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lecture notes in pure and applied mathematics

# partial differential equation methods in control and shape analysis

edited by
Giuseppe Da Prato
Jean-Paul Zolésio

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# Preface

The International Federation for Information Processing (IFIP) working group 7.2 Conference on Control and Shape Optimization was held at Scuola Normale Superiore di Pisa, Italy. The meeting was sponsored by Scuola Normale Superiore di Pisa and CNR Gruppo Nazionale di Analisi Funzionale. The purpose of the workshop was to exchange ideas between the group working on control theory and the group working on shape optimization. It was part of an ongoing collaboration between Scuola Normale Superiore di Pisa and the Centre de Recherche en Mathématiques Appliquées de l'Ecole des Mines de Paris.

Optimization and control theory are recurrent themes in the modeling of real-life systems from many areas: real-time systems, material sciences, lifting profiles, thermal testing, elastic shells, and biodynamics. The Hamilton-Jacobi approach is beginning to play a major role in solving concrete problems where active control is needed, while shape optimization is the tool of choice for passive control problems. The challenge is to bring these two approaches together (e.g., the optimal location of actuators/sensors for tracking improvement, the best shape of a plate for enhancing the stabilizing control). We hope this volume will stimulate further research.

We would like to thank all contributors and Mrs. Caterina D'Elia, at Scuola Normale, for their efforts on behalf of the conference.

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# Shape Control of a Hydrodynamic Wake

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ABSTRACT. This paper propounds a shape variational formulation of a hydrodynamic free interface which appears behind a three dimensional lifting profile. We prove the existence of an optimal wake under Density Perimeter constraints. We derive from this formulation the standard equilibrium condition in the classical case where this interface is a regular surface.

## 1. INTRODUCTION

We consider a "hydrodynamicaly well profiled" body $B$. $B$ has a uniform stationary velocity $U_\infty$. A thin viscous boundary layer is developed around $B$ and in that study we neglect it, in the sense that we consider that the shape $B$ coincide with the shape of the body augmented by its boundary layer, according with the classical boundary layer theory. Then we consider a sliding condition , $U.n = 0$ on $Q = \partial B$ ($V$ being the stationary speed of the fluid). Nevertheless, we cannot completely neglect the vorticity in that flow in view of the modeling of the lifting effect. It is classical in engineering to consider the vorticity of the flow as being supported by a piece of surface $S$ in addition to $Q$. $S$ is called the wake. We assume the flow is governed by Euler's equations in $\Omega \setminus S$, $\Omega$ being the outer domain and $S$ is said "in equilibrium" when the resulting jump of pressure $[\![p]\!]$ across $S$ is zero. The objective of that paper is to solve that free boundary problem whose solution is the couple $(U = \nabla\phi, S)$ with $\phi \in H^1(\Omega \setminus S)$. We develop a new variational formulation on the variables $U$ and $S$. We introduce an energy $J_\varepsilon(S)$ in the form

$$J_\varepsilon(S) = \min_{y\in H^1(\Omega\setminus S)} \int_{\Omega\setminus S} (\frac{\varepsilon}{2}y^2 + \frac{1}{2} \mid \nabla y \mid^2 + i.\nabla y)\, dx$$

and the analysis of the optimality condition for $J_\varepsilon(S)$ makes use of the shape analysis technics. In order to insure the existence of $S$, we introduce a surface tension $\sigma > 0$ via a surface energy for $S$ which is represented by the use of the Density Perimeter which is the adapted perimeter concept for this kind of shape variational problem.

## 2. DEFINITIONS AND MAIN PROPERTIES

$B$ is a bounded domain in $\mathbb{R}^N$ ($N \geq 2$) with boundary $Q$. The fluid will occupy the outer domain. More precisely, we consider a "large" bounded domain $D$ with $\bar{B} \subset D$ and $\partial D$ being lipschitzian. The fluid occupies the domain $\Omega = D \setminus \bar{B}$. The boundary of $\Omega$ is made of two connected components $Q$ and $\partial D$.

The stationary speed field $U$ of the fluid in the domain $\Omega$ is assumed irrotational in $\Omega \setminus S$ where $S$ is a closed subset in $\Omega$ with zero measure and empty interior. Our modeling is assuming that the body $Q$ is "well profiled" in such a way that the support of the $curl(U)$ will be in $S \cup Q$. In this fist paper, we neglect boundary layer effect in the neighborhood of $S \cup Q$. For each closed set $S$ in $\Omega$, we consider the Sobolev space $H^1(\Omega \setminus S)$. The open set $\Omega \setminus S$ is non smooth and $H^1(\Omega \setminus S)$ is defined as

$$H^1(\Omega \setminus S) = \{y \in L^2(\Omega \setminus S),\ \nabla y \in L^2(\Omega \setminus S; \mathbb{R}^N)\}$$

In the case where $S$ is contained in a smooth orientable surface $\Sigma$, the traces of any element $y$ are defined on both sides of $\Sigma$ and may be different functions in $H^{\frac{1}{2}}(\Sigma)$.

In that case, we shall denote by $[\![y]\!]$ the jump of y through the surface $\Sigma$. Of course, $H^1(\Omega \setminus S)$ in not a subspace of $H^1(\Omega)$, but we have $H^1(\Omega) \subset H^1(\Omega \setminus S)$ for any closed set $S$ in $\Omega$. For any $y$ in $H^1(\Omega)$, we have $[\![y]\!] = 0$ on $S$. From irrotationality assumption, we have $U\mid_{\Omega\setminus S}= \nabla\phi$ in $\Omega \setminus S$ for some scalar potential $\phi$ in $H^1(\Omega \setminus S)$.

That $\phi$ defines an element $\phi^o$ of $L^2(\Omega)$ as $S$ has a zero measure. That element $\phi^o$ defines a distribution over $\Omega$, $\phi^o \in \mathcal{D}'(\Omega)$, and we consider its gradient $\nabla\phi^o \in \mathcal{D}'(\Omega)$. In fact, as $\phi^o$ is uniquely associated to $\phi$, the restriction of the distribution $\nabla\phi^o$, element of $\mathcal{D}'(\Omega; I\!\!R^N)$, to the open set $\Omega \setminus S$ is $\nabla\phi$ and $\nabla(\phi^o) = (\nabla\phi)^o + \mu$ where $\mu = \gamma_S^*([\![\phi]\!]\vec{n})$ is a measure, $\mu \in \mathcal{D}^{o\prime}(\Omega; I\!\!R^N)$, supported by $S$. We take $U = (\nabla\phi)^0 = \nabla(\phi^o) - \mu$. In such a situation, we get $curl(U) = curl((\nabla\phi)^0) - curl(\mu)$. The distribution $curl\,\mu$ is supported by $S$ (as was $\mu$), the restriction to $\Omega \setminus S$ of $curl((\nabla\phi)^o)$ is zero (as $curl\nabla = 0$) so that $curl((\nabla\phi)^o) = \mu$ is a distribution of order one supported by $S$. Finally, we get $curl\,U = \gamma_S^*(\vec{n} \wedge \nabla_\Gamma[\![\phi]\!])$ and $divU = \gamma_S^*([\![\frac{\partial\phi}{\partial n}]\!])$

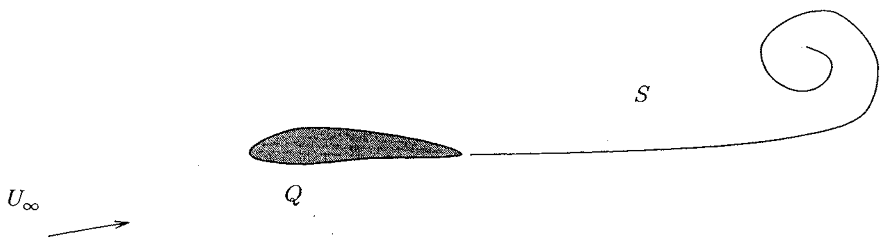


FIGURE 1. Fluid domain

## 3. REDUCTION TO A BOUNDED DOMAIN CONTAINED IN $D$

We introduce the perturbation velocity potential $\varphi$ so that $U = u_\infty(i+(\nabla\varphi)^o)$ in $\Omega$, $U_\infty = u_\infty i$ ; $\phi_M = u_\infty(x_M + \varphi_M)$ with $x_M =< OM, i >_{I\!\!R^3}$ and $\varphi \in H^1(\Omega \setminus S)$

More precisely, we consider a "large" bounded domain $D$ with $\bar{B} \subset D$ and $\partial D$ being lipschitzian.

So, when $D$ is large enough the perturbation speed will be zero out of $D$.

The fluid occupies the domain $\Omega = D \setminus \bar{B}$. The boundary of $\Omega$ is made of two connected components $Q$ and $\partial D$.

## 4. WEAKLY COMPRESSIBLE FLOW

In order to insure the uniformity of the classical Poincaré constant in the non smooth domains $\Omega \setminus S$, we introduce a zero order term in the energy leading to a weakly compressible condition controlled by $\varepsilon$. Given $\varepsilon > 0$, we consider the energy functional

$$E^\varepsilon_{\Omega\setminus S}(y) = \int_{\Omega\setminus S} (\frac{\varepsilon}{2}y^2 + \frac{1}{2} \mid \nabla y \mid^2 + i.\nabla y)\, dx$$

The minimizer $\varphi$ of that functional over $H^1(\Omega \setminus S)$ is the solution of the weak problem

$$\forall y \in H^1(\Omega \setminus S), \int_{\Omega\setminus S} (\varepsilon\varphi\, y + \nabla\varphi.\nabla y + i.\nabla y)\, dx = 0 \tag{4.1}$$

So that, performing by part, we can see that the problem takes the following form

$$\begin{cases} \Delta\varphi = \varepsilon\varphi & \text{in } \Omega \\ \dfrac{\partial\varphi}{\partial n} = -i.n & \text{on } Q \cup S^+ \cup S^- \\ \varphi = 0 & \text{on } \partial D \end{cases}$$

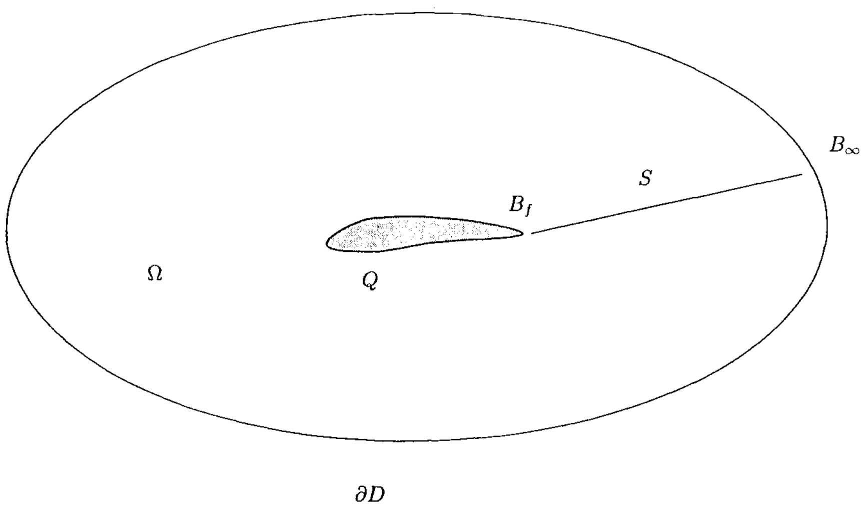


FIGURE 2. Bounded fluid domain

We have the following estimates

**Lemma 1.**

$$\|\nabla\varphi\|_{L^2(\Omega\setminus S;\mathbb{R}^N)} \leq |\,\Omega\,|^{\frac{1}{2}}$$

*Proof.* with $y = \varphi$ in 4.1,

$$\begin{aligned} \int_{\Omega\setminus S} |\,\nabla\varphi\,|^2\,dx &\leq \int_{\Omega\setminus S} |\,i\,|\;|\,\nabla\varphi\,|\,dx \\ &\leq \int_{\Omega\setminus S} |\,\nabla\varphi\,|\,dx \\ &\leq |\,\Omega\,|^{\frac{1}{2}}\,(\int_{\Omega\setminus S} |\,\nabla\varphi\,|^2\,dx)^{\frac{1}{2}} \end{aligned}$$

□

**Lemma 2.**

$$\sqrt{\varepsilon}\,\|\varphi\|_{L^2(\Omega)} \leq |\,\Omega\,|^{\frac{1}{2}}$$

*Proof.* with $y = \varphi$ in (4.1),

$$\begin{aligned}
(\sqrt{\varepsilon}\|\varphi\|_{L^2(\Omega\setminus S)})^2 &= -\|\nabla\varphi\|^2_{L^2(\Omega\setminus S)} - \int_\Omega i.\nabla\varphi\, dx \\
&\leq \int_{\Omega\setminus S} \mid \nabla\varphi \mid \, dx \\
&\leq \mid \Omega \mid^{\frac{1}{2}} (\int_{\Omega\setminus S} \mid \nabla\varphi \mid^2 \, dx)^{\frac{1}{2}} \\
&\leq \mid \Omega \mid
\end{aligned}$$

□

**Lemma 3.**

$$\|\Delta\varphi\|_{L^2(\Omega\setminus S)} \leq \sqrt{\varepsilon} \mid \Omega \mid^{\frac{1}{2}}$$

*Proof.*

$$\begin{aligned}
\|\Delta\varphi\|_{L^2(\Omega\setminus S)} &= \varepsilon \, \|\varphi\|_{L^2(\Omega\setminus S)} \\
&\leq \sqrt{\varepsilon} \mid \Omega \mid^{\frac{1}{2}} \text{ with lemma 2}
\end{aligned}$$

□

In view of that last estimate, we see that $div(U \mid_{\Omega\setminus S}) = u_\infty \, \Delta\varphi$ goes to zero with $\varepsilon$. Then, the flow is almost incompressible.

In the case where $S$ is a smooth surface, we would get, denoting by $\vec{n}$ the normal field on $S$ and performing by part on (4.1).

$$\frac{\partial\varphi_+}{\partial n} = \frac{\partial\varphi_-}{\partial n} = -i.n \text{ on } S$$

For each closed set $S$ in $\Omega$, $\varphi(\Omega \setminus S)$ denoting the solution of problem (4.1), we consider the energy functional, for given $\varepsilon > 0$,

$$J_\varepsilon(S) = \min_{y \in H^1(\Omega\setminus S)} E^\varepsilon_{\Omega\setminus S}(y)$$

**Lemma 4.**

$$J_\varepsilon(S) = -\frac{1}{2}\int_{\Omega\setminus S} (\varepsilon\varphi^2(\Omega \setminus S) + \mid \nabla\varphi(\Omega \setminus S) \mid^2)\, dx$$

*Proof.* with $y = \varphi$ in (4.1),

$$J_\varepsilon(S) = \frac{1}{2}\int_{\Omega\setminus S} < i, \nabla\varphi(\Omega \setminus S) > \, dx = \frac{1}{2} < i, \int_{\Omega\setminus S} \nabla\varphi(\Omega \setminus S) > \, dx >$$

□

**Lemma 5.**

$$0 \geq J_\varepsilon(S) \geq - \mid \Omega \mid$$

*Proof.*

$$J_\varepsilon(S) = -\frac{1}{2}(\varepsilon\|\varphi\|^2_{L^2(\Omega)} + \|\nabla\varphi\|^2_{L^2(\Omega)})$$

with lemma 1 and lemma 2 □

## 5. Deformations of the domains

For any $V \in C^0([0,\tau[; \mathbb{R}^N)$

$$V = 0 \text{ on } Q \cup \partial D$$

we consider the flow mapping $T_t(V) : X \longmapsto x(t, X)$
With $x(t, X)$ solution to the system of ordinary differential equations

$$\begin{cases} \dfrac{d}{dt}x(t,X) = V(t, x(t,X)) \\ x(0,X) = X \end{cases}$$

We know from [3] that $T_t$ is a diffeomorphism from $D \setminus \bar{B}$ onto itself.

## 6. Optimal wake existence

We consider now the extremality of the functional $J_\varepsilon$. The energy associated to $S$ is related to its length. We choose here the density perimeter $P_\gamma(S)$ for a given $\gamma > 0$ which could be related to a surface tension concept, see [1].

$$P_{\gamma,H}(S) \stackrel{def}{=} \sup_{\varepsilon \in (0,\gamma)} [\frac{m(S^\varepsilon)}{2\varepsilon} + H(\varepsilon)]$$

$$S^\varepsilon = \bigcup_{x \in S} B(x, \varepsilon)$$

The main properties of $P_{\gamma,H}$ are

**Proposition 1.**

$$\Omega_n \xrightarrow{H^c} \Omega \Rightarrow P_{\gamma,H}(\partial\Omega) \leq \liminf_{n \to +\infty} P_{\gamma,H}(\partial\Omega_n)$$

**Proposition 2.**

$$\Omega_n \xrightarrow{H^c} \Omega \Rightarrow \Omega_n \xrightarrow{char} \Omega$$

**Proposition 3.**

$$P_{\gamma,H}(\partial\Omega) < \infty \Rightarrow meas(\partial\Omega) = 0$$

$H^c$ is the Hausdorff topology.

$$d_{H^d}(\Omega_1, \Omega_2) = \sup_{x \in \mathbb{R}^N} \mid d_{\Omega_1}(x) - d_{\Omega_2}(x) \mid$$

where $d_{\Omega_1}(x) = \inf_{y \in \Omega_1} \|x - y\|$

$$d_{H^c}(\Omega_1, \Omega_2) = d_{H^d}(\Omega_1^c, \Omega_2^c)$$

$B_f$ (Resp. $B_\infty$) is a closed set in $Q$ (Resp. $\partial D$) with $n-1$ dimensional Hausdorff measure $\mid B_f \mid_{\mathcal{H}^{n-1}} = \mid B_\infty \mid_{\mathcal{H}^{n-1}} = 0$. The admissible family of closed sets $S$ is chosen as

$$\mathcal{S}_0 = \{S = \bar{S},\ meas(S) = 0,\ \bar{S} \supset B_f \cup B_\infty,\ \#(S) = 1\}$$

Where $\bar{S}$ is the closure of $S$ in $\mathbb{R}^N$ and $\#(S)$ is the number of connected components of $S$.

**Proposition 4.** $\forall M > 0$, $\mathcal{S}_0^M = \{S \in \mathcal{S}_0 \mid P_\gamma(S) \leq M\}$ equipped with the Hausdorff metric is a compact metric space.

*Proof.* From [1] we know that given a sequence $S_n$ in $\mathcal{S}_0$ with $P_\gamma(S_n) \leq M$ there exists a subsequence still denoted by $S_n$ such that $S_n \xrightarrow{H} S$ in Hausdorff metric where $S$ is a closed set in $\Omega$. Moreover $\chi_{\Omega\setminus S_n} \longrightarrow \chi_{\Omega\setminus S}$ in $L^2(\Omega)$. So that $meas(S) = 0$. Also, we know ([2]) that $\#$ is lower semi continuous for the Hausdorff topology then $\#S \leq 1$ but as $S_n \supset B_f \cup B_\infty$ we get $S \supset B_f \cup B_\infty$ then $S$ is non empty and then $\#(S) = 1$. Finally $P_\gamma$ is lower semi continuous ([1]), then $P_\gamma(S) \leq M$ □

Given $\sigma > 0$, we consider the optimality problem

$$Min\ \{J_\varepsilon(S) + \sigma\ P_\gamma(S)\ \mid\ S \in \mathcal{S}_0\} \tag{6.1}$$

**Theorem 1.** *For each $\varepsilon > 0$, the problem* (6.1) *has optimal solutions in the family $\mathcal{S}_0$.*

Before showing this theorem, we need the following result

**Lemma 6.** $J_\varepsilon$ is lower semi continuous on $\mathcal{S}_0^M$

*Proof.* Let $S_n \xrightarrow{H} S$, let $\varphi_n = \varphi(\Omega \setminus S_n)$. From Lemma (1) and (2),

$$\begin{aligned} \|(\nabla\varphi_n)^o\|_{L^2(\Omega)} &\leq \ \mid \Omega \mid^{\frac{1}{2}} \\ \|(\varphi_n)^o\|_{L^2(\Omega)} &\leq \ \frac{1}{\sqrt{\varepsilon}} \mid \Omega \mid \end{aligned}$$

Then, after extraction of subsequences

$$\begin{aligned} (\nabla\varphi_n)^o \rightharpoonup f \quad &\text{weakly in} \quad L^2(\Omega; I\!R^N) \\ (\varphi_n)^o \rightharpoonup g \quad &\text{weakly in} \quad L^2(\Omega) \end{aligned}$$

From the Hausdorff convergence of $S_n$ to S, we get: Let $\psi \in \mathcal{D}(\Omega \setminus S)$, $\exists n_\psi = n(d(S,K))$, where $K = supp\psi$. Such that $n \geq n_\psi$ implies $\psi \in \mathcal{D}(\Omega \setminus S_n)$ then, we see easily that $f\ |_{\Omega\setminus S} = \nabla(g\ |_{\Omega\setminus S})$ We set $\varphi = g\ |_{\Omega\setminus S}$ so that $\varphi \in H^1(\Omega \setminus S)$

On the other hand, we have

$$J_\varepsilon(S_n) = \frac{1}{2}\int_{\Omega\setminus S_n} < i, \nabla\varphi_n > dx = \frac{1}{2}\int_\Omega < i, (\nabla\varphi_n)^o > dx$$

Which converges, as $n \longrightarrow \infty$, to

$$\frac{1}{2}\int_\Omega < i, f > dx = \frac{1}{2}\int_{\Omega\setminus S} < i, f > dx = \frac{1}{2}\int_{\Omega\setminus S} < i, \nabla\varphi > dx \geq J_\varepsilon(S)$$

□

*Proof.* [of the theorem] Let $S_n$ be a minimizing sequence for problem (6.1). We assume $J_\varepsilon(S_n) + \sigma\ P_\gamma(S_n)$ monotonically decreasing to the infimum as $n \longrightarrow 0$. Then

$$J_\varepsilon(S_n) + \sigma\ P_\gamma(S_n) \leq J_\varepsilon(S_1) + \sigma\ P_\gamma(S_1) = a$$

$\sigma P_\gamma(S_n) \leq a - J_\varepsilon(S_n)$ then, form lemma 5

$$P_\gamma(S_n) \leq M = \frac{1}{\sigma}(a + \mid \Omega \mid) \tag{6.2}$$

From proposition 4, we can assume that $S_n \longrightarrow S$ in Hausdorff metric with $S \in \mathcal{S}_0$. From [1] we know that $P_\gamma$ is lower semi continuous on $\mathcal{S}_0$ and, as $J_\varepsilon$ is semi continuous inferiorly, the result classically derives. □

## 7. Viscous wake

The perimeter $P_\gamma(S)$ can be considered as a viscous term associated to $S$. We can take in account a more general contribution of the viscous effect. In the flow in $\Omega \setminus S$, we can neglect the viscosity but it is not reasonable on S because of the jump of the speed flow through $S$. Then, we can not neglect a viscous effect and we chose a classical term in the following form:

$$J_\varepsilon(S) + \int_S (\sigma + \nu \| [\![ \nabla_\Gamma \varphi(S) ]\!] \|^2)\, d\Gamma$$

## 8. Necessary optimality condition

**8.1. The smooth case.** We assume that the optimal wake $S$ is smooth enough. Then, using shape sensitivity analysis, we derive the shape gradient of the energy $J_\varepsilon(S)$. We perturb $S$ using a one parameter family of transformation $T_t$ mapping $\Omega$ on itself, $\partial\Omega$ onto $\partial\Omega$, with $T_t(B_f) = B_f$ and $T_t(B_\infty) = B_\infty$

$$J_\varepsilon(S) = \min_{y \in H^1(\Omega \setminus S)} E^\varepsilon_{\Omega\setminus S}(y)$$

so that the wake equilibrium problem (6.1) take the following shape variational form

$$\min_{S \in \mathcal{S}_0} \Big( \min_{y \in H^1(\Omega\setminus S)} E^\varepsilon_{\Omega\setminus S}(y) + \sigma\, P_\gamma(S) \Big) \tag{8.1}$$

We apply the results concerning the derivative of a Minimum with respect to a parameter $s$ [4]. For a given $s > 0$, we set $S_s = T_s(V)(S)$

**Lemma 7.** The family $\mathcal{S}_0$ is stable under transformations $T_s(V)$:

$$\forall V,\ \forall s,\ S_s \in \mathcal{S}_0$$

*Proof.* $T_t : D \longrightarrow D$ is a smooth one to one transformation then $|\, S_s \,| = 0$, $\#(S_s) = \#S$, $S_s$ is closed □

**Lemma 8.** The elements of the Sobolev space are transported by $T_s(V)$:

$$y \in H^1(\Omega \setminus S_s) \iff z = y \circ T_s(V) \in H^1(\Omega \setminus S)$$

Then, problem (8.1) lead to the extremization of the functional

$$J_\varepsilon(S_s) = \min_{z \in H^1(\Omega\setminus S)} E^\varepsilon_{\Omega\setminus S_s}(z \circ T_s(V)^{-1}) \tag{8.2}$$

We set

$$F(s,z) = E^\varepsilon_{\Omega\setminus S}(z \circ T_s(V)^{-1})$$
$$f(s) = J_\varepsilon(S_s)$$

and we make use of the

**Theorem 2.** *Let $K$ be a compact set, $F : [0,\tau] \times K \to I\!R$ a differentiable mapping and let $f(t) = \min\{F(t,y) \mid y \in K\}$. Denote by $K^*$ the subset of $K$ of elements $\varphi$ which realize minimum at t=0.*
*$f$ is side differentiable at $t = 0$ and*

$$\begin{aligned} f'(0,+1) &= \lim_{t\downarrow 0} \frac{f(t) - f(0)}{t} \\ &= \min\{\frac{\partial}{\partial t} F(0,\varphi) \mid \varphi \in K^*\} \end{aligned}$$

In order to apply that result, we need to reduce the minimum to a compact set $K$. This derives from the coercivity of $E^{\varepsilon}_{\Omega\setminus S}$
Effectively, we have

$$\begin{aligned} E^{\varepsilon}_{\Omega\setminus S}(y) &\geq \frac{\varepsilon}{2}\|y\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla y\|^2_{L^2(\Omega)} - \mid \Omega \mid^{\frac{1}{2}} \ \|\nabla y\|_{L^2(\Omega\setminus S)} \\ &\geq \frac{\varepsilon}{2}\|y\|^2_{H^1(\Omega\setminus S)} - \mid \Omega \mid^{\frac{1}{2}} \ \|\nabla y\|_{L^2(\Omega\setminus S)} \\ &\geq \frac{\varepsilon}{4}\|y\|^2_{H^1(\Omega\setminus S)} \end{aligned}$$

as soon as $\|y\|_{H^1(\Omega\setminus S)} \geq \frac{4}{\varepsilon} \mid \Omega \mid^{\frac{1}{2}} = M$
then $\|y\|_{H^1(\Omega\setminus S)} \geq M$ implies $E^{\varepsilon}_{\Omega\setminus S}(y) \geq \frac{4}{\varepsilon} \mid \Omega \mid \geq 0$

But the minimum $J_\varepsilon(S_s)$ being negative, in the minimization problem (8.2) $H^1(\Omega \setminus S)$ can be replaced by

$$K = \{y \in H^1(\Omega \setminus S) \ \mid \ \|y\| \leq M\}$$

$K$ is weakly compact in $H^1(\Omega \setminus S)$

**Lemma 9.** The Eulerian derivative of the domain functional $J_\varepsilon(S)$ in the direction of the vector field $V$ acting on $S$ is

$$dJ_\varepsilon(S;V) = \int_S [\![\frac{1}{2}|\nabla\varphi|^2 + \nabla\varphi.i + \frac{\varepsilon}{2}\varphi^2]\!] V.n d\Gamma$$

*Proof.*

$$f(t) = \int_{\Omega_t} \frac{1}{2}|\nabla(\varphi o T_t^{-1})|^2 + \nabla(\varphi o T_t^{-1}).i \, dx$$

we make use of:

$$\begin{aligned} \frac{\partial}{\partial t}(\int_{\Omega_t} \frac{1}{2}|\nabla(\varphi o T_t^{-1})|^2 dx)_{|_{t=0}} &= \int_\Omega \nabla\varphi.\nabla(-\nabla\varphi.V)\, dx + \int_S \frac{1}{2}[\![|\nabla\varphi|^2]\!] V.n \, d\Gamma \\ &= \int_\Omega \nabla\varphi.V\Delta\varphi \, dx - \int_S [\![\frac{\partial\varphi}{\partial n}\nabla\varphi]\!].V \, d\Gamma + \int_S \frac{1}{2}[\![|\nabla\varphi|^2]\!] V.n \, d\Gamma \\ &= \int_S [\![\nabla\varphi.V]\!]\, i.n \, d\Gamma + \int_S \frac{1}{2}[\![|\nabla\varphi|^2]\!] V.n \, d\Gamma \end{aligned}$$

and

$$\begin{aligned} \frac{\partial}{\partial t}(\int_{\Omega_t} \nabla(\varphi o T_t^{-1}).i \, dx)_{|_{t=0}} &= \frac{\partial}{\partial t}(\int_{\Omega_t} div(\varphi o T_t^{-1} i) dx)_{|_{t=0}} \\ &= \int_\Omega div(-\nabla\varphi.V i)\, dx + \int_S [\![div(\varphi i)]\!] V.n \, d\Gamma \\ &= \int_S [\![-\nabla\varphi.V]\!] i.n \, d\Gamma + \int_S [\![\nabla\varphi.i]\!] \ V.n \, d\Gamma \end{aligned}$$

then

$$f'(0) = \int_S [\![\frac{1}{2}|\nabla\varphi|^2 + \nabla\varphi.i]\!] V.n d\Gamma$$

□

We get now the necessary optimality condition.

**Proposition 5.** Let $S$ be a minimizer for the problem (6.1). Then the pressure is defined on both sides of $S$ and is given by the Bernoulli's equation. Moreover, its jump across $S$ is zero.

$$[\![p]\!] = 0 \text{ on } S$$

*Proof.* On the optimal wake,

$$\int_S [\![\frac{1}{2}|\nabla\varphi|^2 + \nabla\varphi.\vec{U}_\infty + \frac{\varepsilon}{2}\varphi^2]\!]\, V.n\; d\Gamma = 0, \qquad \forall\ V$$

Considering the following Bernoulli's equation on both sides of $S$,

$$\frac{1}{2}(U^2 - u_\infty^2 + \varepsilon\phi^2) + \frac{p}{\rho} + gz = \frac{p_0}{\rho}$$

($p_0 = p_{z=0}$is the atmospheric pressure)

in term of speed perturbation potential, this expression turns to be:

$$u_\infty^2(\frac{1}{2}|\nabla\varphi|^2 + \nabla\varphi.i + \frac{\varepsilon}{2}\varphi^2) + \frac{p}{\rho} + gz = \frac{p_0}{\rho}\text{on } S$$

which permits to conclude. □

**8.2. The non smooth case.** In general, $S$ could be non smooth (as up to now we have derive no smoothness results on $S$). The same shape sensitivity analysis can be performed but avoiding any boundary integral on $S$. Then, taking volume integrals, we give now the necessary condition which will be a relaxed formulation of the previous one.

**Proposition 6.**

$$dJ_\varepsilon(S;V) = \int_{\Omega\backslash S} div\{(\frac{1}{2} \mid \nabla\varphi \mid^2 + \nabla\varphi.i + \frac{\varepsilon}{2}\varphi^2)\ V\}\, dx$$

*Proof.* Using the two following propositions 7 and 8. □

**Proposition 7.**

$$\begin{aligned}\frac{\partial}{\partial t}(\int_{\Omega\backslash S_t} \frac{1}{2}|\nabla(\varphi o T_t^{-1})|^2 dx)_{|t=0} = \ & \frac{1}{2} \int_{\Omega\backslash S} div(\mid \nabla\varphi \mid^2\ V)\, dx \\ & - \int_{\Omega\backslash S} div(V.\nabla\varphi\ \nabla\varphi)\, dx\end{aligned}$$

Before showing this proposition, we need the two following lemmas

**Lemma 10.**

$$\int_{\Omega\backslash S} < \epsilon(V)\nabla\varphi, \nabla\varphi > dx = \int_{\Omega\backslash S} div(V.\nabla\varphi\ \nabla\varphi) - < D^2\varphi\nabla\varphi, V > dx$$

*Proof.*

$$\begin{aligned}\int_{\Omega\backslash S} \partial_i V_j\, \partial_i\varphi\, \partial_j\varphi\; dx &= \int_{\Omega\backslash S} -V_j\, \partial_i(\partial_i\varphi\, \partial_j\varphi) + \partial_i\{V_j(\partial_i\varphi\partial_j\varphi)\}\, dx \\ &= \int_{\Omega\backslash S} - < V, \nabla\varphi > \Delta\varphi - < D^2\varphi\ \nabla\varphi, V > + div(V.\nabla\varphi\ \nabla\varphi)\, dx\end{aligned}$$

□

**Lemma 11.**

$$\int_{\Omega\backslash S} \frac{1}{2} div V\, |\nabla\varphi|^2 dx = \int_{\Omega\backslash S} - < D^2\varphi\nabla\varphi, V > + \frac{1}{2} div(|\nabla\varphi|^2 V)\, dx$$

*Proof.*

$$\int_{\Omega\setminus S} divV\,|\nabla\varphi|^2dx = \int_{\Omega\setminus S} -V.\nabla(|\nabla\varphi|^2)dx + div(|\nabla\varphi|^2)\;V\}dx$$

$$\text{Where}\quad V.\nabla(|\nabla\varphi|^2) =< 2D^2\varphi\nabla\varphi, V >$$

□

*Proof.* [of the proposition 7]

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_{\Omega\setminus S_t}\frac{1}{2}|\nabla(\varphi o T_t^{-1})|^2dx\Big)_{|t=0} &= \frac{\partial}{\partial t}\Big(\int_{\Omega\setminus S}\frac{1}{2} <{}^*DT_t^{-1}\nabla\varphi, {}^*DT_t^{-1}\nabla\varphi > det(DT_t)\,dx\Big)_{|t=0}\\ &= \int_{\Omega\setminus S} < \{\frac{1}{2}I_d\; divV(0) - \epsilon(V(0))\}\nabla\varphi, \nabla\varphi > \,dx\end{aligned}$$

And concluding with lemmas 10 and 11. □

**Proposition 8.**

$$\frac{\partial}{\partial t}\Big(\int_{\Omega_t}\nabla(\varphi o T_t^{-1}).i\,dx\Big)_{|t=0} = \int_{\Omega\setminus S} div(\nabla\varphi.i\;V) + \int_{\Omega\setminus S} div(V.\nabla\varphi\;\nabla\varphi)\,dx$$

*Proof.*

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_{\Omega_t}\nabla(\varphi o T_t^{-1}).i\,dx\Big)_{|t=0} &= \int_{\Omega\setminus S} < -DV^*(0)\nabla\varphi, i > + < \nabla\varphi, i > divV(0)\,dx\\ &= \int_{\Omega\setminus S} div(\nabla\varphi.i\;V) - < D^2\varphi\, i, V > \,dx\\ &\quad - \int_{\Omega\setminus S} div(V.\nabla\varphi\; i) + < D^2\varphi\; V, i > \,dx\\ &= \int_{\Omega\setminus S} div(\nabla\varphi.i\;V) + \int_{\Omega\setminus S} div(V.\nabla\varphi\;\nabla\varphi)\,dx\end{aligned}$$

□

## References

1. D. BUCUR and J.P. ZOLESIO, *Pseudo courbure dans l'optimisation de forme*, C.R. Acad. Sci. Paris tome 320, Serie I, 1995.
2. M. DELFOUR and J.P. ZOLESIO, *Shape Analysis via Oriented Distance Functions*, Journal of Functional Analysis, 1994.
3. J. SOKOLOWSKI and J.P. ZOLESIO, *Introduction to Shape Optimisation: Shape sensitivity analysis*, Computational Mathematics, vol. 16, Springer-Verlag, New York, Berlin, Heidelberg, 1992.
4. J.P. ZOLESIO, *The material derivative (or speed) method for shape optimization*, E.J. Haug and J.Cea, Sijthoff and Noordhoff, Alphen aan den Rijn, 1981, pp 1089-1151.

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# On Some Inverse Geometrical Problems

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**I Introduction**

Recent efforts have focused on an industrial process : Nondestructive thermal testing of materials. They are generated by a growing interest in the detection and location of structural internal flaws. These methods give rise to a class of identification problems : Inverse geometrical problems defined by overspecified data.

These kind of problems are posed as follows : Consider a material occupying a domain $\Omega$ in $\mathrm{IR}^n$, $n \geq 2$. and let $\Gamma$ be the unknown geometry.

One wishes to determine $\Gamma$ by injecting a heat flux $\Phi$ (or a current flux in the case of electrical testing) across $\partial\Omega$ and measuring the temperature $f$ (or the voltage) on an open subset of $\partial\Omega$ : M.

The temperature field $u$ satisfies the steady state heat conduction problem :

$$\begin{cases} -\Delta u = 0 \text{ in } \Omega_\Gamma \\ \dfrac{\partial u}{\partial n} = \Phi \text{ on } \partial\Omega_\Gamma \end{cases}$$

$$\left( \int_{\partial\Omega_\Gamma} u = 0 \text{ and } \int_{\partial\Omega_\Gamma} \Phi = 0 \right)$$

Thus the problem is to know if $\Gamma$ can be determined by one choice of the heat flux $\Phi$ (and the corresponding measurement f).

The determination of $\Gamma$ consists in seeking the solution of three questions.

1) The uniqueness : Does $\Phi$ (and f) uniquely determine the unknown $\Gamma$.

2) The stability : Because of the error in measurements and in view of numerical treatments one has to study the variation of the geometry with respect to a variation of the measurement.

3) The inversion process : The goal of the problem is the determination of the unknown geometry by finding an inversion process which can be explicit or iterative.

The uniqueness question has been widely studied by many authors for different kind of geometrical flaws. In the case of inclusions [13] proved that when the inclusion D is a priori known to be a convex polyhedron , the shape and the location of D are determined by one measurement only. In the case where the unknown $\Gamma$ is a part of the outer part of $\partial\Omega$, one heat flux with its correspondant measurement, suffices to determine $\Gamma$ [5].

In the case of a buried insulated crack [14] showed that two specific current fluxes together with correspondant voltage suffice to determine the crack. Furthermore, they proved that this result is the best one possible. This result was extended in the case of a family of n cracks [9 ], it was proved that a family of n+1 fluxes with their corresponding voltage suffices to establish the uniqueness result. Recently [2] improved this result showing that two specific fluxes suffice (and are necessary) to establish the identifiability.

Notice that in all these works the crucial step towards the identifiability result rests on the knowledge of the shape of the level lines and therefore one can point out the bidimensional character of the proofs. It is shown here that in the case of a crack with a known emerging point on the boundary, one specific heat flux (or current flux) with the corresponding temperature field (or voltage) suffice to determine the crack .

The second question (the stability) can be viewed as the continuity of the mapping that associates the geometry to the data.

For that purpose, and in the cases of unknown boundaries as well as in the case of segment cracks with an emerging point on the boundary, compact metric spaces of admissible geometries are constructed. The stability result is derived from the uniqueness theorems and the continuity of the direct problems.The method given in[8] for inclusions is followed to reach this result.
A more precise stability result of Bellout-Friedman type[8] is also obtained : We prove that the mapping that associates the geometry to the data is locally lipshitzian. This result is optimal. Notice that the stability can be interpreted as the variation of the geometry with respect to the variation of the measurements, which suggests the main tool used : The domain derivative theory [16].
The last section of this presentation is devoted to numerical treatments. The identification process is based on the minimisation of an error functional initially introduced by Kohn and Vogelius in the case of parameter identification.
In this work, this functional is interpreted in the case of line segment cracks identification. Notice that this method can be applied to more general inverse geometrical problems[4].

**I Uniqueness results**

As pointed out earlier, this work is concerned by inverse geometrical problems defined by Laplacian equation and corresponding to overdetermined data. In this case these data correspond to the trace $f$ of the solution on $M$ , and to its normal derivative $\Phi = \frac{\partial u}{\partial n}$ . To establish the uniqueness result one has to answer to the following question :

Does the pair$(\Phi, f)$ uniquely determine the unknown geometry?

To prove the uniqueness, one has to compare two harmonic functions, defined in two different domains and having the same Cauchy data on $M$ : The main tool towards the result is the Holmgren uniqueness theorem. This tool has been widely used in this kind of problems : In the case of an a priori known convex polyhedron by Friedman and Isakov[13], in the case of $C^{2,\alpha}$ inclusions by Bellout and Friedman[8] , in the case of $C^2$ cracks in Friedman and Vogelius[14] and Bryan and Vogelius[9].

This section is devoted to uniqueness results concerning inverse geometrical problems. Two kinds of inverse geometrical problems are studied : the problem of the identification of inaccessible smooth boundary which is supposed to be islated (this kind of problems can be incoutered in thermal testing of composite materials). The second problem is the identification of a $C^2$ crack with an a priori known emerging point on the boundary.

**II.1 Case of a smooth boundary**

Let $\Omega$ be an open simply connected set of $IR^n$ with a $C^{1,1}$ boundary $\partial\Omega$ .
$\Gamma_\Phi$ , $M$ and $\Gamma$ a partition of $\partial\Omega$ . $M$ is supposed to be $C^2$. $\Gamma$ designates the inaccessible part of $\partial\Omega$ .
Denote by $\Omega_\Gamma$ the open set $\Omega$ . On $\partial\Omega_\Gamma$ a flux $\Phi$ ;$\Phi \not\equiv 0$ is imposed ($\int_{\partial\Omega}\Phi = 0$). Furthermore one supposes that supp $\Phi \subset \Gamma_\Phi$ .

Consider the direct problem corresponding to an unknown isolated part of the boundary :

$$(II.1)\quad \begin{cases} -\Delta u = 0 & in\ \Omega_\Gamma & (1)\\ \dfrac{\partial u}{\partial n} = \Phi & on\ \partial\Omega_\Gamma \setminus \Gamma & (2)\\ \dfrac{\partial u}{\partial n} = 0 & on\ \Gamma & (3)\\ \int_{\partial\Omega} u = 0 & & (4)\end{cases}$$

*THEOREM*

*let $\Gamma_1$ and $\Gamma_2$ be two possible $C^{1,1}$ boundaries to identify , $\Gamma_1$ and $\Gamma_2$ having the same endpoints .*

*$u_i$ designates the solution of (II.1) for $\Gamma = \Gamma_i$ $i = 1, 2$.*

*if $u_{1|M} = u_{2|M}$ then $\Gamma_1 = \Gamma_2$*

**Proof :**

The proof is achieved into two steps :

Step1

let $w = u_1 - u_2$ ; then $w$ satisfies the following Cauchy problem :

$$\begin{cases} -\Delta w = 0 & in\ \Omega_{\Gamma_1} \cap \Omega_{\Gamma_2}\\ w = 0\ \ on M & \\ \dfrac{\partial w}{\partial n} = 0 & on M\end{cases}$$

By the unique continuation theorem :

$$w \equiv 0 \text{ dans } \Omega_{\Gamma_1} \cap \Omega_{\Gamma_2}$$

And therefore :

$$u_1 = u_2 \text{ et } \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} \text{ sur } \partial(\Omega_{\Gamma_1} \cap \Omega_{\Gamma_2})$$

Step2

Consider now the open set : $0 = \Omega_{\Gamma_1} \cup \Omega_{\Gamma_2} \setminus \overline{\Omega_{\Gamma_1} \cap \Omega_{\Gamma_2}}$ Suppose that $0$ is non empty and let $0_1$ be one connected componant of $0$.

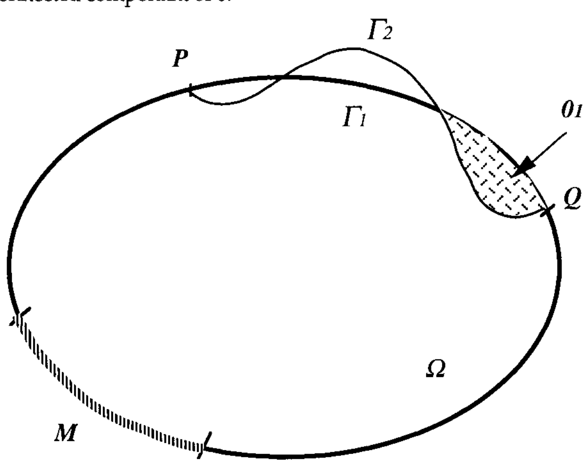



Figure 1

$\partial 0_1$ is constituted from parts of $\Gamma_1$ and $\Gamma_2$. Suppose for instance that $0_1 \subseteq \Omega_{\Gamma_1} \setminus \Omega_{\Gamma_2}$. One has $\partial 0_1 \cap \Gamma_2 \subseteq \partial(\Omega_{\Gamma_1} \cap \Omega_{\Gamma_2})$ and therefore :

$u_1 = u_2$ et $\dfrac{\partial u_1}{\partial n} = \dfrac{\partial u_2}{\partial n}$ sur $\partial 0_1 \cap \Gamma_2$ (where n designates the outer normal to $\Omega_{\Gamma_2}$)

Therefore $u_1$ extends $u_2$ across $\partial 0_1 \cap \Gamma_2$ and then

$$\frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n}$$

Since $\partial 0_1 \cap \Gamma_2$ is a smooth part of $\Omega_{\Gamma_1}$, $\dfrac{\partial u_1}{\partial n}$ is continuous across $\partial 0_1 \cap \Gamma_2$, it follows that:

$$\frac{\partial u_1}{\partial n} = 0 \text{ on } \partial 0_1 \cap \Gamma_2$$

That is, $u_1$ is on $0_1$ a solution of the following problem:

$$\begin{cases} -\Delta u_1 = 0 & in\ 0_1 \\ \dfrac{\partial u_1}{\partial n} = 0 & on\ \partial 0_1 \cap \Gamma_1 \\ \dfrac{\partial u_1}{\partial n} = 0 & on\ \partial 0_1 \cap \Gamma_2 \end{cases}$$

and therefore $u_1$ = cte on $\Omega_{\Gamma_1}$ by the unique continuation theorem. This is in contradiction with $\Phi \not\equiv 0$. That is $0_1 = \varnothing$ and it follows that $0 = \varnothing$.

Remark:

The same result is proven when the inaccessible boundary is supposed to satisfy a boundary condition of Signorini type, the proof is based on the same ideas. [10]

### II.2 Case of a crack initiated at the boundary

The body occupies a simply connected domain, one supposes that $\Omega$ contains exactly one crack $\sigma$ which has a known emerging point S on the boundary $\partial\Omega$ (a crack is a $C^2$ non selfintersecting curve) $\partial\Omega$. $\partial\Omega$ is parametrized by the arclength $s$ with S as origin.
Consider P,Q,R 3 points of $\partial\Omega$, such that :

$$0<s(R)<s(Q)<s(P)$$

and the flux $\Phi$ given by :

$$\Phi = \begin{cases} 1 & on RQ \\ -\dfrac{|RQ|}{|PR|} & on\ QP \\ 0 & elsewhere \end{cases}$$

The corresponding direct problem is therefore given by :

$$(II.2) \qquad \begin{cases} -\Delta u_\sigma = 0 & in\ \Omega \setminus \sigma \\ \dfrac{\partial u_\sigma}{\partial n} = \Phi & on\ \partial\ \Omega \\ \dfrac{\partial u_\sigma}{\partial n} = 0 & on\ \sigma \end{cases}$$

and one supposes the temperature $u_\sigma$ being measured on a curve $M$ (mes($M$) > 0)

*THEOREM*

*Let $\sigma$ and $\sigma'$ be two $C^2$ curves modelising two cracks having S as an endpoint .One supposes that these two cracks lead to the same measurement on M , for the flux$\Phi$ defined previously , then $\sigma = \sigma'$.*

**Proof :**

Let $u_\sigma$ be the solution corresponding to a crack $\sigma$ having S.as an endpoint.$u_\sigma \in H^1(\Omega \setminus \sigma)$, Denote by $\tau_\sigma$ the vector $\nabla u_\sigma$; $\tau_\sigma$ is divergence free in $L^2(\Omega \setminus \sigma)$, by the trace theorem $\tau_\sigma$ has a normal trace on the two sides of $\sigma$; $\tau_\sigma . n^+$ and $\tau_\sigma . n^-$.
but

$$\nabla u_\sigma . n^+ = \nabla u_\sigma . n^- = 0 \text{ on } \sigma$$

and therefore

$$\tau_\sigma . n^+ = \tau_\sigma . n^- = 0 \quad \text{on } \sigma$$

now $\nabla . \tau_\sigma = 0$ in $\Omega$, in the sense of distributions, therefore there exists a function $\omega_\sigma \in H^1(\Omega)$ such that :

$$\tau_\sigma = -(\nabla \omega_\sigma)^\perp = (\frac{\partial \omega_\sigma}{\partial x_2}, -\frac{\partial \omega_\sigma}{\partial x_1})$$

$x_1$ and $x_2$ designates the cartesian coordinates.
$\omega_\sigma$ is uniquely determined up to a constant.
Furthermore, one has :

$$\frac{\partial \omega_\sigma}{\partial \tau} = -(\nabla \omega_\sigma)^\perp . n = \Phi \text{ on } \partial \Omega$$
$$\omega_\sigma = K_\sigma \text{ on } \sigma .$$

That is $\omega_\sigma$ satisfies :

$$\begin{cases} -\Delta \omega_\sigma = 0 & \text{in } \Omega \setminus \sigma \\ \omega_\sigma = K_\sigma & \text{on } \sigma \\ \omega_\sigma = \varphi & \text{on } \partial \Omega \end{cases}$$

and $\omega_\sigma \in H^1(\Omega)$ (because $\omega_\sigma$ is continuous across$\sigma$ )
For $\sigma'$ one has also :

$$\begin{cases} -\Delta \omega_{\sigma'} = 0 & \text{in } \Omega \setminus \sigma' \\ \omega_{\sigma'} = K_{\sigma'} & \text{on } \sigma' \\ \omega_{\sigma'} = \varphi & \text{on } \partial \Omega \end{cases}$$

Denote by $\omega$ the field $\qquad \omega = \omega_\sigma - \omega_{\sigma'}$
$\omega$ is harmonic in $\Omega \setminus (\sigma \cup \sigma')$ , and satisfies :

$$\omega \equiv 0 \qquad \text{on } M$$

and $\frac{\partial \omega}{\partial n} = -(\nabla \omega_\sigma)^\perp . \tau + (\nabla \omega_{\sigma'})^\perp . \tau = \tau_{\sigma'} . \tau - \tau_\sigma . \tau = 0$ on $M$

Since $u_\sigma$ and $u_{\sigma'}$ have the same trace on $M$.
It comes that $\omega \equiv 0$ , in the exterior connected component $\Omega \setminus (\sigma \cup \sigma')$ , denoted by$\Omega_e$ .
By the specific choice made of $\varphi$ ( that is of the flux $\Phi$ ), $\varphi(S)$ *is the* minimum of$\varphi$ ,and $\varphi$ is constant on the arc PR of$\partial\Omega$ and is equal to $\varphi(S)$ .

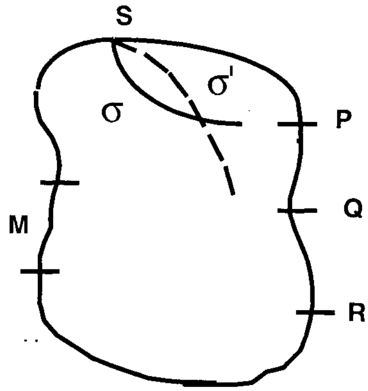



Figure 2

Suppose now that $\sigma$ and $\sigma'$ do not coincide, then there exists $z \in \sigma' \backslash \sigma$ (ou $\sigma \backslash \sigma'$). $z$ an interior point of $\Omega \backslash \sigma$ where the minimum of $\omega_\sigma$ is achieved( the minimum of $\omega_\sigma$ is exactly the minimum of $\varphi$ ) and therefore $\omega_\sigma \equiv$ cte, this is in contradiction whith the hypothesis $\Phi \neq 0$.

♦

**III-STABILITY**

**III.1 Statement**

In this section the problems (II.1) and (II.2)) are reconsidered. In these two cases the overspecified data are supposed to be accessible on an open set **M** of the boundary $\partial\Omega$. Since the measurements are given by experiments, they usually are subjected to errors. The goal of this section is to study the stability of the inverse geometrical problems under consideration, that is, roughly speaking to study if small perturbation in measurements lead to a geometry in the vicinity of the actual geometry.

To formalize this idea, consider:

$\Gamma_{ad}$ a set of admissible geometries (lcracks or smooth boundaries) the operator $\eta$ defined for a fixed identifying flux $\Phi$ .

$$\eta : \Gamma_{ad} \to L^2(M)$$
$$\Gamma \to f = u_{\Gamma\,|M}$$

By the previous section, the operator $\eta$ is injective, consider the mapping (for simplicity also called $\eta$).

$$\eta : \Gamma_{ad} \to \eta(\Gamma_{ad})$$
$$\Gamma \to f = u_{\Gamma\,|M}$$

$\eta$ is therefore invertible. The stability will be established if one proves the continuity of $\eta^{-1}$.

So $\Gamma_{ad}$, as well as ( $L^2(M)$) has to be equipped with an appropriate topology.

The main resultts of this section are a global "weak " stability result, that is the continuity of the operator $\eta^{-1}$ the methods followed here to perform these results walk for the smooth unknown boundary problem as well as the line segment crack one. For the reader convenience, this section focuses on the line segment crack problem. The results are completely shown in this case.

**III.2. "Weak" stability results**

One can see that the compactness of the set of unknown geometries and the uniqueness result lead to the stability. This seems to be general : It was proved for buried cracks in [14], for monotone inclusions in [1]. The next theorem is devoted to this kind of result in the case of straight cracks having an apriori known endpoint on the boundary.

The set $\Sigma$ is chosen to be compact for the Hausdorff metric :

$$d(\sigma,\sigma') = \left( \overset{s}{\max_{x \in \sigma}} \; \min_{y \in \sigma'} |x-y| \right)$$

( $L^2(M)$) is equipped with the $L^2$- norm. And consider :

$$\eta : \Sigma \to \eta(\Sigma)$$
$$\sigma \to f = u_{\sigma \ |M}$$

*Theorem :*

*The operator $\eta^{-1}$ is continuous*

Proof :

Let $\sigma_n$, $\sigma \in \Sigma$ such that the corresponding data $f_{\sigma_n} \to f_\sigma$ in $L^2(M)$

By compactness, $\sigma_n$ has a subsequence $\sigma_{p(n)}$converging to $\tilde{\sigma} \in \Sigma$

Then $f_{\sigma_{p(n)}} \to f_{\tilde{\sigma}}$ (stability of the direct problem) and therefore $f_{\tilde{\sigma}} = f_\sigma$

By the uniqueness result $\tilde{\sigma} = \sigma$

Then $\sigma = \tilde{\sigma}$ is the unique adherence value of $\sigma_n$ and d $(\sigma_n, \sigma) \to 0$. ♦

Remark : the same result occurs in the case of an unknown smooth boundary.

Notice that since the set of admissible cracks has been chosen compact, the previous theorem establishes actually that $\eta^{-1}$ has a modulos of continuity [11]. That is there exists an increasing mapping $\varphi$ :

$$\varphi : \overline{IR}^+ \to \overline{IR}^+$$

$\varphi$ continuous in 0 and $\varphi(0) = 0$

and $$| f_\sigma - f_{\sigma'} |_{L^2(M)} \leq \varphi(d(\sigma,\sigma'))$$

The goal of the next section is to have more information on $\varphi$, that is to "quantify" the continuity of $\eta^{-1}$. Actually, one proves that $\varphi$ is locally lipschitzian. Since the stability is estimating the deviation of the geometry in terms of the deviation of the measurements. This, suggests the tool to use to perform the local stability result : the domain derivative.

### III.2. Local lipschitzian stability

### III.2. 1 Domain derivative

The method followed to establish the results concerning the domain derivative is based on the results of Murat and Simon [ 16].

Consider a family of diffeomorphims $F_h$ mapping $\Omega \setminus \sigma_h$ onto $\Omega \setminus \sigma$. The open sets$\Omega \setminus \sigma_h$ are coisen in such a way that $\sigma_h$belong $\Sigma$. As in [ 16] , are chosen as perturbations of the identity :

$$F_h = Id + h\,\theta$$

For h "small" enough, $F_h$ is a set of diffeomorphisms. $\theta \in W^{1,\infty}(\Omega \setminus \sigma)$ and $\theta \equiv 0$ on $\partial\Omega$.

The next proposition gives the lagrangian first derivation of the solution of (I) with respect to a variation of the domain.

***Proposition***

*The scalar fild $u^h_{\sigma_h}$defined on $\Omega \setminus \sigma$, has in $H^1(\Omega \setminus \sigma)$, the asymptotic expansion:*

$$u^h_{\sigma_h} = u^0_\sigma + hu^1 \text{ .in } H^1(\Omega \setminus \sigma)$$

*where* $u_\sigma^0$ *is the solution of (I),* $u^1$ *is the solution of the problem:*

$$\int_{\Omega\setminus\sigma} \nabla u^1, \nabla v = \int_{\Omega\setminus\sigma}\left({}^t\frac{\partial\theta}{\partial M} + \frac{\partial\theta}{\partial M}\right)\nabla u_\sigma \, \nabla v - \int_\Omega (\nabla u_\sigma, \nabla v)\, div\, \sigma$$

$$\forall v \in H^1(\Omega)$$

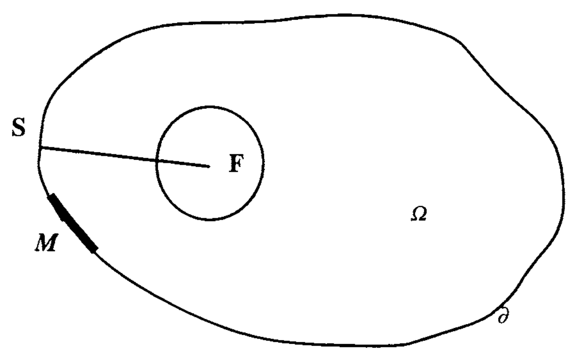
**Proof**:
The proof of this result is similar of the one given in [12] in the case of elasticity. ♦

By this particular choice of $F_h$ , one has :

$$|f_h - f|_M = |u_{\sigma h} - u|_M = |u_{\sigma h} o F_h - u|_M$$

Then in ordre to prove a local stability result, it suffices to prove that $u^1$cannot vanish all over $M$ .

**III.2.2 Stability with respect to a length variation**
Let σ be a line segment crack with S as an endpoint denote by F the end point of σ belonging to Ω and $\sigma_h$ a line set cracks $\sigma_h \subset$ (S,F). Such that $|\sigma_n| = (1+h)|\sigma|$.
((S,F) line crossing S and F) $|\sigma|$ denote the length of σ.




***Theorem***
*Let* $f_h$ *(respectively) be the trace of the solution of (I) in* $\Omega\setminus\sigma_h$ *(respectively in* $\Omega\setminus\sigma$*)*
*Under the assumption :*
***(H)*** *the singularity coefficient of* $u_\sigma$ *at* ***F*** *(*$\sigma = [S,F]$*) is different of 0*
*one has :*

$$\lim_{h\to 0} \frac{|f_h - f|}{h} > 0$$

To prove this theorem, one needs some preliminary results.

**PRELIMINARY RESULTS**
The next result relates the derivative with respect to the crack length of the potential energy to the domain derivative of the heat field $u_\sigma$.
Recall that in the case of an insulated crack the solution $u_\sigma$ is know to be composed by a somooth part $u_\sigma^S$ and a singular part [15]