent at each of their points to the right eigenvector of the matrix A. The number of the simple wave solutions is equal to the number of linearly independent eigenvectors $\mathbf{r}$. Since the system is hyperbolic, this number is equal to n.

Let us consider one of the Riemann waves corresponding to a simple root of the characteristic equation. On the integral curve, the vector $\mathbf{U}$ is a function of a single parameter θ. This parameter can be chosen arbitrarily depending on our needs and tastes. This can be an arc length in the $\mathbf{U}$-space measured from the initial value $\theta_0 = \theta(x, 0)$, one of the components U_k of the vector $\mathbf{U}$ or a characteristic velocity λ. The only requirement is that the chosen parameter must be monotone along the part of the integral curve under consideration. On choosing θ, we can determine its value from Eq. (1.4.14),

$$\frac{\partial\theta}{\partial t} + \lambda(\mathbf{U}(\theta))\,\frac{\partial\theta}{\partial x} = 0. \tag{1.4.17}$$

A scalar function of the vector here is regarded as a function of its components.

On the other hand, the characteristics

$$\frac{dx}{dt} = \lambda(\mathbf{U}(\theta))$$

of Eq. (1.4.17) coincide with the chosen characteristic family of the system (1.4.1). Along each of the characteristic lines of (1.4.17), we have $d\theta/dt = 0$, that is, $\theta =$ const and, hence, all the components U_k of the vector $\mathbf{U}$ and λ itself are constant. This means that in the x–t plane the characteristics are represented by straight lines and their slopes can be determined even at $t = 0$. The remaining families of characteristics are generally curvilinear.

If we wish the solution to be in the form of a Riemann wave, initial conditions for $\mathbf{U}$ must also be represented by some function of θ. This means that only one arbitrary function $\theta(x, 0) = \theta_0(x)$ must occur in them. In addition, the solution contains $n - 1$ constants, which are necessary to single out the integral curve of Eq. (1.4.13). As shown above, the solution in the form of a Riemann wave can be constructed uniquely only in the part of the x–t plane where characteristics do not intersect each other.

The above consideration shows that simple waves generalize small-perturbation waves governed by Eq. (1.4.9). In fact, each element $d\mathbf{U} = (d\mathbf{U}/d\theta)d\theta$ of the Riemann wave varies proportionally to the right eigenvector of the coefficient matrix of the system, exactly in the same way as in the case of propagation of a small perturbation. The velocities of propagation are also the same. A simple wave can therefore be represented as a series of small perturbations, each moving in a wake behind the foregoing one. Depending on the character of the characteristic velocity variation along the integral curve, the profile of the wave can suffer deformations.

If $\lambda(\theta) =$ const on the integral curve, the characteristics in the x–t plane are straight lines (see Fig. 1.2b) and never intersect each other. In this case the Riemann wave is a travelling wave with

$$\mathbf{U} = \mathbf{U}(\theta), \quad \theta = \theta_0(x - \lambda t).$$

If λ is not constant, in the monotonicity intervals of the characteristic velocity we can choose it as a parameter θ. Thus, we obtain

$$\frac{\partial \lambda}{\partial t} + \lambda \frac{\partial \lambda}{\partial x} = 0$$

with the initial condition $\lambda(x, 0) = \lambda_0(x)$.

Let the initial profile be represented by the function shown in Fig. 1.2. As shown above for the scalar equation (1.4.3), on the interval with $d\lambda/dx > 0$, the solution can be uniquely determined. This is caused by the fact that preceding parts of the profile move faster than those following them. On the other hand, on the interval with $d\lambda/dx < 0$ the elements closer to the maximum of $\lambda_0(x)$ move faster than foregoing ones and sooner or later catch them. This process is known by the name of the *wave steepening*. It finally results in the wave breaking (see Fig. 1.2e). Since nonunique solutions are mostly disregarded in continuum mechanics, it is adopted that at the moments of the characteristic intersection the classical solution ceases to exist and a discontinuity originates.

There is a class of equations possessing solutions in the form of Riemann waves in which the characteristic velocity λ is constant. This occurs if λ is constant along each integral curve of the wave. In this case the Riemann wave is a traveling wave, that is, $U_k = U_k(x - \lambda t)$, $\lambda =$ const. Such waves that propagate without changing their shape will be called nondeforming waves. If the function θ is initially discontinuous, it will remain discontinuous for all subsequent t. Each discontinuity of this type has a counterpart discontinuity with the opposite quantity variation. Discontinuities that are at the same time Riemann waves will be called reversible. As examples of nondeforming Riemann waves and reversible discontinuities one can indicate rotational (Alfvén) Riemann waves and rotational discontinuities in magnetohydrodynamics (Landau and Lifshitz 1984).

Let us note an important particular case of Riemann waves, namely, self-similar Riemann waves. Initial condition at $t = 0$ for such waves is a piecewise-constant function θ with a discontinuity at the point taken as the origin $x = 0$. If $\lambda|_{x>0} > \lambda|_{x<0}$, a fan of rectilinear characteristics $x/t = \lambda(\theta)$ corresponding to the considered wave starts from the origin. Since the quantities U_k are constant along the characteristics, the characteristic velocity $\lambda = x/t$ itself can be taken as a parameter θ.

The properties of classical solutions described in this subsection show that, since the general system of quasilinear hyperbolic equations cannot be written for the Riemann invariants, the domain of definition of the solution can only be found simultaneously with the solution itself. In addition, a classical solution and its derivatives do not remain bounded.

1.4.2 Generalized solutions. The above reasoning leads us to introduce the notion of a generalized solution. We shall call the vector function *a generalized solution* of the system (1.4.1) if it satisfies the system of (1.2.14) for an arbitrary piecewise-smooth contour ∂V bounding the volume V. It is clear that all classical solutions form a subset of generalized solutions. On the other hand, $\mathbf{U}(x, t)$ can only be piecewise-continuous with continuous first derivatives within each continuity interval.

The surface on which the function $\mathbf{U}(x, t)$ is discontinuous is called a *surface of strong discontinuity*, or a *shock surface*. If only the first derivatives of $\mathbf{U}$ are discontinuous, then we have a *weak discontinuity*.

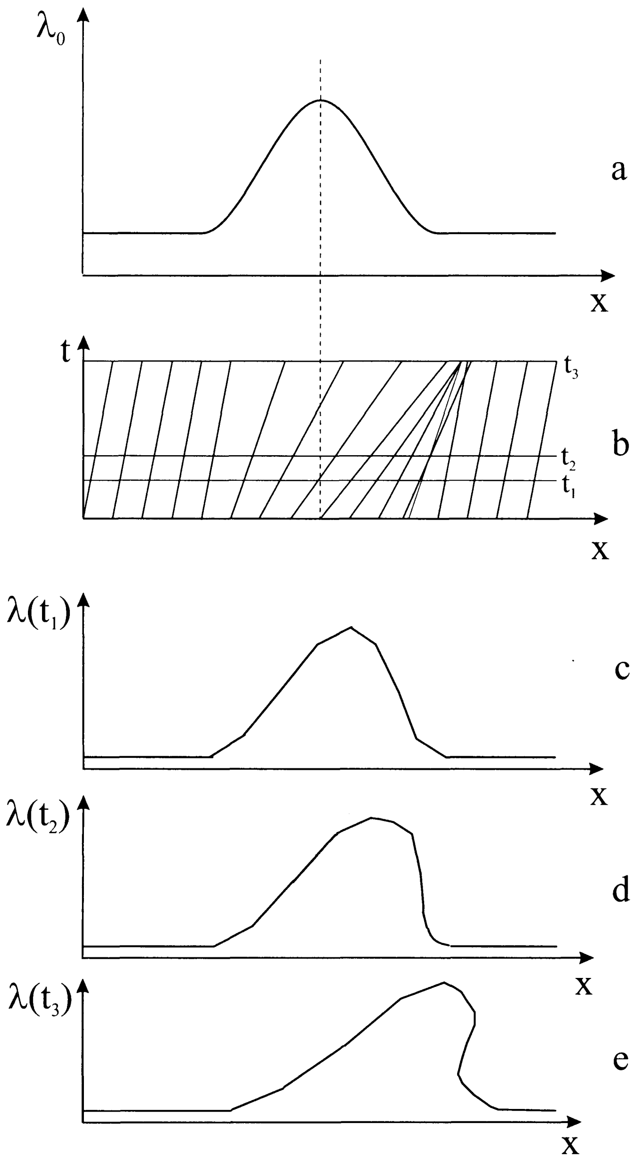

Figure 1.2 Breaking of the smooth profile.

Shock relations

Let us find the formulas relating unknown functions on the shock. Although such relations can be obtained directly in the multidimensional form (see, e.g., Godlewski and Raviart 1996), for the sake of simplicity we shall present their derivation for the system of the two variables x and t. In this case the system (1.2.14) transforms into

$$\frac{d}{dt}\int_{x_1}^{x_2} U_i(\mathbf{u})dx + F_i(\mathbf{u})|_{x=x_2} - F_i(\mathbf{u})|_{x=x_1} = 0. \tag{1.4.18}$$

Let the vector $\mathbf{u}$ be discontinuous on the line $x = X(t)$ in the x–t plane while remaining continuous on both sides of this line. Here we assumed that the shock moves from the left to the right and its velocity is

$$W = \frac{dX}{dt}. \tag{1.4.19}$$

If we denote as $u_i^{\mathrm{R}} = u_i(X+0,t)$ and $u_i^{\mathrm{L}} = u_i(X-0,t)$ the values of the functions u_i ahead of and behind the discontinuity, respectively, then for the fixed time instant from Eq. (1.4.18) we can easily obtain

$$\frac{dX}{dt}(U_i^{\mathrm{R}} - U_i^{\mathrm{L}}) + \int_{x_1}^{X} \frac{\partial U_i}{\partial t}dx + \int_{X}^{x_2} \frac{\partial U_i}{\partial t}dx + F_i(\mathbf{U}(x_2)) - F_i(\mathbf{U}(x_1)) = 0, \tag{1.4.20}$$

where $U_i^{\mathrm{R}} = U_i(\mathbf{u}^{\mathrm{R}})$, $U_i^{\mathrm{L}} = U_i(\mathbf{u}^{\mathrm{L}})$, and x_1 and x_2 are constant.

If we let $x_1 \to X+0$ and $x_2 \to X-0$, the integrals in Eq. (1.4.20) vanish and we obtain

$$W\{U_i\} = \{F_i\}. \tag{1.4.21}$$

Here by definition

$$\{f\} = f^{\mathrm{R}} - f^{\mathrm{L}}.$$

Note that $W = 0$ in the coordinate system attached to the shock and, hence, $\{F_i\} = 0$. This means that the flux vector remains constant in this frame. Relations (1.4.21), by analogy with gas dynamics, are called the Hugoniot relations.

Uniqueness of generalized solutions

Once we know that generalized solutions of hyperbolic systems can be discontinuous, the question arises whether solutions of this type are unique. It is well known (see, e.g., Rozhdestvenskii and Yanenko 1983) that satisfaction of the conservation equations and initial conditions is not sufficient to determine a unique solution of the hyperbolic system. This can be seen from the simple model equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0 \tag{1.4.22}$$

with the initial condition

$$u(x,0) = u_0(x) = \begin{cases} u^{\mathrm{L}} & \text{for } x < 0, \\ u^{\mathrm{R}} & \text{for } x > 0. \end{cases} \tag{1.4.23}$$

Since we are interested in the generalized solution, we seek it in the class of piecewise-continuous functions satisfying the integral equation

$$\oint_{\partial V} u\,dx - \tfrac{1}{2}u^2\,dt = 0 \tag{1.4.24}$$

and the initial condition (1.4.23).

The straightforward solution involving a single shock is

$$u_1(x,t) = \begin{cases} u^{\mathrm{L}} & \text{for } x < Wt, \\ u^{\mathrm{R}} & \text{for } x > Wt \end{cases} \tag{1.4.25}$$

with the shock speed $W = \frac{1}{2}(u^{\mathrm{L}} + u^{\mathrm{R}})$.

Let $u^{\mathrm{L}} < u^{\mathrm{R}}$. Then we can construct another solution to the problem (1.4.22)–(1.4.23) in the form

$$u_2(x,t) = \begin{cases} u^{\mathrm{L}} & \text{for} \quad x \le u^{\mathrm{L}}t, \\ \dfrac{x}{t} & \text{for } u^{\mathrm{L}}t \le x \le u^{\mathrm{R}}t, \\ u^{\mathrm{R}} & \text{for} \quad x \ge u^{\mathrm{R}}t. \end{cases} \tag{1.4.26}$$

Thus, we encountered the fact of the solution nonuniqueness, whereas one would expect a unique solution of the Cauchy problem in the class of discontinuous functions.

To determine the unique solution, the following assumptions (see Rozhdestvenskii and Yanenko 1983) can be adopted:

- any classical solution, if exists, is also a solution in the generalized sense;
- limits of classical solutions are also solutions of the integral conservation laws in the class of discontinuous functions.

While the first assumption is fairly natural, the second one implies continuous dependence of the Cauchy problem solutions on initial data. Thus, the above assumptions are based on the well-posedness of the Cauchy problem. Later on we shall discuss the requirements to generalized solutions in detail. The above two assumptions, however, allow us to make a choice between the solutions u_1 and u_2 of the Cauchy problem (1.4.22)–(1.4.23).

In Figs. 1.3a and 1.3b the behavior of characteristics $x = x_0 + u(x,t)\,t$ for the solutions u_1 and u_2, respectively, are presented. If we smear the initial data (1.4.23) in the vicinity of the shock and use the obtained function $u_\delta(x)$ as a new initial profile (see Fig. 1.4a) that coincides with $u_0(x)$ outside the interval $|x| \le \delta$, the characteristic behavior represented in Fig. 1.4b will tend to that shown in Fig. 1.3b as $\delta \to 0$. Thus, in accordance with our assumptions, only u_2 is a valid solution to the Cauchy problem under consideration. In fact, the first solution is completely artificial, since the shock inherent in it occurs at the intersection points of characteristics that originate at infinity rather than on the axis $t = 0$ of the initial data. If on the contrary $u_{\mathrm{L}} > u_{\mathrm{R}}$, then the shock will be characteristically consistent with the initial data and the solution u_1 will be valid. The principles formulated by Rozhdestvenskii and Yanenko (1983) are of great importance in view of the shock-capturing numerical approach, which is frequently used to obtain discontinuous solutions of hyperbolic systems. The discontinuities in this approach are represented by sharp gradients of appropriate functions on the computational mesh. We can hope that the solutions obtained

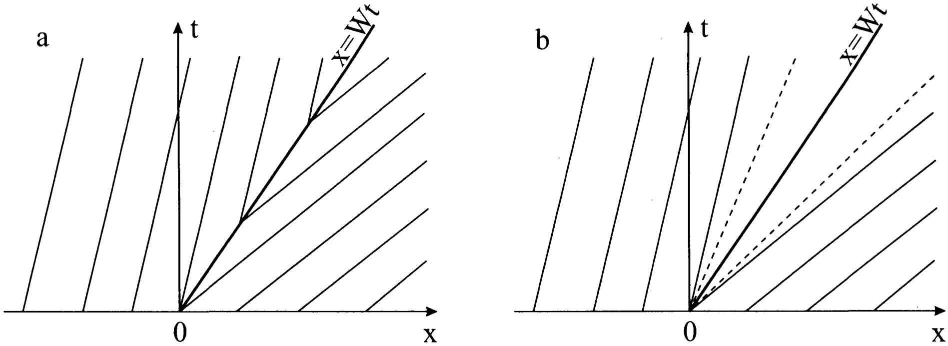

Figure 1.3 Characteristics corresponding to the generalized solutions u_1 (a) and u_2 (b).

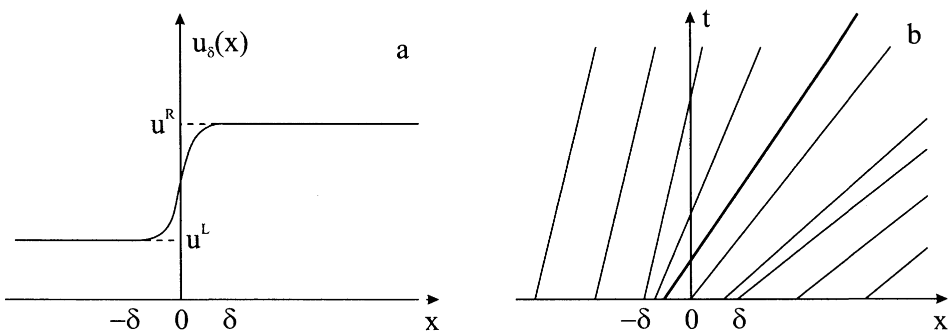

Figure 1.4 A smeared initial profile (a) and corresponding characteristics (b).

by this approach will tend to truly discontinuous solutions of the hyperbolic system as we refine the mesh. On the other hand, the zones of high gradients occurring in these solutions are obviously governed by numerical viscosity. Thus, we can say that a shock is admissible if it can be obtained by steepening of a corresponding viscous profile.

1.4.3 Small-amplitude shocks.

We shall now consider the discontinuities satisfying the Hugoniot relations (1.4.21) and assume that

$$U^{\mathrm{L,R}} = \lim_{\epsilon\to 0} U(X(t) \mp \epsilon, t).$$

In what follows the superscript L will be omitted. The equations describing continuous smooth solutions have a standard quasilinear form resulting from Eq. (1.4.1),

$$\frac{\partial U_i}{\partial t} + F_{ij}\frac{\partial U_j}{\partial x} = 0, \qquad F_{ij} = \frac{\partial F_i(\mathbf{U})}{\partial U_j}. \tag{1.4.27}$$

For any fixed initial values of U_i^{R}, Eq. (1.4.21) determines a curve in the U_i-space and the value of W on it. This curve passes through the initial point U_i^{R} and is called a shock adiabatic curve, or a Hugoniot curve. Consider small-amplitude discontinuities for which the quantities $\{U_i\}$ are small (Lax 1957). Expanding $\{F_i\}$ in a power series of $\{U_j\}$ and retaining only the first two terms, we obtain

$$\{F_i\} = F_{ij}^{\mathrm{R}}\{U_j\} + \frac{1}{2}\left(\frac{\partial^2 F_i}{\partial U_j \partial U_k}\right)^{\mathrm{R}} \{U_j\}\{U_k\}. \tag{1.4.28}$$

The expansion coefficients are calculated ahead of the shock.

In accordance with (1.4.28) we can rewrite Eq. (1.4.21) in the form

$$(F_{ij}^{*} - W\delta_{ij})\{U_j\} = 0, \tag{1.4.29}$$

where

$$F_{ij}^{*} = F_{ij}^{\mathrm{R}} + \frac{1}{2}\left(\frac{\partial F_{ij}}{\partial U_k}\right)^{\mathrm{R}} \{U_k\}.$$

The matrix F_{ij}^{*} is calculated at the point $U_i^{\mathrm{R}} + \frac{1}{2}\{U_i\}$, which is the middle of the chord connecting the initial point U_i^{R} with a point U_i on the Hugoniot curve.

It follows from Eq. (1.4.29) that (i) W coincides with the characteristic velocity at the middle of the above-mentioned chord and (ii) the chord direction coincides with that of the corresponding eigenvector $\mathbf{r}^*$ of the matrix F_{ij} at the chord middlepoint.

Let $\{U_i\} \to 0$. Then from Eq. (1.4.29) we obtain that the velocity W of an infinitely weak shock is equal to the characteristic velocity λ^{R} and the vector $\mathbf{t}^{\mathrm{R}}$ tangent to the Hugoniot curve at the initial point coincides with the right eigenvector $\mathbf{r}^{\mathrm{R}}$ of the matrix F_{ij}^{R}. Assuming that in a small vicinity of the initial point U_i^{R} the characteristic velocity λ up to small correction terms is a linear function of U_i, from (ii) we obtain that

$$W = \lambda^* = \frac{1}{2}(\lambda^{\mathrm{R}} + \lambda) + O(\{\lambda\}^2). \tag{1.4.30}$$

Let us show that the curvature of the shock curve at the initial point coincides with the curvature of the integral curve of the Riemann wave passing through this point. For the unit vectors $\mathbf{t}$ and $\mathbf{r}$ tangent to the Hugoniot curve and to the integral curve of the Riemann wave, respectively, the following formulas are valid:

$$\mathbf{t}\left(\frac{l}{2}\right) = \mathbf{t}^{\mathrm{R}} + \left(\frac{\partial \mathbf{t}}{\partial l}\right)^{\mathrm{R}} \frac{l}{2} + O(l^2), \tag{1.4.31}$$

$$\mathbf{r}\left(\frac{l}{2}\right) = \mathbf{r}^{\mathrm{R}} + \left(\frac{\partial \mathbf{r}}{\partial l}\right)^{\mathrm{R}} \frac{l}{2} + O(l^2), \tag{1.4.32}$$

where l is the distance from the initial point U_i^{R} to U_i along the corresponding curve. The derivatives $(\partial\mathbf{t}/\partial l)^{\mathrm{R}}$ and $(\partial\mathbf{r}/\partial l)^{\mathrm{R}}$ are the curvatures of the corresponding curves at the initial point. The tangent to the Hugoniot curve at the point $l/2$ with an accuracy to $O(l^2)$ is directed along the chord of the arc of length l. According to (ii), this direction to the same accuracy is defined by the eigenvector $\mathbf{r}^*$ of the matrix $F_{ij}^{*} = F_{ij}(U_k^{\mathrm{R}} + \frac{1}{2}\{U_k\})$, that is, by the matrix F_{ij} evaluated at the middle of the chord.

As the integral curve of the Riemann wave and the Hugoniot curve are tangent to each other at the initial point, the points corresponding to $l/2$ on both curves and the middle of the chord of the Hugoniot curve with the arc length equal to l are separated by the distance of the order of $O(l^2)$. Thus, we can conclude that the left-hand side of (1.4.32) also coincides with $\mathbf{r}^*$ to the same order of accuracy.

Since the left-hand sides of Eqs. (1.4.31)–(1.4.32) and the first terms on their right-hand sides coincide, the curvatures of the Hugoniot curve and the Riemann wave integral curve are also the same at the initial point, that is,

$$\left(\frac{\partial \mathbf{t}}{\partial l}\right)^{\mathrm{R}} = \left(\frac{\partial \mathbf{r}}{\partial l}\right)^{\mathrm{R}}$$

As apparent from (1.4.30), the variation of W and λ occurs in the same direction both along the shock curve and along the corresponding Riemann wave. In particular, the segment of the Hugoniot curve with growing W and the segment of the Riemann wave with decreasing λ can be combined at the initial point into one curve with a continuous tangent and curvature. This will be used later for the construction of solutions of certain self-similar problems.

1.4.4 Evolutionary conditions for shocks. In the general case, discontinuities of solutions are surfaces on which the conditions are imposed that relate the quantities on both sides of the discontinuities. These conditions usually involve the discontinuity velocity W. For hyperbolic systems in the conservation-law form, these relations have the form (1.4.21). Note that $\mathbf{U} = [U_i]$ and $\mathbf{F} = [F_i]$ are the conservative variables and the fluxes of them through a unit area of the discontinuity surface.

The evolutionary conditions are necessary conditions for resolvability of the problem of the discontinuity interaction with small disturbances depending on the x-coordinate normal to the discontinuity surface. Consider small disturbances $\delta U_i^{\mathrm{L,R}}$ propagating through the states $U_i^{\mathrm{L,R}}$ behind and ahead of the discontinuity. Linearizing Eq. (1.4.21), we obtain n relations for δU_i^{L}, δU_i^{R}, and the disturbance δW of the shock velocity. For hyperbolic equations, linear one-dimensional disturbances can be represented as a superposition of n waves, each one being a travelling wave propagating at the characteristic velocity $\lambda_i^{\mathrm{L,R}}$. This allows us to subdivide all these waves into incoming and outgoing ones, depending on the sign of the difference $\lambda_i^{\mathrm{L,R}} - W$. Incoming waves are fully determined by the initial conditions, while outgoing ones must be determined from the linearized boundary conditions on the shock.

Each of the linear waves is described by a single quantity w_i called the amplitude, see Eq. (1.4.12). The disturbances of all quantities can be expressed in terms of these amplitudes. It is obvious that w_i and δU_i are related by a linear invertible transformation.

Performing the same transformation in the linearized relations on the discontinuity, we obtain n linear equations relating $2n + 1$ quantities $w_i^{\mathrm{L}}, w_i^{\mathrm{R}}$, and δW with the coefficients depending on W, U_i^{L}, and U_i^{R}. According to the above considerations, only δW and the amplitudes w_i^{L} and w_i^{R}, which correspond to outgoing waves, are to be determined from this system.

Let s^{R} and s^{L} be the numbers of rightward and leftward outgoing waves, respectively. The number of quantities to be determined from the linearized system of the Hugoniot relations is thus equal to $s^{\mathrm{R}} + s^{\mathrm{L}} + 1$. If this number is equal to the number of equations, that is,

$$s^{\mathrm{R}} + s^{\mathrm{L}} + 1 = n, \tag{1.4.33}$$

then, in the general case of a nonzero determinant of the coefficient matrix, the problem is uniquely resolvable. This means that small incoming disturbances generate small outgoing disturbances and small δW. Equality (1.4.33) is called the Lax condition (Lax 1957).

If (1.4.33) is satisfied, that is, if the number of weak outgoing disturbances of different types is equal but one to the number of boundary conditions at the discontinuity, the discontinuity is called *evolutionary* (Landau and Lifshitz 1987). Otherwise, it is nonevolutionary.

If

$$s^{\mathrm{R}} + s^{\mathrm{L}} + 1 > n, \tag{1.4.34}$$

that is, if the number of unknown quantities to be determined is greater than the number of boundary conditions, these quantities cannot be found uniquely and depend on one or more arbitrary functions of time. This implies that such discontinuities do not exist or the conditions on them are underdetermined and there are physical reasons to impose additional, independent of (1.4.21), boundary conditions that make the discontinuity evolutionary (see Chapter 7).

If

$$s^{\mathrm{R}} + s^{\mathrm{L}} + 1 < n, \tag{1.4.35}$$

then the linearized boundary conditions cannot be satisfied in the general case by means of the quantities to be determined. Thus, the problem of the discontinuity interaction with small disturbances has no solution in the linear approximation. Since we expect a well-posed physical problem to have a solution, this means that finite (not small) deviations from the initial state must occur. Previous studies of various physical problems show that interaction of nonevolutionary discontinuities with small disturbances results in their disintegration into two or more evolutionary discontinuities (see Chapter 7).

The evolutionary condition (1.4.33) can be rewritten in the form of inequalities relating the shock velocity W and the velocities $\lambda_i^{\mathrm{L,R}}$ of small disturbances. Let us enumerate the characteristic velocities on both sides of the discontinuity as follows:

$$\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n. \tag{1.4.36}$$

Condition (1.4.33) can then be rewritten in the form

$$\lambda_k^{\mathrm{R}} \leq W \leq \lambda_{k+1}^{\mathrm{R}}, \qquad \lambda_{k-1}^{\mathrm{L}} \leq W \leq \lambda_k^{\mathrm{L}}, \tag{1.4.37}$$

where $k = 1, \ldots, n$. To avoid misinterpreting, we must put here $\lambda_0^{\mathrm{L}} = -\infty$ and $\lambda_{n+1}^{\mathrm{R}} = \infty$.

Inequalities (1.4.37) allow us to divide all evolutionary shocks into n types depending on the value of k. The shock satisfying the relation (1.4.37) is called k-shock. The above relations can be represented as an evolutionary diagram (Akhiezer, Lyubarskii and Polovin 1958). The values of W and λ_k^{R} on the horizontal axis of this diagram are represented in the real scale, whereas the values of W and λ_k^{L} on the vertical axis are arbitrarily scaled with retaining the inequalities between the quantities. The straight lines parallel to the axes and passing through the points λ_k^{L} and λ_k^{R} divide the plane into several rectangles (see Fig. 1.5). If the point (W, W) lies in one of the outlined rectangles, the evolutionary inequalities (1.4.37) are satisfied.

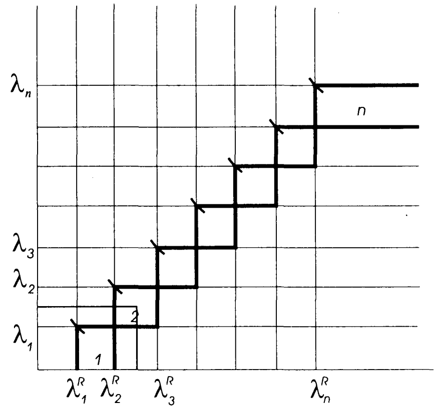

Figure 1.5 Evolutionary diagram.

Let us check whether small-amplitude shocks are evolutionary. According to Section 1.4.3, the k-shock velocity in this case is

$$W = \frac{1}{2}(\lambda_k^{\mathrm{L}} + \lambda_k^{\mathrm{R}}), \tag{1.4.38}$$

under the assumption that the difference $\lambda_k^{\mathrm{L}} - \lambda_k^{\mathrm{R}}$ is small for a chosen k. The differences $\lambda_j - W$ for $j \neq k$ are not considered small and therefore cannot change the sign when crossing the discontinuity. According to (1.4.38), the inequality $\lambda_k^+ > W$ holds for $W > \lambda_k^-$ and the characteristics of the same (kth) family approach the discontinuity from both its sides. The remaining $n - 1$ characteristics arrive at the discontinuity from one side and leave it from another. Thus, there are $n - 1$ outgoing characteristics and such discontinuity is evolutionary.

If $W < \lambda_k^{\mathrm{R}}$, then we have $W > \lambda_k^{\mathrm{L}}$ and the characteristics of the kth type leave the discontinuity on both its sides. We have $n+1$ outgoing characteristics and this discontinuity is nonevolutionary. In Fig. 1.5 weak shock waves are shown as the segments of the curves passing through the points $(\lambda_k^{\mathrm{R}}, \lambda_k^{\mathrm{L}})$, $k = 1, \ldots, n$.

The shocks whose velocity coincides with one of the characteristic velocities can also be evolutionary. Evolutionarity of these limiting shocks (we shall call them Jouget discontinuities, or Jouget shocks), however, must be checked separately in each individual case. For example, weak shocks for which

$$\lambda_k^{\mathrm{R}} = W = \lambda_k^{\mathrm{L}}$$

are evolutionary. This a reason for using the equality signs in the relations (1.4.37).

1.4.5 Entropy behavior on discontinuities. The concept of entropy plays an important role in continuum mechanics. Godunov (1961, 1978) introduced an important class of hyperbolic partial differential equations expressing the conservation laws for which the notion of entropy is defined. It is assumed that the conservative system (1.4.1) results in one additional conservation equation for entropy in the form

$$\frac{\partial S(\mathbf{U})}{\partial t} + \frac{\partial F(\mathbf{U})}{\partial x} = 0. \tag{1.4.39}$$

It can be obtained from (1.4.1) if we multiply the latter by factors q_i and perform summation over i. Choosing q_i as independent variables and introducing the functions

$$T = S - q_i U_i, \qquad H = F - q_i F_i, \tag{1.4.40}$$

we can rewrite Eqs. (1.4.1) and (1.4.39) in the form

$$\frac{\partial}{\partial t}\left(\frac{\partial T}{\partial q_i}\right) + \frac{\partial}{\partial x}\left(\frac{\partial H}{\partial q_i}\right) = 0, \tag{1.4.41}$$

$$\frac{\partial}{\partial t}\left(T - q_i\frac{\partial T}{\partial q_i}\right) - \frac{\partial}{\partial x}\left(H - q_i\frac{\partial H}{\partial q_i}\right) = 0. \tag{1.4.42}$$

Equations (1.4.41)–(1.4.42) represent a canonical form of the Godunov system. It is apparent that Eq. (1.4.39) can result from Eq. (1.4.1) only if

$$dS = q_i dU_i, \qquad dF = q_i dF_i.$$

We see that in the one-dimensional case only two unknown functions T and S occur in it. The function $T(\mathbf{q})$ is a convex function if the function $S(\mathbf{q})$ is convex, since $-S$ and T are related via the Legendre transform

$$S = T - q_i\frac{\partial T}{\partial q_i}, \qquad F = H - q_i\frac{\partial H}{\partial q_i}.$$

In mechanics of continuous media S is interpreted as entropy and F as its flux.

Let us introduce the entropy production at the discontinuity as the difference between the entropy inflow and outflow

$$\{P(q_i, W)\} = \{WS - F\} = W\{T - q_i T_i\} - \{H - q_i H_i\}, \tag{1.4.43}$$

where

$$T_i = \frac{\partial T}{\partial q_i}, \qquad H_i = \frac{\partial H}{\partial q_i}. \tag{1.4.44}$$

According to the second law of thermodynamics, $\{P\}$ must be nonnegative.

Let us estimate the variation of $\{P\}$ along the Hugoniot curve. This curve for a given state q_i^- ahead of the discontinuity is described by the equation

$$\{H_i\} - W\{T_i\} = 0 \tag{1.4.45}$$

corresponding to the conservation law (1.4.41).

Differentiating Eqs. (1.4.43) and (1.4.45) under assumption $q_i^{\mathrm{R}} = \mathrm{const}$ and eliminating dq_i, we obtain

$$d\{P\} = [T - T^{\mathrm{R}} - T_i^{\mathrm{R}}(q_i - q_i^{\mathrm{R}})]\, dW. \tag{1.4.46}$$

If the function $T(\mathbf{q})$ is convex, the sign of the right-hand side of Eq. (1.4.46) coincides with the sign of dW. Discontinuities with $W \geq \lambda^{\mathrm{R}}$, which correspond to the Hugoniot curve points adjacent to the initial point and belonging to the segment with nondecreasing W, do not in this case contradict the second law of thermodynamics.

1.5 Disintegration of a small arbitrary discontinuity

Let us consider the Riemann problem describing the disintegration of an arbitrary initial discontinuity, or simply the Riemann problem, for brevity. This is the Cauchy problem for the system

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{0}$$

with special initial conditions in the form (Riemann 1860)

$$\mathbf{U}(x, 0) = \begin{cases} \mathbf{U}^{\mathrm{R}} & \text{for } x > 0, \\ \mathbf{U}^{\mathrm{L}} & \text{for } x < 0. \end{cases} \tag{1.5.1}$$

Let us find the form of the solution of this problem for $t > 0$ and for small differences $U_i^{\mathrm{R}} - U_i^{\mathrm{L}}$, where U_i are the components of the vector of conservative variables $\mathbf{U}$. The Riemann problem is self-similar and its solution must depend on x/t and consist of the Riemann waves, regions of constant U_i, and discontinuities.

Examine, first, the solution of a linearized problem assuming that all characteristic velocities λ_i are different. Since we have a discontinuity at $t = 0$ and $x = 0$, the solution consists of n characteristic waves (1.4.12). Each of them is a half-infinite step profile and has a jump at its right end. The discontinuities propagate at their own velocities λ_i. The variation of parameters in the wave is proportional to the right eigenvector $\mathbf{r}^i$ of the matrix

$$F_{lm} = \frac{\partial F_l}{\partial U_m},$$

which is assumed constant in the linear approximation. The proportionality factors w_i will be called wave amplitudes, provided that after proper normalization the eigenvectors have unit lengths. The problem of determining the wave amplitudes reduces to decomposition of the vector $\mathbf{U}^{\mathrm{L}} - \mathbf{U}^{\mathrm{R}}$ in terms of the eigenvectors $\mathbf{r}^i$,

$$\mathbf{U}^{\mathrm{L}} - \mathbf{U}^{\mathrm{R}} = \sum_i w_i \mathbf{r}^i.$$

Since eigenvectors corresponding to different eigenvalues are linearly independent, the problem is resolvable. The amplitudes w_i can be considered as a coordinate system in the vicinity of the initial point. As follows from the above discussion, the Jacobian of the coordinate transformation matrix is not equal to zero. For $t = \text{const} > 0$, the motion from the right to the left along the x-axis corresponds in the U_i-space to a sequence of jumps, distributed in the order of decreasing λ_i and located along the coordinate lines w_i. They form a broken line connecting the points U_i^{R} and U_i^{L}

If nonlinearity must be taken into account, we still seek the solution of the Riemann problem as a sequence of n waves. The rightmost wave corresponds to the highest characteristic velocity λ_n. Depending on the direction of the quantity variation in the wave, it can be either an expanding Riemann wave with λ_n decreasing from the forward front to the backward one, or an evolutionary shock with $\lambda_n^{\mathrm{R}} < W < \lambda_n^{\mathrm{L}}$. The latter inequality excludes coexistence of the n-shock and the n-Riemann wave. The state behind the nth wave is the point in the U_i-space on the curve composed of the part of the Riemann wave integral curve with decreasing λ_n and of the evolutionary segment of the Hugoniot curve along which the shock velocity grows in the direction from the initial point. As shown earlier, this is the curve with the continuous tangent and curvature at the initial point.

The next change occurs in the wave of the $(n-1)$th type, which can also be either a Riemann or a shock wave. If n- and $(n-1)$-shocks are weak, their velocities are close to the corresponding characteristic velocities, and these shocks turn out to be separated on the x-axis by the region of parameters independent of x. The state behind the $(n-1)$-wave belongs to the curve in the U_i-space composed of the evolutionary part of the Hugoniot curve and the segment of the Riemann wave integral curve corresponding to an expanding wave.

Proceeding with the construction of the solution, we obtain a broken line whose ith segment is the segment of a curve corresponding to the ith wave. The lengths and directions of the segments must be chosen in a way such that the broken line connects the points U_i^{R} and U_i^{L}. The lengths of the segments together with the signs determining their directions can be considered as new coordinates in the vicinity of the point U_i^{R} in the U_i-space.

In an infinitely small vicinity of the initial point this coordinate transformation reduces to that considered above when constructing the solution of the linearized problem (the segments of the curve in this case are replaced by the elements of their tangents and the variation in the eigenvector directions is disregarded). Thus, the Jacobian of the transformation from U_i to the new coordinate system is not equal to zero at the initial point. By continuity, this means that it is not equal to zero in a vicinity of this point. If U_i^{L} belongs to this vicinity, the Riemann problem can be uniquely resolved.

It is worth noting that the described classical behavior of the solution to the Riemann problem can be violated for non-small $\mathbf{U}^{\mathrm{L}} - \mathbf{U}^{\mathrm{R}}$. There are a few reasons for this violation. One of them is in the nonuniqueness of the transformation from U_i to the variables characterizing the wave amplitudes. Another one is in the appearance of new types of discontinuities with additional relations to be satisfied on them. These cases will be considered in detail in Chapter 7.

Chapter 2
Numerical Solution of Quasilinear Hyperbolic Systems

In this chapter we describe basic approaches to constructing shock-capturing and shock-fitting methods for solving multidimensional quasilinear hyperbolic systems of general form. Among numerical methods we mainly select those that are based on the exact or approximate solution of the corresponding one-dimensional Riemann problem of disintegration of an arbitrary discontinuity or can be interpreted as based on this solution. Such methods are called Godunov methods. They proved to be extremely fruitful in numerous applications. This is due to the fact that the Godunov-type methods are based on the fundamental properties of hyperbolic systems.

The numerical algorithms described below can adequately predict the propagation of discontinuities, which are common for quasilinear hyperbolic systems, and simulate monotone profiles of grid variables in the vicinity of discontinuities.

A method for solving the Riemann problem will be referred to as a Riemann problem solver or simply a solver. We present numerical schemes using both the exact solver and several approximate Riemann problem solvers. The approximate Riemann problem solvers include the Courant–Isaacson–Rees (CIR), Roe and Osher solvers. In the Osher solver the solution is constructed of the Riemann waves only. The CIR and Roe solvers are based on the solution of the Riemann problem for a linearized hyperbolic system of equations. In this case the solution contains only travelling discontinuities, since in the linearized problem there is no difference between a Riemann wave and a travelling discontinuity, see Chapter 1. Indeed, all Riemann waves are represented in this problem by step-functions dividing the regions of constant parameters. The solvers to be described permit one to construct finite-difference and finite-volume schemes for both conservative and nonconservative hyperbolic systems.

We describe also some specific issues of reconstruction of discrete mesh functions, the generalized Riemann problem, additional monotonization procedures, algorithms for selecting physically admissible solutions, and others.

The basic methods presented in this chapter are written for the general hyperbolic system. In Chapters 3–6 they will be applied to the construction of specific numerical algorithms and Riemann problem solvers for gas dynamic equations, shallow water equations, and equations of magnetohydrodynamics and solid dynamics.

2.1 Introduction

The construction of new numerical methods and modification of known methods in order to improve their efficiency have always been topical problems of computational sciences. This is connected both with practical demand to obtain numerical solutions of new complicated problems and the logic of development of numerical methods as a theoretical branch of mathematical sciences.

It is well known that solutions of various problems of mathematical physics described by hyperbolic systems can be smooth in one subdomain and discontinuous in another (Rozhdestvenskii and Yanenko 1983; Petrovskii 1991; Godlewski and Raviart 1996), see Chapter 1. Note that discontinuous solutions can arise even from smooth initial data. Such properties of solutions result in contradictory requirements on algorithms of numerical calculations. The algorithms must preserve the monotonicity of the unknown functions in subdomains where these functions have large gradients and simultaneously ensure high order of accuracy in subdomains where the solution varies smoothly. Godunov's theorem (1959) states that within the framework of linear finite-difference schemes, these two requirements cannot be met simultaneously.

To overcome this difficulty, *shock-fitting* finite-difference methods can be applied, which are based on a direct fitting of discontinuities in the solution. This fitting is produced by appropriate generation of a discrete mesh associated with discontinuities. In particular, the method of characteristics can be used here, see Zhukov (1960) and Richardson (1964). As far as the shock-fitting methods are concerned, we can subdivide them into several groups. One of them is represented by genuinely shock-fitting methods. They are applied if the internal structure of the solution, as well as the number and type of each discontinuity are known in advance. The location and velocity of the discontinuities are to be determined. In this case, one makes an initial guess about the location of a discontinuity and organizes a numerical process so that in the calculation of the derivatives, using finite differences crossing the discontinuity is not allowed. This implies that the numerical grid must be adjusted to the discontinuity surface. This can be done if we always have grid points on this surface. Alternatively, in the finite volume methods computational cell surfaces must coincide with discontinuities. Note that the initial conditions may not satisfy the discontinuity relations; nevertheless, in this case the discontinuities will move in order to finally adjust themselves to these relations. The steady-state solution is obtained if all discontinuities have zero velocity.

It is obvious that approximation of derivatives in the vicinity of a discontinuity must only involve one-sided differences. This may require invoking the characteristic properties of the hyperbolic system to choose the correct direction of the wave propagation. The relations on discontinuities occurring in solutions of hyperbolic equations are satisfied exactly in shock-fitting methods. Note also that, since we perform numerical approximation of derivatives only in smooth regions, requirements on the choice of a particular numerical scheme are not so strict as in the case of uniform methods, known also as *shock-capturing* methods. These methods smear all discontinuities over a length scale determined by the numerical dissipation of the scheme and transform the discontinuities into narrow domains with large gradients. The widths of these domains are smaller for higher-order numerical schemes. On the other hand, spurious oscillations, inevitable in this case, manifest themselves mainly in the vicinity of discontinuities and must be damped by an artificial viscosity. For example, either a linear or a quadratic viscosity (von Neumann and Richtmyer 1950) can be introduced; for details see Richtmyer and Morton (1967), Roache (1976), and Wilkins (1980). It is worth mentioning that the use of the artificial viscosity may essentially change the solution (Latter 1955), and the numerical results must be thoroughly verified. In smooth regions nonmonotone low-viscosity central numerical schemes can also be applied (Lax and Wendroff 1960, 1964; MacCormack 1969). Some applications of genuinely shock-fitting methods will be described in Chapter 3, which deals with equations of gas dynamics.

Theoretically, shock-fitting methods fit all discontinuities, although this seems to be possible only in the one-dimensional case. As far as the two- and three-dimensional cases are concerned, the fitting of all discontinuities can encounter substantial difficulties. In these cases, one can fit only a few main surfaces of discontinuities. The numerical modelling in domains between these surfaces can be carried out by a uniform (shock-capturing) finite-difference or finite-volume scheme. Such an approach with partial fitting of discontinuities is widely used, see Moretti (1963), Moretti and Abbett (1966), Moretti and Bleich (1967), Richtmyer and Morton (1967), Lyubimov and Rusanov (1970), and Roache (1976).

Another group of shock-fitting methods will be referred to as *floating shock-fitting* methods. These methods are designed to fit all discontinuities that originate with time. This requires the development of algorithms for their detection and further tracking as boundaries of smooth subregions of numerical calculation. Algorithms of this sort are becoming more and more complicated if the number of discontinuities to be fitted increases. In Sections 2.9 and 3.5 we outline this approach and give appropriate references.

It is clear that in order to avoid spurious oscillations near discontinuities, one should add viscosity in their vicinity. On the other hand, a higher-order approximation is preferable in smooth regions. Another approach to the numerical investigation of a solution to a hyperbolic system of equations with different properties in different subdomains is the use of *hybrid* shock-capturing schemes, or schemes of varying order of accuracy. The hybridity means that the numerical scheme can locally change its properties, for example, the order of accuracy. In particular, the hybridity permits one to carry out shock-capturing calculations within the framework of the second or higher order of accuracy of the scheme in subdomains with a smooth solution and within the framework of a first-order monotone scheme in subdomains where the solution has large gradients. This approach permits one to combine the positive properties of different methods in a shock-capturing algorithm.

One can simultaneously apply shock-fitting and shock-capturing approaches. A combination of a discontinuity-fitting technique with shock-capturing schemes in domains between the discontinuity surfaces can also be quite useful. Also it seems favorable to use shock-capturing numerical schemes based on moving (dynamic) adaptive meshes (McRae and Lafli 1999; Zegeling 1999; Ivanenko 1999; Azarenok and Ivanenko 1999, 2000).

The basic methods of the fitting technique were created 30 to 40 years ago and are topical to the present day. Today, new fitting methods appear very rarely. In contrast, the shock-capturing numerical schemes are under constant development.

Primarily, shock-capturing numerical methods of a fixed order of accuracy were devised. First-order methods were developed by Courant, Isaacson, and Rees (1952), Lax and Friedrichs (Lax 1954), and Godunov (1959). Subsequently, methods of the second order were suggested by Lax and Wendroff (1960, 1964), MacCormack (1969), and Kutler, Lomax, and Warming (1973). The methods of third order of accuracy were developed by Rusanov (1968, 1970), Burstein and Mirin (1970), Abarbanel and Zwas (1971), and Kutler, Lomax, and Warming (1973). Schemes of the fourth order of accuracy were created by Abarbanel and Zwas (1971), Abarbanel and Gottlieb (1973), and Abarbanel, Gottlieb, and

Turkel (1975). In parallel with the practical application of the above numerical methods, comparative reviews of different methods were published, see Emery (1968), Taylor, Ndefo, and Masson (1972), and Anderson (1974). Note also the reviews of reviews published by Srinivas, Gururaja, and Krishra (1976) and Sod (1978). Those papers and practical requirements stimulated creation of the hybrid schemes, or schemes with the variable order of accuracy.

The hybrid difference schemes were the first stage in the development of the shock-capturing schemes of variable order of accuracy. In the simplest case, a hybrid scheme is a combination of two schemes. This combination has the form $gS_1 + (1 - g)S_2$, where S_1 is a first-order scheme, S_2 a second-order scheme, and g the hybridity coefficient, $0 \leq g \leq 1$. The first hybrid scheme was presented by Fedorenko (1962) for an advection equation. He suggested a hybrid difference scheme and a rule of local switching between two basic schemes S_1 and S_2 on the basis of an analysis of the ratio between the second difference of the solution and its first difference. Gol'din, Kalitkina, and Shishova (1965) developed several hybrid numerical schemes for linear and nonlinear advection equations with smooth switching between two schemes of the first and second order of accuracy. Their hybridity coefficient g depended on the gradient of the solution. The first hybrid scheme for a system of equations was presented by Harten and Zwas (1972a, 1972b). In particular, Harten and Zwas (1972a) combined the Lax–Friedrichs scheme (Lax 1954) of the first order of accuracy with the second-order scheme by Lax and Wendroff (1960, 1964). van Leer (1973, 1974) presented a special algorithm for the monotonization of the Lax–Wendroff method, see also a generalization of the monotonization by van Leer (1977a, 1977b). A hybridization of the Godunov method (1959) was first presented by Kolgan (1972, 1975). For the hybridization he used piecewise linear functions and was the first to apply a version of the minmod limiter. Kutler, Lomax, and Warming (1972), and Beam and Warming (1976) hybridized a symmetric and a nonsymmetric scheme. In their scheme the hybridity coefficient depended on the Mach number.

Boris and Book (1973, 1975, 1976) developed a hybrid method that permits one to increase the order of accuracy by a special procedure of flux corrected transport (FCT). At the first stage a numerical solution is calculated by a monotone first-order scheme. The second stage must modify the numerical solution and provide the second order of accuracy in time and space. This stage must not generate any new extrema in numerical solution and must not lead to an increase (or decrease) in the maxima (or minima) that already exist. Note that these requirements are equivalent to the condition of boundedness of the total variation of the numerical solution. Thus, the FCT method contains elements of the total variation diminishing (TVD) schemes (Harten 1983). To make the solution satisfy the TVD property, an instrument of piecewise linear (polynomial) function reconstruction was developed. The slopes of the function being reconstructed are limited by special functions called limiters. The limiters depend on finite differences. A detailed analysis of the properties of modern limiters was given by Sweby (1984) and, on a different basis, by Roe (1985), see also Yee (1989), Hirsch (1990), and Toro (1997).

Today the term *hybridity* is rarely used in the context of difference schemes. However, schemes of variable order of accuracy still exist and represent the main instrument of numerical simulation. The hybrid difference schemes have been transformed into modern schemes of variable order of accuracy by eliminating from consideration some formal and semiempirical, although interesting and sophisticated, approaches. In particular, the modern schemes of variable order of accuracy are based mainly on the piecewise-polynomial reconstruction of discrete mesh functions. However the simplest hybrid difference schemes based on taking into account specific features of hyperbolic systems may be useful as a first step in the numerical investigation of the hyperbolic system. Some of such schemes will be described below. Note that these schemes can be used for both conservative and nonconservative forms of equations.

Our consideration is based on a special selection of numerical schemes that allow us to give a clear physical interpretation of the schemes and establish their connection with the solution of the Riemann problem. This class of schemes is called Godunov methods. They are currently under intensive development. This is due to their efficiency in a number of numerical applications. This is accounted for by the fact that the Godunov-type methods are based on the fundamental properties of hyperbolic systems.

It should be noted that the knowledge of the solution to the Riemann problem may be very important by itself. The use of the Riemann problem solution may add to the reliability and accuracy of known numerical methods for hyperbolic systems. Recent examples of this are the works by Monaghan (1997) and Parshikov (1999), who used some elements of the linearized gas dynamic Riemann problem solution to improve the smooth particle hydrodynamics (SPH) method (Monaghan and Gingold 1983; Benz 1988; Monaghan 1989; Stellingwerf and Wingate 1993; Chow and Monaghan 1997; and V. D. Ivanov et al. 1999).

Hyperbolic systems of equations can also be applied in grid generation problems, see Steger and Chaussee (1980), Thompson, Warsi, and Mastin (1985), and Chan (1999). For the marching noniterative generation of orthogonal grids, Semenov (1995c, 1995d, 1996) applied a system which is hyperbolic only in the extended form (Courant and Lax 1949); see also Section 1.2.1. Matsuno (1999) developed a high-order accurate TVD upwind grid-generation method for both two- and three-dimensional grid generation problems.

2.2 Methods based on the exact solution of the Riemann problem

We shall consider the one-dimensional Riemann problem (see, for details, Section 1.4) for a quasilinear hyperbolic conservation law of the form

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} = \mathbf{0}, \quad A = \frac{\partial \mathbf{F}}{\partial \mathbf{U}}. \tag{2.2.1}$$

Here $\mathbf{U} = \mathbf{U}(t,x) = [U_1,\dots,U_n]^{\mathrm{T}}$, $\mathbf{F}(\mathbf{U}) = [F_1,\dots,F_n]^{\mathrm{T}}$, $t \geq 0$, $-\infty < x < \infty$, and $\mathbf{U}(0,x) = \mathbf{U}_0(x)$. The vector of the initial data, $\mathbf{U}_0$, is a vector step function,

$$\mathbf{U}_0(x) = \begin{cases} \mathbf{Q}_+, & x > 0, \\ \mathbf{Q}_-, & x < 0, \end{cases} \tag{2.2.2}$$

where $\mathbf{Q}_+$ and $\mathbf{Q}_-$ are constant vectors.

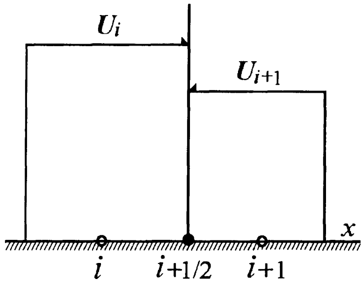

Figure 2.1 Piecewise constant distribution of **U**.

2.2.1 The Godunov method of the first order. In 1959, Godunov suggested a numerical scheme for solution of gas dynamic equations. This scheme is essentially based on either the exact or approximate solution of the Riemann problem (2.2.1)–(2.2.2). For the system (2.2.1) this method can be formulated as follows.

Let us introduce a discrete uniform mesh with size Δx. Let $\mathbf{U}_i$ be the mesh function values, where the integer subscript $i = 1, 2, \ldots$ refers to the center of the ith computational cell, see Fig. 2.1. The half-integer subscript $i \pm 1/2$ refers to the boundary between the cells with the numbers i and $i \pm 1$. Assume that all mesh functions are constant inside each space cell. Let the integer superscript $k = 0, 1, 2, \ldots$ indicate the time layer and Δt be the time increment. Then for the cell boundary $i + 1/2$ and for each time step we can solve the Riemann problem with the following initial data: $\mathbf{U}_i^k = \text{const}$ for $x < x_{i+1/2}$ and $\mathbf{U}_{i+1}^k = \text{const}$ for $x > x_{i+1/2}$. Let $\mathbf{U}_{i+1/2}$ be a solution of this problem. In the same way we can calculate $\mathbf{U}_{i-1/2}$ for the boundary $i - 1/2$. The Godunov finite-volume explicit scheme has the form

$$\frac{\mathbf{U}_i^{k+1} - \mathbf{U}_i^k}{\Delta t} + \frac{\mathbf{F}_{i+1/2} - \mathbf{F}_{i-1/2}}{\Delta x} = \mathbf{0}, \qquad \mathbf{F}_{i\pm 1/2} = \mathbf{F}(\mathbf{U}_{i\pm 1/2}). \tag{2.2.3}$$

This scheme is of the first order of accuracy with respect to the space variable and time.

The spectral analysis (Richtmyer and Morton 1967) of the linearized equations (2.2.3) leads to the stability condition

$$\max_p |C_p| \leq 1, \quad C_p = \lambda_p \frac{\Delta t}{\Delta x}, \tag{2.2.4}$$

where λ_p are the eigenvalues of Jacobian matrix A of (2.2.1). Inequality (2.2.4) is known as the CFL condition (Courant, Friedrichs, and Lewy 1928) and C_p is the CFL number corresponding to λ_p.

Scheme (2.2.3) can easily be extended to a nonuniform space mesh, self-adjusting and moving grids, shock-fitting calculations, two- or three-dimensional cases, etc. Let us consider, for instance, the two-dimensional hyperbolic system

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{U})}{\partial x} + \frac{\partial \mathbf{E}(\mathbf{U})}{\partial y} = \mathbf{0}. \tag{2.2.5}$$

The scheme for this system for a uniform Cartesian space mesh can be written as

$$\frac{\mathbf{U}_{i,j}^{k+1} - \mathbf{U}_{i,j}^{k}}{\Delta t} + \frac{\mathbf{F}_{i+1/2,j} - \mathbf{F}_{i-1/2,j}}{\Delta x} + \frac{\mathbf{E}_{i,j+1/2} - \mathbf{E}_{i,j-1/2}}{\Delta y} = \mathbf{0}, \tag{2.2.6}$$
$$\mathbf{F}_{i\pm1/2,j} = \mathbf{F}(\mathbf{U}_{i\pm1/2,j}), \quad \mathbf{E}_{i,j\pm1/2} = \mathbf{E}(\mathbf{U}_{i,j\pm1/2}),$$

where Δx and Δy are the mesh sizes in the x- and y-direction, respectively. The double integer subscripts (i,j) refer to the centers of two-dimensional space cells and the half-integer subscripts refer to the corresponding boundaries of the cells. The quantities $\mathbf{U}_{i+1/2,j}$, $\mathbf{U}_{i-1/2,j}$, $\mathbf{U}_{i,j+1/2}$, and $\mathbf{U}_{i,j-1/2}$ are solutions of the corresponding Riemann problem. That is, we must solve the one-dimensional Riemann problem for each cell boundary. The scheme thus constructed is a two-dimensional finite-volume Godunov scheme for a uniform mesh.

It is not difficult to generalize the Godunov scheme (2.2.6) to an arbitrary space grid. Let us rewrite Eq. (2.2.5) in integral form as

$$\frac{d}{dt}\left(\iint_G \mathbf{U}\, dG\right) + \oint_S (\mathbf{F}\, dy - \mathbf{E}\, dx) = \mathbf{0}. \tag{2.2.7}$$

Here G is a domain in the two-dimensional (x, y) space, $dG = dx\, dy$ is the area element, and S is the boundary of G.

Let us construct the explicit Godunov finite-volume scheme for equations written in the integral form (2.2.7). We discretize the computational domain by constructing a grid of arbitrary convex polygons having areas G_i, $i = 1, 2, \ldots$, with $m = m(i)$ sides having lengths S_j, $j = 1, \ldots, m(i)$; $\mathbf{S}_j = \mathbf{n}_j S_j \equiv [S_x, S_y]_j^{\mathrm{T}} = [S_{xj}, S_{yj}]^{\mathrm{T}}$, where $\mathbf{n}_j$ is the outward normal to S_j. On each polygon the above integral equations can be approximated as follows:

$$G_i \frac{\mathbf{U}_i^{k+1} - \mathbf{U}_i^k}{\Delta t} + \sum_{j=1}^{m(i)} (\mathbf{F}_j S_{xj}^k + \mathbf{E}_j S_{yj}^k) = \mathbf{0}. \tag{2.2.8}$$

The integer subscript i in Eq. (2.2.8) denotes the values of grid variables calculated at the center of mass of the ith polygon, and the subscript j denotes their values on the middle of jth side of the polygon. The quantities $\mathbf{F}_j$ and $\mathbf{E}_j$ are calculated by solving the corresponding Riemann problem in the direction of the jth outward normal.

The Godunov finite-volume scheme in the case of general moving grids can be written out as

$$\frac{(G\mathbf{U})_i^{k+1} - (G\mathbf{U})_i^k}{\Delta t} + \sum_{j=1}^{m(i)} \left[(\mathbf{F}_j - D_{xj}\mathbf{U}_j) S_{xj}^{k+1/2} + (\mathbf{E}_j - D_{yj}\mathbf{U}_j) S_{yj}^{k+1/2}\right] = \mathbf{0}, \tag{2.2.9}$$

$$\frac{G_i^{k+1} - G_i^k}{\Delta t} - \sum_{j=1}^{m(i)} (\mathbf{D}_j \cdot \mathbf{S}_j^{k+1/2}) = 0. \tag{2.2.10}$$

Here $\mathbf{D}_j = [D_x, D_y]_j^{\mathrm{T}} = [D_{xj}, D_{yj}]^{\mathrm{T}}$ is the velocity at the jth cell side center and $\mathbf{a} \cdot \mathbf{b}$ stands for the scalar product of the vectors $\mathbf{a}$ and $\mathbf{b}$. The quantities $\mathbf{F}_j$, $\mathbf{E}_j$, and $\mathbf{U}_j$ are defined by solving the corresponding Riemann problem and are calculated for the cell side moving with the velocity $\mathbf{D}_j$. The vectors $\tilde{\mathbf{F}} = \mathbf{F} - D_x\mathbf{U}$ and $\tilde{\mathbf{E}} = \mathbf{E} - D_y\mathbf{U}$ are the modified function fluxes along the x- and y-direction, respectively. We see that now the quantities

G_i and $\mathbf{S}_j$ are functions of time. The half-integer subscript $k+1/2$ in Eqs. (2.2.9)–(2.2.10) denotes the values for the time instant $t+\frac{1}{2}\Delta t$. Equation (2.2.10) is a discrete equation that describes the evolution of the computational cell volume G_i. The important property of this approximation is that the uniform-flow solutions $\mathbf{U} = \mathbf{U}_0 = \text{const}$ to Eq. (2.2.5) are also solutions to the discrete equations (2.2.9)–(2.2.10). This condition is of great importance in using arbitrary moving curvilinear coordinate systems. For given approximation of $\mathbf{S}$ in time one can use Eq. (2.2.10) to calculate the areas G_i^{k+1}.

The numerical scheme (2.2.8) is stable on a uniform Cartesian grid if

$$\max(|C_x| + |C_y|) \leq \max|C_x| + \max|C_y| \leq 1, \tag{2.2.11}$$

where C_x and C_y are the CFL numbers that correspond to the x-axis and y-axis and depend on the eigenvalues of the matrices $\partial\mathbf{F}/\partial\mathbf{U}$ and $\partial\mathbf{E}/\partial\mathbf{U}$, respectively.

By analogy we can construct the Godunov method for multidimensional cases.

2.2.2 Exact solution of the Riemann problem.

In this section we describe the main approaches and algorithms for constructing the general exact solution of the Riemann problem. One must first construct two basic elementary solutions. Then the general solution—particularly, in gas dynamics, shallow water equations, etc.—can be represented as a combination of these solutions.

Elementary solution 1: Strong discontinuity

The first elementary solution is a moving discontinuity. For obtaining the discontinuity relations let us integrate of Eq. (2.2.1) over t and x and consider their integral form

$$\oint_L (\mathbf{U}\,dx - \mathbf{F}\,dt) = \mathbf{0}, \tag{2.2.12}$$

where L is the boundary of a region in the (t,x) plane. Let us search for a solution of Eq. (2.2.1) in the form of a travelling wave $f(t,x) = f(\zeta) \equiv f(x-Wt)$, where $W = \text{const}$ is a wave velocity. Consider Eq. (2.2.12) in orthogonal coordinates (ζ,τ) associated with discontinuity, where $\zeta = x - Wt$, and $\tau = Wx + t$. The coordinate ζ is normal and coordinate τ is tangential with respect to the discontinuity. Using transformation

$$x = \frac{\zeta + W\tau}{1+W^2}, \quad t = \frac{-W\zeta + \tau}{1+W^2}$$

we can rewrite Eq. (2.2.12) as

$$\frac{1}{1+W^2}\oint_L [(W\mathbf{U} - \mathbf{F})\,d\tau + (\mathbf{U} + W\mathbf{F})\,d\zeta] = \mathbf{0}. \tag{2.2.13}$$

Let us integrate (2.2.13) over a rectangular region $\tau_0 - \delta\tau \leq \tau \leq \tau_0 + \delta\tau$, $\zeta_0 - \delta\zeta \leq \zeta \leq \zeta_0 + \delta\zeta$, where $\zeta = \zeta_0$ corresponds to the discontinuity. We can find

$$\begin{aligned} &(W\mathbf{U} - \mathbf{F})_1 2\delta\tau - (W\mathbf{U} - \mathbf{F})_2 2\delta\tau = \mathbf{0} \\ \Longrightarrow\quad & W\{\mathbf{U}\} - \{\mathbf{F}\} = \mathbf{0}, \end{aligned} \tag{2.2.14}$$

where $\{q\} \equiv q_1 - q_2$, and indices 1 and 2 denote the variables on the left- and right-hand side of the discontinuity.

Let the piecewise constant initial data $\mathbf{U}_0(x)$ be defined as follows: $\mathbf{U}_0(x) = \mathbf{U}_1 = \text{const}$ for $x < 0$ and $\mathbf{U}_0(x) = \mathbf{U}_2 = \text{const}$ for $x > 0$. Both $\mathbf{U}_1$ and $\mathbf{U}_2$ satisfy Eq. (2.2.1). Suppose that this initial strong discontinuity moves with a velocity of W and is a solution of the Riemann problem. Then, Eq. (2.2.14) relates $\mathbf{U}_1, \mathbf{U}_2$, and W. If these relations are satisfied, then the moving discontinuity is a formal solution of Eq. (2.2.1).

The above solution is self-similar with respect to the variable $\zeta = x - Wt$, where $W = \text{const}$. This solution is also self-similar with respect to the variable $\xi = x/t$. In fact, a moving discontinuity in the (t, x) coordinates is a straight line that satisfies the relation $\xi = W = \text{const}$.

Elementary solution 2: Riemann wave

There also exists a continuous elementary solution of the Riemann problem, which is called a Riemann wave. Let us seek a continuous solution in the form $f(t, x) = f(\xi) \equiv f(x/t)$ and consider the nonconservative form of Eq. (2.2.1),

$$\mathbf{U}_t + A\mathbf{U}_x = \mathbf{0}. \tag{2.2.15}$$

Equation (2.2.15) can also be rewritten in terms of some other variables $\mathbf{u}$ such that $\mathbf{U} = \mathbf{U}(\mathbf{u})$:

$$\mathbf{u}_t + B\mathbf{u}_x = \mathbf{0}, \tag{2.2.16}$$

where $B = M^{-1}AM$, $M = \partial\mathbf{U}/\partial\mathbf{u}$. The subscripts t and x in Eqs. (2.2.15)–(2.2.16) denote the respective partial derivatives. Further we will transform this system of equations into the characteristic form.

Consider the hyperbolic system (2.2.16) and multiply it by a matrix Ω_{L} composed of the left eigenvectors of the matrix B. Then the characteristic form of system (2.2.16) is given by

$$\Omega_{\mathrm{L}}\mathbf{u}_t + \Lambda\Omega_{\mathrm{L}}\mathbf{u}_x = \mathbf{0},$$

or in the expanded form,

$$\sum_{k=1}^{n} \Omega_{\mathrm{L}pk}\frac{\partial u_k}{\partial t} + \lambda_p \sum_{k=1}^{n} \Omega_{\mathrm{L}pk}\frac{\partial u_k}{\partial x} = 0, \qquad p = 1, \dots, n;$$

where $\mathbf{u} = [u_1, \dots, u_n]^{\mathrm{T}} = [u_k]^{\mathrm{T}}$, $k = 1, \dots, n$; $\Omega_{\mathrm{L}} = [\Omega_{\mathrm{L}pk}]$ and $\Lambda = [\lambda_p\delta_{pk}]$. Let us search for an exact continuous solution as a function of $\xi = x/t$. Substituting it in the characteristic form, we obtain

$$(\lambda_p - \xi)\left[\sum_{k=1}^{n} \Omega_{\mathrm{L}pk}\frac{\partial u_k}{\partial \xi}\right] = 0, \qquad p = 1, \dots, n.$$

Thus, if there exists an exact solution, then it must satisfy one of the n systems of equations

$$\lambda_\alpha - \xi = 0, \tag{2.2.17}$$

$$(\lambda_\beta - \xi)\left[\sum_{k=1}^{n} \Omega_{\mathrm{L}\beta k}\frac{\partial u_k}{\partial \xi}\right] = 0 \quad \Longrightarrow \quad \sum_{k=1}^{n} \Omega_{\mathrm{L}\beta k}\frac{\partial u_k}{\partial \xi} = 0 \ \text{ if } \ \lambda_\beta \neq \lambda_\alpha. \tag{2.2.18}$$