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Civil Engineer's REFERENCE BOOK

FOURTH EDITION

L S Blake

Civil Engineer's Reference Book

Fourth Edition

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Civil Engineer's Reference Book

Fourth Edition

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Preface to the fourth edition

The aim of this edition, as of those that preceded it, is to give civil engineers a concise presentation of theory and practice in the many branches of their profession. The book is primarily a first point of reference which, through its selective lists of references and bibliographies, will enable the user to study a subject in greater depth. However, it is also an important collection of state-of-the-art reports on design and construction practices in the UK and overseas.

First published in 1951, the book was last revised in 1975. Although civil engineering is not normally regarded as involving fast-moving technologies, so many advances have occurred in the theory and practice of most branches of civil engineering during the past decade or so that the preparation of a fourth edition became essential. Some of these advances have taken the form of improvements in earlier practices, for example in surveying, geotechnics, water management, project management, underwater working, and the control and use of materials. Other radical changes have resulted from the evolving needs of clients for almost all forms of construction, maintenance and repair. Another major change has been the introduction of new national and Euro-codes based on limit state design covering most aspects of structural engineering.

The fourth edition incorporates these advances and, at the same time, gives greater prominence to the special problems relating to work overseas, with differing client requirements and climatic conditions.

As before, careful attention has been given to the needs of the different categories of readers. Students and graduates at the start of their careers need guidance on the practice of design and construction in many of the fields of civil engineering covered in Chapters 11 to 44. The engineer in mid-career will also find these chapters valuable as presentations of the state of the art by acknowledged experts in each field, in addition to the references and bibliographies they contain for deeper study of specific problems. Chapters 1 to 10 provide engineers, at all levels of development, with up-to-date 'lecture notes' on the basic theories of civil engineering.

Although the book was primarily prepared for civil engineers in the UK and elsewhere in the world, members of other professions involved in construction—architects, lawyers, mechanical engineers, insurers and clients—will also benefit by referring to it.

I am most grateful to the authors who have contributed chapters. They are all engineers of considerable standing—consultants, contractors, research workers or academics—who have devoted a substantial amount of time to presenting their expert knowledge and experience for the benefit of the profession.

L.S. Blake
Bournemouth
November 1988

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B C Best BSc
Consultant

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MATHEMATICS

1.1 Algebra

1.1.1 Powers and roots

The following are true for all values of indices, whether positive, negative or fractional:

$$a^p \times a^q = a^{p+q}$$
$$(a^p)^q = a^{pq}$$
$$(a/b)^p = a^p/b^p$$
$$(ab)^p = a^p b^p$$
$$a^p/a^q = a^{p-q}$$
$$a^{-p} = (1/a)^p = 1/a^p$$
$$\sqrt[p]{a} = a^{1/p}$$
$$a^0 = 1$$
$$0^p = 0$$

1.1.2 Solutions of equations in one unknown

1.1.2.1 Linear equations

Generally $ax+b=0$
of which there is one solution or root $x=-b/a$

1.1.2.2 Quadratic equations

Generally $ax^2+bx+c=0$
of which there are two solutions or roots

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a} \qquad (1.1)$$

where, if $b^2>4ac$, the roots are real and unequal, $b^2=4ac$, the roots are real and equal, and $b^2<4ac$, the roots are conjugate complex.

It is worth attempting to rearrange equations as, often, they can be put into a more familiar form simply by rearrangement, e.g.:

$$ax^{2m}+bx^m+c=0$$

is a quadratic equation in x^m

while $a/x^2+b/x+c=0$

is the quadratic $cx^2+bx+a=0$

1.1.2.3 Cubic equations

Generally $x^3+bx^2+cx+d=0$
If the substitution: $x=y-b/3$ is made the equation

becomes $y^3+ey+f=0$

where $e=(3c-b^2)/3$

and $f=(2b^3-9bc+27d)/27$

now define

$$A=\left[-\frac{f}{2}+\left(\frac{f^2}{4}+\frac{e^3}{27}\right)\right]^{1/3}$$

$$B=\left[-\frac{f}{2}-\left(\frac{f^2}{4}+\frac{e^3}{27}\right)\right]^{1/3}$$

and the three roots, in terms of y are:

$$y_1=[A+B]$$

$$y_{2,3}=[-(A+B)/2\pm\sqrt{-3}(A-B)/2]$$

and in terms of x the three roots are:

$$x_{1,2,3}=y_{1,2,3}-\frac{b}{3}$$

1.1.2.4 Equations of higher degree

Equations of degree higher than the second (quadratic equations) are not solvable directly as the method of solving the cubic equation above shows. Generally recourse must be had to either graphical or numerical techniques.

If the equation be of the form:

$$F(x)=0$$

e.g. $a_n x^n + a_{n-1}x^{n-1} \ldots + a0 = 0$

then plot the graph of $y=F(x)$ the values of x at which $y=0$ are the roots or solutions to the equation. Frequently this graphical approach may be used fairly roughly (and therefore quickly) to obtain an estimate of a root. This estimate can then be improved by numerical means. For instance, values of $F(x)$ may be calculated for values of x close to that given as a root by the graphical method. The difficulty (which is not serious for hand calculations) is guessing by how much to adjust x to get $F(x)$ nearer to 0.

1.1.3 Newton's method

This is a method of step-by-step iteration in which an estimate of a root is refined.

Suppose that a_1 is an approximation to a root of an equation then, for small q:

$$F(a_1+q)\simeq F(a_1)+qF(a_1)$$

So that if we assume (a_1+q) to be the better solution we are seeking, i.e.:

$$F(a_1+q)=0 \qquad (1.2)$$

then:

$$q=\frac{-F(a_1)}{F'(a_1)} \qquad (1.3)$$

and $a_2=a_1+q$ is a second and better approximation.

This is well illustrated by drawing a curve cutting the x-axis, assuming a value a_1 of x near to the intersection to have been found, drawing the ordinate to the curve $x=a_1$ and then constructing the tangent to the curve $y=F(x)$ at the point $x=a_1$.

The point $x=a_2$ where this tangent cuts the axis is plainly a better estimate of the intersection than is a_1.

This technique can be used successfully in automatic calculation on a computer. The problem then becomes that of determining when to stop the iteration process:

$$a_1, a_2, a_3 \ldots$$

which may be best done by stopping when the change between successive approximations, a_n and a_{n+1} becomes less than some small preset amount.

Graphical and numerical methods will generally be required to deal with transcendental equations although in some cases it may be more convenient to find the intersections of two graphs rather than try to compute where a more complicated graph cuts an axis

e.g. $x-\sin x=0$

is best solved by plotting:

$y=x$

and $y=\sin x$

to find the intersection which will give an estimate which can be refined numerically.

1.1.4 Progressions

(1) Arithmetic progressions in which the difference between consecutive terms is a constant amount. Thus, the terms may be:

$a,\ a+d,\ a+2d,\ a+3d\ \ldots$

The nth term is $a+(n-1)d$ and the sum to n terms,

$$S_n=\frac{n}{2}\{2a+(n-1)\ d\} \tag{1.4}$$

(2) Geometrical progressions in which the ratio between consecutive terms is a constant. Generally terms are:

$a,\ ar,\ ar^2,\ ar^3\ \ldots$

The nth term is ar^{n-1} and the sum of n terms is:

$$S_n=\frac{a(1-r^n)}{1-r} \tag{1.5}$$

If r is strictly smaller than 1 so $-1<r<1$, then r^n tends to zero as n becomes larger so that for such geometric progressions we can find the 'sum to infinity' of the series:

$$S_\infty=\frac{a}{1-r} \tag{1.6}$$

The geometric mean of a set of n numbers is the nth root of their product.

If we limit consideration to non-negative numbers then the arithmetic mean of a set of numbers will be greater than or equal to their geometric mean.

1.1.5 Logarithms

Logarithms, which, short of calculating machinery of some form, are probably the greatest aid to computation are based on the properties of indices.

Thus, if we consider logarithms to base a we have the following results:

$a^x=P$ is equivalent to $\log_a P=x$

$a^1=a$ is equivalent to $\log_a a=1$

$a^0=1$ is equivalent to $\log_a 1=0$

So that using rules for powers given on page **1**/3:

If: $a^x=P$ and $a^y=Q$

then: $PQ=a^{x+y}H$

so: $\log_a PQ=x+y=\log_a P+\log_a Q$

Similarly: $\log_a(P/Q)=\log_a P-\log_a Q$

Also: $P^n=a^{nx}$

so: $\log aP^n=nx=n\log_a P$

In computation, it is generally convenient to use as base the number 10, i.e. in the expressions given above $a=10$. However, in fundamental work or integration natural logarithms (also known as Napierian or hyperbolic logarithms) are generally used. These are logarithms to base e a transcendental number given approximately by:

$$e=2.7182\,8 \tag{1.7}$$

and whose definition can be taken as: 'The value of the solution of the differential equation $dy/dx=y$ for $x=1$.'

(Note the solution of $dy/dx=y$ is $y=e^x$.)

1.1.6 Permutations and combinations

If, in a sequence of N events, the first can occur in n_1 ways, the second in n_2, etc. then the number of ways in which the whole sequence can occur is:

$n_1n_2n_3\ldots n_N$

1.1.6.1 Permutations

The number of permutations of n different things taken r at a time means the number of ways in which r of these n things can be arranged *in order*. This is denoted by:

$${}^nPr=n(n-1)(n-2)\ldots(n-r+1)=\frac{n!}{(n-r)!} \tag{1.8}$$

where $n!=n(n-1)(n-2),\ \ldots 3.2.1$ is called factorial n.

It is clear that:

${}^nPn=n!$

and that:

${}^nP_1=n$

If, of n things taken r at a time p things, are to occupy fixed positions then the number of permutations is given by:

$${}^{n-p}Pr-p \qquad (1.9)$$

If in the set of n things, there are g groups each group containing n_1, n_2 ... n_g things which are identical then the number of permutations of all n things is:

$$\frac{n!}{n_1!n_2!\ldots n_g!}$$

1.1.6.2 Combinations

The number of combinations of n different things, into groups of r things at a time is given by:

$${}^{n}Cr=\frac{n!}{r!(n-r)!}=\frac{{}^{n}Pr}{r!} \qquad (1.10)$$

It is important to note that, whereas in permutations the order of the things does matter, in combinations the order does not matter. From the general expression above, it is clear that:

$${}^{n}Cn=1$$

$${}^{n}C_1=n \qquad (1.11)$$

If, of n different things taken r at a time p are always to be taken then the number of combinations is:

$${}^{n-p}Cr-p \qquad (1.12)$$

If, of n different things taken r at a time p are never to occur the number of combinations is:

$${}^{n-p}Cr \qquad (1.13)$$

Note that combinations from an increasing number of available things are related by:

$${}^{n+1}Cr={}^{n}Cr+{}^{n}Cr-1 \qquad (1.14)$$

also $\quad {}^{n}Cr={}^{n}Cn-r \qquad (1.15)$

1.1.7 The binomial theorem

The general form of expansion of $(x+a)^n$ is given by:

$$(x+a)^n={}^{n}C_0x^n+{}^{n}C_1\times{}^{n-1}a^r+{}^{n}C_2x^{n-2}a^2\ldots \qquad (1.16)$$

Alternatively this may be written as:

$$(x+a)^n=x^n+nx^{n-1}a+\frac{n(n-1)}{1.2}x^{n-2}a^2+\frac{n(n-1)(n-2)}{1.2.3}x^{n-3}a^3 \qquad (1.17)$$

It should be noted that the coefficients of terms equidistant from the end are equal (since ${}^{n}Cr={}^{n}Cn-r$).

1.2 Trigonometry

The *trigonometric functions* of the angle a (see Figure 1.1) are defined as follows:

$\sin a=y/r$	$\operatorname{cosec} a=r/y$
$\cos a=x/r$	$\sec a=r/x$
$\tan a=y/x$	$\cot a=x/y$

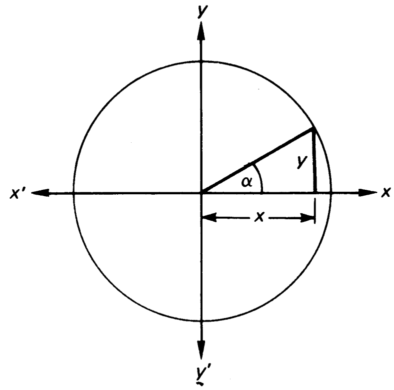

Figure 1.1 Trigonometric functions

These functions satisfy the following identities:

$$\sin^2 a+\cos^2\alpha=1$$
$$1+\tan^2 a=\sec^2 a$$
$$1+\cot^2 a=\operatorname{cosec}^2 a$$

1.2.1 Positive and negative lines

In trigonometry, lines are considered positive or negative according to their location relative to the coordinate axes xOx', yOy', (see Figure 1.2).

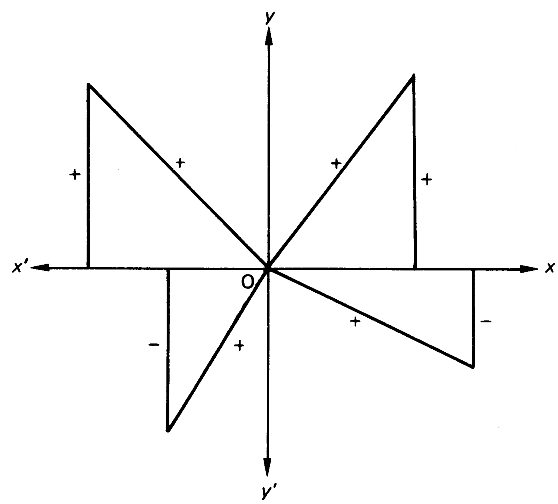

Figure 1.2 Positive and negative lines

1.2.1.2 Positive lines

Radial: any direction.
Horizontal: to right of yOy'.
Vertical: above xOx'.

1.2.1.3 Negative lines

Horizontal: to left of yOy'.
Vertical: below xOx'.

1.2.2 Positive and negative angles

Figure 1.3 shows the convention for signs in measuring angles. Angles are positive if the line OP revolves anti-clockwise from

Ox as in Figure 1.3a and are negative when OP revolves clockwise from Ox.

Signs of trigonometrical ratios are shown in Figure 1.4 and in Table 1.1.

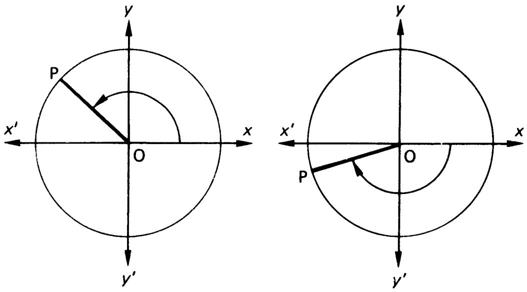

Figure 1.3 (a) Positive (b) negative angle

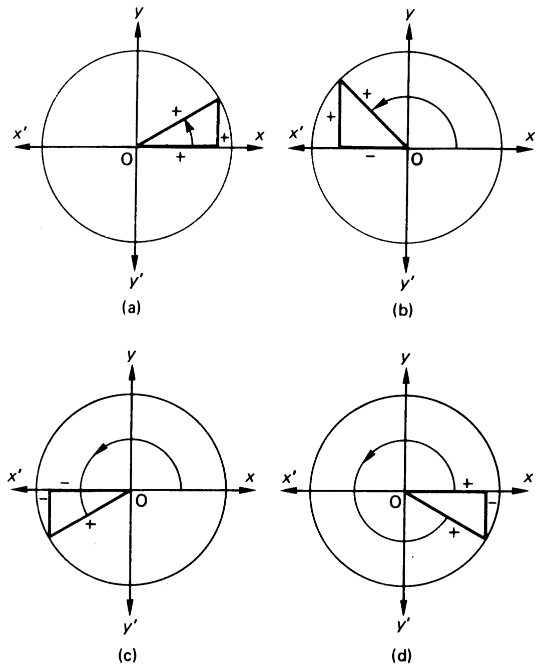

Figure 1.4 (a) Angle in first quadrant; (b) angle in second quadrant; (c) angle in third quadrant; (d) angle in fourth quadrant

1.2.3 Trigonometrical ratios of positive and negative angles

Table 1.1

Quadrant	*Sign of ratio*	
	positive	*negative*
First	sin	
	cos	
	tan	
	cosec	
	sec	
	cot	
Second	sin	cos
	cosec	sec
		tan
		cot
Third	tan	sin
	cot	cosec
		cos
		sec
Fourth	cos	sin
	sec	cosec
		tan
		cot

1.2.4 Measurement of angles

1.2.4.1 English or sexagesimal method

1 right angle $= 90°$ (degrees)
$1°$ (degree) $= 60'$ (minutes)
$1'$ (minute) $= 60''$ (seconds)

This convention is universal.

1.2.4.2 French or centesimal method

This splits angles, degrees and minutes into 100th divisions but is not used in practice.

1.2.4.3 The radian

This is a constant angular measurement equal to the angle subtended at the centre of any circle by an arc equal in length to the radius of the circle as shown in Figure 1.5.

$$\pi \text{ radians} = 180°$$

$$1 \text{ radian} = \frac{180}{\pi} = \frac{180}{3.141\,6} = 57°\,17'\,44'' \text{ approximately}$$

Table 1.2

$\sin(-\alpha)$	$= -\sin\alpha$	$\tan(-\alpha)$	$= -\tan\alpha$	$\sec(-\alpha)$	$= \sec\alpha$
$\cos(-\alpha)$	$= \cos\alpha$	$\cot(-\alpha)$	$= -\cot\alpha$	$\text{cosec}(-\alpha)$	$= -\text{cosec}\,\alpha$
$\sin(90°-\alpha)$	$= \cos\alpha$	$\tan(90°-\alpha)$	$= \cot\alpha$	$\sec(90°-\alpha)$	$= \text{cosec}\,\alpha$
$\cos(90°-\alpha)$	$= \sin\alpha$	$\cot(90°-\alpha)$	$= \tan\alpha$	$\text{cosec}(90°-\alpha)$	$= \sec\alpha$
$\sin(90°+\alpha)$	$= \cos\alpha$	$\tan(90°+\alpha)$	$= -\cot\alpha$	$\sec(90°+\alpha)$	$= -\text{cosec}\,\alpha$
$\cos(90°+\alpha)$	$= -\sin\alpha$	$\cot(90°+\alpha)$	$= -\tan\alpha$	$\text{cosec}(90°+\alpha)$	$= \sec\alpha$
$\sin(180°-\alpha)$	$= \sin\alpha$	$\tan(180°-\alpha)$	$= -\tan\alpha$	$\sec(180°-\alpha)$	$= -\sec\alpha$
$\cos(180°-\alpha)$	$= -\cos\alpha$	$\cot(180°-\alpha)$	$= -\cot\alpha$	$\text{cosec}(180°-\alpha)$	$= \text{cosec}\,\alpha$
$\sin(180°+\alpha)$	$= -\sin\alpha$	$\tan(180°+\alpha)$	$= \tan\alpha$	$\sec(180°+\alpha)$	$= -\sec\alpha$
$\cos(180°+\alpha)$	$= -\cos\alpha$	$\cot(180°+\alpha)$	$= \cot\alpha$	$\text{cosec}(180°+\alpha)$	$= -\text{cosec}\,\alpha$

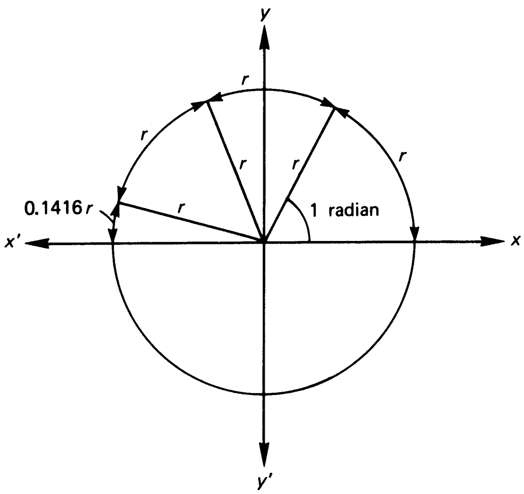

Figure 1.5 The radian

1.2.4.4 Trigonometrical ratios expressed as surds

Table 1.3

Angle in radians	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
Angle in degrees	0°	30°	45°	60°	90°
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞

Table 1.3 gives these ratios for certain angles.

1.2.5 Complementary and supplementary angles

Two angles are complementary when their sum is a right angle; then either is the complement of the other, e.g. the sine of an angle equals the cosine of its complement. Two angles are supplementary when their sum is two right angles.

1.2.6 Graphical interpretation of the trigonometric functions

Figures 1.6 to 1.9 show the variation with α of $\sin\alpha$, $\cos\alpha$, $\tan\alpha$ and $\operatorname{cosec}\alpha$ respectively. All the trigonometric functions are periodic with period 2π radians (or 360°).

1.2.7 Functions of the sum and difference of two angles

$$\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$$
$$\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$$
$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\pm\tan A\tan B}$$

1.2.8 Sums and differences of functions

$$\sin A+\sin B=2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\sin A-\sin B=2\cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\cos A+\cos B=2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B)$$
$$\cos A-\cos B=-2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B)$$
$$\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$$
$$\cos^2 A-\cos^2 B=-\sin(A+B)\sin(A-B)$$
$$\cos^2 A-\sin^2 B=\cos(A+B)\cos(A-B)$$

1.2.9 Functions of multiples of angles

$$\sin 2A=2\sin A\cos A$$
$$\cos 2A=\cos^2 A-\sin^2 A=2\cos^2 A-1=1-2\sin^2 A$$
$$\tan 2A=2\tan A/(1-\tan^2 A)$$
$$\sin 3A=3\sin A-4\sin^3 A$$
$$\cos 3A=4\cos^3 A-3\cos A$$
$$\tan 3A=(3\tan A-\tan^3 A)/(1-3\tan^2 A)$$
$$\sin pA=2\sin(p-1)A\cos A-\sin(p-2)A$$
$$\cos pA=2\cos(p-1)A\cos A-\cos(p-2)A$$

1.2.10 Functions of half angles

$$\sin A/2=\sqrt{\left(\frac{1-\cos A}{2}\right)}=\frac{\sqrt{(1+\sin A)}}{2}-\frac{\sqrt{(1-\sin A)}}{2}$$

$$\cos A/2=\sqrt{\left(\frac{1+\cos A}{2}\right)}=\frac{\sqrt{(1+\sin A)}}{2}+\frac{\sqrt{(1-\sin A)}}{2}$$

$$\tan A/2=\frac{1-\cos A}{\sin A}=\frac{\sin A}{1+\cos A}=\sqrt{\left(\frac{1-\cos A}{1+\cos A}\right)}$$

1.2.11 Relations between sides and angles of a triangle (Figures 1.10 and 1.11)

$$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$$

$$a=b\cos C+c\cos B$$

$$c^2=a^2+b^2-2ab\cos C \tag{1.18}$$

$$\sin A=\frac{c}{bc}\sqrt{\{s(s-a)(s-b)(s-c)\}} \tag{1.19}$$

where $2s=a+b+c$

Area of triangle $\triangle=\tfrac{1}{2}ab\sin C=\sqrt{\{s(s-a)(s-b)(s-c)\}}$

$$\tan\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}$$

$$\cos\frac{A}{2}=\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$$

$$\sin\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$$

$$\tan\frac{B-C}{2}=\frac{(b-c)}{(b+c)}\cot\frac{A}{2}$$

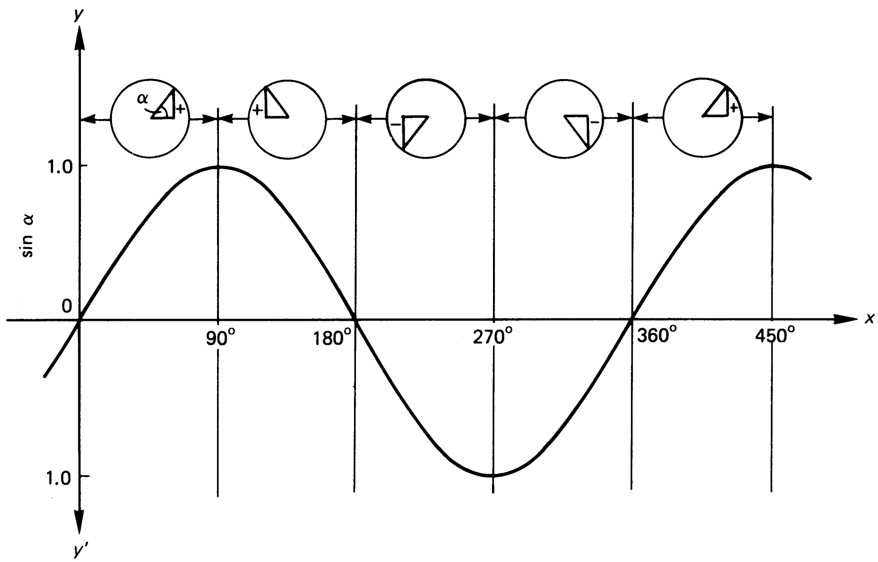

Figure 1.6 Sin α

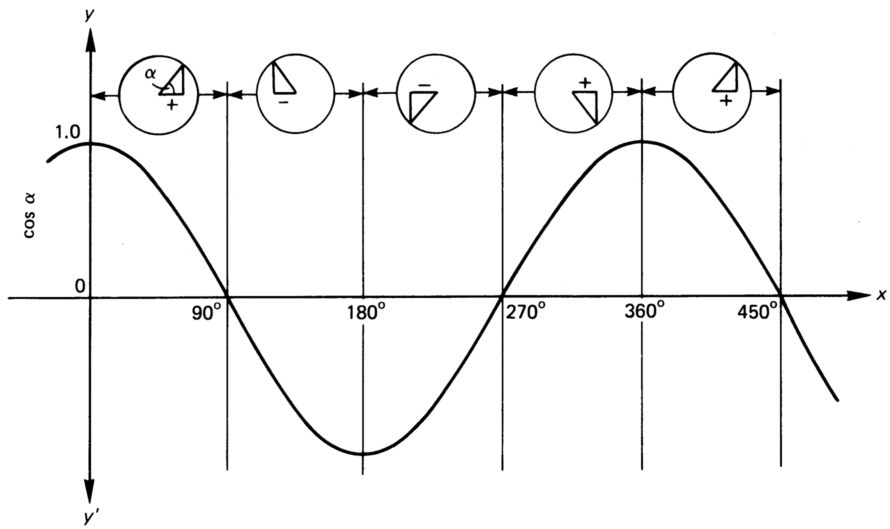

Figure 1.7 Cos α

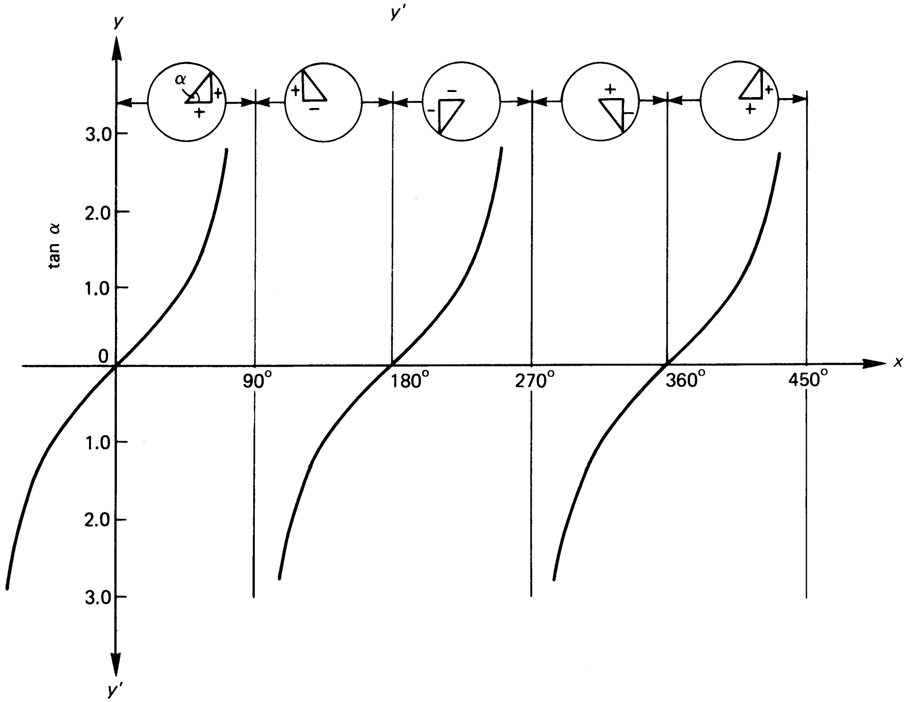

Figure 1.8 Tan α

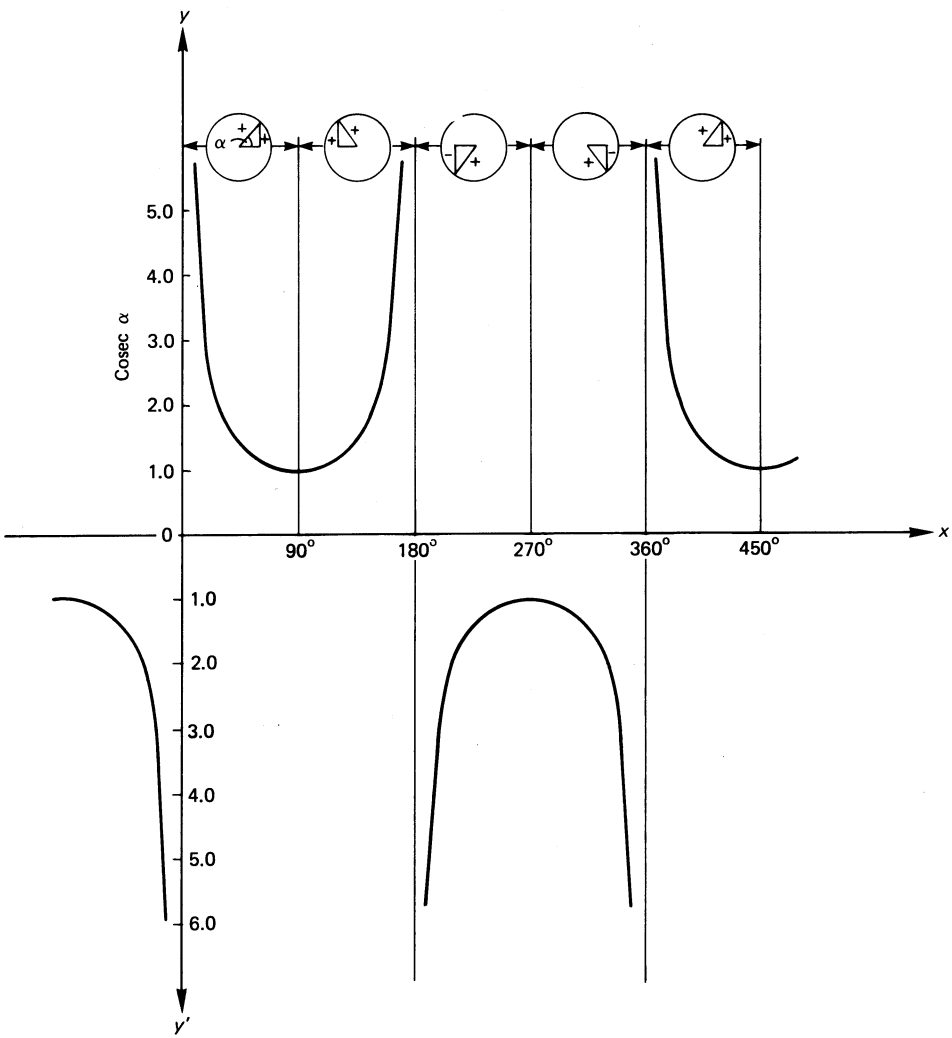

Figure 1.9 Cosec α

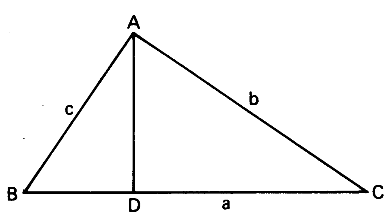

Figure 1.10

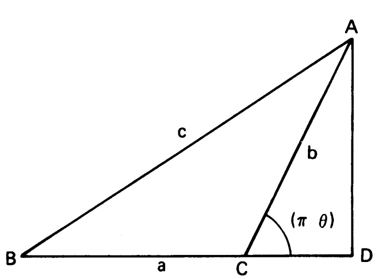

Figure 1.11

1.2.11.1 Any right angled triangle (Figure 1.12)

$a^2+b^2=c^2$; $A+B=90°$; $\sin A=\cos B$; $\cot A=\tan B$ etc.

Area of $\triangle$ $ABC=\frac{1}{2}ab=\frac{1}{2}bc\sin A=\frac{1}{2}ac\sin B$

1.2.11.2 Any equilateral triangle (Figure 1.13)

$a=b=c$; $A=B=C=60°=\pi/3$

$h=\dfrac{b\sqrt{3}}{2}$; area $=\dfrac{b^2\sqrt{3}}{4}$

Circumscribed circle radius $R=\dfrac{b\sqrt{3}}{3}$

Inscribed circle, radius $r=\dfrac{b\sqrt{3}}{6}$

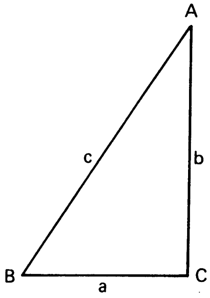

Figure 1.12

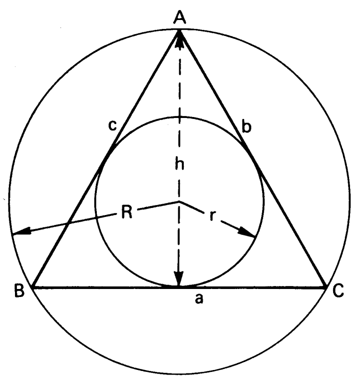

Figure 1.13

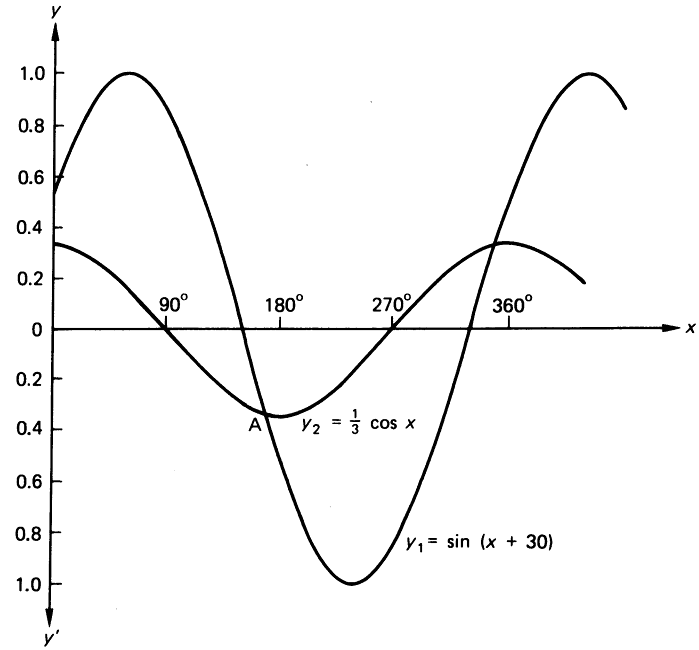

Figure 1.14 Solution of trigonometrical equations showing the intersection between $x=10$ and $x=\pi$ as $x=169°$ approximately

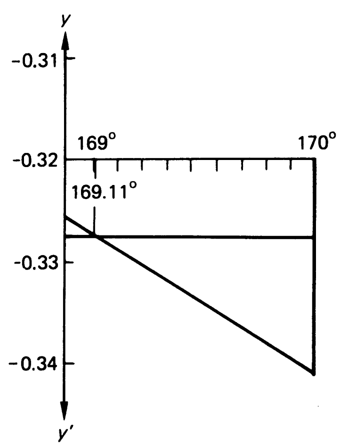

Figure 1.15 Enlargement at A of Figure 1.14

1.2.12 Solution of trigonometric equations

The method best suited to the solution of trigonometric equations is that described in the section on algebra which deals with the method of solving transcendental equations by means of graphs. The expression to be solved is arranged as two identities and two graphs drawn as shown in Figure 1.14. The points of intersection of the curves projected on to the coordinate axes give the values which will satisfy the trigonometric equation.

Example 1.1 Solve $\sin(x+30)=\frac{1}{3}\cos x$ for x between 0 and 2π.

Assigning values to x in Table 1.4 and calculating the corresponding values for $y=\sin(x+30)$ and $y=\frac{1}{3}\cos x$ gives the readings for plotting the curves in Figure 1.14.

Plotting the curves between $x=169°$ and 170° shows that the intersection is at $x=169.11°$ to the second approximation. Greater accuracy can be obtained by continuing the small range large scale plots of the type in Figure 1.15.

There is one further value of x between $x=300°$ and 360° which will satisfy the equation as can be seen on Figure 1.14.

1.2.13 General solutions of trigonometric equations

Due to the periodic nature of the trigonometric functions there is an infinite number of solutions to trigonometric equations. Having obtained the smallest positive solution, α, the general solution for θ is then given by:

if	$a=\sin^{-1}x$	then	$\theta=n\pi+(-1)^n a$
	$a=\cos^{-1}x$		$\theta=2n\pi\pm a$
	$a=\tan^{-1}x$		$\theta=n\pi+a$

where θ and a are measured in radians and n is any integer.

1.2.14 Inverse trigonometric functions

Inverse functions of trigonometric variables may be simply defined by the example: $y=\sin^{-1}\frac{1}{2}$ which is merely a symbolic way of stating that y is an angle whose sine is $\frac{1}{2}$, i.e. y is actually 30° or $\pi/6$ in radian measure but need not be quoted if written as $\sin^{-1}\frac{1}{2}$.

1.3 Spherical trigonometry

1.3.1 Definitions

Referring to Figure 1.16, representing a sphere of radius r:

Small circle The section of a sphere cut by a plane at a section not on the diameter of the sphere, e.g. EFGH.

Table 1.4

x	0	30	60	90	120	150	180
$y_1 = \sin(x+30)$	0.5	0.866	1.0	0.866	0.5	0	−0.5
$y_2 = \frac{1}{3}\cos x$	0.333	0.289	0.166 7	0	−0.166 7	−0.289	−0.333

x	210	240	270	300	330	360
$y_1 = \sin(x+30)$	−0.866	−1.0	−0.866	−0.5	0	+0.5
$y_2 = \frac{1}{3}\cos x$	−0.289	−0.166 7	0	0.166 7	0.289	+0.333

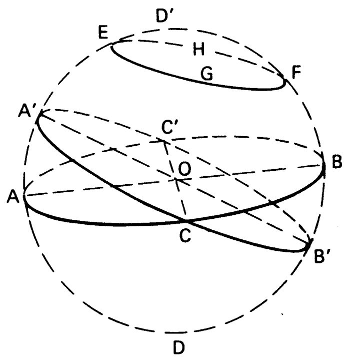

Figure 1.16 Sphere illustrating spherical trigonometry definitions

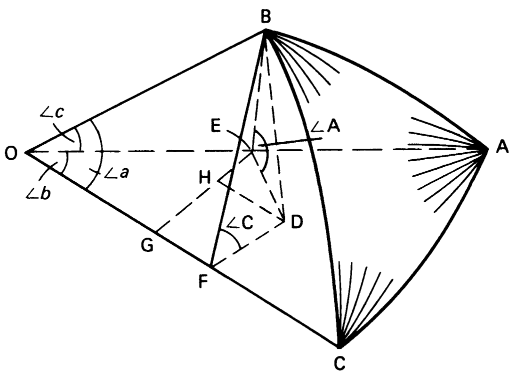

Figure 1.17 Spherical triangles

Great circle The section of a sphere cut by a plane through any diameter, e.g. ACBC′.

Poles Poles of any circular section of a sphere are the ends of a diameter at right angles to the section, e.g. D and D′ are the poles of the great circle ACBC′.

Lunes The surface areas of that part of the sphere between two great circles; there are two pairs of congruent areas, e.g. ACA′C′A; CBC′B′C and ACB′C′A; A′CBC′A′.

Area of lune If the angle between the planes of two great circles forming the lune is θ (radians), its surface area is equal to $2\theta r^2$.

Spherical triangle A curved surface included by the arcs of three great circles, e.g. CB′B is a spherical triangle formed by one edge BB′ on part of the great circle DB′BA the second edge B′C on great circle B′CA′C′ and edge CB on great circle ACBD′. The angles of a spherical triangle are equal to the angles between the planes of the great circles or, alternatively, the angles between the tangents to the great circles at their points of intersection. They are denoted by the letters C, B', B for the triangle CB′.

Area of spherical triangle $\mathrm{CB'B} = (B' + B + C - \pi)r^2$.

Spherical excess Comparing a plane triangle with a spherical triangle the sum of the angles of the former is π and the spherical excess E of a spherical triangle is given by $E = B' + B + C - \pi$; hence, area of a spherical triangle can be expressed as $(E/4\pi) \times$ surface of sphere.

Spherical polygon A spherical polygon of n sides can be divided into $(n-2)$ spherical triangles by joining opposite angular points by the arcs of great circles.

$$\text{Area of spherical polygon} = [\text{sum of angles} - (n-2)\pi]r^2$$

$$= \frac{E}{4\pi} \times \text{surface of sphere.}$$

Note that $(n-2)\pi$ is the sum of the angles of a plane polygon of n sides.

1.3.2 Properties of spherical triangles

Let ABC, in Figure 1.17, be a spherical triangle; BD is a perpendicular from B on plane OAC and $\mathrm{O\hat{E}D}$, $\mathrm{O\hat{F}D}$, $\mathrm{O\hat{E}B}$, $\mathrm{O\hat{F}B}$, $\mathrm{O\hat{G}E}$, $\mathrm{D\hat{H}G}$ are right angles; then $\mathrm{B\hat{E}D} = A$ and $\mathrm{B\hat{F}D} = C$ are the angles between the planes OBA, OAC and OBC, OAC respectively. $\mathrm{D\hat{E}H} = \mathrm{C\hat{O}A} = b$ also $\mathrm{C\hat{O}B} = a$, $\mathrm{A\hat{O}B} = c$, and since $\mathrm{OB} = \mathrm{OA} = \mathrm{OC} =$ radius r of sphere, $\mathrm{OF} = r\cos a$, $\mathrm{OE} = r\cos c$; then

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$
$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$
$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

Also the sine formulae are:

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$$

and the cotangent formulae are:

$$\sin a \cot c = \cos a \cos B + \sin B \cot C$$
$$\sin b \cot c = \cos b \cos A + \sin A \cot C$$
$$\sin b \cot a = \cos b \cos C + \sin C \cot A$$
$$\sin c \cot a = \cos c \cos B + \sin B \cot A$$
$$\sin c \cot b = \cos c \cos A + \sin A \cot B$$
$$\sin a \cot b = \cos a \cos C + \sin C \cot B$$

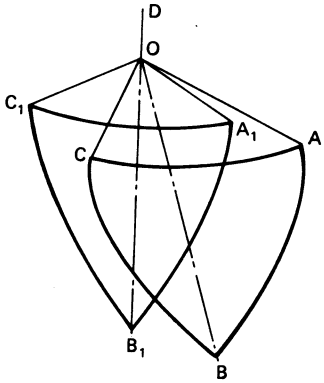

Figure 1.18 Polar triangles

In Figure 1.18, ABC, $A_1B_1C_1$ are two spherical triangles in which A_1, B_1, C_1 are the poles of the great circles BC, CA, AB respectively; then $A_1B_1C_1$ is termed the polar triangle of ABC and vice versa. Now OA_1, OD are perpendicular to the planes BOC and AOC respectively; hence $A_1\hat{O}D$ = angle between planes BOC and AOC $= C$. Let sides of triangle $A_1B_1C_1$ be denoted by $a_1b_1c_1$ then $c_1 = A_1\hat{O}B_1 = \pi - C$ also $a_1 = \pi - A$ and $b_1 = \pi - B$; $c = \pi - C_1$; $a = \pi - A_1$; $b = \pi - B_1$ and from these we get

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C} \qquad (1.20)$$

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \qquad (1.21)$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} \qquad (1.22)$$

1.3.2.1 Right-angled triangles

If one angle A of a spherical triangle ABC is 90° then $\cos a = \cos b \cos c = \cot B \cot C$

$$\cos B = \frac{\tan c}{\tan a}; \quad \cos C = \frac{\tan b}{\tan a}; \quad \sin B = \frac{\sin b}{\sin c};$$

$$\sin C = \frac{\sin c}{\sin a}; \quad \tan B = \frac{\tan b}{\sin c}; \quad \tan C = \frac{\tan c}{\sin b};$$

$$\cos B = \cos b \sin C; \quad \cos C = \cos c \sin B.$$

1.4 Hyperbolic trigonometry

The hyperbolic functions are related to a rectangular hyperbola in a manner similar to the relationship between the ordinary trigonometric functions and the circle. They are defined by the following exponential equivalents:

$$\sinh\theta = \frac{e^{\theta} - e^{-\theta}}{2} \qquad \operatorname{cosech}\theta = \frac{1}{\sinh\theta}$$

$$\cosh\theta = \frac{e^{\theta} + e^{-\theta}}{2} \qquad \operatorname{sech}\theta = \frac{1}{\cosh\theta}$$

$$\tanh\theta = \frac{\sinh\theta}{\cosh\theta} \qquad \coth\theta = \frac{1}{\tanh\theta}$$

1.4.1 Relation of hyperbolic to circular functions

$$\begin{aligned}
\sin\theta &= -i \sinh i\theta \\
\cos\theta &= \cosh i\theta \\
\tan\theta &= i \tanh i\theta \\
\operatorname{cosec}\theta &= i \operatorname{cosech} i\theta \\
\sec\theta &= \operatorname{sech} i\theta \\
\cot\theta &= i \coth i\theta \\
\sinh\theta &= -i \sin i\theta \\
\cosh\theta &= \cos i\theta \\
\tanh\theta &= -i \tan i\theta \\
\operatorname{cosech}\theta &= i \operatorname{cosec} i\theta \\
\operatorname{sech}\theta &= i \sec i\theta \\
\coth\theta &= i \cot i\theta
\end{aligned}$$

1.4.2 Properties of hyperbolic functions

$$\cosh^2\theta - \sinh^2\theta = 1$$

$$\operatorname{sech}^2\theta = 1 - \tanh^2\theta$$

$$\sinh 2\theta = 2\sinh\theta\cosh\theta$$

$$\cosh 2\theta = \cosh^2\theta + \sinh^2\theta$$

$$\operatorname{cosech}^2\theta = \coth^2\theta - 1$$

$$\tanh 2\theta = \frac{2\tanh\theta}{1 + \tanh^2\theta}$$

$$\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh y$$

$$\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$$

$$\tanh(x \pm y) = \frac{\tanh x \pm \tanh y}{1 \pm \tanh x \tanh y}$$

$$\sinh x + \sinh y = 2\sinh\tfrac{1}{2}(x+y)\cosh\tfrac{1}{2}(x-y)$$

$$\sinh x - \sinh y = 2\cosh\tfrac{1}{2}(x+y)\sinh\tfrac{1}{2}(x-y)$$

$$\cosh x + \cosh y = 2\cosh\tfrac{1}{2}(x+y)\cosh\tfrac{1}{2}(x-y)$$

$$\cosh x - \cosh y = 2\sinh\tfrac{1}{2}(x+y)\sinh\tfrac{1}{2}(x-y)$$

1.4.3 Inverse hyperbolic functions

As with trigonometric functions, we define the inverse hyperbolic functions by $y = \sinh^{-1} x$ where $x = \sinh y$:

Therefore: $x = (e^y - e^{-y})/2$

Rearranging and adding x^2 to each side:

$$e^{2y} - 2x.e^y + x^2 = x^2 + 1$$

or: $e^y - x = \sqrt{(x^2 + 1)}$

and therefore: $y = \sinh^{-1} x = \log_e [x + \sqrt{(x^2 + 1)}]$ (1.23)

The other inverse functions may be treated similarly. We find:

$\sinh^{-1} x = \log[x + \sqrt{(x^2 + 1)}]$;

$\cosh^{-1} x = \log[x + \sqrt{(x^2 - 1)}]$;

$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}$;

$$\operatorname{cosech}^{-1} x = \log \frac{1+\sqrt{(1+x^2)}}{x}$$

$$\operatorname{sech}^{-1} x = \log \frac{1+\sqrt{(1-x^2)}}{x}$$

$$\coth^{-1} x = \tfrac{1}{2} \log \frac{x+1}{x-1}$$

The relationships with the corresponding inverse trigonometric functions are as follows:

$$\begin{aligned}
\sinh^{-1} x &= -i \sin^{-1} ix \\
\cosh^{-1} x &= i \cos^{-1} x \\
\tanh^{-1} x &= -i \tan^{-1} ix \\
\sin^{-1} x &= -i \sinh^{-1} ix \\
\cos^{-1} x &= -i \cosh^{-1} x \\
\tan^{-1} x &= i \tanh^{-1} ix
\end{aligned}$$

1.5 Coordinate geometry

1.5.1 Straight-line equations

The equation of a straight line may be expressed as:

(1) $ax+by+c=0$ or $y=-\frac{a}{b}x-\frac{c}{b}=mx+n$ (1.24)

where a, b and c are constants and m is the slope of the line as shown in Figure 1.19.

(2) $\frac{x}{k}+\frac{y}{l}=1$ (1.25)

where k is the intercept on the x axis and l is the intercept on the y axis.

(3) $x\cos a+y\sin a=p$ (1.26)

where p = length of the perpendicular from the origin to the line and a the inclination of this perpendicular to Ox in Figure 1.20.

The length d of a perpendicular (see Figure 1.21) from any point $(x'y')$ to a straight line is given by $(ax'+by'+c)/\sqrt{(a^2+b^2)}$ if the straight line equation is as given in (1), or $(x'\cos a+y'\sin a-p)$ if the straight line equation is as given in (3).

The equation of a straight line through one given point $(x'y')$ is $y-y'=m(x-x')$.

The equation of a straight line through two given points (Figure 1.22) $(x_1y_1)(x_2y_2)$ is:

$$\frac{y-y_1}{y_2-y_1}=\frac{x-x_1}{x_2-x_1} \tag{1.27}$$

The angle ψ between two straight lines (Figure 1.23) $y=m_1x+n_1$ and $y=m_2x+n_2$ is given by:

$$\tan\psi=\frac{m_1-m_2}{1+m_1m_2} \tag{1.28}$$

For lines which are parallel $m_1=m_2$.
For lines at right angles $1+m_1m_2=0$.

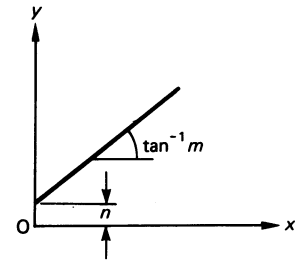

Figure 1.19 Straight-line equation $y=mx+n$

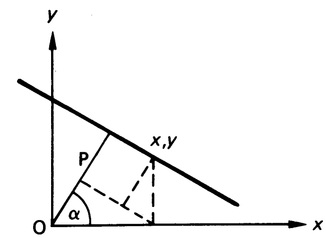

Figure 1.20 Straight-line equation $x\cos\alpha+y\sin\alpha=p$

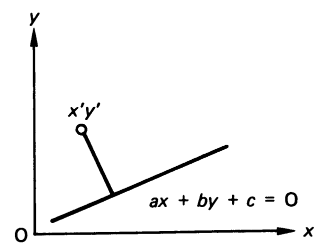

Figure 1.21 Perpendicular to straight line

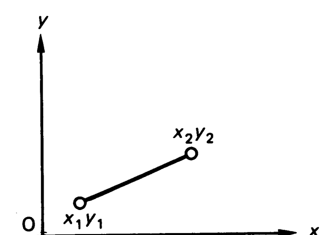

Figure 1.22 Straight line through two points

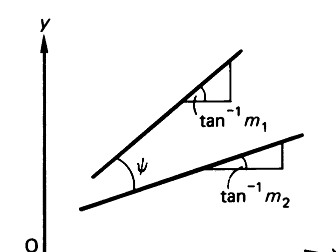

Figure 1.23 Angle ψ between two straight lines

1.5.2 Change of axes

Let the equation of the curve be $y=f(x)$ referred to coordinate axes Ox, Oy; then its equation relative to axes $O'x'$, $O'y'$ parallel to Ox, Oy with origin O' at point (r, s) is given by $y+s=j(x+r)$ in which x and y refer to the new axes.

If the equation of a curve is given by $y=f(x)$ referred to coordinate axes Ox, Oy, then if these axes are each rotated an angle ψ anti-clockwise about O, the equation of the curve referred to the rotated axes is given by $x \sin\psi + y\cos\psi = f(x\cos\psi - y\sin\psi)$.

1.5.2.1 Tangent and normal to any curve $y=f(x)$

The tangent PT and the normal PN at any point $x_1 y_1$ on the curve $y=f(x)$ in Figure 1.24 are given by the following equations:

Tangent: $y-y_1=\dfrac{dy}{dx}(x-x_1)$ where $\dfrac{dy}{dx}=m=$ the slope of the curve at P

Normal: $(y-y_1)\dfrac{dy}{dx}+(x-x_1)=0$

If ϕ be the angle which the tangent at P makes with the axis of x, then:

$$\tan\phi=\frac{dy}{dx};\ \cos\phi=\frac{dx}{ds};\ \sin\phi=\frac{dy}{ds}$$

where s is the distance measured along the curve.

1.5.2.2 Tangent and normal to any curve $f(xy)=0$

The function is implicit in this case so that partial differential coefficients are employed in the equations for the tangent and for the normal at x_1 y_1.

Tangent: $(y-y_1)\dfrac{\partial f}{\partial y}+(x-x_1)\dfrac{\partial f}{\partial x}=0$

Normal: $\dfrac{(y-y_1)}{(\partial f/\partial y)}=\dfrac{(x-x_1)}{(\partial f/\partial x)}$

where $\dfrac{dy}{dx}=-\dfrac{\partial f}{\partial x}\Big/\dfrac{\partial f}{\partial y}$

1.5.2.3 Subtangent and subnormal to any curve $y=f(x)$

The subtangent is TQ and the subnormal is QN at any point $P(x_1 y_1)$ on the curve $y=f(x)$ in Figure 1.24. Their lengths are given by:

Subtangent, $TQ=y_1\Big/\left(\dfrac{dy}{dx}\right)_1$

and subnormal, $QN=y_1\left(\dfrac{dy}{dx}\right)_1$

Example 1.2 Find the equation of the tangent and of the normal where $x=p$ on the curve $y=\cos \pi x/(2p)$

$\dfrac{dy}{dx}=-\dfrac{\pi}{2p}\sin\dfrac{\pi x}{2p}$ and when $x=p$, $\sin\dfrac{\pi x}{2p}=1$,

i.e. $\dfrac{dy}{dx}=-\dfrac{\pi}{2p}$ and $y=0$

Therefore:

the required equation of the tangent is $y=-\dfrac{\pi}{2p}(x-p)$

and the equation of the normal is $y=\dfrac{2p}{\pi}(x-p)$

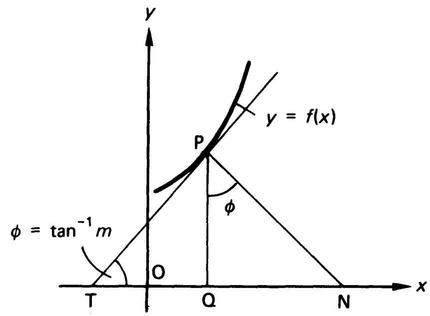

Figure 1.24 Tangent, normal, subtangent and subnormal to curve

1.5.3 Polar coordinates

The polar coordinates of any point P in a plane are given by r, θ where r is the length of the line joining P to the origin O and θ is the inclination of OP, the radius vector relative to the axis Ox (see Figure 1.25).

The relations between the rectangular coordinates x and y and the polar coordinates r and θ are:

$x=r\cos\theta$, $y=r\sin\theta$;

$r=\sqrt{(x^2+y^2)}$, $\theta=\tan^{-1} y/x$

If PT is a tangent to the curve at point P then:

$\tan\phi=r d\theta/dr$; $\cot\phi=(1/r)(dr/d\theta)$;

$\sin\phi=r d\theta/ds$ and $\cos\phi=dr/ds$

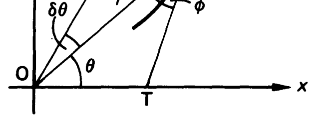

Figure 1.25 Polar coordinates

1.5.3.1 Polar subtangent and subnormal

In Figure 1.26 the polar subtangent is OR and the polar subnormal is OQ where QR is perpendicular to OP and their lengths are given by: polar subtangent $=r^2 d\theta/dr$; polar subnormal $=dr/d\theta$.

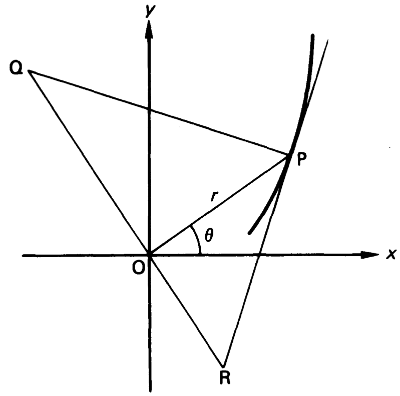

Figure 1.26 Polar subtangent and subnormal

1.5.3.2 Curvature

Let PQ in Figure 1.27 represent an elemental length δs of a given curve and PS, QT the tangents at the points P, Q then:

Curvature at P $= d\beta/ds$. For a circle centre at C, radius ρ, $ds = \rho\, d\beta$, i.e. curvature $= 1/\rho$.

Therefore: $$\rho = \frac{ds}{d\beta} = \text{radius of curvature} \qquad (1.29)$$

Putting $\beta = \tan^{-1}\left(\frac{dy}{dx}\right)$ and differentiating:

$$\text{Curvature} = \frac{d\beta}{ds} = \frac{1}{\rho} = \frac{\dfrac{d^2y}{dx^2}}{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}$$

$$\text{Radius of curvature } \rho = \frac{\left[1+\left(\dfrac{dy}{dx}\right)^2\right]^{3/2}}{\dfrac{d^2y}{dx^2}}$$

Where dy/dx is small (as in the bending of beams), the radius of curvature is given by:

$$\rho = \frac{1}{d^2y/dx^2} \qquad (1.30)$$

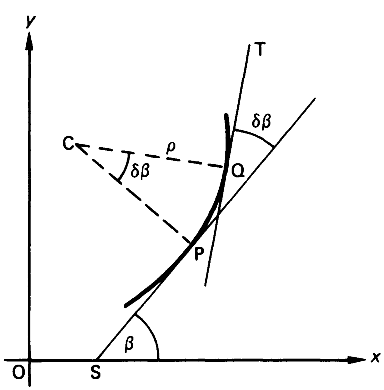

Figure 1.27 Curvature

Example 1.3 Find the radius of curvature at any point at the curve $y = a\cos x/a$:

$$\frac{dy}{dx} = \sinh\frac{x}{a}$$

Therefore:

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = \left[1+\sinh^2\frac{x}{a}\right]^{3/2}$$

$$= \left(\cosh^2\frac{x}{a}\right)^{3/2} = \cosh^3\frac{x}{a}$$

$$\frac{d^2y}{dx^2} = \frac{1}{a}\cosh\frac{x}{a}$$

Therefore:

$$\rho = \frac{a\cosh^3\dfrac{x}{a}}{\cosh\dfrac{x}{a}} = a\cosh^2\frac{x}{a} = \frac{y^2}{a}$$

1.5.4 Lengths of curves

1.5.4.1 General theory

From Figure 1.28:

$$ds^2 = dx^2 + dy^2$$

Hence:

$$ds = \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx = \sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\,dy$$

Therefore: $$s = \int_a^b \sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}\,dx$$

or: $$s = \int_c^d \sqrt{\left\{1+\left(\frac{dx}{dy}\right)^2\right\}}\,dy$$

For the evaluation of s for any given continuous function, use the first formula if x is single-valued, i.e. if one value of x corresponds to one point only in the function, e.g. Figure 1.29. If more than one point on the curve corresponds to one value of x, the second formula for a curve of the form shown in Figure 1.30, should be used.

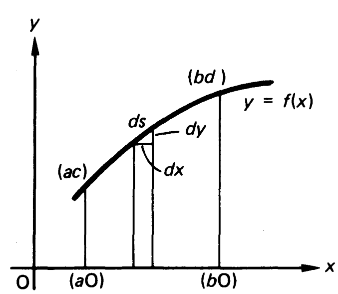

Figure 1.28

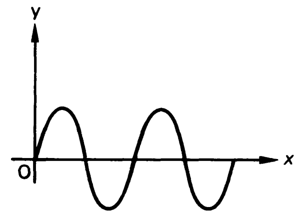

Figure 1.29

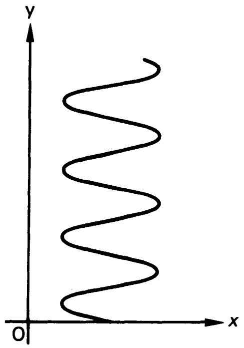

Figure 1.30

For polar coordinates, from Figure 1.31

$$ds=\sqrt{\{(\rho d\theta)^2+(d\rho)^2\}}=\sqrt{\left\{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right\}}\,d\theta$$

$$s=\int_{\theta_1}^{\theta_2}\sqrt{\left\{\rho^2+\left(\frac{d\rho}{d\theta}\right)^2\right\}}\,d\theta \qquad (1.32)$$

or:

$$s=\int_{\rho_1}^{\rho_2}\sqrt{\left\{1+\left(\rho\frac{d\theta}{d\theta\rho}\right)^2\right\}}\,d\rho \qquad (1.33)$$

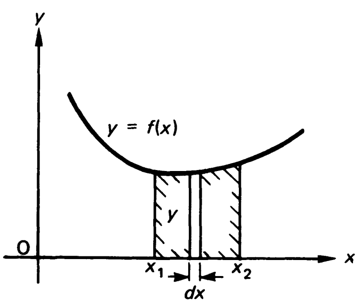

Figure 1.31

1.5.5 Plane areas by integration

See Figures 1.32 and 1.33.

1.5.5.1 General theory

From Figure 1.32, $A=\int_{x_1}^{x_2} y\,dx=\int_{x_1}^{x_2} f(x)\,dx$

1.5.5.2 Polar coordinates

From Figure 1.33, $dA=\frac{1}{2}\rho^2\,d\theta$

Therefore: $$A=\tfrac{1}{2}\int\rho^2 d\theta=\tfrac{1}{2}\int\{f(\theta)\}^2 d\theta \qquad (1.34)$$

(*Note*. For curve cutting x axis, equate $f(x)$ to zero, find values of x for $y=0$ and integrate between these values for the area cut off by the x axis.)

When the area lies above and below the x axis integrate the positive and negative areas separately and add algebraically.

Where the area does not extend to the x axis in the case of cartesian coordinates, or to the origin in the case of polar coordinates, then double integration must be used.

Thus: $$A=\iint dx.dy.\iint\rho.d\rho.d\theta \qquad (1.35)$$

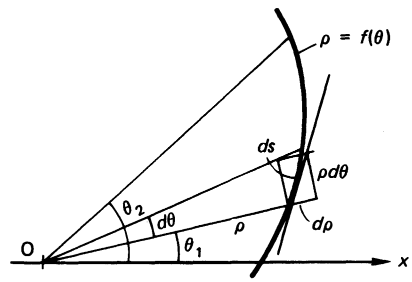

Figure 1.32

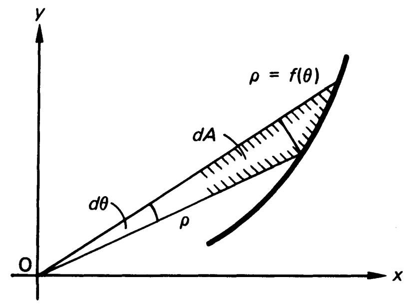

Figure 1.33

1.5.6 Plane area by approximate methods

See Figure 1.34.

1 *Trapezoidal rule*:

$$A=\frac{h}{2}\{y_0+2(y_1+y_2+\ \ldots\ +y_{n-1})+y_n\} \qquad (1.36)$$

(2) *Durand's rule*:

$$A=h(0.4y_0+1.1y_1+y_3+\ \ldots\ +y_{n-2}+1.1y_{n-1}+0.4y_n) \qquad (1.37)$$

(3) *Simpson's rule (*n *made even)*

$$A=\frac{h}{3}(y_0+4y_1+2y_2+4y_3+2y_4+\ \ldots\ +\ 2y_{n-2}+4y_{n-1}+y_n) \qquad (1.38)$$

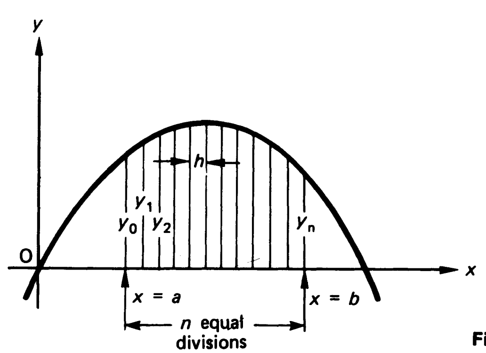

Figure 1.34

Of these, Simpson's is the most accurate. The accuracy is increased in all cases by increasing the number of divisions. Areas can often be determined more rapidly by plotting on squared paper and 'counting the squares' or by the use of a planimeter.

1.5.7 Conic sections

Conic sections refer to the various profiles of sections cut from a pair of cones vertex to vertex when intersected by a plane. Figure 1.35 shows a pair of cones generated by two intersecting straight lines AB, CD about the bisector EF of the angle between the lines.

Two straight lines. A section through the axis EF.

Circle. A section b–b parallel to the base of a cone.

Ellipse. A section c–c not parallel to the base of a cone and intersecting one cone only.

Parabola. A section d–d parallel to the side of a cone.

Hyperbola. A section e–e inclined to the side of a cone and intersecting both cones.

1.5.8 Properties of conic sections

A conic section is defined as the locus of a point P which moves so that its distance from a fixed point, the focus, bears a constant ratio, the eccentricity, to its perpendicular distance from a fixed straight line, the directrix.

Referring to Figure 1.36: the vertex of the curve is at V, the focus of the curve is at F, the directrix of the curve is the line DD parallel to yy'; the latus rectum is the line LR through the focus parallel to DD, $FL=FR=l$; the eccentricity of the curve is the ratio $FP/PQ=FV/VS=e$.

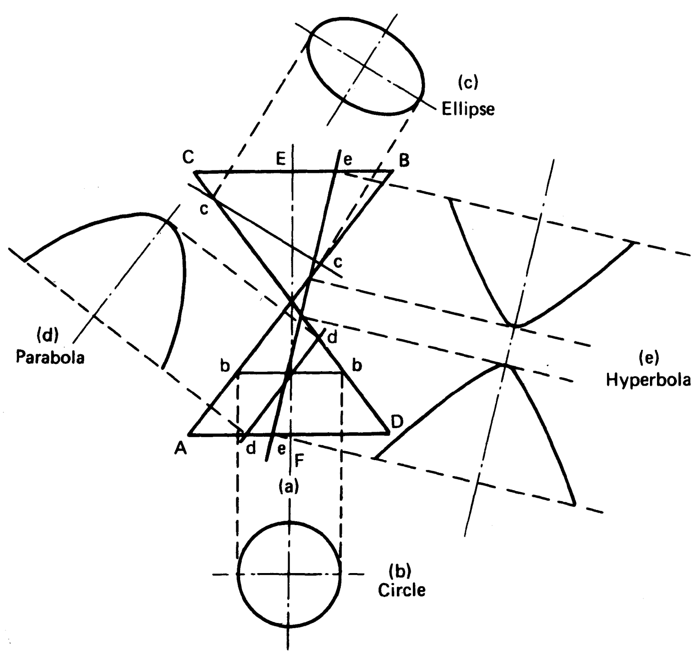

Figure 1.35 Circular cones generated by two intersecting straight lines

Then the curve is a parabola if $e=1$, an ellipse if $e<1$; and a hyperbola if $e>1$. A circle is a particular case of an ellipse in which $e=0$.

The polar equation of a conic is given by $l=\rho(1-e\cos\theta)$ where ρ is the radius vector of any point P on the curve, θ the angle the vector makes with VX and l the semi latus rectum.

Parabola ($e=1$) (see Figure 1.36).

1.5.8.1 Equations

With origin at V and putting $a=VS=VF$ then for P at (x, y): $(x-a)^2+y^2=(x+a)^2$, i.e. $y^2=4ax$.

1.5.8.2 Tangents

Let PT be a tangent at any point P $(x_1\ y_1)$ then the equation of PT is given by:

$$y-y_1=m(x-x_1)=(2a/y_1)(x-x_1)$$

or $yy_1=2a(x+x_1)$

since $d/dx(y^2)=2y\ dy/dx=4a$,

i.e. $m=dy/dx=2a/v_1$ at P $(x_1\ y_1)$.

Alternatively, if any straight line $y=mx+c$ meets the parabola $y^2=4ax$ then $(mx+c)^2=4ax$ at the points of intersection and this expression will satisfy the condition for tangency if the roots of $m^2x^2+2(mc-2a)x+c^2=0$ are equal, i.e. if $4(mc-2a)^2=4m^2c^2$ or $c=a/m$ so that the equation for the tangent may be expressed as $y=mx+a/m$ for all values of m where $m=dy/dx$, and tangency occurs at the point $(a/m^2, 2a/m)$.

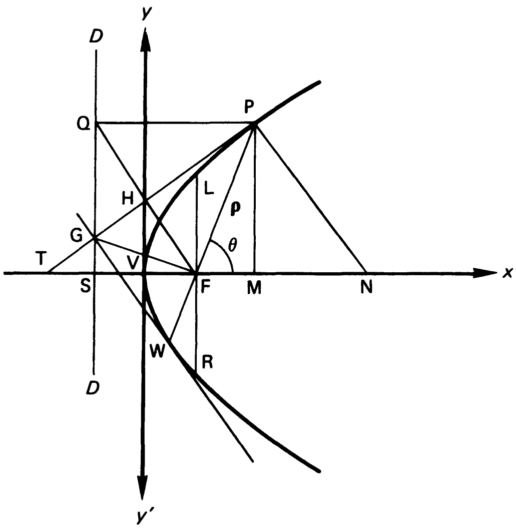

Figure 1.36 Properties of a conic section

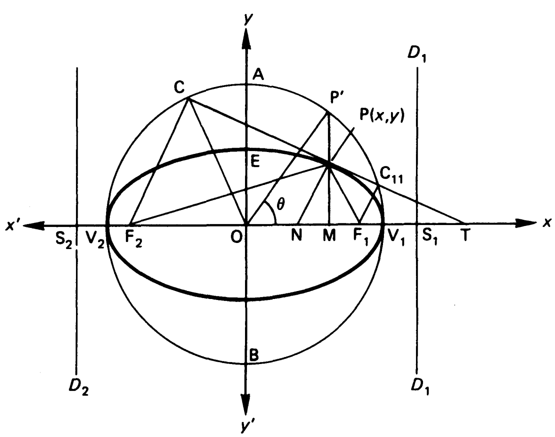

Figure 1.37 Ellipse in cartesian coordinates

1.5.8.3 Normal

Let PN be the normal at any point P $(x_1\ y_1)$; then the equation of PN is given by:

$$y-y_1=-(y_1/2a)(x-x_1) \tag{1.39}$$

1.5.8.4 General properties

Tangents:

(1) The tangent PT bisects F$\hat{\text{P}}$Q.
(2) The tangents PG, GW where PW is a focal chord intersect at G on DD.
(3) The tangent PT intersects the axis of the parabola at a point T where TV = VM; TF = SM = PF.
(4) The angles GFP, PHQ and PGW are right angles.

Normals: any normal PN intersects VX at N where FT = FN.

Subnormals: the subnormal MN is a constant length, i.e. MN = FS = 2*a*.

1.5.8.5 Ellipse (e < 1)

Referring to Figure 1.37, F_1, F_2 and the foci; D_1D_1, D_2D_2 the directrices.

$$e=\frac{F_1V_1}{S_1V_1}=\frac{F_1V_2}{S_1V_2}=\frac{F_2V_1}{S_2V_1}=\frac{F_1P}{MS_1}=\frac{F_2P}{MS_2}=\frac{OF_1}{OV_1}=\frac{OF_2}{OV_2}=\frac{F_1F_2}{V_1V_2}$$

Let OV_1 the semi-major axis $=a$ and OE the semi-minor axis $=b$,

then $OF_1=OF_2=ae$ and $OS_1=OS_2=\frac{a}{e}$

also $F_1P=a-ex$; $F_2P=a+ex$ $\therefore F_1P+F_2P=2a$

$F_1E=eOS_1=a$; $(OE)^2=b^2=(F_1E)^2-(OF_1)^2=a^2(1-e^2)$,

or $e^2=1-\frac{b^2}{a^2}$

Hence, as OM $=x$ and PM $=y$ we have the following.

1.5.8.5 Equation of ellipses

$y^2=a^2(1-e^2)-x^2(1-e^2)$

or $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in cartesian coordinates.

Substituting $\rho\cos a$ for x and $\rho\sin a$ for y (see Figure 1.38) in the above equation for an ellipse we have for the polar equation for an ellipse:

$$\frac{1}{\rho^2}=\frac{\cos^2 a}{a^2}+\frac{\sin^2 a}{b^2} \tag{1.40}$$

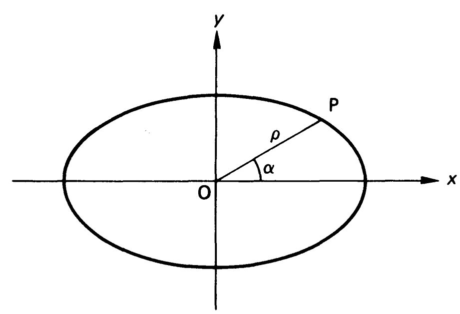

Figure 1.38 Ellipse in polar coordinates

1.5.8.6 Tangent

At any point *P* $(x_1\ y_1)$ on the ellipse
Let $f(xy)=0$ represent the curve, then:

$$\frac{dy}{dx}=-\frac{\partial f}{\partial x}\ \frac{\partial f}{\partial y} \tag{1.41}$$

Therefore $\partial f/\partial x=2x/a^2$ and $\partial f/\partial y=2y/b^2$ so that dy/dx at point $(x_1\ y_1)$ is given by $-b^2x_1/a^2y_1=m$. Substituting this value of m in the equation of a straight line $(y-y_1)=m(x-x_1)$ we have the equation of tangent PT: $xx_1/a^2+yy_1/b^2=1$.

Alternatively, the straight line $y=mx+c$ is a tangent to the ellipse $x^2/a^2+y^2/b^2=1$ when the roots of $x^2/a^2+(mx+c)^2/b^2-1=0$ are equal, i.e. when $c^2=a^2m^2+b^2$. Substituting we have for the equation of a tangent at any point P: $y=mx+\sqrt{(a^2m^2+b^2)}$.

The equation to the tangent may also be written in the form:

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1. \tag{1.42}$$

The coordinates of the point of contact are $(a\cos\theta,\ b\sin\theta)$, θ being known as the eccentric angle (see Figure 1.37).

1.5.8.7 Normal

Substituting the value of m above in the general equation for the normal PN to a curve at point P $(x_1\ y_1)$ given by: $(y-y_1)m+(x-x_1)=0$ we have as the equation for the normal $(y-y_1)b^2/y_1=(x-x_1)a^2/x_1$.

1.5.8.8 General properties

(1) The circle AV_2BV_1 is termed the auxiliary circle (Figure 1.37).
(2) $OM\times OT=a^2$.
(3) $F_2N=eF_2P$.
(4) $F_1N=eF_1P$.
(5) PN bisects $\angle F_1PF_2$.
(6) The perpendiculars from F_1, F_2 to any tangent meet the tangent on the auxiliary circle.

1.5.8.9 Circle (c=0)

The circle may be regarded as a particular case of the ellipse (see above). The equation of a circle of radius a with centre at the origin is $x^2+y^2=a^2$ or, in polar coordinates, $\rho=a$.

The equation of the tangent at the point $(x_1\ y_1)$ is $xx_1+yy_1=a^2$, or, $y=mx+a\surd(1+m^2)$. The equation of the normal is $xy_1-yx_1=0$.

1.5.8.10 Hyperbola (e>1)

This is shown in Figure 1.39 where F_1 F_2 are the foci, D_1D_1 and D_2D_2 the directrices and:

$$e=\frac{F_1V_1}{S_1V_1}=\frac{F_1P}{MS_1}=\frac{F_2P}{MS_2}=\frac{F_1V_2}{S_1V_2}=\frac{F_2V_1}{S_2V_1}=\frac{V_1V_2}{S_1S_2}=\frac{OV_1}{OS_1}=\frac{OV_2}{OS_2}$$

where O is the origin of the axes x and y.

Putting $OV_1=OV_2=a$ then $OF_1=OF_2=ea$ and $OS_1=OS_2=a/e$; also $F_1P=ex-a$ and $F_2F=ex+a$. Now $(F_1P)^2=(PM)^2+(F_1M)^2$, so $(ex-a)^2=y^2+(x-ae)^2$ which becomes $y^2=(e^2-1)x^2-(e^2-1)a^2$. Putting $(e^2-1)a^2=b^2$ then $y^2=(b^2/a^2)x^2-b^2$; therefore the equation of the hyperbola is given by $x^2/a^2-y^2/b^2=1$ in cartesian coordinates, or:

$$\frac{1}{\rho^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2} \tag{1.43}$$

in polar coordinates.

Rearranging we have $y=b\surd(x^2/a^2-1)$, i.e. y is imaginary when $x^2<a^2$ and $y=0$ for $x=\pm a$. y is real when $x>a$ and there are two values for y of opposite sign.

1.5.8.11 Conjugate axis

The conjugate axis lies on yy' and is given by CC′ where $OC=OC'=\pm b$.

1.5.8.12 Tangents

Let the straight line $y=mx+c$ meet the hyperbola $x^2/a^2-y^2/b^2=1$; then $x^2/a^2-(mx+c)^2/b^2-1=0$ will give the points of intersection. The condition for tangency is that the roots of this equation are equal, i.e. $c=\surd(a^2m^2-b^2)$ and the equation of the tangent is given by $y=mx+\surd(a^2m^2-b^2)$ at any point. Alternatively, the tangent to the hyperbola at $(x_1\ y_1)$ is given by $xx_1/a^2-yy_1/b^2=1$.

1.5.8.13 Normal

The equation for the normal at any point $(x_1\ y_1)$ on the curve is given by:

$$(y-y_1)b^2/y_1+(x-x_1)a^2/x_1=0. \tag{1.44}$$

1.5.8.14 Asymptotes

The tangent to the hyperbola becomes an asymptote when the roots of the equation $x^2/a^2-(mx+c)^2/b^2-1=0$ are both infinite, i.e. when $b^2-a^2m^2=0$ and $a^2mc=0$. Therefore: $m=\pm b/a$ and $c=0$. Substituting for m in $y=mx+c$ we have as the equation for an asymptote $y=\pm(b/a)x$. The combined equation for both asymptotes is given by:

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \tag{1.45}$$

The equation of the hyperbola referred to its asymptotes as oblique axes is:

$$X.Y=\frac{a^2+b^2}{4} \tag{1.46}$$

1.5.8.15 General properties

(1) $F_2P-F_1P=2a$.
(2) The product of the perpendiculars from any point on a hyperbola to its asymptotes is constant and equal to $a^2b^2/(a^2+b^2)$.

1.5.8.16 Rectangular hyperbola

When the transverse axis V_1V (Figure 1:39) is equal to the conjugate axis CC′ the hyperbola is a rectangular hyperbola, i.e. $a=b$ and the equation for the curve is given by $x^2-y^2=a^2$.

The equation for the asymptotes then becomes $y=\pm x$ which represents two straight lines at right angles to each other. The equation of the rectangular hyperbola referred to its asymptotes as axes of coordinates is given by $xy=$constant.

1.5.8.17 General equation of a conic section:

The general equation of a conic section has the form:

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \tag{1.47}$$

Let $D=\begin{array}{c|c|c} a & h & g \\ h & b & f \\ g & f & c \end{array}$ and $d=\begin{vmatrix} a & b \\ h & b \end{vmatrix}$

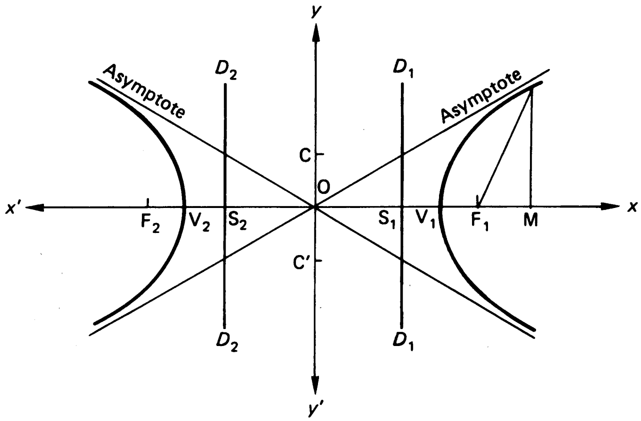

Figure 1.39 Hyperbola

Then, the general equation represents:

(1) An ellipse if $d>0$;
(2) A parabola if $d=0$;
(3) A hyperbola if $d<0$;
(4) A circle if $a=b$ and $h=0$;
(5) A rectangular hyperbola if $a+b=0$;
(6) Two straight lines (real or imaginary) if $D=0$;
(7) Two parallel straight lines if $d=0$ and $D=0$.

The centre of the conic (x_0y_0) is determined by the equations: $ax_0-hy_0+g=0$, $hx_0+by_0+f=0$.

1.6 Three-dimensional analytical geometry

1.6.1 Sign convention

1.6.1.1 Cartesian coordinates

This is shown in Figure 1.40, there being eight compartments formed by the right-angled intersection of three planes. The signs of x, y, z follow the convention that these are positive when measured in the directions Ox, Oy, Oz of the coordinate axes and negative when measured in the directions Ox', Oy', Oz' respectively.

1.6.1.2 Polar coordinates

The location of any point P in space (see Figure 1.41) is fully located by the radius vector $\boldsymbol{\rho}$ and the two angles θ and ϕ thus $(\boldsymbol{\rho}\theta\phi)$. From Figure 1.41:

$$\mathrm{OP}=\boldsymbol{\rho}=\sqrt{[(\mathrm{OD})^2+(\mathrm{OB})^2+(\mathrm{OC})^2]}=\sqrt{(x^2+y^2+z^2)}$$

and $\quad x=\boldsymbol{\rho}\sin\theta\cos\phi$; $y=\boldsymbol{\rho}\sin\theta\sin\phi$; $z=\boldsymbol{\rho}\cos\phi$.

1.6.1.3 Cylindrical coordinates

In this system the point P (Figure 1.41) is located by its perpendicular distance, z, from the x–y plane and the polar coordinates of the foot, A, of that perpendicular in the x–y plane. P is the point r, ϕ, z where OA $=r$.

1.6.1.4 Direction-cosines of a straight line

If the direction of the line OP in Figure 1.42 is determined by α, β, γ then the projections of a unit length of OP on to the axes Ox, Oy, Oz are given by $\cos\alpha$, $\cos\beta$, $\cos\gamma$ respectively, termed direction-cosines. Let $l=\cos\alpha$, $m=\cos\beta$, $n=\cos\gamma$ and CP $=\rho$; then $\rho^2(l^2+m^2+n^2)=x^2+y^2+z^2=\rho^2$, i.e. $l^2+m^2+n^2=1$.

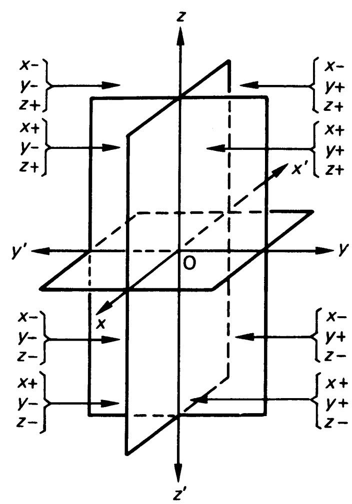

Figure 1.40 Sign conventions in analytical solid geometry

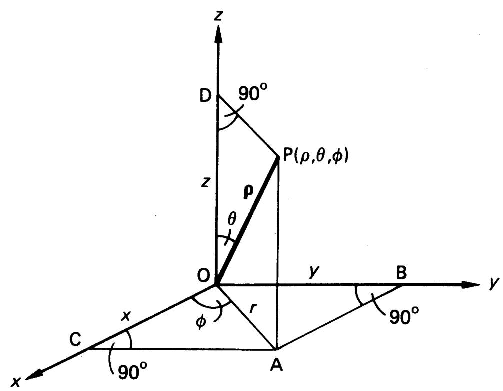

Figure 1.41 Polar coordinates in three dimensions

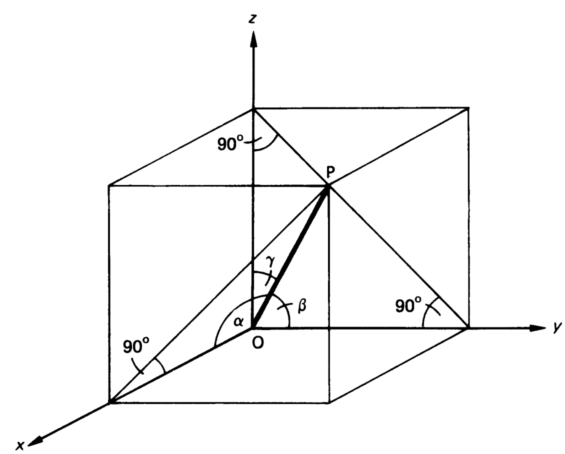

Figure 1.42 Direction-cosines

Also:

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = (1 - l^2) + (1 - m^2) + (1 - n^2) = 2$$

Again if:

$$l:m:n = s:t:u \text{ then } \frac{l^2}{s^2} = \frac{m^2}{t^2} = \frac{n^2}{u^2} = \frac{l^2+m^2+n^2}{s^2+t^2+u^2} = \frac{1}{s^2+t^2+u^2}$$

i.e.:

$$l = \frac{s}{\sqrt{(s^2+t^2+u^2)}};$$

$$m = \frac{t}{\sqrt{(s^2+t^2+u^2)}};$$

$$n = \frac{u}{\sqrt{(s^2+t^2+u^2)}}$$

1.6.1.5 General equations

The expression $F(xyz) = 0$ represents a surface of some kind and if we put $x = 0$ the resulting equation is for a curve in the y–z plane; similarly, for $y = 0$ the curve is in the x–z plane, etc. In general, any two simultaneous equations, $F(xyz) = 0$, $F'(xyz) = 0$ represent a line (either straight or curved) being the intersection of two surfaces. Any three such simultaneous equations represent a point (or several points).

1.6.2 Equation of a plane

The general equation of a plane is given by the expression $ax + by + cz + d = 0$ ($abcd$ being constants). By putting $y = 0$, $z = 0$ then $x = -d/a = a'$ which is the intercept of the plane on the x axis at a distance a' from the origin. Similarly, the intercepts on the y and z axes are b' and c'. Hence $a = -d/a'$; $b = -d/b'$; $c = -d/c'$ and substituting these values in the general equation for the plane we have the intercept equation for a plane as $x/a' + y/b' + z/c' = 1$.

In Figure 1.43 let P be any point on the plane ABC and let OQ of length p be at 90° to the plane ABC; then if l, m, n are the direction cosines of OQ we have $p = lx + my + nz$, which is the perpendicular form of the equation to a plane. The various forms of the equation to a plane are interchangeable since:

$$p = -\frac{d}{\sqrt{(a^2+b^2+c^2)}} = la' = mb' = nc'$$

and

$$\frac{1}{a'^2} + \frac{1}{b'^2} + \frac{1}{c'^2} = \frac{1}{p^2} \qquad (1.48)$$

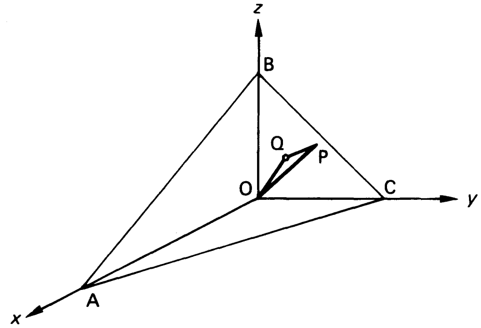

Figure 1.43 Equation of a plane

1.6.3 Distance between two points in space

Let the two points be $P(x_1y_1z_1)$; $Q(x_2y_2z_2)$. Assume origin shifted to P and axes kept parallel to original axes, then coordinates of Q relative to P are $(x_2 - x_1)$, $(y_2 - y_1)$, $(z_2 - z_1)$ and the length $PQ = r$, i.e. $r = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$ and the locus of Q is a sphere if r is constant.

1.6.4 Equations of a straight line

Using direction cosines for PQ, $l = (x_2 - x_1)/r$; $m = (y_2 - y_1)/r$; $n = (z_2 - z_1)/r$. If Q is taken as any point then the symmetrical equation of a straight line is given by $r = (x - x_1)/l = (y - y_1)/m = (z - z_1)/n$ and the coordinates of any point on the line are given by $x = x_1 + rl$; $y = y_1 + rm$; $z = z_1 + rn$. For a line through the origin this becomes:

$$r = \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

The equation of the straight line through the points $(x_1y_1z_1)$ and $(x_2y_2z_2)$ is:

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

1.6.4.1 Angle between two lines of known direction cosines

Let PA, QB be any two lines in space (Figure 1.44) and let P'O, Q'O be parallel to PA, QB respectively and having direction cosines $l_1m_1n_1$; $l_2m_2n_2$ respectively then $\cos \alpha = l_1l_2 + m_1m_2 + n_1n_2$ where $\alpha = P'\hat{O}Q'$

1.6.4.2 The angle between two planes

Let the equations of the planes be:

$$a_1x + b_1y + c_1z + d_1 = 0$$

and:

$$a_2x + b_2y + c_2z + d_2 = 0$$

then the direction-cosines of the normals to these planes are:

$$\frac{a_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}};\ \frac{b_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}};\ \frac{c_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

and:

$$\frac{a_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}};\ \frac{b_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}};\ \frac{c_2}{\sqrt{(a_2^2+b_2^2+c_2^2)}}$$

If θ is the angle between the planes, this is equal to the angle between the normals to these planes, i.e.:

$$\cos \theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{[(a_1^2+b_1^2+c_1^2)(a_2^2+b_2^2+c_2^2)]}} \qquad (1.49)$$

The planes are perpendicular to each other if $a_1a_2 + b_1b_2 + c_1c_2 = 0$. They are parallel if $a_1/a_2 = b_1/b_2 = c_1/c_2$.

1.6.4.3 The angle between a plane and a straight line

The angle θ between the plane $l_1x + m_1y + n_1z = p$ and the line $(x - x_1)/l_2 = (y - y_1)/m_2 = (z - z_1)/n_2$ is given by $\sin \theta = (l_1l_2 + m_1m_2 + n_1n_2)$.

1.6.4.4 Length of the perpendicular from a point $x_1y_1z_1$ to a plane

(1) Where the equation of the plane is the perpendicular form $lx+my+nz=p$ then the equation of a plane containing the point $(x_1y_1z_1)$ and parallel to the given plane is given by $lx+my+nz=p'$ where p and p' are the lengths of perpendiculars from the origin. Therefore required length of perpendicular is $p'-p=lx_1+my_1+nz_1-p$, since the point $(x_1y_1z_1)$ lies on the second plane.

(2) Where the equation of the plane takes the general form $ax+by+cz+d=0$ the length of perpendicular from point $x_1y_1z_1$ is given by:

$$\frac{ax_1+by_1+cz_1+d}{\sqrt{(a^2+b^2+c^2)}}$$

In the above the equation of the perpendicular is given by:

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

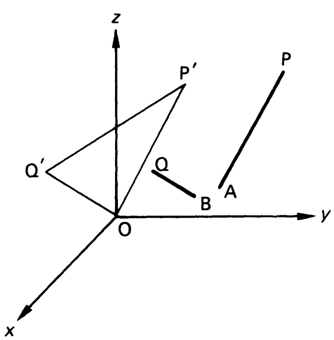

Figure 1.44 Angle between two lines of known direction-cosines

1.7 Calculus

The calculus deals with quantities which vary and with the rate at which this variation takes place.

Variables may be denoted by u, v, w, x, y, z and increments of these variables are denoted by $du, dv \ldots dz$. A simple example concerns the slope of a curve. Suppose that the curve is defined by some function:

$$y=f(x) \tag{1.50}$$

The slope at the point $x=x_1$ may be approximated to as follows. Let x_2 be close in value to x_1; then, provided the curve $f(x)$ is well behaved in the region of x_1, the line joining $f(x_1)$ to $f(x_2)$ is an approximation to the tangent to the curve at $x=x_1$. As x_2 is moved closer to x_1 the approximation becomes better and better, until, in the limit, when x_2 reaches x_1 the tangent (instead of the secant) is obtained and thereby the slope of the curve $y=f(x)$ is found at the point $x=x_1$. This process is known as 'differentiation'.

1.7.1 Differentiation

This process is used to find the value of dy/dx.
For combinations of functions u, and v of x:

$$\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx} \tag{1.51}$$

$$\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v(du/dx)-u(dv/dx)}{v^2} \tag{1.52}$$

For polynomial functions, $y=ax^n$:

$$\frac{dy}{dx}=nax^{n-1} \tag{1.53}$$

The differentiation process may be carried out more than once. Thus:

$$\frac{d}{dx}\frac{(dy)}{dx}=\frac{d^2y}{dx^2}\text{ etc.} \tag{1.54}$$

As an example, if $y=f(x)=ax^4+bx^3+cx^2+dx+c$

then: $\dfrac{dy}{dx}=f'(x)=4ax^3+3bx^2+2cx+d$

$$\frac{d^2y}{dx^2}=f''(x)=12ax^2+6bx+2c$$

$$\frac{d^3y}{dx^3}=f'''(x)=24ax+6b$$

$$\frac{d^4y}{dx^4}=f^{iv}(x)=24a$$

$$\frac{d^5y}{dx^5}=f^{v}(x)=0$$

It is often convenient, when dealing with long, complicated expressions, to substitute a symbol for a part of a compound expression. Suppose we have:

$$y=f(x) \tag{1.55}$$

a complicated expression and we choose to make a substitution u then the differential, $f'(x)$ can be found from the rule:

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$

e.g.: $y=(x^4+a^2)^6$

The substitution

$u=x^4+a^2$ is appropriate

so: $y=u^6$

thus: $\dfrac{dy}{dx}=6u^5$ and $\dfrac{du}{dx}=4x^3$

so: $\dfrac{dy}{du}=\dfrac{dy}{du}\dfrac{du}{dx}=6(x^4+a^2)^5 4x^3$

$$=24x^3(x^4+a^2) \tag{1.56}$$

In the case of trigonometric functions the differentiation process can be obtained via the expressions for multiple angles (see section 1.2).

For, suppose:

$$y=\sin\theta$$

We let:

$$y+\delta y=\sin(\theta+\delta\theta)$$

$$=\sin\theta\cos\delta\theta+\cos\theta\sin\delta\theta$$

$$=\sin\theta+\cos\theta.\delta\theta$$

So:

$$\delta y=\cos\theta.\delta\theta$$

$$\frac{\delta y}{\delta\theta}=\cos\theta \text{ or } \frac{\delta y}{\delta\theta}=\cos\theta \tag{1.57}$$

In cases where inverse trigonometric functions are involved, the principle of substitution is employed, for suppose:

$$y=\sin^{-1}u$$

then:

$$u=\sin y$$

$$\text{so:}\quad \frac{du}{dy}=\cos y=(1-\sin^2 y)=(1-u^2)$$

$$\text{so:}\quad \frac{dy}{du}=1 \quad \frac{du}{dy}=\frac{1}{(1-u^2)} \tag{1.58}$$

In cases where exponentiation is involved, the principle of substitution may again be employed:

For, suppose $y=e^{3x^2/4}$

we write $y=e^u$

where $u=3x^2/4$

$$\text{and}\quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u 6x/4$$

$$=\frac{3x}{2}e^{3x^2/4} \tag{1.59}$$

1.7.2 Partial differentiation

The dependent variable u may be a function of more than one independent variable, x and y, and we wish to find the rates of changes of u with respect to u and v separately. These rates of change, the partial differentials with respect to x and y are denoted by:

$$\frac{du}{dx} \text{ and } \frac{du}{dy}$$

In these processes the normal rules of differentiation are followed except that in finding du/dx, y is treated as a constant and in finding du/dy, x is treated as a constant.

The total differential of a function:

$$u=f(x,y)$$

where both x and y are functions of t is given by:

$$\frac{du}{dt}=\frac{du}{dx}\frac{dx}{dt}+\frac{du}{dy}\frac{dy}{dt} \tag{1.60}$$

1.7.3 Maxima and minima

Maxima and minima of functions occur when the function has zero slope or first differential. Thus, in order to determine a maximum or minimum of a function $y=f(x)$:

$$\text{we set:}\quad \frac{dy}{dx}=f'(x)=0 \tag{1.61}$$

and solve this equation, say $x=x_1$.

To distinguish between maxima and minima it is necessary to evaluate:

$$\frac{d^2y}{dx^2} \text{ at the point } x_1$$

$$\text{For a maximum:}\quad \frac{d^2y}{dx^2}<0$$

$$\text{For a minimum:}\quad \frac{d^2y}{dx^2}>0$$

1.7.4 Integration

Integration is generally the reverse of the process of differentiation. It may also be regarded as equivalent to a process of summing a number of finite quantities but, in the limit the number of quantities becomes infinite and their size becomes infinitesimal.

By the reverse of the differentiation process the integral

$$\int ax^n.dx=\frac{ax^{n+1}}{n+1}+c \tag{1.62}$$

the c being an arbitrary constant which, for shortness, is frequently not written. This is called an indefinite integral as no range over which the integration is to be performed has been specified. If such a range is specified then we obtain the case of a definite integral, e.g.:

if $F(x)\,dx=f(x)$

$$\text{then}\quad \int_b^a F(x)\,dx=f(b)-f(a) \tag{1.63}$$

In geometrical terms, this integral represents the area bounded by the curve $y=F(x)$, the x axis, and the two lines $x=a$, $x=b$.

1.7.5 Successive integration

This is the reverse process from that of successive differentiation, each cycle of operations consisting of the integration of the function resulting from the immediately previous integration. In general terms instructions to carry out successive integration are expressed thus: $y=\iiiint f(x)\ dx,\ dx,\ dx,\ dx$, which means integration is to be successively carried out 4 times with respect to x.

Another form of successive integration is $v=\iiint f(x, y, z)\, dx, dy, dz$; referred to as a volume integral. A surface integral would take the form $s=\iint f(x, y)\, dx, dy$.

Example 1.4 Find a general expression for the deflection of a simple span girder of span l loaded uniformly by a load w per unit length of span given that $w=EI\, d^4y/dx^4$ and taking E and I as constant and x as measured from one end.
Load:

$$EI\frac{d^4y}{dx^4}=w$$

Shear:

$$EI\int\frac{d^4y}{dx^4}dx=EI\frac{d^3y}{dx^3}=wx+c_1=wx-\frac{wl}{2}$$

$$\left(\text{Shear}=-\frac{wl}{2}\text{ for }x=0\right)$$

Bending moment:

$$EI\int\frac{d^3y}{dx^3}dx=EI\frac{d^2y}{dx^2}=\frac{wx^2}{2}-\frac{wlx}{2}+c_2=\frac{wx^2}{2}-\frac{wlx}{2}$$
(B.M. $=0$ for $x=0$)

Slope:

$$EI\int\frac{d^2y}{dx^2}dx=EI\frac{dy}{dx}=\frac{wx^3}{6}-\frac{wlx^2}{4}+c_3=\frac{wx^3}{6}-\frac{wlx^2}{4}+\frac{wl^3}{24}$$

$$\left(\text{Slope}=0\text{ for }x=\frac{l}{2}\right)$$

Deflection:

$$EI\int\frac{dy}{dx}dx=EIy=\frac{wx^4}{24}-\frac{wlx^3}{12}+\frac{wl^3x}{24}$$

(Deflection $=0$ for $x=0$)

i.e. at any distance x from one end the deflection

$$y=\frac{1}{24}\frac{w}{EI}(x^4-2lx^3+l^3x)$$

$$\iiiint\frac{d^4y}{dx^4}dx\,dx\,dx\,dx=\frac{w}{24EI}(x^4-2lx^3+l^3x)$$

which is in the general form.
For the mid-span deflection the range of integration is from $x=0$ to $x=\frac{1}{2}l$.

Hence:

$$\iiiint_0^{l/2}\frac{d^4y}{dx^4}dx.dx.dx.dx=\frac{wl^4}{24EI}\left(\tfrac{1}{16}-\tfrac{1}{4}+\tfrac{1}{2}\right)=5\frac{wl^4}{384EI}$$

1.7.6 Integration by substitution

The integration of functions can often be simplified by substituting a new variable for a part or the whole of the original function, thereby reducing it to one of the standard forms.

Example 1.5 Find the value of $\int\sqrt{(3+x)}\,dx$

Let $3+x=u$

Therefore $dx=du$

so that
$\int\sqrt{(3+x)}\,dx=\int u^{\frac{1}{2}}\,du=\frac{2}{3}u^{3/1}=\frac{2}{3}(3+x)^{3/2}$ *or* $\frac{2}{3}\sqrt{(3+x)^3}$

Example 1.6 Find the value of

$$2\int\frac{dx}{e^{3x}+c^{-3x}}$$

Let $e^{3x}=v$; then $3e^{3x}.dx=dv$, or $dx=dv/3v$.

Substituting

$$2\int\frac{dx}{e^{3x}+e^{-3x}}=\tfrac{2}{3}\int\frac{dv}{v(v+1/v)}=\tfrac{2}{3}\int\frac{dv}{v^2+1}=\tfrac{2}{3}\tan^{-1}v=\tfrac{2}{3}\tan^{-1}e^{3x}.$$

Example 1.7 Find the value of $\int\sqrt{(1-x^2)}\,dx$.
Put $x=\sin\theta$; then $\sqrt{(1-x^2)}=\cos\theta$

Therefore
$\int\sqrt{(1-x^2)}\,.dx=\int\cos\theta.d\sin\theta=\int\cos^2\theta.d\theta$

$$=\int\frac{1+\cos 2\theta}{2}\cdot d\theta$$

$$=\frac{\theta}{2}+\frac{\sin 2\theta}{4}=\frac{1}{2}\{\sin^{-1}x+x\sqrt{(1-x^2)}\}$$

1.7.7 Integration by transformation

The integration of trigonometric functions can often be simplified by transformation into a standard form of integral.

TYPE

$$\int\sin^m\theta\cos^n\theta d\theta$$

Case 1: $m=$ positive odd integer, $n=$ any positive integer.

TRANSFORMATIONS

$$\int\sin^{m-1}\theta\sin\theta\cos^n\theta d\theta$$

$$=\int(1-\cos^2\theta)^{(m-1)/2}\sin\theta\cos^n\theta d\theta$$

$$=-\int(1-\cos^2\theta)^{(m-1)/2}\cos^n\theta d(\cos\theta)$$

Example 1.8 Solve $\int\sin^3\theta\cos^2\theta d\theta$

$$\int\sin^3\theta\cos^2\theta d\theta$$

$$=\int(1-\cos^2\theta)\sin\theta\cos^2\theta d\theta$$

$$=\int\cos^2\theta\sin\theta d\theta-\int\cos^4\theta\sin\theta d\theta$$

$$= -\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}$$

Case 2: m = any positive integer, n = positive odd integer.

TRANSFORMATION

$\int \sin^m\theta \cos^{n-1}\theta d\theta = \int (1-\sin^2\theta)^{(n-1)/2} \sin^m \theta d(\sin\theta)$.

Example 1.9 Solve $\int \sin^2\theta \cos^3\theta d\theta$

$\int \sin^2\theta \cos^3\theta d\theta = \int (1-\sin^2\theta) \sin^2\theta \cos\theta d\theta$

$= \int \sin^2\theta \cos\theta d\theta - \int \sin^4\theta \cos\theta d\theta$

$$= \frac{\sin^3\theta}{3} - \frac{\sin^5\theta}{5}$$

TYPE

$\int \tan \theta d\theta$ where n is an integer >1.

TRANSFORMATION

$\int \tan^{n-2}\theta \tan^2\theta d\theta = \int \tan^{n-2}\theta (\sec^2\theta - 1) d\theta$

$= \int \tan^{n-2}\theta . \tan\theta - \int \tan^{n-2}\theta . d\theta$

TYPE

$\int \cot^n \theta d\theta$ where n is an integer >1.

TRANSFORMATION

$\int \cot^{n-2}\theta \cot^2\theta d\theta = \int \cot^{n-2}\theta(\text{cosec}^2\theta - 1) d\theta$

$= -\int \cot^{n-2}\theta . d\cot\theta - \int \cot^{n-2}\theta . d\theta$.

TYPE

$\int \sec^n \theta d\theta$ where n is positive and even.

TRANSFORMATION

$\int \sec^{n-2}\theta \sec^2\theta d\theta = \int (\tan^2\theta + 1)^{(n-2)/2} d\tan\theta$.

TYPE

$\int \text{cosec}^n \theta d\theta$ where n is positive and even.

TRANSFORMATION

$\int \text{cosec}^{n-2}\theta \,\text{cosec}^2\theta d\theta = \int -(\cot^2\theta + 1)^{(n-2)/2} d\cot\theta$

TYPE

$\int \tan^m\theta \sec^n\theta d\theta$ where n is positive and even.

TRANSFORMATION

$\int \tan^m\theta \sec^{n-2}\theta \sec^2\theta d\theta = \int \tan^m\theta (\tan^2\theta + 1)^{(n-2)/2} d\tan\theta$.

TYPE

$\int \cot^m\theta \,\text{cosec}^n\theta d\theta$ where n is positive and even.

TRANSFORMATION

$\int \cot^m\theta \,\text{cosec}^{n-2}\theta \,\text{cosec}^2\theta d\theta = \int -\cot^m\theta (\cot^2\theta + 1)^{(n-2)/2} d\cot\theta$.

TYPE

$\int \tan^m\theta \sec^n\theta d\theta$ where m and n are odd.

TRANSFORMATION

$\int \tan^{m-1}\theta \tan\theta \sec^{n-1}\theta \sec\theta d\theta$
$= \int (\sec^2\theta - 1)^{(m-1)/2} . \sec^{n-1}\theta . d\sec\theta$.

1.7.8 Integration by parts

The integration of functions can often be simplified by breaking up the function into two parts u and dv where u and v are the substituted variables in $\int u.dv = u.v - \int v.\text{du}$, the fundamental formula for integration by parts, $\int u.dv$ representing the function to be integrated. In applying this method of integration $\int v.\text{du}$ should not be more complex than $\int u.dv$. The integration of logarithmic, exponential, inverse trigonometric and products of algebraic expressions may be simplified by this procedure.

Example 1.10 $\int w \sin w\, dw$
Let $u = w$ and $dv = \sin w\, dw$ then $du = dw$ and $v = -\cos w$

Therefore:

$$\int w \sin w . dw = \int u.dv = -w\cos w + \int \cos w\, dw$$
$$= -w\cos w + \sin w$$

Example 1.11 $\int xe^x dx$
Let $u = x$ and $dv = c^x dx$ then $du = dx$ and $v = e^x$

Therefore:

$$\int xe^x dx = \int u.dv = xe^x - \int e^x dx = xe^x - e^x = e^x(x-1)$$

Example 1.12 $\int \cos^2\theta d\theta$
Let $u = \cos\theta$ and $dv = \cos\theta d\theta$ then $du = -\sin\theta d\theta$ and $v = \sin\theta$

Therefore:

$$\int \cos^2\theta d\theta = \int u.dv = \cos\theta \sin\theta + \int \sin^2\theta d\theta$$
$$= \frac{\sin 2\theta}{2} + \int (1 - \cos^2\theta) d\theta$$

i.e.:

$$2\int \cos^2\theta d\theta = \frac{\sin 2\theta}{2} + \theta \text{ hence } \int \cos^2\theta d\theta = \frac{\sin 2\theta}{4} + \frac{\theta}{2}$$

Example 1.13 $\int \sec^3\theta d\theta$
Let $u = \sec\theta$ and $dv = \sec^2\theta d\theta$ then $du = \sec\theta \tan\theta d\theta$ and $v = \tan\theta$

Therefore:

$$\int \sec^3\theta d\theta = \sec\theta \tan\theta - \int \tan^2\theta \sec\theta d\theta$$

$$= \sec\theta \tan\theta - \int \sec^3\theta d\theta + \int \sec\theta d\theta$$

i.e.:

$$2\int \sec^3\theta d\theta = \sec\theta\tan\theta + \log_e\left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}$$

Therefore:

$$\int \sec^3\theta d\theta = \frac{1}{2}\left[\sec\theta\tan\theta + \log_e\left\{\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}\right]$$

1.7.9 Integration of fractions

The integration of functions consisting of rational algebraic fractions is best carried out by first splitting the function into partial fractions. It is assumed that the numerator is of lower degree than the denominator; if not, this should first be achieved by dividing out. It may be shown that the prime real factors of any polynomial are either quadratic or linear in form. This leads to four distinct types of partial fraction solutions which are now described.

1.7.9.1 Fractions type 1

The denominator can be factored into real linear factors all different. The partial fractions are then of the form $a/(bx+c)$.

Example 1.14 $\int \frac{2x+3}{x^2-4}dx$

Now

$$\frac{2x+3}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

i.e.: $2x+3 = A(x+2)+B(x-2)$

i.e. $A=\frac{7}{4}$ and $B=\frac{1}{4}$

Therefore:

$$\int\frac{2x+3}{x^2-4}dx = \frac{7}{4}\int\frac{dx}{x-2} + \frac{1}{4}\int\frac{dx}{x+2} = \frac{7}{4}\log_e(x-2)+\frac{1}{4}\log_e(x+2)$$

1.7.9.2 Fractions type 2

The prime factors of the denominator include quadratic functions and all factors are different. The partial fractions then include expressions of the form $(ax+b)/(cx^2+dx+e)$.

Example 1.15 $\int\frac{7x^2-3}{2x^3-3x^2+4x-6}\cdot dx$

Put $$\frac{7x^2-3}{2x^3-3x^2+4x-6} = \frac{Ax+B}{x^2+2}+\frac{C}{2x-3}$$

i.e.: $7x^2-3 = (Ax+B)(2x-3)+C(x^2+2)$

$$= (2A+C)x^2+(2B-3A)x-(3B-2C)$$

and therefore $A=2$, $B=3$, $C=3$

Therefore:

$$\int\frac{7x^2-3}{2x^3-3x^2+4x-6}\cdot dx = \int\frac{2x+3}{x^2+2}\cdot dx + \int\frac{3}{2x-3}\cdot dx$$

$$= \int\frac{2x}{x^2+2}\cdot dx + \int\frac{3}{x^2+2}\cdot dx + \int\frac{3}{2x-3}\cdot dx$$

$$= \log_e(x^2+2) + \frac{3}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + \frac{3}{2}\log_e(2x-3)$$

1.7.9.3 Fractions type 3

The denominator can be factored into real linear factors, some of which are repeated. The partial fractions then include expressions of the form $a/(bx+c)^n$.

Example 1.16 $\int\frac{3x^2+8x+16}{x^3+3x^2-4}dx = \int\frac{f(x)}{F(x)}dx$

$$F(x) = (x-1)(x+2)^2 \text{ and } \frac{f(x)}{F(x)} = \frac{A}{x-1}+\frac{B}{(x+2)}+\frac{C}{(x+2)^2}$$

Hence:

$$3x^2+8x+16 = A(x+2)^2+B(x-1)(x+2)+C(x-1)$$

putting $x=1$ then $A=3$; $x=-2$ then $C=-4$; substitution gives $B=0$

Therefore:

$$\int\frac{f(x)}{F(x)}dx = 3\int\frac{dx}{x-1} - 4\int\frac{dx}{(x+2)^2} = 3\log_e(x-1)+\frac{4}{(x+2)}$$

1.7.9.4 Fractions type 4

The prime factors of the denominator include quadratic functions some of which are repeated. The partial fractions then include expressions of the form $(ax+b)/(cx^2+dx+e)^n$.

Example 1.17 $\int\frac{12x-1}{(x^2+1)^2(x+2)}dx = \int\frac{f(x)}{F(x)}dx$

$$\int\frac{f(x)}{F(x)}dx = \frac{Ax+B}{(x^2+1)^2}+\frac{Cx+D}{(x^2+1)}+\frac{E}{(x+2)}$$

i.e.

$$12x-1 = (Ax+B)(x+2)+(Cx+D)(x^2+1)(x+2)+E(x^2+1)^2.$$

Put $x=-2$; then $E=-1$.

Therefore:

$$x^4+2x^2+12x = Cx^4+(D+2C)x^3+(A+2D+C)x^2+(2A+B+D+2C)x+2(B+D)$$

Equating coefficients, we find $C=1$, $D=-2$, $B=2$ and $A=5$

Therefore:

$$\int\frac{f(x)}{F(x)}dx = \int\frac{5x+2}{(x^2+1)^2}dx + \int\frac{x-2}{x^2+1}dx - \int\frac{dx}{x+2}$$

$$= \frac{5}{2}\int \frac{d(x^2+1)}{(x^2+1)^2} + 2\int \frac{dx}{(x^2+1)^2} + \frac{1}{2}\int \frac{d(x^2+1)}{x^2+1}$$
$$-2\int \frac{dx}{x^2+1} - \int \frac{dx}{x+2}$$

$$= -\frac{5}{2}\int \frac{1}{x^2+1} + \frac{x}{x^2+1} + \int \frac{dx}{x^2+1} + \frac{1}{2}\log_e(x^2+1)$$
$$-2\tan^{-1}x - \log_e(x+2)$$

$$= \frac{2x-5}{2(x^2+1)} + \log_e \frac{\sqrt{(x^2+1)}}{x+2} \tan^{-1}x$$

1.8 Matrix algebra

A matrix is an array of mn numbers in m rows and n columns

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The element in the ith row and jth column a_{ij} is called the (i, j)th element and the matrix is often denoted by $[a_{ij}]$ or A. When $m=n$ the matrix is square. An $m \times 1$ matrix is called a column vector or column matrix.

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \qquad (1.64)$$

A $1 \times n$ matrix is called a row vector

$$Y = [y_1, y_2 \ldots y_n] \qquad (1.65)$$

1.8.1 Addition of matrices

Two matrices may be added if and only if they are of the same order $m \times n$.

Then: $A+B=[a_{ij}]+[b_{ij}]=[(a_{ij}+b_{ij})]$

i.e. the sum is formed by adding corresponding elements.

1.8.2 Multiplication of matrices

(1) By a scalar.
Any matrix may be multiplied by a scalar.

Then $\lambda A = \lambda[a_{ij}] = [(\lambda a_{ij})] = A\lambda$

i.e. all the elements of A are multiplied by λ.

(2) Multiplication of two matrices.
Two matrices may be multiplied (A times B in that order) only if the number of columns of A is equal to the number of rows of B. If A is $[a_{ij}]$ of order $m \times n$ and B is $[b_{ij}]$ of order $n \times p$ then

$$AB = [a_{ij}][b_{ij}] = \left[\left(\sum_{k=1}^{n} a_{ik} b_{kj} \right) \right]$$

is of order $m \times p$.
It should be noted that, in general $AB \neq BA$.

1.8.3 The unit matrix

The unit matrix is a square matrix I for which:

$a_{ij}=0$ for $i \neq j$
$a_{ij}=1$ for $i=j$

1.8.4 The reciprocal of a matrix

The reciprocal matrix A^{-1} of A exists only if the determinant of A is nonzero and is given by:

$$AA^{-1} = I = A^{-1}A$$

1.8.5 Determinants

The determinant of a square matrix is defined as:

$$|A| = \|a_{ij}\| = \sum(\pm a_{1\alpha} a_{2\beta} \ldots a_{n\nu})$$

the summation of $n!$ terms being over all the arrangements $(\alpha, \beta, \ldots \nu)$ of the column suffixes and the sign $\pm$ being chosen according to whether the arrangement is even or odd.
In the simplest case,

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

and from this can be developed the expressions for the expansion of determinants of higher order than the second. The minor of a_{ij} in A is the determinant of the matrix obtained by deleting the ith row and jth column of A. The cofactor A_{ij} of a_{ij} in A is $(-1)^{i+j} \times$ minor of a_{ij}.
Now the expression for a determinant is given by:

$$|A| = a_{i1}|A_{i1}| + a_{i2}|A_{i2}| \ldots + a_{in}|A_{in}|$$

The value of a determinant is unaltered by interchanging rows with columns. Interchanging either two rows or two columns changes the sign of a determinant. Thus, if either two rows or two columns are identical the determinant is zero.

1.8.6 Simultaneous linear equations

Simultaneous linear equations can be arranged in matrix form and their solution obtained via determinants

$$\begin{matrix} a_{11}x_1 + a_{12} & x_2 + & \ldots & + a_{in}x_n = b_1 \\ \vdots & & & \vdots \quad \vdots \\ a_{m1}x_1 + \ldots & & & + a_{mn}x_n = b_m \end{matrix}$$

may be written $AX=B$

and now $X=A^{-1}B$

alternatively $x_j = \dfrac{|D_j|}{|D|}$

where D denotes the determinant $|a_{ij}|$ and D_j denotes the determinant D with the elements $a_{ij}a_{2j} \ldots a_{mj}$ replaced by $b_1 b_2 \ldots bm$.

STATISTICS

1.9 Introduction

Statistical techniques are used in engineering mainly in connection with the quality control of manufacturing of produced material and with the checking for compliance of such products, with whatever specifications or clauses are contained in the contracts covering their purchase and sale. In order to exercise quality control or to check for compliance it is necessary to make measurements of one sort or another. Now it is well established that the result of repeating a measurement (or of repeating an experiment) does not generally repeat the observation or original result. Further, repeat measurements will lead to further results and so appears the problem of variability.

Generally speaking, the variation in results arises both because the subjects of the measurement are themselves different and also because of errors introduced by the experiment or the measuring technique. Such variation is common experience in the measurement of, for example, the strengths of materials. It will often be desirable (if only from an economic viewpoint) to reduce the variation to as small an amount as can conveniently be arranged. However, it is not generally possible to reduce such variation to an unimportantly small value and so it becomes necessary to deal with the problem posed by the obtaining of different results from apparently identical experiments. It is to deal with the evaluation of such scattered experimental results that statistical techniques have been developed.

It is supposed that, were it possible to continue the experiments indefinitely, the results so obtained would cluster around some fixed value which would be the required value. (It is an implicit assumption that the indefinite series of experiments be conducted under identical conditions.) Since it is not possible to conduct indefinitely long experiments the problem becomes that of trying to determine, from a finite series of experiments, that fixed value (which is presumably the true value) about which the indefinite series of results would cluster. This attempt to determine is known as 'estimating', and while the use of that particular word does not imply that there has been any guesswork in obtaining it, there is an implication of uncertainty about the result. In statistical methods this uncertainty is calculated and specified in terms of confidence limits. A result obtained after statistical calculations should generally be given in terms of an estimate surrounded by confidence limits. Of course the more nearly certain we wish to be that the confidence limits contain the true value the wider those limits must be. In cases where the experiment or test is not aimed at the estimating of some particular quantity, the form of the estimate and confidence limit changes to one that such and such a result would not have arisen 'by chance' more than on so many per cent of occasions in an indefinitely long series of trials.

It is important that statistical results should be properly presented in the form of estimate and confidence limits: having decided upon such a form it is then sensible to use an appropriate precision for reporting the values. For example, when estimating the strength of concrete where an estimate might be of the form: $42 \pm 5\ \mathrm{N/mm^2}$ (at 95% confidence) there is clearly no point in reporting the result to several decimal places.

When an estimate of some quantity has been obtained, the interval between the confidence limits may be wider than it is desired they should be, in which case the interval may be narrowed by accepting a lower confidence. If this is not desirable it will be necessary to: (1) take more observations; or (2) improve the experimental techniques used to reduce the variability of the results.

It is important that the question of what is required by way of precision should be considered prior to an experiment so that the number of observations necessary to obtain the required precision may be assessed. In making that assessment it will be necessary to have information about the variability of parts of the experiment. This information may be available from previous experience, but if not it must be obtained by a pilot experiment.

It will be clear from the foregoing that any result which is obtained, being subject to error, may cause a wrong decision to be taken. Thus when dealing with, for instance, material to be checked for strength the contract for the supply of the material should indicate a test scheme to determine whether the strength of the material is correct or not.

Such a test scheme will involve experiments, and the possibilities for a wrong decision are:

(1) That the test will wrongly show as unsatisfactory, material with the correct strength. (This is known as the manufacturer's or supplier's risk.)
(2) That the test will wrongly show as satisfactory, material with an incorrect strength. (This is known as the consumer's risk.)

The performance of a test scheme is defined by its power and is represented by a graph showing, on one axis, the true value of the parameter in question (e.g. the strength of the material) plotted against the probability that material will pass the test and so be accepted. The calculation of such graphs is not a simple matter and requires full information about all aspects of the test scheme under consideration. The power curves of two test schemes represent, however, the only way in which the performance of the two schemes may be compared.

In the following sections are presented definitions of some of the terms used in statistical work, descriptions of statistical techniques and tests which may be used as a part of the experimenter's armoury of techniques and a description of central charts as a method of quality control. In the final section the references have, in the first cases, been selected for their readability as well as for their coverage of any particular point. Thus the works by Moroney[1] and Neville and Kennedy[2] are especially recommended as initial reading for anyone interested in statistical problems and techniques.

1.10 Definitions of elementary statistical concepts

1.10.1 Statistical unit or item

One of a number of similar articles or parts each of which may possess several different quality characteristics.

Example 1.18 A piece of glass tubing taken from a large number produced in quantity for which the diameter and other characteristics may be measured; a concrete cube for which the strength may be measured.

1.10.2 Observation – observed value

The value of a quality characteristic measured or observed on a unit.

Example 1.19 The diameter in millimetres of a piece of tubing; the strength of a concrete cube.

1.10.2.1 Sample

A portion of material or a group of units taken from a larger number which is used to obtain estimates of the properties of the larger quantity.

Example 1.20 Forty-eight pieces of tubing sampled from all the pieces produced during a day; the concrete cube made from a batch of concrete.

1.10.2.2 Random sample

A sample selected in such a manner that every item has an equal chance of inclusion.

1.10.2.3 Representative sample

A sample whose selection requires planned action to ensure that proportions of it are taken from different subportions of the whole.

Example 1.21 The forty-eight pieces of tubing selected two from every hour's production in one day; concrete cubes made, one from every batch, of a lot consisting of several batches.

1.10.2.4 Population

A large collection of individual units from one source. In particular circumstances this may be, for example, an output or batch: the bulk of material (concrete) or total collection of units (pieces of tube) produced by a set of machines or a factory in a specified time.

Example 1.22 Pieces of tubing made in a particular factory during a month; the concrete produced by a single plant during 1 day.

1.10.2.5 Statistic

A statistic is a quantity computed from the observations of a sample.

1.10.2.6 Parameter

A parameter is a quantity computed from the observations made on a sample. Thus, the value of a parameter for a population is estimated by the appropriate statistic for the sample.

1.11 Location

1.11.1 Measures

1.11.1.1 Arithmetic mean

The arithmetic mean, often called the 'mean' or the 'average', is the sum of all the observations divided by the number of observations:

$$\bar{x}=\frac{1}{n}\sum_{i=1}^{n} x_i \tag{1.66}$$

Example 1.23 0.20 (2.540 + 2.538 + 2.547 + 2.544 + 2.541) = 2.542.

1.11.1.2 Median

The value which is greater than one-half of the values and less than one-half of the values.

Example 1.24 The value 2.541 is the median of the above five numbers. (Had there been an even rather than an odd number of numbers the median is the average of the two numbers either side of the median position.)

1.11.1.3 Midpoint or midrange

The value which lies half way between the extreme values.

Example 1.25 Using the numbers above the mid point is

0.5 (2.538 + 2.547) = 2.542 5

1.12 Dispersion

1.12.1 Measures

1.12.1.1 Range

The difference between the largest and the smallest values.

Example 1.26 2.547 − 2.538 = 0.009.

1.12.1.2 Deviation

The difference between a value and the mean of all the values.

1.12.1.3 Variance

The variance of a set of values is the mean squared deviation of the individual values and is normally represented by σ^2.

$$\sigma^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 \tag{1.67}$$

where μ is the mean value.

A frequently occurring problem is that of estimating the main properties (the mean to describe the location and the variance to describe the dispersion) of a population by measurements (x_i) taken on a sample. From the measurements on the sample we can calculate the sample mean, $\bar{x}$ which is an estimate of the population mean μ. The sum of the squared deviations is smallest about the arithmetic mean; thus, for the population an estimate of variance using the sample mean and sample variance will be an underestimate. In cases where we wish to estimate population parameters from sample observations, a correction is made by using $(n-1)$ as divisor instead of n. Thus, the estimate of the population variance from observations x_i on a sample is:

$$\sigma^2=\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2 \tag{1.68}$$

1.12.1.4 Standard deviation

The standard deviation is the square root of the variance.

$$s=\left[\frac{1}{n-1}\sum_{i=1}^{n}(x_i-\bar{x})^2\right]^{1/2} \tag{1.69}$$

As in the case of variance, the divisor n is replaced by $(n-1)$ when working with sample observations to estimate a population standard deviation. The standard deviation has the same units as the original observations and their mean $\bar{x}$.

When carrying-out hand calculations, the identity:

$$\sum_{i=1}^{n}(x_i-\bar{x})^2=\sum_{i=1}^{n}(x_i)^2-n\bar{x}^2 \tag{1.70}$$

frequently saves effort. However, this method is not recommended for use on computers because of the danger of loss of accuracy when n is large and x_i has several significant figures.

1.12.1.5 Coefficient of variation

The coefficient of variation is the standard deviation expressed as a percentage of the mean. This is useful for dealing with properties whose standard deviation rises in proportion to the mean, for instance the strengths of concrete as measured by compressive tests on cubes.

1.12.1.6 Standard error

The standard error is the standard deviation of the mean (or of any other statistic). If in repeated samples of size n from a population the sample means are calculated, the standard deviation calculated from these means is expected to have a value:

$$Sm = \sigma/\sqrt{n} \tag{1.71}$$

where σ is the standard deviation of the population.

An important result is that whatever the distribution of the parent population (normal or not) the distribution of the sample mean tends rapidly to normal form as the sample size increases.

1.13 Samples and population

1.13.1 Representations

1.13.1.1 Frequency

The number of observations having values between two specified limits. It is often convenient to group observations by dividing the range over which they extend into a number of small, equal, intervals. The number of observations falling in each interval is then the frequency for that interval. This allows a convenient representation of the information by means of a histogram.

1.13.1.2 Histogram or bar chart

A diagram in which the observations are represented by rectangles or bars with one side equal to the interval over which the observations occurred and the other equal to the frequency of occurrence of observations within that range (Figure 1.45).

1.13.1.3 Distribution curve

The result of refining a histogram by reducing the size of the intervals and correspondingly increasing the total number of observations. In the limit, when the intervals become infinitesimally small and the number of observations infinitely large, the tops of the rectangles of a histogram become a distribution curve (Figure 1.46).

1.13.1.4 Normal distribution (or Gaussian distribution)

A particular type of distribution curve given by:

$$y(x) = \frac{1}{\sigma(2\pi)^{\frac{1}{2}}} \exp\left\{\frac{-\frac{1}{2}(x-\mu)^2}{\sigma^2}\right\} \tag{1.72}$$

where x is the observational scale value, μ the population mean and σ the population standard deviation.

These parameters of the distribution are estimated by the sample mean $\bar{x}$ and standard deviation s.

It has been found that a great many frequency distributions met with in practice fit quite closely to the normal distribution. However, one should beware of thinking that there is any law which says that this shall be so; it is simply a matter of experience. In circumstances where the observed frequency distribution does not appear to be normal it is often possible to transform the original data (e.g. by taking logarithms, square roots or squares) so that the transformed data is nearly normal. These two facts explain why so much of the effort in statistical theory has been devoted to treatment of normal-distribution problems.

For normal distributions the percentage of observations (in large samples) lying within certain limits of the observational scale are given in Table 1.5 and Figure 1.47.

1.14 The use of statistics in industrial experimentation

As has been stated, in experimental work units in a sample drawn from a parent population and the observations made on them are subject to error, and our task for which we use statistics is to make useful statements about the properties of the parent population. To achieve this, the most important statistics are the mean and the standard deviation. This section, therefore, considers the obtaining of sample means and standard deviations and confidence limits for them in situations where the parent population is normally distributed. Tests of significance for comparisons of means and variances are also described. Inevitably, only brief summaries are given and a study of standard works is advised before using the techniques on any important matters. As an alternative, the help of the statistical expert should be sought. If such assistance is to be obtained, it cannot be emphasized too strongly that it should be acquired right at the outset of the problem. It is rarely of much help to anyone (even though it happens only too frequently) for the statistician to be asked: 'Please tell me what these numbers show: they must mean something, I've collected so many, and they cost a great deal to obtain.'

1.14.1 Confidence limits for a mean value

If the form of the distribution were known together with the true mean μ and the standard deviation σ, then it is easy to make statements about the mean of a number of observations. If the population is normal then the mean $\bar{x}$ of a sample size n drawn randomly will, on average, satisfy:

$$\mu - \frac{3\sigma}{\sqrt{n}} < \bar{x} < \mu + \frac{3\sigma}{\sqrt{n}}$$

997 times out of 1000 (see Table 1.5).

Thus, if μ is actually unknown (and we are trying to estimate it) we may assert:

$$\bar{x} - \frac{3\sigma}{\sqrt{n}} < \mu < \bar{x} + \frac{3\sigma}{\sqrt{n}}$$

with 99.7% confidence. By this, we mean that if we go on making such assertions indefinitely we shall be wrong only 3 times in every 1000. We can make the containing interval narrower by reducing confidence so that we assert with 95% confidence that the limits for μ are $\bar{x} \pm 1.96\sigma/\sqrt{n}$ (Figure 1.48).

Very often we may be concerned with a limit on only one side, for instance we may require assurance that μ is greater than a certain value. Now, the probability of $\bar{x}$ falling above $\mu + 2\sigma/\sqrt{n}$

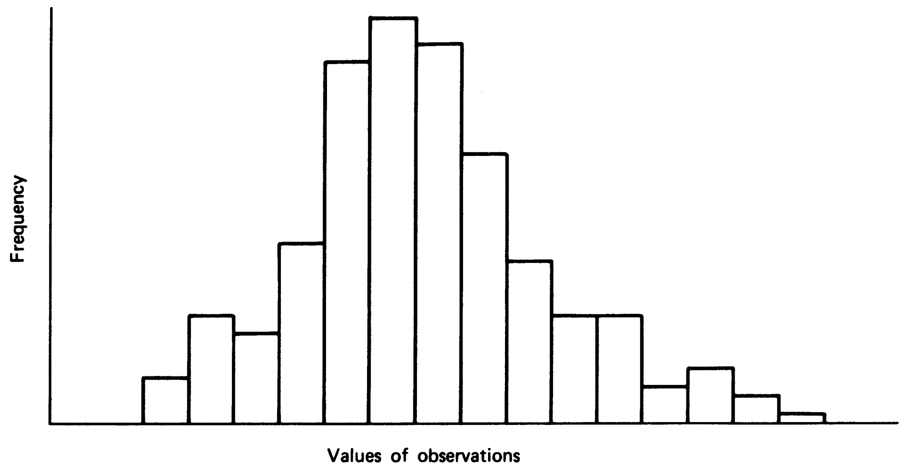

Figure 1.45 A histogram of observations from a sample

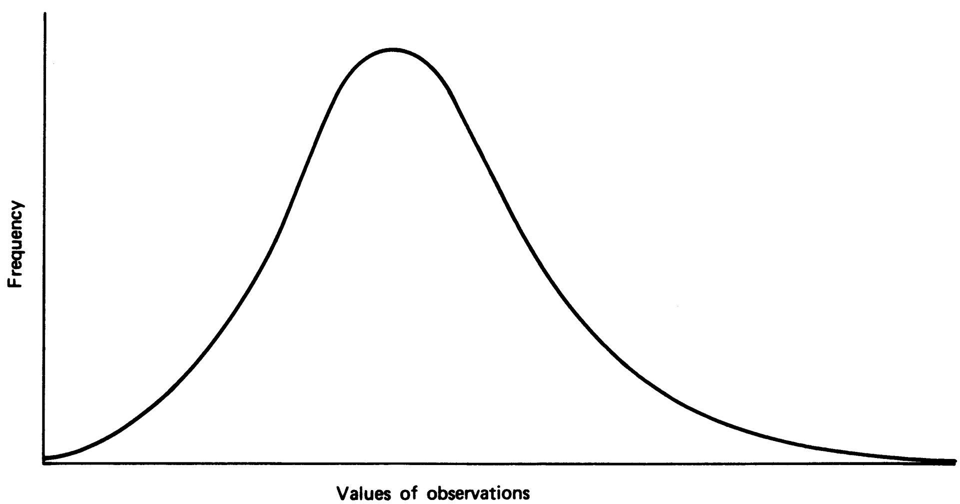

Figure 1.46 A continuous distribution curve

Table 1.5 The normal distribution

Range	*Observations within range* (%)
$\mu \pm \sigma$	68.27
$\mu \pm 2\sigma$	95.45
$\mu \pm 3\sigma$	99.73
$\mu \pm 1.96\sigma$	95
$\mu \pm 3.09\sigma$	99.8

is 2.27%. Thus, we may assert with 97.73% confidence that μ does not lie below $\bar{x}-2\sigma/\sqrt{n}$.

Generally, the proportion of sample means $\bar{x}$ which exceed $\mu+u_a\sigma/\sqrt{n}$ is equal to a where u_a is the value given in a table of the normal distribution for a specified probability, say P. Because the distribution is symmetrical, α is also the proportion of sample means which are exceeded by $\mu-u_a\sigma/\sqrt{n}$. Thus, the whole range of values which μ may take is divided into three parts and three assertions can be made, to correspond one with each part:

(1) $\mu \geqslant \bar{x}-u_a\sigma/\sqrt{n}$ with confidence $100(1-a)\%$.
(2) $\mu \leqslant \bar{x}+u_a\sigma/\sqrt{n}$ with confidence $100(1-a)\%$.
(3) $\bar{x}-u_a\sigma/\sqrt{n} \leqslant \bar{x}+u_a\sigma/\sqrt{n}$ with confidence $100(1-2\alpha)\%$.

This shows two sorts of statement, the single-sided (cases 1 and 2) and the double-sided (case 3). When using statistical tables it is important to check whether the tabulation is for single-tailed testing or two-tailed testing. (This description arises because cases 1 and 2 are, in the practical cases where a useful level of confidence is being used, representable as the two tails of a curve shaped like the normal distribution curve.)

In the discussion of confidence limits for the mean value μ of a population estimated by the mean of the sample $\bar{x}$ above it was assumed that the population standard deviation was known. Generally this will not be the case and μ will have to be estimated as s, a sample standard deviation and used in place of μ in the calculations above. The confidence limits for μ are now

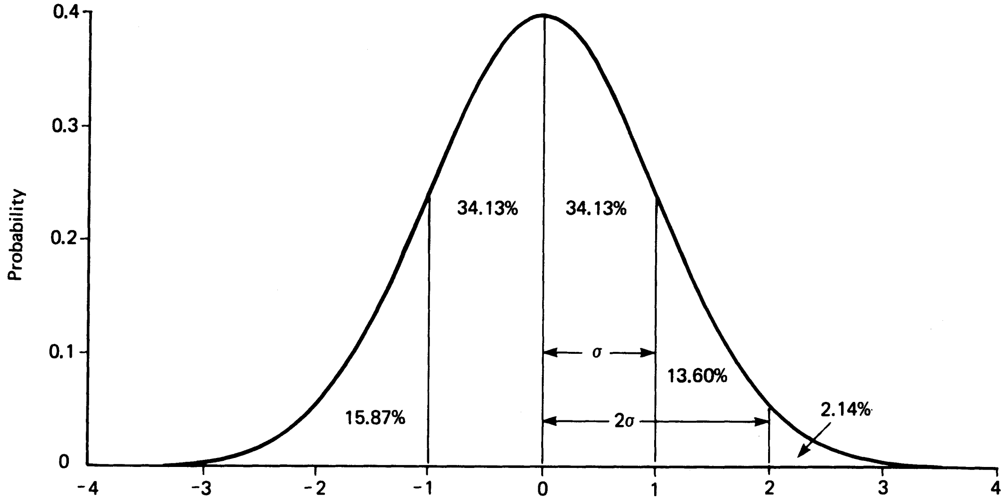

Figure 1.47 The normal distribution with limits

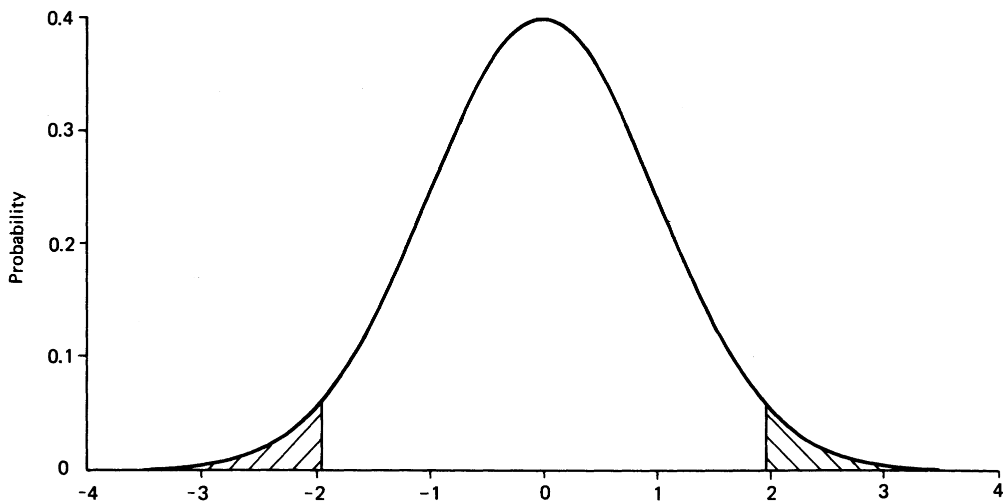

Figure 1.48 Diagrammatic representation of confidence limits (with 95% limits shown)

wider because of the uncertainty about s and, instead of using u from the normal distribution curve it becomes necessary to use tables of student's t. The particular value of t to be used depends on how good the estimate s of σ is, which in turn depends upon the number of degrees of freedom in making the estimate. In the case of a standard deviation of n observations, the number of degrees of freedom is $(n-1)$. The number of degrees of freedom is generally denoted by ϕ. Some values of t are given in Table 1.6. In using this table the $100(1-2a)\%$ confidence limits are:

(1) Lower limit $\quad \bar{x}-t_a s/\sqrt{n}$.
(2) Upper limit $\quad \bar{x}+t_a s/\sqrt{n}$.

using the value of t_a for the appropriate number of degrees of freedom.

Table 1.6 Significance points of the t-distribution (single-sided)

ϕ	*Probability: P*				
	0.1	0.05	0.025	0.01	0.005
1	3.08	6.31	12.70	31.80	63.70
2	1.89	2.92	4.30	6.96	9.92
5	1.48	2.01	2.57	3.36	4.03
10	1.37	1.81	2.23	2.76	3.17
20	1.32	1.72	2.09	2.53	2.85
40	1.30	1.68	2.02	2.42	2.70
∞	1.28	1.64	1.96	2.33	2.58

1.14.2 The difference between two mean values

A problem which arises frequently is that of determining if the difference between two means has occurred by chance because of natural variation or whether there is a real difference. A real difference can only be asserted in the form of a statistical statement that the difference is significant at a certain level, i.e. there is a probability that there is a real difference. This is done by calculating a t statistic from information about the samples and comparing the result with the tabulated t values. The means, standard deviations and number of observations of the two tests are denoted by $\bar{x}_1$, $\bar{x}_2$, s_1, s_2, n_1 and s_2.

Calculate $t=(\bar{x}_1-\bar{x}_2)/s_p$

where $$s_p=\sqrt{\left[\left(\frac{1}{n_1}+\frac{1}{n_2}\right)\left(\frac{v_1s_1^2+v_2s_2^2}{v_1+v_2}\right)\right]}$$

and $v_1=n_1-1$, $v_2=n_2-1$

(*Note*: s_p is the pooled standard deviation for the samples 1 and 2.).

If this calculated value exceeds a tabulated value of t (for $\phi=v_1+v_2$) then the difference is significant at the level determined by the probability heading the column of the t table.

As an example, consider the comparison of two testing machines for crushing concrete cubes. The machines are to be compared by making a single batch of twelve concrete cubes and testing six cubes on each machine. The results obtained are:

Machine 1	39.2	38.4	44.7	41.0	41.0	44.1
Machine 2	41.1	33.8	42.4	36.8	32.0	40.1

From these observations we can calculate:

Sample sizes	$n_1=6$, $n_2=6$
Sample means	$\bar{x}_1=41.4$, $\bar{x}_2=37.7$
Sample standard deviations	$s_1=2.54$, $s_2=4.19$

$$s_p=\sqrt{\left\{\left(\frac{1}{6}+\frac{1}{6}\right)\left(\frac{5\times2.54^2+5\times4.19^2}{5+5}\right)\right\}}=2.0$$

so: $t=(41.4-37.7)/2.0=1.85$

The number of degrees of freedom $\phi=v_1+v_2=10$.

From Table 1.6 it is seen that for $\phi=10$ the single-sided 2.5% (or 0.025) point of the t distribution is 2.23 and so the calculated t value is not significant at the $2\times2.5\%$ level.

(*Note*: It is necessary to double the probability value from the table because the question: 'Are the testing machines different?' requires a two-sided test to be carried out. By contrast the question: 'Is mean 1 greater than mean 2?' would require a single-sided test.)

1.14.3 The ratio between two standard deviations

In a similar way in which it may be desired to compare two means, it may be desired to compare two standard deviations. Whereas means are compared by calculating their difference, standard deviations are compared by calculating the ratio of the variances (the square of the standard deviation) and comparing the ratio with tabulated values in an F test. In all such calculations the value obtained for the ratio must be greater than unity so that the larger standard deviation (say s_1) must be placed over the smaller (s_2) where s_1 and s_2 are sample standard deviations and so estimates of the population standard deviations. Since we have of necessity $s_1\geqslant s_2$ when we calculate

$$F=\frac{s_2^1}{s_2^2}$$

the F test is a one-sided test.

Values of F for comparison with the calculated value from the observed standard deviations are given in most statistical books.[2] Such tables are presented generally with one table for each specified probability and within such a single table the column headings are the values of v_1 (the number of degrees of freedom, n_1-1, of the smaller standard deviation estimate s_2).

By way of illustration, consider the example used above for the comparison of means. In that example the observations lead to:

$n_1=n_2=6$

so: $v_1=v_2=5$

$s_1=2.54$ $s_2=4.19$

In this example $s_2>s_1$ so the calculation of F is:

$$F=\left(\frac{4.19}{2.54}\right)^2=2.72$$

In the tables (e.g. in Neville and Kennedy[2]) the tabulated 1% confidence point of F is 10.97 and the 5% point is 5.05 both found for $v_1=v_2=5$. Since the calculated F ratio is not greater than the tabulated values the conclusion to be drawn is that there is not strong evidence that population standard deviations are different.

1.14.4 Analysis of variance

If a manufacturing process or a testing scheme involves a number of independent factors, each of which contributes to the variability of the results, then the variance of the whole system is equal to the sum of the component variances. (Note that the variance must be added, not the standard deviation.) This additive property permits the technique of analysis of variance, which can take many forms depending on the structure of the process which is being analysed. One of the major difficulties of analysis of variance lies in deciding what form of structure is appropriate to the process being modelled by the analysis of variance. In the majority of cases which are not both simple and short it will be sensible for the arithmetic to be performed by computer. However, in simple and short situations the calculations may reasonably be undertaken by hand.

Probably the most commonly occurring simple situation is that of analysis to determine variance between and within batches. The methods are best described by an example. The example will be one in which concrete cubes are made batches (each of three cubes) and strength tested at (say) 28 days. The first step is to define the statistical model which is being used:

$$Y_{ij}=Y+A_i+E_{ij} \tag{1.73}$$

where there are i batches each of j cubes, Y_{ij} is the observed strength of the jth cube in the ith batch, Y is the average strength (averaged over all tests), A_i is the difference between Y and the average strength of batch i, and E_{ij} is the difference between the jth cube of batch i and the average strength $Y+A_i$ of that batch.

If the data for four batches of cubes is:

19.8	21.1	19.8	(batch 1)
21.8	22.0	21.0	(batch 2)
21.2	21.5	21.2	(batch 3)
21.4	21.4	21.0	(batch 4)

it is found that $Y=21.1$.

From this can now be found the sums of squares of the A_i and E_{ij}. Associated with each sum of squares is a number of degrees of freedom (as usual one less than the number of occurrences) so that dividing the sums of squares by the appropriate number of degrees of freedom gives the mean square. Thus is constructed an analysis of variance table as shown in Table 1.7.

The method of test is by F ratio so that the larger variance (the average of the sums of squares of errors) is divided by the smaller. Here, $1.07/0.23=4.61$ with 3 degrees of freedom for the column heading and 8 for the row heading when comparing with the tabulated F values. For $v_1=3$, $v_2=8$ the tabulated F value at 1% confidence level is 7.59. The observed value does not exceed this and so there is no assertion that can be made at the 1% level. However, the tabulated F value at the 5% confidence level is 4.07. The observed value exceeds this and so a result significant at the 5% level has been obtained. Thus, although there is not strong evidence there is some evidence of a real difference between batches.

In an example so small as this one the necessary arithmetic (especially if properly organized) may reasonably be tackled by hand. However, as can be deduced from examination of F tables, it is not always easy to get significant results with small experiments. Thus, the use of the technique will in many cases imply the use of a computer for handling the arithmetic. In such circumstances the engineer is likely to be using an existing computer program and need only concern himself with correctly presenting the data for the program to analyse and then with the interpretation of results and comparisons with tabulated F values. He has no need therefore to develop great skills in short-cut arithmetic methods.

Table 1.7

Model term	*Sum of squares*	*Degrees of freedom*	*Mean square*
A_i	3.2	3	1.07
E_{ij}	1.9	8	0.23

1.14.5 Straight-line fitting and regression

Experiments may be designed to examine whether two parameters are related. The circumstances may involve the effect on a property of a product of some parameter in the production process. In the experiment the parameter will be controlled or constrained to take a number n of prescribed values x_i over some range and the consequential observations y_i will be paired with them. The question now arises as to the 'best straight line' through the points x_i, y_i. It is assumed that the x_i values are error-free but that the observations y_i are subject to error. The method of obtaining the 'best' straight line in such circumstances is to choose the two parameters m and c of the straight line:

$$y=mx+c \tag{1.74}$$

in such a way that the sum of the squares of the errors in the y direction is a minimum. This is achieved by making:

$$m=\frac{n\Sigma xy-\Sigma x\Sigma y}{n\Sigma x^2-(\Sigma x)^2} \tag{1.75}$$

and:

$$c=\frac{\Sigma x^2\Sigma y-\Sigma x\Sigma xy}{n\Sigma x^2-(\Sigma x)^2} \tag{1.76}$$

This line is called the line of regression of y on x and one of its properties is that it passes through the centroid $\bar{\bar{x}}$, $\bar{y}$ of the observed points. The usual statistical question now arises concerning the confidence limits which should be applied to the calculated line which is an estimage of a relationship. To examine this problem the errors or deviations must be calculated. At every observation point x_i y_i which does not actually lie on the calculated line there is an e_i. The variance of y estimated by the regression line is then:

$$s_y^2=\frac{\Sigma e_i^2}{v} \tag{1.77}$$

where v is the number of degrees of freedom.

Since calculation of m and c impose two restraints the value of v is given by:

$$v=n-2 \tag{1.78}$$

The variance of the mean value $\bar{y}$ is given by:

$$s_{\bar{y}}^2=\frac{s_y^2}{n} \tag{1.79}$$

so that the confidence limits for $\bar{y}$ *are:*

$$\bar{y}\pm ts_{\bar{y}} \tag{1.80}$$

where, just as for a sample mean, the value of t is found from tables using the appropriate number of degrees of freedom.

The variance of the slope m is given by:

$$s_m^2=\frac{s_y^2}{\Sigma(x-\bar{x})^2} \tag{1.81}$$

and the confidence band for slope is given by:

$$m\pm ts_m \tag{1.82}$$

It may be necessary to compare one regression line with another, theoretical one, to see if there is any significant difference between the theoretical slope, m_0, and the observed slope m. This test is performed by calculating a t statistic:

$$t=\frac{m-m_0}{s_b} \tag{1.83}$$

and comparing with the tabulated values. Just as in the case of comparison by means of samples we can compare the slopes of two observed lines by replacing $m-m_0$ by m_1-m_2 in Equation (1.83) and using a pooled standard deviation from the variances of the slopes of both lines in place of s_b. The number of degrees of freedom used in the t table will be n_1+n_2-4.

1.15 Tolerance and quality control

Material is often manufactured for supply according to a specification which will include compliance clauses for the performance of the product. As an example, CP 110[3] lays down (in Section 6.8) certain strength requirements and also suggests a testing plan. The *Handbook* to that code[4] discusses the problems of compliance and shows how different forms of testing plan after the operating characteristic of a test plan and so charge the risks run by the producer and by the customer. The customer has, in theory, the opportunity of reducing his risk by adopting a more vigorous testing plan. This, however, is likely to cost more and a customer may well deem this not worth while. The producer, on the other hand, must expect to have to meet the compliance clauses and needs to arrange his production methods so as to make a profit taking account of whatever limits or penalties may be imposed on him by the compliance clauses under which he has to operate. Thus, the manufacturer or producer is faced with a problem of how to control his product.

One example of a technique for exercising this control is shown by a system advocated for controlling the strength of ready-mixed concrete[5] by means of the cumulative sum chart which is an improved form of control chart especially developed and adapted to the problems of concrete manufacture.

In the process of manufacture and measurement of some property of the product natural variation will cause the results obtained to be distributed in some way. The problems facing the manufacturer are:

(1) To maintain adequate control over the process so that the variation in results does not become so large that an uneconomic number fall outside the specified tolerances.
(2) To detect any trend for the observations obtained to be moving out of the specified limits, sufficiently early to take useful corrective action.

As usual, samples are taken to estimate the properties of the parent population. To do this comparatively, many samples (25 or more) of comparatively small (but not less than about four and all the same) size are tested and the mean of the means $\bar{\bar{x}}$ used to estimate the population mean. The population standard deviation is estimated from the variance within samples, the average sample standard deviation from the average sample range.

Thus: $$\bar{\bar{x}} = \frac{\bar{x}_1 + \bar{x}_2 \ldots + \bar{x}_k}{k} \tag{1.84}$$

$$\bar{x} = s/\sqrt{n}$$

for k samples of size n.

Now a chart is drawn with time or sample number in the horizontal axis and observation values on the vertical axis. A line drawn at $\bar{\bar{x}}$ represents the target performance of the process and two surrounding lines at $\bar{\bar{x}} \pm 1.96/(\sqrt{n})s$ represent warning levels for the process while surrounding lines at $\bar{\bar{x}} \pm 3.09/(\sqrt{n})s$ can be regarded as action levels.

The choice of the figures 1.96 and 3.09 has been made on the assumption that the process is functioning in such a way that the specified tolerance limits are reasonable, i.e. they are not so stringent that the chance of the product meeting the requirements is not high while on the other hand the process is not so 'good' (in which case it may be unnecessarily expensive) that all the results obtained lie well within limits.

The design and use of control charts is a valuable use of statistical methods. Generally they are robust in the sense that their usefulness is little affected by factors such as non-normality of the basic data. However, for their efficient use in some area experience of the particular technology is desirable and for a better understanding of the possibilities of the techniques the reader is recommended to works by the British Ready Mixed Concrete Association[5] and Davies and Goldsmith.[6]

COMPUTERS

Computers and computing have made a substantial impact on most walks of life, civil engineering not excepted. The pace of development in computing is substantially greater than for any other area of activity in the engineering world. Although other subject areas are subject to bursts of activity from time to time, when research or some specific project provides the necessary spur, computers are developing rapidly all the time, whether the engineering world wishes it or not. In consequence a great many organizations find it difficult to keep abreast of what is available or of what might actually be of benefit to them in their work. This difficulty is not eased by the wide discrepancy between the useful life of most civil engineering work and the life of computers.

Although, in principle, computers are simple machines which can perform simple arithmetic and make simple decisions (according to a set of coded instructions—the program) that fact is ever more frequently masked by the use of sophisticated techniques which appear to make computers behave more and more like human beings, and able to undertake tasks previously the province of human effort.

1.16 Hardware and software

One of the most important distinctions which must be understood when considering computers is the difference between *hardware* and *software*. A simple criterion is to imagine that the hardware consists of the material pieces which one can see—boxes, wires, screens, discs, chips etc.—while the software comprises the instructions which the hardware obeys. It is in the nature of the general developments in society that the cost of making the hardware is, in real terms, falling all the time. This fall in cost comes about through better design of components, automated manufacturing techniques and so on. All this is similar to the developments which have been taking place in other fields of manufacturing.

The software, on the other hand, consumes human effort and imagination very intensively. It is not easy to improve the techniques of manufacture here! In consequence, the total cost of a computer installation—if it is regarded, for simplicity, as being composed of the two elements of hardware and of software—has changed considerably. In the early days, when computers were harnessed in working offices, the cost of the hardware was the major consideration and the software, if considered at all, tended to be something of an afterthought. Now we are recognizing that the software is, or ought to be, the major consideration. Once the major details of the software suitable for the envisaged tasks have been settled it is logical to search for the 'best' hardware solution which will accommodate the chosen software.

It is, of course, unlikely that any office, let alone organization, will wish to 'computerize' just one activity. It is normal for a great multitude of tasks to benefit from being done by machine rather than by man. In this event the choice of hardware will be constrained by a, perhaps wide, variety of software. This emphasizes the fact that computers should be thought of (in the hardware sense) as general-purpose machines.

1.17 Computers

The changes which have taken place in recent years encompass the change from remote 'batch' computing to 'personal' computing where every person who needs one in order to do his job appears to have access to one on his desk. In truth, this revolution has come about via an intermediate stage, i.e. the change from the large mainframe machines which, while they could perform several tasks apparently concurrently, had to be run by dedicated operators remote from the users, to the mini machines operated via remote terminals. With this scheme, the many users all feel (most of the time) that the computer is dedicated to them alone while, actually, the centre of the machine is servicing up to some tens of users and it is only the terminal which is dedicated to the individual user.

The development of the 'personal' computer has had a bigger impact on this situation than is at first sight obvious. Personal computers came about because the huge improvements in computer technology allowed the production of a machine which can sit, complete, on a desk, but which has power greater than the mainframe machines of a decade ago. (Those mainframes had required a large dedicated room, and air-conditioning, as well as operating staff.)

The presence of the computer on the desk, with an impressive array of available software, encouraged a situation in which the users were often repeating tasks being performed by colleagues (especially the inputting of data). One description of the development as it affected the functioning of an organization was that it was leading to near anarchy with little managerial control of what was happening to the benefit of the organization.

This independence of the personal computers has therefore been both a benefit and a source of difficulty. Trying to get the best of all possible worlds has led to much emphasis on communications. Here is meant the communication between different computers, generally communicating with other computers within the same organization, but sometimes further afield. Increasingly, for organizations of a certain size, the plan followed is one with a major computer at the heart of operations with a network of personal computers around it. These personal computers can be connected to the 'heart machine', or can be operated in stand-alone mode, at the will of the user. In these circumstances it becomes possible for the heart machine to be the repository of the valuable corporate data (which should then be held once only) to which the individual users can have access as and when their work demands it. The individual users can then use their own 'personal' data and run the programs of interest to them on their personal computer without affecting anyone else. Should an individual user have available something (be it data or be it a program) to which other users require access, this is arranged via the heart machine.

The organization and control of such an arrangement is not simple and produces interesting problems of a managerial and human nature. But it is now possible to make arrangements which seem to be getting near to providing a situation in which men, machines, and the organization can all work reasonably efficiently.

The providers of computing solutions (hardware and software) are, of course, in business. They will therefore advise potential customers of the benefits of the particular solutions they purvey. It is not easy for the (computing) lay person to judge the advice received from such quarters. It seems likely, therefore, that many organizations will be well advised to adopt the strategy of ensuring that they have in-house expertise to judge such matters. This notwithstanding the fact that, as time goes by, the purveyors of computing solutions are making greater emphasis of the idea that their solution needs no computer expertise. As in other walks of life, the lack of expertise in an activity in which one is engaged is likely to be costly.

Since the advent of the personal computer (able to double as a terminal) sitting on the desk, has come the mobile or portable computer. This is depicted as sitting on the knees of the user and working off batteries, thus freeing the user from the need for access to mains electricity. While an obvious early use was, for example, for salesmen to enter their transactions, other applications are being found. The collection of technical data on site is an obvious parallel to those activities in other fields, so such machines are proving useful to the engineer. It can reasonably be said that the use of the fruits of computer invention are limited only by human imagination. Though trite, this statement has considerable importance. It can be very difficult indeed to think of a really new way of achieving some objective: the straitjacket of 'we've always done it this way' can be extremely strong.

1.17.1 The use of computers by civil engineers

Although engineers have appeared, at times, to lag behind in the use of computers, they actually began using them at a very early stage. The first 'obvious' application lay in the solution of the many simultaneous linear equations to which many problems of structural analysis can be reduced. This mathematical problem had received much attention in an effort to speed, refine, and make more reliable, hand methods. The ability of the computer to perform repetitive tasks reliably shifted the search to making the preparation, and input, of the data describing the problem more robust. This search was hampered for some time by limitations of the hardware. However, the availability of substantially increased computing power eventually allowed the problem of the data to be encompassed as well as the problem of solving the equations and presenting the results.

This theme of the availability of increasing computing power allowing new tasks to be tackled has recurred frequently in the history of computing.

The use of computers for structural analysis represented the limit of activity in engineering for some time. However, developments of other machines for drawing or plotting prompted an attack on another phase of engineering design activity. Conceptually, the operation of a design project can be split into four stages: (1) the concept and choice of solution; (2) the analysis of the whole structure; (3) the design of individual members; and (4) the preparation of detailed drawings.

The contribution computers can provide to (1) above has only comparatively recently become apparent in terms of rearranging scheme drawings and the holding of base data. Stage (2) was covered by the early computing endeavours and (4), the detail drawings, became possible when the plotters, developed for aero work, for example, became cheap enough for use in civil engineering. The development of a package for the production of drawings of reinforced concrete details was a major breakthrough. In use the detailer has available the information arising from the design of members. Using a desktop computer, for example, he supplies data of the basic dimensions of the member and then, via a question-and-answer dialogue, supplies information to define the reinforcement detail. At all stages the information supplied by the user is checked for logical consistency, geometric compatibility and compatibility with appropriate code or standard documents. If an attempt is made to do something impossible or contrary to regulation, then the user is not permitted to proceed until the error has been rectified. By contrast, an attempt to do something which, according to standards incorporated with the program, is unusual will result in a message which the user can heed, or ignore, at will. Such interactive programs were impossible with the earlier batch machines.

Having been developed in modern environments, such a program is now expected to be very 'user friendly'. To take a cynical view, a user-friendly program is one for which the user has no need to consult the (written) user manual!

The effect of using such an aid is that a small team of detailers can become very much more productive, producing many more drawings per week than by manual means. Further, the fact that the data defining the drawings is stored means that, in the event of changes becoming necessary, the revised drawings can be produced very much more quickly (and reliably) than if done by hand.

An interesting sideline to the development of such a tool is the attitude taken to drawings. While some drawings are required to make an impression, and so are treated as works of art, the drawing of a reinforcement detail is just a technical necessity and it is used only by technical people and therefore may be less impressive. In consequence, detail drawings, produced on comparatively cheap dot matrix printers, have become quite acceptable. Only a few years ago even these technical drawings were also treated as works of art.

The third stage of the design office exercise, that of designing the individual members has also been solved. More than one approach has been adopted but this has the benefit of providing potential purchasers and users with competitive choice.

If we consider the three technical stages of the design office activity and the solutions listed above, we find the appearance of some more common occurrences such as the requirements for compatibility and the transfer of information between different stages of the work. In this example, the information from the global structural analysis is required by both the design and the detailing activities. Similarly, the information from the member design is required for the detailing phase. There are different views about the extent to which this information transfer can, or indeed should, be made automatic. At the time of writing the general feeling is to limit the amount of automatic transfer, it being held that the contribution of man is too difficult to codify and too valuable to lose. Such views have held sway before and have, eventually, been overturned. It seems probable that the developments in expert systems and other advanced computer technology may have the same overturning effect here in due course.

The production of the software for such systems represents a very substantial expenditure and it is important that the solutions developed should not be excessively dependent on particular hardware. In fact the drawing part of the solution has used plotters and more recently dot matrix printers. (A likely change is that laser printing will be a practical tool for such work.) The actual computers used have covered a wide range although the operating system used by the computer has been important. This is another area where the general developments in computing towards standardized operating and filing systems will make the transport of software solutions from machine-to-(often successive)-machine a comparatively painless task.

This example of the solution of an engineering set of problems—analysis, design, detailing—has been described at some length because it typifies the problems which will require consideration in some form almost whenever a computer solution to a problem is being sought. It is foolish to underestimate the benefits which the computer can bring, but it is important to be aware of, and consider properly, the problems and side issues which can arise. Proper treatment of such matters can sometimes bring unexpected benefits.

1.17.2 Nontechnical computing

Although, when first invented, computers were largely used by technical people to perform technical tasks, it has long been the case that the bulk of computer sales and use have been in commercial fields. For some time this affected the design of computers but the picture now is one of much more general application as computers are becoming the user's workhorse. It is not practicable to have many different computers to perform the many different tasks which an individual may tackle.

1.17.2.1 Spreadsheet

One example of the change of use to which software can be put is the spreadsheet. A spreadsheet is essentially a rectangular array of boxes identified by cartesian-type coordinates. A box may contain either a value or a formula. Such formulae may relate to the values held in other boxes. After entry of data and formulae the user will request that calculation of values be performed. For whatever reason an item of data, or a formula, may need to be altered. After the change a recalculation can be requested and will usually seem to be performed almost instantaneously. It does not matter, of course, to the computer, whether the change was a correction of an error or a change of mind on the part of the user. The traditional use of spreadsheets has been in financial areas where the slogan about answering 'What if?' questions was meant to appeal to those with responsibility for profit margins, etc. However, a spreadsheet is nothing more than a general-purpose organization of calculation: there is no reason why a spreadsheet should not be used to calculate a set of sine or of logarithm tables. Increasingly, engineers are finding that, if they think about a problem from a different angle, a different solution tool may come to their aid. The spreadsheet is one example. The use of an improved solution tool will not come about unless knowledge of the tool and its capabilities exists together with a knowledge of the tasks tackled by the organization. As has been mentioned above, the best results will come only when there is a proper awareness of requirements and capabilities.

1.17.2.2 Word processing

Probably the most widespread computer application now is that of word processing. Although, traditionally, an author has passed his original (e.g. manuscript or dictated tape) to another person who has sole responsibility for production of the typed form, this may well change. With the increasing use of computers many workers who at some stage take on the role of author, are becoming more or less keyboard-competent. Now, while it would be too much to claim that such authors can key as well as a professional typist, there is an increasing number of these 'sometime' authors who can type quickly enough to keep up with their own creative thought processes. Also, by using a few of the more basic capabilities of the word processing system the author can produce a good result that is well ordered and cogent (even if the spelling and layout may leave something to be desired) more quickly than with the older techniques. Even the problems of spelling and of layout can, in part, be tackled by the computer. It seems hardly likely that the secretary is under serious threat from such developments of author capability but the notion of the copy-typing task may well be one that will disappear. Provided reasonable control can be exercised over aspects of detail and over the proper use of an individual's time, there may well soon be a substantial increase in the number of engineers producing their own reports.

1.17.2.3 Networks

It is this chameleon-like behaviour of the user at his desk, wanting to be structural analyst, financial analyst, typist, etc. which makes the proper arrangement of personal computers that are able to double as terminals so important. While, in some circumstances, it may be suitable to have these personal computer networks connected to one another without the existence of any 'heart' machine, as described above, it seems likely that the more frequent situation will be one in which the central computer is needed to provide not only backup facilities but also the corporate data (details of cost rates for example) which many users may require.

1.17.3 Specific vs. general-purpose software

When seeking software there is sometimes a choice between a program which has been designed for a strictly limited and well-defined purpose on the one hand and a program designed to be general-purpose on the other hand. To illustrate this, consider the problem of producing drawings of reinforcement details as discussed above. The end product drawing is nothing more than a set of lines drawn on a piece of paper. Some of these lines are straight, some curved and some are in the form of characters. Thus, a general-purpose drawing package, capable of putting lines on the paper according to data instructions defining, for example, cartesian coordinates of end points of line segments, could produce the required drawing if the data are prepared. On the other hand, the special-purpose detailing program will require far less data in order to produce the same end result. It will, partly in consequence, be very much easier for humans to understand the data of the special-purpose version at a glance. They are thereby more able to spot mistakes and correct them.

On the other hand, the detailing program will be no use for the production of general-arrangement drawings or for a host of other tasks. It will generally be the case that the general-purpose program will feel, to the user, much more cumbersome, than a program built to specific purpose. When this occurs, most operators begin to feel that they are not properly in control and thereby become a little careless, allowing mistakes to creep in.

It is not practicable to produce special-purpose programs for all problems; there are too many problems. Indeed, even a special-purpose program will be, to some extent, general-purpose. (A detail drawing program will, for instance, be capable of drawing a wide variety of beam types, though it may not be capable of drawing a column.)

There is no universal answer to this choice problem. It is a question which can be resolved only by harnessing a proper awareness.

1.17.4 Computers and information

The last half decade has seen an explosion in the amount of data stored in computers. (Technically this has become possible as the cost of unit storage has reduced.) However, there is no point in storing data in a computer if it cannot be accessed with both speed and ease, and then manipulated to meet the need.

Computers traditionally have been regarded as 'unintelligent' so that information would, of necessity, be stored only in carefully prearranged patterns in order to permit subsequent location and retrieval. The argument has been that, if there is no pattern, retrieval will be impossible, so do not store. (There have inevitably been 'squirrels' who have adopted the policy of storing everything, in case it may be useful. This philosophy has not been regarded as generally cost-effective.)

Information, which can be expensive to collect and to keep, has typically been stored in large databases to which accredited users can gain access. These databases have generally been in very well-defined structural forms. In consequence access has, in general, been rapid. However, the design of the database envisages 'all' possible accesses which might be used in future. Now, however, increased machine speeds and the production of 'intelligent' software is cutting across these restrictions. The way ahead is not clear, nor is it likely to be quick because of the sheer volume of information which is available to man. However, there will be movement towards making information, generally, more easily available.

1.17.5 Computers and management

The size of projects in which mankind engages has increased enormously; so has the complexity. The management of projects (and the training of managers) has become a major problem. Early tools to come to the aid of management have included bar-chart techniques, etc. The problem with most of these techniques is the volume of work necessary to cope with the inevitable alterations to the original plan. These alterations are liable to occur throughout the life of the project. Ideally, the manager would like to examine the effect of the change forced on him and then consider possible effects of changing his own plan for proceeding. Of course, the calculation power of the computer is the facility which makes such possibilities realistic.

However, this is only dealing with the techniques. There is also the problem of training managers, preferably without the trainees making mistakes (with very large cost consequences) on a real job. Developments in universities and research organizations have played a major part here.

1.17.5.1 Training games

These developments take the form of the simulation of a construction project, e.g. the construction of a manhole. This simulation is incorporated in a 'training game'. The game is set up by the tutor and included in it are details of the project and rates, e.g. for crane hire. Some of this information is made available to the player, who is invited to manage the construction. For each day's work his management will take the form of ordering types of labour and/or materials. The player has options, e.g. a cheap but not too reliable crane hire company, as opposed to a more reliable, more expensive one. The simulation makes available weather forecasts for the following day at the stage when the player is ordering. As in real life the weather is *generally* similar to the forecasts but differences do occur. With the labour and materials he has ordered, a certain amount of construction will get done in the day, and for this the player earns credit or payment. On the other side the labour and materials will be expensive. The actual progress of the work is subject to statistical interruptions whose level of occurrence is set by the tutor in advance. The objective for the players is to make a profit that is as large as possible.

The use of this training tool seems to be most effective when the player is actually a small team of about four students. The element of competition provided by three other teams working at the same time (but independently so they suffer different statistical 'accidents') increases the learning by sharing complementary experiences. There is clear evidence that this training is effective: it is certainly cheaper than making mistakes on a real job.

1.17.5.2 Project planning models

A further illustration of the way computers assist with tackling the unknown is to be found in a tool to be used in advance planning. For this application the project is modelled as a fairly conventional bar chart (possibly at more than one level). However, the model is not deterministic, i.e. it is recognized that when the chart is constructed, a bar is only a best advance guess and that it is subject, in the event, to variation. For many items (e.g. weather effects, rate of bricklaying, etc.) data is available about the variations which occur in practice. These variations are incorporated with the basic data. The best-guess bars represent just one way in which the project might be built. Changing one bar (within its allowed variation) produces another way the project might be built. What the computer does is to 'build' the project many hundreds of times allowing all the bars to vary stochastically. The result is an envelope of possible construction routes. Some will be quicker, others will be slower; some will be cheaper, others will be more expensive. Overall, however, the envelope will highlight potential holdups caused by delays, indicate cashflow requirements, etc. Clearly, the system can be run not only prior to construction but also during construction (when work already complete is, of course, no longer subject to variation). The tool ideally should be used collaboratively between client and contractor in a noncompetitive manner. At present, the world is far from ideal but there may be sufficient benefit for this route to appeal, especially to those involved in the very large projects for which it is best used. It is interesting that this potentially valuable exercise demands nothing more expensive than a fairly run-of-the-mill desktop personal computer in order to produce useful results.

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2 Strength of Materials

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2.1 Introduction

The subject 'Strength of Materials' originates from the earliest attempts to account for the behaviour of structures under load. Thus the problems of particular interest to the first investigators, Galileo and Hooke in the seventeenth century, and Euler and Coulomb in the eighteenth,[1] were the very practical problems associated with the behaviour of beams and columns; at a somewhat later stage, general mathematical investigations of the behaviour of elastic bodies were made by Navier (1821) and Cauchy (1822). The theory of structures has subsequently developed so that it now includes many different and sophisticated fields of interest. Nevertheless, the topic 'Strength of Materials' traditionally covers those aspects of the theory that were the subject of the original research: the theory of bars and the general theory of elasticity. This chapter, therefore, is essentially a review of the main features of these two somewhat disparate theories, and contains some of the results that are of immediate importance to civil engineers.

2.2 Theory of elasticity

2.2.1 Internal stress

Internal stress is the name given to the intensity of the internal forces set up within a body subject to loading. Consider such a body shown in Figure 2.1(a) and an imaginary plane surface within the body passing through a point P. The internal forces exerted between atoms across this surface are represented in the expanded view of Figure 2.1(b). They are described by stress vectors (having the dimensions of force per unit area), and the particular vectors at P give a measure of the intensity of the internal forces at this point. They are denoted by σ and called *internal stress vectors*. If they are directed away from the material as in Figure 2.1(c) they are called *tensile*, and if towards the material *compressive*.

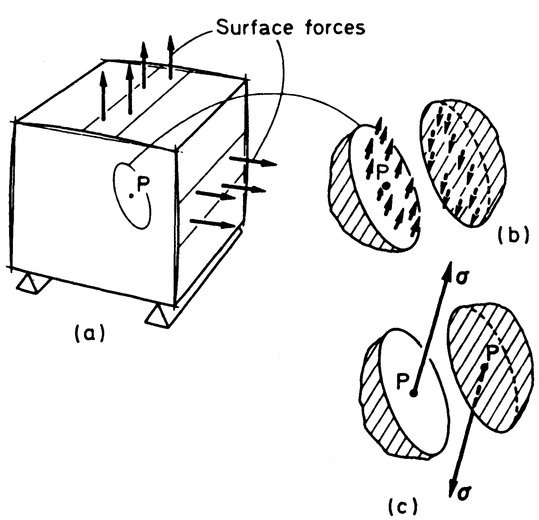


Figure 2.1

2.2.1.1 Components of stress

The complete state of stress at P is defined in terms of the internal stress vectors acting on three particular surfaces at P called the *positive coordinate surfaces*. (The positive x coordinate surface is the surface parallel to the *y–z* plane of an *x*, *y*, *z* coordinate system, with the material situated so that a vector directed outwards from the material and normal to the surface is in the positive direction of the *x* coordinate line as in Figure 2.2.)

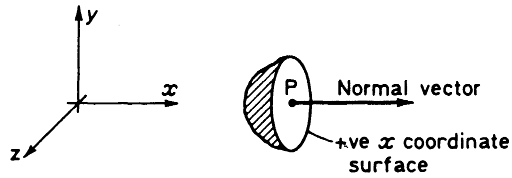


Figure 2.2

These internal stress vectors are distinguished by appropriate subscripts. Thus $\boldsymbol{\sigma}_x$ acts on the positive x coordinate surface, while $\boldsymbol{\sigma}_y$ and $\boldsymbol{\sigma}_z$ respectively act on the *y* and *z* surfaces. Their scalar components† are then denoted by two subscripts. Thus the components of $\boldsymbol{\sigma}_x$ are σ_{xx}, σ_{xy} and σ_{xz} and are shown in Figure 2.3(a). Similarly the components of $\boldsymbol{\sigma}_y$ are σ_{yx}, σ_{yy}, σ_{yz} and of $\boldsymbol{\sigma}_z$ are σ_{zx}, σ_{zy}, σ_{zz} as shown in Figure 2.3(b) and (c). σ_{xx}, σ_{yy} and σ_{zz} are called the *direct stress components* at P in the *x*, *y* and *z* directions respectively, while σ_{xy}, σ_{xz}, σ_{yx}, σ_{yz}, σ_{zx}, and σ_{zy} are called the *shear stress components*.

While the above notation is strictly logical and clarifies the basic concepts of stress, conventional engineering notation is somewhat different and emphasizes the physical differences between the components. Thus the direct stress components are written σ_x, σ_y and σ_z, while the shear stress components are written τ_{xy}, τ_{xz}, τ_{yx}, τ_{yz}, τ_{zx}, τ_{zy}. Except in section 2.2.1.3 (below), this latter notation is employed in the remainder of this chapter.

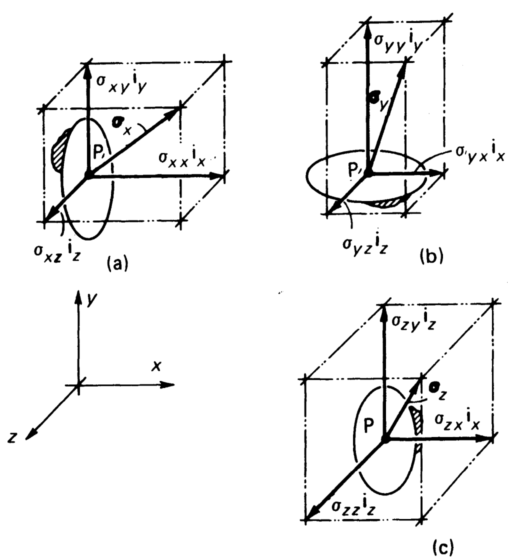


Figure 2.3

2.2.1.2 Stress on an arbitrary surface

Suppose a plane surface through P is defined in terms of the components n_x, n_y and n_z of the outward unit normal vector **n**, as in Figure 2.4. The stress vector $\boldsymbol{\sigma}_n$ acting on this surface is

† A vector **F** at P is equal to $F_x\mathbf{i}_x + F_y\mathbf{i}_y + F_z\mathbf{i}_z$, where F_x, F_y and F_z are the scalar components of **F**, and $\mathbf{i}_x$, $\mathbf{i}_y$ and $\mathbf{i}_z$ are unit base vectors parallel respectively to the *x*, *y* and *z* coordinate lines at P.

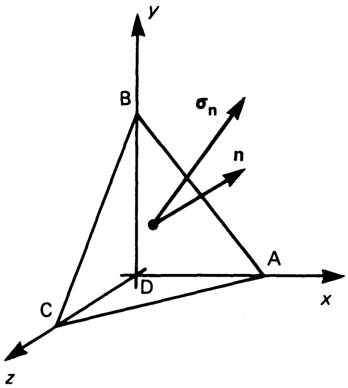


Figure 2.4

obtained in terms of the basic stress components defined in the previous section by considering the linear equilibrium of the differentially small trapezoidal element ABCD shown in the figure. Thus:

$$\sigma_{nx}=\sigma_x n_x+\tau_{yx}n_y+\tau_{zx}n_z \tag{2.1}$$

$$\sigma_{ny}=\tau_{xy}n_x+\sigma_y n_y+\tau_{zy}n_z \tag{2.2}$$

$$\sigma_{nz}=\tau_{xz}n_x+\tau_{yz}n_y+\sigma_z n_z \tag{2.3}$$

where σ_{nx}, σ_{ny} and σ_{nz} are the components of $\boldsymbol{\sigma}_n$.

2.2.1.3 Transformation of stress

Considering a new coordinate system x', y', z' rotated relative to the x, y and z system as in Figure 2.5, then the components of stress in the new system are defined as in section 2.2.1.1, so that $\tau_{x'y'}$ $(=\sigma_{x'y'})$, for example, is the component in the y' direction of the stress vector acting on the positive x' coordinate surface.

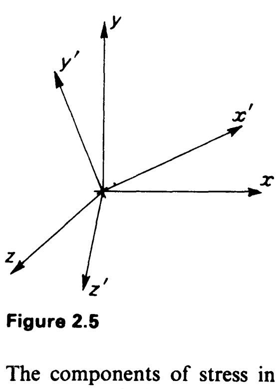


Figure 2.5

The components of stress in the two systems are related by equations of the following type (where for conciseness we employ the original notation of section 2.2.1.1):

$$\begin{aligned}\sigma_{x'y'}=&\frac{\partial x}{\partial x'}\frac{\partial x}{\partial y'}\sigma_{xx}+\frac{\partial x}{\partial x'}\frac{\partial y}{\partial y'}\sigma_{xy}+\frac{\partial x}{\partial x'}\frac{\partial z}{\partial y'}\sigma_{xz}\\ &+\frac{\partial y}{\partial x'}\frac{\partial x}{\partial y'}\sigma_{yx}+\frac{\partial y}{\partial x'}\frac{\partial y}{\partial y'}\sigma_{yy}+\frac{\partial y}{\partial x'}\frac{\partial z}{\partial y'}\sigma_{yz}\\ &+\frac{\partial z}{\partial x'}\frac{\partial x}{\partial y'}\sigma_{zx}+\frac{\partial z}{\partial x'}\frac{\partial y}{\partial y'}\sigma_{zy}+\frac{\partial z}{\partial x'}\frac{\partial z}{\partial y'}\sigma_{zz}\end{aligned} \tag{2.4}$$

Equation (2.4) and eight similar equations formed by permuting x', y' and z' are called the *transformation equations of stress*. The partial derivatives in Equation (2.4) are called direction cosines, since $\partial y/\partial x'$, for example, is equal to the cosine of the angle between the y and x' coordinate lines.

2.2.1.4 Principal stresses

For a particular orientation of x', y' and z' it is found that all the shear stress components vanish, i.e. that the stress vectors $\boldsymbol{\sigma}_{x'}$, $\boldsymbol{\sigma}_{y'}$ and $\boldsymbol{\sigma}_{z'}$ are directed at right angles to their respective coordinate surfaces. Calling this coordinate system X, Y and Z, the matrix of stress components takes the form:

$$\begin{matrix}\sigma_X & 0 & 0\\ 0 & \sigma_Y & 0\\ 0 & 0 & \sigma_Z\end{matrix}$$

The direct stresses σ_X, σ_Y and σ_Z are called the *principal stresses* at P, while the X, Y and Z coordinate lines are called the *principal directions of stress*.

The values of the principal stresses in terms of the stress components in the x, y and z system are equal to the three roots of the equation:

$$\sigma^3-I_1\sigma^2+I_2\sigma-I_3=0 \tag{2.5}$$

where

$$I_1=\sigma_x+\sigma_y+\sigma_z \tag{2.6}$$

$$I_2=\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x-\tau_{xy}^2-\tau_{yz}^2-\tau_{zx}^2 \tag{2.7}$$

$$I_3=\sigma_x\sigma_y\sigma_z+2\tau_{xy}\tau_{yz}\tau_{zx}-\sigma_x\tau_{yz}^2-\sigma_y\tau_{zx}^2-\sigma_z\tau_{xy}^2 \tag{2.8}$$

The direction cosines of the Y coordinate line say, relative to the x, y and z coordinate lines (λ_{Yx}, λ_{Yy}, λ_{Yz}), are found by solving the equations

$$\begin{bmatrix}(\sigma_x-\sigma_Y) & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & (\sigma_y-\sigma_Y) & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & (\sigma_z-\sigma_Y)\end{bmatrix}\begin{bmatrix}\lambda_{Yx}\\ \lambda_{Yy}\\ \lambda_{Yz}\end{bmatrix}=0 \tag{2.9}$$

$$(\lambda_{Yx})^2+(\lambda_{Yy})^2+(\lambda_{Yz})^2=1 \tag{2.10}$$

(Note that the three equations represented by Equation (2.9) are not independent.)

2.2.1.5 Internal equilibrium equations

Consideration of the equilibrium of a differentially small parallelepiped element of material surrounding an internal point P, leads to three equations of linear equilibrium:

$$\frac{\partial\sigma_x}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+F_x=0 \tag{2.11}$$

$$\frac{\partial\sigma_y}{\partial y}+\frac{\partial\tau_{zy}}{\partial z}+\frac{\partial\tau_{xy}}{\partial x}+F_y=0 \tag{2.12}$$

$$\frac{\partial\sigma_z}{\partial z}+\frac{\partial\tau_{xz}}{\partial x}+\frac{\partial\tau_{yz}}{\partial y}+F_z=0 \tag{2.13}$$

and three equations of rotational equilibrium:

$$\tau_{xy} = \tau_{yx} \tag{2.14}$$

$$\tau_{yz} = \tau_{zy} \tag{2.15}$$

$$\tau_{zx} = \tau_{xz} \tag{2.16}$$

In Equations (2.11 to 2.13), F_x, F_y and F_z are the components of any body force vector **F** (units: force per unit volume) acting at P. Note, for example, that a body force vector of magnitude (ρg)/unit volume is exerted by the Earth at all points within a body situated in its gravitational field, ρ being the local density of the body and g being the acceleration due to gravity.

The shear stress components τ_{xy} and τ_{yx} being equal, are called *complementary shear stresses*. It is apparent from Equations (2.14 to 2.16) that if a body is in equilibrium then only six of the nine stress components can take different values at any point.

2.2.1.6 Plane stress

For structures made of elements whose dimensions in the z direction are much smaller than the dimensions in the x and y directions, such as thin plate girders, slabs, shear walls, etc., the following assumptions can be made: (1) the stress components σ_z, τ_{yz}, τ_{xz} can be ignored; and (2) the stress components are uniform across the thickness of the element. That is, they are independent of z.

Such a state of stress is called *plane stress*.

For plane stress, the transformation Equations (2.4) take a simple and important form. Suppose the x', y', z' system is formed by a rotation of $\alpha°$ about the z axis anticlockwise from the reader's viewpoint, as in Figure 2.6. The transformation equations between σ_x, σ_y, τ_{xy} and $\sigma_{x'}$, $\sigma_{y'}$, $\tau_{x'y'}$, are then as follows:

$$\sigma_{x'} = \tfrac{1}{2}(\sigma_x + \sigma_y) + \tfrac{1}{2}(\sigma_x - \sigma_y)\cos(2\alpha) + \tau_{xy}\sin(2\alpha) \tag{2.17}$$

$$\sigma_{y'} = \tfrac{1}{2}(\sigma_x + \sigma_y) - \tfrac{1}{2}(\sigma_x - \sigma_y)\cos(2\alpha) - \tau_{xy}\sin(2\alpha) \tag{2.18}$$

$$\tau_{x'y'} = -\tfrac{1}{2}(\sigma_x - \sigma_y)\sin(2\alpha) + \tau_{xy}\cos(2\alpha) \tag{2.19}$$

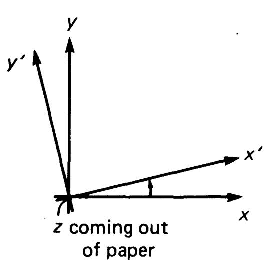


Figure 2.6

These equations can then be represented by the following graphical construction. Two axes are drawn, the vertical representing shear stress and the horizontal, direct stress, and a circle is constructed whose centre is at $(\sigma_x + \sigma_y)/2$ on the direct stress axis, and which passes through the point (σ_x, τ_{xy}) as in Figure 2.7. The line through the centre of the circle at an angle $2\alpha°$ *clockwise* to the line joining the centre and (σ_x, τ_{xy}) then intersects the circle at ($\sigma_{x'}$, $\tau_{x'y'}$). Produced backwards, it intersects the circle at a point whose abscissa is $\sigma_{y'}$. This construction was devised by Otto Mohr in 1882 and the circle is called *Mohr's circle of stress*.

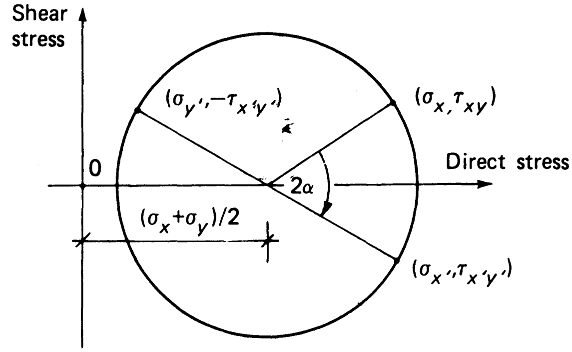


Figure 2.7 Mohr's circle of stress

2.2.2 Strain

Strain is the general name given to the deformation of a body subject to loading.

2.2.2.1 Displacements

A particular point P in a body before loading, occupies its initial position P_i say, and after loading its final position P_f. The line joining P_i to P_f is a vector which is denoted by **u** and called the *displacement vector* at P. In general, this vector varies continuously from point to point in the body, and its three components u_x, u_y and u_z are continuous functions of the coordinates of P.†

Consider two neighbouring points P(x, y, z) and P*(x+dx, y+dy, z+dz) in the body. Then

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy + \frac{\partial u_x}{\partial z}dz \tag{2.20}$$

$$du_y = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy + \frac{\partial u_y}{\partial z}dz \tag{2.21}$$

$$du_z = \frac{\partial u_z}{\partial x}dx + \frac{\partial u_z}{\partial y}dy + \frac{\partial u_z}{\partial z}dz \tag{2.22}$$

where the differentials du_x, du_y and du_z are the differences between the components of **u** at the two points. As such, these differentials can be regarded as the components of the vector giving the displacement of P* *relative* to P.

2.2.2.2 Components of strain

In order to obtain a concise description of the deformation of the material at P it is convenient to define nine dimensionless components ε_{xx}, ε_{yy}, ε_{zz}, ε_{xy}, ε_{yz}, ε_{zx}, ω_{xy}, ω_{yz}, ω_{zx} by the following equations, called the *strain–displacement relations*:

$$\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \quad \varepsilon_{yy} = \frac{\partial u_y}{\partial y}, \quad \varepsilon_{zz} = \frac{\partial u_z}{\partial z} \tag{2.23, 2.24, 2.25}$$

$$\varepsilon_{xy} = \frac{1}{2}\left(\frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x}\right), \quad \varepsilon_{yz} = \frac{1}{2}\left(\frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y}\right),$$

$$\varepsilon_{zx} = \frac{1}{2}\left(\frac{\partial u_z}{\partial x} + \frac{\partial u_x}{\partial z}\right) \tag{2.26, 2.27, 2.28}$$

† In most cases **u** is so small that the coordinates of P do not change appreciably during the loading.

$$\omega_{xy}=\frac{1}{2}\left(\frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x}\right),\quad \omega_{yz}=\frac{1}{2}\left(\frac{\partial u_y}{\partial z}-\frac{\partial u_z}{\partial y}\right),$$

$$\omega_{zx}=\frac{1}{2}\left(\frac{\partial u_z}{\partial x}-\frac{\partial u_x}{\partial z}\right) \qquad (2.29,\ 2.30,\ 2.31)$$

The physical meaning of these components is clarified by considering the deformation of the rectangular element of material containing P shown in Figure 2.8(a) for each component in turn.

Thus if $\varepsilon_{xx}\neq 0$, $\varepsilon_{yy}=\varepsilon_{zz}=\varepsilon_{xy}=\varepsilon_{yz}=\varepsilon_{zx}=\omega_{xy}=\omega_{yz}=\omega_{zx}=0$ then, by using Equations (2.20 to 2.22), it can be shown that the element deforms as in Figure 2.8(b). ε_{xx}, corresponding to this type of longitudinal deformation is called the *direct strain component* in the x direction at P. If it is positive it is called *tensile* and the element lengthens and if negative, it is called *compressive* and the element shortens. Similarly, the components ε_{yy} and ε_{zz} corresponding respectively to longitudinal deformation in the y and z directions are called the direct strain components in these directions.

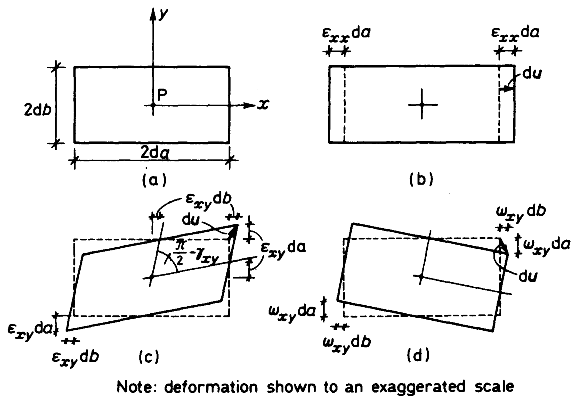


Figure 2.8

If $\varepsilon_{xy}\neq 0$, $\varepsilon_{xx}=\varepsilon_{yy}=\varepsilon_{zz}=\varepsilon_{yz}=\varepsilon_{zx}=\omega_{xy}=\omega_{yz}=\omega_{zx}=0$ then $\partial u_x/\partial y=\partial u_y/\partial x=\varepsilon_{xy}$ and again by using Equations (2.20 to 2.22) it can be shown that the element deforms into a lozenge shape as in Figure 2.8(c). Deformation of this type is called shear strain, and ε_{xy} is called the *mathematical shear strain component* at P. The adjective 'mathematical' is used to distinguish between this and the engineering shear strain at P, which is denoted by γ_{xy} and is equal to the closure in radians of the angle between the x and y coordinate lines. From the geometry of Figure 2.8(c) we have

$$\gamma_{xy}=2\varepsilon_{xy} \qquad (2.32)$$

Similarly, the components ε_{yz} and ε_{zx} correspond to shear strain in the y–z and z–x planes respectively.

Finally, if $\omega_{xy}\neq 0$, $\varepsilon_{xx}=\varepsilon_{yy}=\varepsilon_{zz}=\varepsilon_{xy}=\varepsilon_{yz}=\varepsilon_{zx}=\omega_{yz}=\omega_{zx}=0$ then $\partial u_x/\partial y=-\partial u_y/\partial x=\omega_{xy}$ and it can be shown that the element rotates without deformation about the z coordinate line as in Figure 2.8(d). ω_{xy} is called the *rotation* at P. Similarly ω_{yz} and ω_{zx} correspond respectively to local rotations about the x and y coordinate lines through P. These rotations are necessary in the theoretical discussion in order to define the displacement derivatives in Equations (2.20 to 2.22). However, since they do not define *deformation* directly, they are not considered further in elastic analysis.

As in the case of stresses, the conventional engineering notation for the strain components is somewhat different from the above and the direct strain components are written ε_x ε_y and ε_z. Except in section 2.2.2.4 (below), this latter notation is employed in the remainder of the chapter, and the shear strains are described in terms of γ_{xy}, γ_{yz} and γ_{zx}.

In the majority of civil engineering structures, the strain components are very small, of the order of magnitude 10^{-3}. Thus, the deformation of the elements in Figure 2.8 is exaggerated. The strain–displacement relations in Equations (2.23 to 2.28) assume that the *displacements* are small. If this is not the case, nonlinear terms involving the products of the derivatives are included.[2] These nonlinear terms are significant in defining the buckling characteristics of thin elements in compression.[3,4]

2.2.2.3 Uniform strain

If the displacement components u_x, u_y and u_z are linear functions of the coordinates of P then the corresponding strains given by Equations (2.23 to 2.28) are uniform. The overall changes in the geometry of a body are then simply related to the strain components. Thus consider, for example, a line AB in or on the surface of the body which originally coincides with an x coordinate line. If the original length of AB is l and its increase in length is Δl, then:

$$\varepsilon_x=\Delta l/l \qquad (2.33)$$

2.2.2.4 Transformation of strain

Considering again a new coordinate system x', y', z' rotated relative to the x, y and z system as in Figure 2.5, then the components of strain in this new system are defined by strain–displacement relations similar to Equations (2.23 to 2.28). Thus $\gamma_{x'y'}(=2\varepsilon_{x'y'})$, for example, is given by:

$$\gamma_{x'y'}=\left(\frac{\partial u_{x'}}{\partial y'}+\frac{\partial u_{y'}}{\partial x'}\right) \qquad (2.34)$$

where $u_{x'}$, $u_{y'}$ and $u_{z'}$ are the components of the displacement vector **u** relative to x', y' and z'. The components of strain in the two systems are related by equations of the same type as Equation (2.4) (where again for conciseness we employ the original notation of section 2.2.2.2). Thus:

$$\varepsilon_{x'y'}=\frac{\partial x}{\partial x'}\frac{\partial x}{\partial y'}\varepsilon_{xx}+\frac{\partial x}{\partial x'}\frac{\partial y}{\partial y'}\varepsilon_{xy}+\frac{\partial x}{\partial x'}\frac{\partial z}{\partial y'}\varepsilon_{xz}$$

$$+\frac{\partial y}{\partial x'}\frac{\partial x}{\partial y'}\varepsilon_{yx}+\frac{\partial y}{\partial x'}\frac{\partial y}{\partial y'}\varepsilon_{yy}+\frac{\partial y}{\partial x'}\frac{\partial z}{\partial y'}\varepsilon_{yz}$$

$$+\frac{\partial z}{\partial x'}\frac{\partial x}{\partial y'}\varepsilon_{zx}+\frac{\partial z}{\partial x'}\frac{\partial y}{\partial y'}\varepsilon_{zy}+\frac{\partial z}{\partial x'}\frac{\partial z}{\partial y'}\varepsilon_{zz} \qquad (2.35)$$

The nine equations formed by permuting x', y' and z' in Equation (2.35) are called the *transformation equations of strain*.

2.2.2.5 Principal strains

For a particular orientation of x', y' and z', all the shear strain components vanish, and in most materials this orientation is the same as that of the principal directions of stress discussed in section 2.2.1.4. Calling the coordinate system X, Y and Z as before, the direct strains ε_X, ε_Y and ε_Z are called the *principal strains* at P.

The values of the principal strains are equal to the three roots of the equation:

$$\varepsilon^3 - E_1\varepsilon^2 + E_2\varepsilon - E_3 = 0 \tag{2.36}$$

where:

$$E_1 = \varepsilon_x + \varepsilon_y + \varepsilon_z \tag{2.37}$$

$$E_2 = \varepsilon_x\varepsilon_y + \varepsilon_y\varepsilon_z + \varepsilon_z\varepsilon_x - \tfrac{1}{4}(\gamma_{xy}^2 + \gamma_{yz}^2 + \gamma_{zx}^2) \tag{2.38}$$

$$E_3 = \varepsilon_x\varepsilon_y\varepsilon_z + \tfrac{1}{4}(\gamma_{xy}\gamma_{yz}\gamma_{zx} - \varepsilon_x\gamma_{yz}^2 - \varepsilon_y\gamma_{zx}^2 - \varepsilon_z\gamma_{xy}^2) \tag{2.39}$$

2.2.2.6 Compatibility equations

The three displacement components u_x, u_y and u_z can be eliminated from the six strain–displacement relations in Equations (2.23 to 2.28) to produce three equations called the *compatibility equations*, which must be satisfied by the strain components. This elimination can be done in different ways to produce different sets of equations. Two such are:

$$\frac{\partial^2\varepsilon_x}{\partial y^2} + \frac{\partial^2\varepsilon_y}{\partial x^2} - \frac{\partial^2\gamma_{xy}}{\partial x\partial y} = 0 \tag{2.40}$$

$$\frac{\partial^2\varepsilon_y}{\partial z^2} + \frac{\partial^2\varepsilon_z}{\partial y^2} - \frac{\partial^2\gamma_{yz}}{\partial y\partial z} = 0 \tag{2.41}$$

$$\frac{\partial^2\varepsilon_z}{\partial x^2} + \frac{\partial^2\varepsilon_x}{\partial z^2} - \frac{\partial^2\gamma_{zx}}{\partial z\partial x} = 0 \tag{2.42}$$

$$\frac{2\partial^2\varepsilon_x}{\partial y\partial z} - \frac{\partial}{\partial x}\left(\frac{\partial\gamma_{xy}}{\partial z} - \frac{\partial\gamma_{yz}}{\partial x} + \frac{\partial\gamma_{zx}}{\partial y}\right) = 0 \tag{2.43}$$

$$\frac{2\partial^2\varepsilon_y}{\partial z\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial\gamma_{yz}}{\partial x} - \frac{\partial\gamma_{zx}}{\partial y} + \frac{\partial\gamma_{xy}}{\partial z}\right) = 0 \tag{2.44}$$

$$\frac{2\partial^2\varepsilon_z}{\partial x\partial y} - \frac{\partial}{\partial z}\left(\frac{\partial\gamma_{zx}}{\partial y} - \frac{\partial\gamma_{xy}}{\partial z} + \frac{\partial\gamma_{yz}}{\partial x}\right) = 0 \tag{2.45}$$

2.2.2.7 Plane strain

Plane strain is said to exist when the strain components ε_z, ε_{yz} and ε_{zx} are equal to zero. It occurs when $u_z = 0$ at every point within a region of a body. From symmetry this is the case in the central region of a body which: (1) is very long in the z direction; (2) is of uniform cross-section; and (3) is subjected to loading in the z plane that is uniformly distributed along its length (Figure 2.9). It can therefore occur in structures such as gravity dams, tunnel linings or retaining walls.

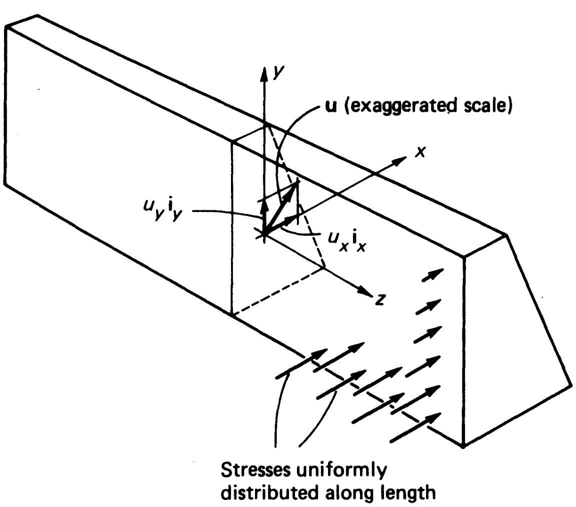


Figure 2.9

Considering again the new coordinate system x', y', z' formed by a rotation of $\alpha°$ anticlockwise about the z axis as in Figure 2.6, the transformation equations between ε_x, ε_y, ε_{xy} and $\varepsilon_{x'}$, $\varepsilon_{y'}$, and $\varepsilon_{x'y'}$ take the same form as Equations (2.17 to 2.19). These transformation equations are represented by a graphical construction called *Mohr's circle of strain*, whose function is the same as that of Mohr's circle of stress.

2.2.3 Elastic stress–strain relations

The relationship between the stress and strain components at a point in a body is a property of the particular material making up the body. For an isotropic elastic material the stress–strain relations are linear and are independent of the orientation of the x, y, z coordinate system. They take the following form:

$$\varepsilon_x = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)] + \alpha\Delta T \tag{2.46}$$

$$\varepsilon_y = \frac{1}{E}[\sigma_y - \nu(\sigma_z + \sigma_x)] + \alpha\Delta T \tag{2.47}$$

$$\varepsilon_z = \frac{1}{E}[\sigma_z - \nu(\sigma_x + \sigma_y)] + \alpha\Delta T \tag{2.48}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy},\ \gamma_{yz} = \frac{1}{G}\tau_{yz},\ \gamma_{zx} = \frac{1}{G}\tau_{zx} \qquad (2.49, 2.50, 2.51)$$

where ΔT is the temperature change from some initial state. E and G are constants having the dimensions of force per unit area and are called *Young's modulus* and the *shear modulus* respectively, ν is a dimensionless constant called *Poisson's ratio* and α is a constant having the dimensions °C^{-1} and is called the *temperature coefficient of expansion*. G in fact is related to E and ν by the following equation:

$$G = E/2(1+\nu) \tag{2.52}$$

Values of E, ν and α for a variety of practical materials are given in Table 2.1.

The corresponding inverse stress–strain relations are found by solving Equations (2.46 to 2.51) for the stresses and are as follows:

$$\sigma_x = 2\mu\varepsilon_x + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) - (3\lambda + 2\mu)\alpha\Delta T \tag{2.53}$$

$$\sigma_y = 2\mu\varepsilon_y + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) - (3\lambda + 2\mu)\alpha\Delta T \tag{2.54}$$

$$\sigma_z = 2\mu\varepsilon_z + \lambda(\varepsilon_x + \varepsilon_y + \varepsilon_z) - (3\lambda + 2\mu)\alpha\Delta T \tag{2.55}$$

$$\tau_{xy} = \mu\gamma_{xy},\ \tau_{yz} = \mu\gamma_{yz},\ \tau_{zx} = \mu\gamma_{zx} \qquad (2.56, 2.57, 2.58)$$

where for conciseness we employ the *Lamé constants* λ and μ defined in terms of E and ν by the equations:

$$\lambda = \nu E/(1+\nu)(1-2\nu) \tag{2.59}$$

$$\mu = E/2(1+\nu) \tag{2.60}$$

Table 2.1 Properties of materials (representative)

Material	*Density* (kg/m³)	*E* (GN/m²)	*μ*	*α* (°C^{-1})	*Limit of proportionality* (MN/m²)	*Ultimate stress* (MN/m²)	*Uniform elongation*
Mild steel	7840	200	0.31	1.25×10^{-5}	280	370	0.30
High-strength steel	7840	200	0.31	1.25×10^{-5}	770	1550	0.10
Medium-strength aluminium alloy	2800	70	0.30	2.3×10^{-5}	230	430	0.10
Titanium alloy	4500	120	0.30	0.9×10^{-5}	385	690	0.15
Magnesium alloy	1800	45	0.30	2.7×10^{-5}	155	280	0.08
Concrete	2410	25	0.20	1.2×10^{-5}	—	3 (tension) 30 (compression)	—
Timber (Douglas fir)	576	7 (with grain)		0.6×10^{-5}	43 (compression with grain)	52 (compression with grain)	—
Glass	2580	60	0.26	0.7×10^{-5}	—	1750	—
Nylon	1140	2	—	10×10^{-5}	77	90	1.00
Polystyrene (not expanded)	1050	4	—	10×10^{-5}	46	60	0.03
High-strength glass-fibre composite	2000	60	—	—	—	1600	—
Carbon fibre composite	1600	170	—	—	—	1400	—

The stress–strain relations hold for a wide range of stresses in most practical materials. They become invalid when the inter-atomic bonds in the materials break down, this process being called *yielding* or *fracture*. Yielding in steel can be demonstrated by the tensile test, where a known stress system $\sigma_x\neq0$, $\sigma_y=\sigma_z=\tau_{xy}=\tau_{yz}=\tau_{zx}=0$, called *uniaxial stress*, is induced in a specimen and the corresponding strain ε_x is measured. A typical plot of σ_x versus ε_x for a mild steel tensile specimen then takes the form shown in Figure 2.10(a). The initial straight section of the curve of slope equal to E corresponds to Equation (2.46), but at a certain stress of the order of 250 MN/m², the strain increases dramatically with little or no increase of load. This stress is called the *uniaxial yield stress* of mild steel. Subsequently, the stress–strain curve indicates that the specimen supports larger stresses up to a maximum value of the order of 400 MN/m² which is called the *ultimate tensile stress*. The uniaxial stress–strain curve for an aluminium alloy specimen shown in Figure 2.10(b) does not display a marked yield stress and the material is linear elastic up to a stress called the *limit of proportionality* which again is of the order of 250 MN/m². Two other properties frequently quoted in engineering literature, the 0.2% *proof stress* and the *uniform elongation*, are shown in the figure. Values for the limit of proportionality, ultimate stress and uniform elongation are included in Table 2.1.

For accounts of yield criteria and plastic stress–strain relations corresponding to more general stress systems see, for example, Bisplinghoff *et al*,[5] and Prager and Hodge.[6]

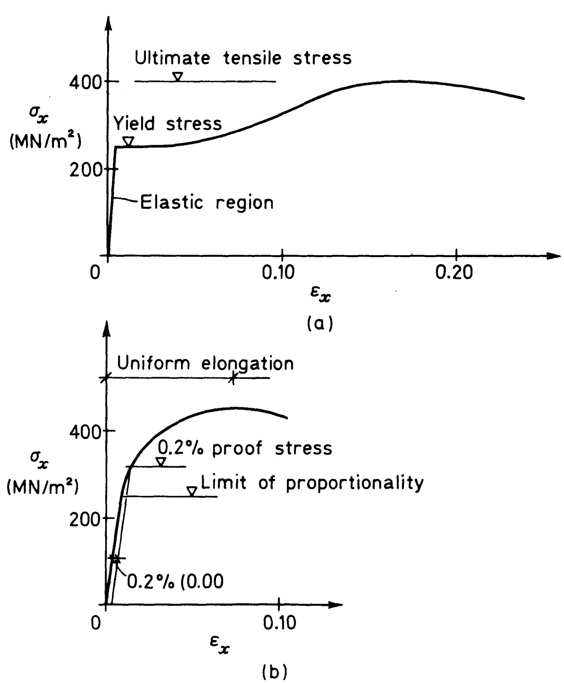


Figure 2.10 Definitions of material properties

2.2.4 Analysis of elastic bodies

The internal equilibrium Equations (2.11 to 2.16), strain–displacement relations Equations (2.23 to 2.28) and the stress–strain relations Equations (2.46 to 2.51) are eighteen differential equations in the unknowns of the analysis problem, namely the nine stress components, the six strain components and the three displacement components. These equations must be satisfied subject to boundary conditions.

2.2.4.1 Boundary conditions

The boundary conditions at a point P on the surface of a body are expressed in terms of the components S_x, S_y and S_z of the surface stress vector **S** acting at P, and the components u_x, u_y and u_z of the displacement vector **u** of P. They are of three types, as follows.

Static boundary conditions. The three stress vector components at P are specified. Thus at an unloaded point on the boundary $S_x=S_y=S_z=0$, while at a loaded point $S_x=k_1$, $S_y=k_2$, $S_z=k_3$, where k_1, k_2 and k_3 are known values at P.

Kinematic boundary conditions. The three displacement components at P are specified. Thus at a rigid support $u_x=u_y=u_z=0$, while at a point whose displacements are constrained by, say, a screw jack $u_x=j_1$, $u_y=j_2$, $u_z=j_3$, where j_1, j_2 and j_3 are known values at P.

Mixed boundary conditions. Certain displacement and certain stress-vector components at P are specified simultaneously. For example, at the point P on the roller support shown in Figure 2.11, $S_x=0$ and $u_y=u_z=0$.

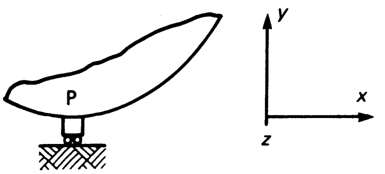


Figure 2.11

2.2.4.2 Solution in terms of displacements

A straightforward solution method involves treating the displacement components as the basic unknowns. The three linear equilibrium Equations (2.11 to 2.13) are expressed in terms of the displacements by using the stress–strain relations followed by the strain–displacement relations. The resulting differential equations in u_x, u_y and u_z are called the *Navier equations*. They are as follows:

$$\mu\nabla^2 u_x+(\lambda+\mu)\frac{\partial\Phi}{\partial x}+F_x=0 \tag{2.61}$$

$$\mu\nabla^2 u_y+(\lambda+\mu)\frac{\partial\Phi}{\partial y}+F_y=0 \tag{2.62}$$

$$\mu\nabla^2 u_z+(\lambda+\mu)\frac{\partial\Phi}{\partial z}+F_z=0 \tag{2.63}$$

where

$$\nabla^2 u_x=\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\partial^2 u_x}{\partial z^2} \tag{2.64}$$

and:

$$\Phi=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z} \tag{2.65}$$

In order to solve these equations, the boundary conditions must all be expressed in terms of the displacements of the surface points. In the case of the static boundary conditions, equations for the internal stress components are obtained using Equations (2.1 to 2.3) with the components of σ_n replaced by the components of S. These are then converted to differential boundary conditions in displacements by again using the stress–strain and the strain–displacement relations. Thus at each internal point and each boundary point there are three simultaneous differential equations in the unknowns u_x, u_y and u_z. In most cases, a direct solution is obtainable only by a numerical procedure such as the finite-difference method.[7]

2.2.4.3 Solution in terms of stresses

A second solution method involves treating the nine internal stress components as the basic unknowns. Six equations in these unknowns are immediately available from the internal equilibrium Equations (2.11 to 2.16). A further three equations are obtained from the compatibility Equations (2.40 to 2.42) or (2.43 to 2.45) by using the stress–strain relations to express them in terms of the stress components. The resulting equations are called the *Beltrami–Michell* equations and are as follows:

$$\nabla^2\sigma_x+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x^2}=\frac{-\nu}{(1-\nu)}\Psi-\frac{2\partial F_x}{\partial x} \tag{2.66}$$

$$\nabla^2\sigma_y+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y^2}=\frac{-\nu}{(1-\nu)}\Psi-\frac{2\partial F_y}{\partial y} \tag{2.67}$$

$$\nabla^2\sigma_z+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial z^2}=\frac{-\nu}{(1-\nu)}\Psi-\frac{2\partial F_z}{\partial z} \tag{2.68}$$

or:

$$\nabla^2\tau_{xy}+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial x\partial y}=-\left(\frac{\partial F_x}{\partial y}+\frac{\partial F_y}{\partial x}\right) \tag{2.69}$$

$$\nabla^2\tau_{yz}+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial y\partial z}=-\left(\frac{\partial F_y}{\partial z}+\frac{\partial F_z}{\partial y}\right) \tag{2.70}$$

$$\nabla^2\tau_{zx}+\frac{1}{(1+\nu)}\frac{\partial^2\Theta}{\partial z\partial x}=-\left(\frac{\partial F_z}{\partial x}+\frac{\partial F_x}{\partial z}\right) \tag{2.71}$$

where

$$\Theta=\sigma_x+\sigma_y+\sigma_z \tag{2.72}$$

$$\Psi=\frac{\partial F_x}{\partial x}+\frac{\partial F_y}{\partial y}+\frac{\partial F_z}{\partial z} \tag{2.73}$$

The only problems than can be solved directly in terms of stresses conveniently are those in which all the boundary conditions are static boundary conditions. In such problems, three equations in the internal stress components are obtained using Equations (2.1 to 2.3) and these, together with the three equations of rotational equilibrium and the three compatibility equations, provide the required nine equations at the boundary. In problems where displacements are specified at various boundary points, the corresponding boundary stresses cannot usually be obtained in advance of the solution except for those special cases where the body is externally statically determinate.

Direct solutions in terms of stresses can in principle be obtained using numerical procedures. However, many solutions, especially to two-dimensional problems,[2,8] have been obtained using stress functions which automatically satisfy the equilibrium equations.

2.2.5 Energy methods

2.2.5.1 Virtual work

Consider a body which is in equilibrium under surface stresses **S** over part of its surface and body forces **F**. Suppose the corresponding internal stress system is given by σ_x, σ_y, σ_z, τ_{xy}, τ_{yz}, τ_{zx}. This is called an *equilibrium force system.*

Next consider an entirely independent system of displacements $\mathbf{u}^*$ which vary continuously from point to point in the body and satisfy the kinematic boundary conditions. Suppose the corresponding strain system is given by ε_x^*, ε_y^*, ε_z^*, γ_{xy}^*, γ_{yz}^*, γ_{zx}^*. This is called a *compatible displacement system.*

The virtual work W_e^* done by the external forces **S** and **F**, supposing they were to move through $\mathbf{u}^*$, is as follows:

$$W_e^*=\int_A(S_xu_x^*+S_yu_y^*+S_zu_z^*)\,\mathrm{d}A+\int_V(F_xu_x^*+F_yu_y^*+F_zu_z^*)\,\mathrm{d}V \tag{2.74}$$

where $\int_A(\ )\,dA$ represents an integral taken over the loaded surface of the body, and $\int_V(\ )\,dV$ represents an integral taken over its volume. By a purely mathematical operation[5] it can be shown that

$$W_e^* = W_i^* \tag{2.75}$$

where W_i^* is a quantity called the *internal virtual work* and is given by

$$W_i^* = \int_V (\sigma_x\varepsilon_x^* + \sigma_y\varepsilon_y^* + \sigma_z\varepsilon_z^* + \tau_{xy}\gamma_{xy}^* + \tau_{yz}\gamma_{yz}^* + \tau_{zx}\gamma_{zx}^*)\,dV \tag{2.76}$$

Equation (2.75) is called the *equation of virtual work*. Note that its derivation is independent of the nature of the stress–strain relations of the material making up the body.

2.2.5.2 Strain energy

Consider the body in equilibrium under **S** and **F** and suppose differential changes in the loading d**S** and d**F** occur causing corresponding differential changes in the *real* displacements d**u**. d**u** and the strains $d\varepsilon_x$, $d\varepsilon_y$, $d\varepsilon_z$, $d\gamma_{xy}$, $d\gamma_{yz}$, $d\gamma_{zx}$ can be regarded as a compatible system of displacements in Equation (2.75). The work terms on either side of Equation (2.75) are then differential quantities of real work caused by the loading change. In particular the internal work is given by

$$dW_i = \int_V (\sigma_x\,d\varepsilon_x + \sigma_y\,d\varepsilon_y + \sigma_z\,d\varepsilon_z + \tau_{xy}\,d\gamma_{xy} + \tau_{yz}\,d\gamma_{yz} + \tau_{zx}\,d\gamma_{zx})\,dV \tag{2.77}$$

Using the elastic stress–strain relations it is possible to integrate Equation (2.77) to obtain the total internal work done on an elastic body from the initial state with zero stress to the final state with stresses corresponding to **S** and **F**. This internal work is found to be independent of the loading path to the final state and is called the *elastic strain energy U*. It can be expressed in three forms:

$$\begin{aligned} U &= \int_V \frac{1}{2E}\Big[(\sigma_x^2+\sigma_y^2+\sigma_z^2) - 2\nu(\sigma_x\sigma_y+\sigma_y\sigma_z+\sigma_z\sigma_x) \\ &\quad + 2(1+\nu)(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2)\Big]\,dV \\ &= \int_V \tfrac{1}{2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\sigma_z\varepsilon_z+\tau_{xy}\gamma_{xy}+\tau_{yz}\gamma_{yz}+\tau_{zx}\gamma_{zx})\,dV \\ &= \int_V \Big[\mu(\varepsilon_x^2+\varepsilon_y^2+\varepsilon_z^2) \\ &\quad + \frac{\lambda}{2}(\varepsilon_x+\varepsilon_y+\varepsilon_z)^2 + \frac{\mu}{2}(\gamma_{xy}^2+\gamma_{yz}^2+\gamma_{zx}^2)\Big]\,dV \end{aligned} \tag{2.78}$$

2.2.5.3 Principle of stationary total potential energy

The external work done by the loading in the previous subsection is given by:

$$dW_e = \int_A (S_x\,du_x + S_y\,du_y + S_z\,du_z)\,dA + \int_V (F_x\,du_x + F_y\,du_y + F_z\,du_z)\,dV \tag{2.79}$$

If the loading is *conservative*, so that all the loads on the body are independent of the displacements, it is possible to define a function V as follows:

$$V = U - \int_A (S_xu_x + S_yu_y + S_zu_z)\,dA - \int_V (F_xu_x + F_yu_y + F_zu_z)\,dV \tag{2.80}$$

so that the equation of virtual work for the differential change of the body in equilibrium takes the form:

$$d\Phi = 0 \tag{2.81}$$

Φ is called the *total potential energy* of the system of the body plus loads.

Equation (2.81) is the mathematical statement of the Principle of Stationary Total Potential Energy. Thus, *for a body in equilibrium, the total potential energy is stationary with respect to small changes in the actual displacements of the body*. This is the most important energy principle, and its method of application for the solution of structures involves expressing all the displacements of the structure in terms of a (usually limited) number of degrees of freedom. (This can be done exactly for frameworks, but only approximately for structures such as slabs.) The stationary position of the total potential energy is found by equating to zero the derivatives of Φ with respect to the degrees of freedom. The resulting equations are analogous to the stiffness equations in the stiffness method of structural analysis. They are solved for the degrees of freedom to yield the exact or approximate displacements of the structure corresponding to equilibrium.

If the structural displacements are assumed to be small so that the linear strain–displacement relations in Equations (2.23 to 2.28) are applicable, then it can be shown that the potential energy is a *minimum* for a structure in equilibrium.[9] The equilibrium is then said to be *stable*. If the displacements are not small, and the non-linear strain–displacement relations are used to obtain Φ, the equilibrium potential energy can either be a *minimum* or a *maximum*. In the latter case the equilibrium is said to be *unstable*. For certain values of load called the critical loads or *eigenvalues*, the equilibrium is *neutral*. This is indicated mathematically when the determinant of the coefficient matrix in the stiffness equations is zero. Extensive treatments of the eigenvalue problem have been given in many texts, e.g. by Croll and Walker[10] and by Thompson and Hunt.[11]

2.2.6 Measurement of stress and strain

2.2.6.1 Surface strain

The measurement of strain is usually limited to obtaining direct strains tangential to the surfaces of structures by means of mechanical or electrical strain gauges. If the complete state of tangential strain at a surface point is to be determined, then separate measurements of direct strain have to be obtained in three distinct directions at the point. In interpreting these measurements, we then use the fact that two of the principal directions of stress and strain are tangential to the surface whilst the third is normal to it. Thus using, for example, a 45° strain-gauge rosette, producing strain measurements ε_1, ε_2 and ε_3 as shown in Figure 2.12, it can be shown that the principal direction X is at $\theta°$ anticlockwise to the x coordinate line where:

$$\tan(2\theta) = \frac{(2\varepsilon_2 - \varepsilon_1 - \varepsilon_3)}{(\varepsilon_1 - \varepsilon_3)} \tag{2.82}$$

The two principal surface strains ε_X and ε_Y are then given by:

$$\varepsilon_X = \frac{(\varepsilon_1+\varepsilon_3)}{2} + r \qquad \varepsilon_Y = \frac{(\varepsilon_1+\varepsilon_3)}{2} - r \tag{2.83, 2.84}$$

where

$$r = \tfrac{1}{2}[(\varepsilon_1-\varepsilon_3)^2 + (2\varepsilon_2-\varepsilon_1-\varepsilon_3)^2]^{1/2} \tag{2.85}$$

Example 2.1. The strains measured by the three gauges of the 45° rosette shown in Figure 2.12 are respectively:

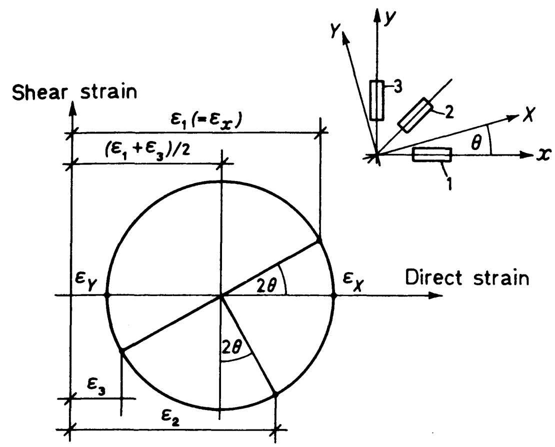


Figure 2.12

$$\varepsilon_1 = -5.0 \times 10^{-4} \quad \varepsilon_2 = +3.0 \times 10^{-4} \quad \varepsilon_3 = +1.0 \times 10^{-4}$$

What are the principal strains at the point and the orientation of the principal direction X, to the x coordinate line?

From Equation (2.85):

$$r = \tfrac{1}{2}[(-5.0 - 1.0)^2 + (2 \times 3.0 + 5.0 - 1.0)^2]^{1/2} \times 10^{-4}$$

$$= 5.8 \times 10^{-4}$$

Thus:

$$\varepsilon_X = 3.8 \times 10^{-4} \quad \varepsilon_Y = -7.8 \times 10^{-4}$$

From Equation (2.82):

$$\tan(2\theta) = -1.667$$

Thus:

$$2\theta = -59.0^\circ \quad \text{or} \quad 121.0^\circ$$

The ambiguity in the expression for θ is resolved by examining the position of the strains on the Mohr's circle of strain for the surface plane (Figure 2.13). Thus, it is clear that in this example, 2θ must be greater than 90°. The X coordinate line is therefore directed at 60.5° anticlockwise to the x coordinate line.

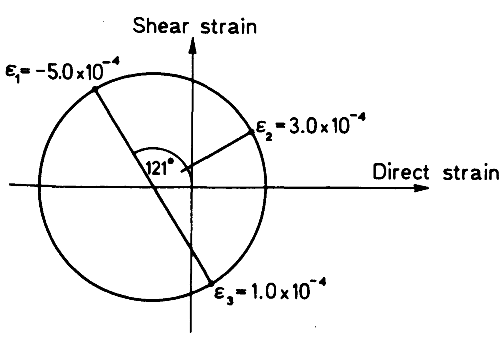


Figure 2.13

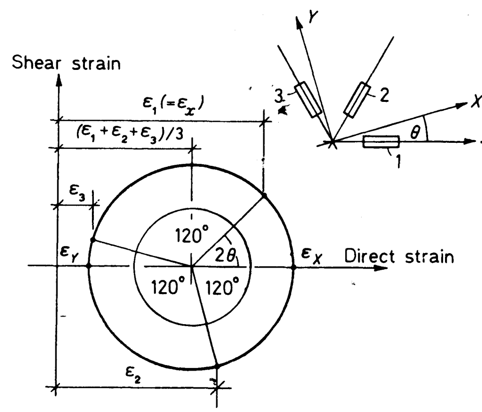


Figure 2.14

Another common layout for strain gauges is the 60° rosette shown in Figure 2.14. The principal direction X is then at θ° anticlockwise to the x coordinate line where:

$$\tan(2\theta) = \sqrt{3}(\varepsilon_2 - \varepsilon_3)/(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) \tag{2.86}$$

while the principal surface strains ε_X and ε_Y are given by

$$\varepsilon_X = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} + r \quad \varepsilon_Y = \frac{\varepsilon_1 + \varepsilon_2 + \varepsilon_3}{3} - r \qquad (2.87, 2.88)$$

where

$$r = \tfrac{2}{3}(\varepsilon_1^2 + \varepsilon_2^2 + \varepsilon_3^2 - \varepsilon_1\varepsilon_2 - \varepsilon_2\varepsilon_3 - \varepsilon_3\varepsilon_1)^{1/2} \tag{2.89}$$

The complete state of surface stress corresponding to the strains measured above can be found from the stress–strain relations, noting that in the absence of surface loading the state of stress is one of plane stress.

2.2.6.2 The photoelastic method[12,13]

A good indication of the internal stresses in model structures can be obtained by making use of the property of certain materials such as glass and plastics, that they become double-refracting when subject to stress.

The apparatus for photoelastic stress analysis consists essentially of a light source L (Figure 2.15), a *polarizer* P, and an *analyser* A and the model M of photoelastic material, which is held in a reaction frame and subjected to loads. The lenses L_1 and L_2 are arranged so that a parallel beam of light passes through the model. An image containing bands of different colours then appears on the ground glass screen, these colours representing regions of equal principal stress difference $(\sigma_X - \sigma_Y)$ in the model. For further experimental and theoretical details see, for example, Hendry.[2]

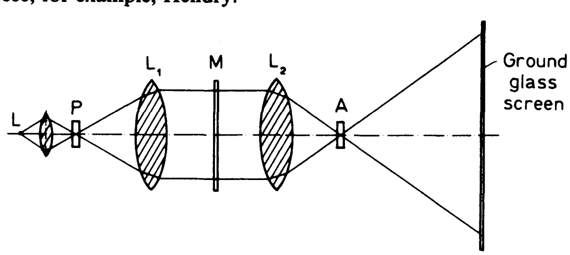


Figure 2.15

2.3 Theory of bars (beams and columns)

2.3.1 Introduction

A great many engineering structures contain components whose dimensions in two coordinate directions are small compared with their dimensions in the third. These components can be called *bars* as a means of general classification, although they are often given other names to denote the particular way they are loaded in structures. Thus if they are subjected to tensile forces they are called *ties*, to compressive forces they are called *struts* or *columns*, to lateral forces they are called *beams*, while if they are subjected to both compressive and lateral forces they are called *beam-columns*.

Structures completely composed of bars are called *frames*, and are either two-dimensional *plane frames*, or three-dimensional *space frames*.

This section reviews the engineering theory of straight bars of uniform cross-section.

2.3.2 Cross-section geometry

2.3.2.1 First moment of area

Consider a bar of some particular cross-sectional shape shown in Figure 2.16, and the two orthogonal axes y and z. (The choice of axes with y horizontal and z downwards, is quite arbitrary but has two advantages when applied to beams: (1) the displacements of a beam are usually vertically downwards, and therefore in the positive direction of z; and (2) as shown in section 2.3.5, a positive bending moment about the y axis causes tension on the bottom of the beam; and therefore positive stresses occur at points in the beam defined by positive values of z.) The first moment of area of the cross-section about the y axis G_y, is defined as the sum of the products obtained by multiplying each element of cross-sectional area dA by its distance z from the y axis. Thus:

$$G_y=\int_A z\,\mathrm{d}A \tag{2.90}$$

Similarly:

$$G_z=\int_A y\,\mathrm{d}A \tag{2.91}$$

The position of the *centroid* of the cross-section is such that the first moment of area about any axis passing through it is zero. Thus if C is the centroid in Figure 2.16, then

$$G_y=G_z=0$$

From this it is clear that C must lie on any axis of symmetry of the section. The centroid can be located in general by selecting any two orthogonal axes y' and z'. The coordinates of the centroid relative to this system, y'_c and z'_c, are then given by:

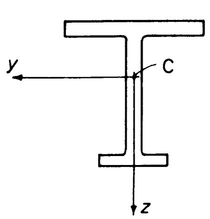


Figure 2.16

$$y'_c=G_{z'}/A \qquad z'_c=G_{y'}/A \tag{2.92, 2.93}$$

where A is the area of the cross-section. The positions of the centroids of various cross-sectional shapes are shown in Table 2.2.

The *longitudinal axis* of the bar is defined as the line passing through the centroids of its cross-sections.

2.3.2.2 Moments of inertia

The *moment of inertia*† of the cross-section about the y axis I_y, is defined as the sum of the products obtained by multiplying each element of cross-sectional area dA by the square of its distance z from the y axis. Thus:

$$I_y=\int_A z^2\,\mathrm{d}A \tag{2.94}$$

Similarly:

$$I_z=\int_A y^2\,\mathrm{d}A \tag{2.95}$$

The *product of inertia*, I_{yz} is defined as:

$$I_{yz}=\int_A yz\,\mathrm{d}A \tag{2.96}$$

where y and z are the respective distances of each element of area dA from the z and y axes.

The *polar moment of inertia* of the cross-section about the x axis, I_p, is defined as:

$$I_p=\int_A r^2\,\mathrm{d}A \tag{2.97}$$

where r is the distance of each element of cross-sectional area dA from the x axis. Note that since $r^2=(y^2+z^2)$

$$I_p=\int_A (y^2+z^2)\,\mathrm{d}A=I_z+I_y \tag{2.98}$$

If y' is an axis parallel to the centroidal axis y and distance c from it, then:

$$I_{y'}=I_y+Ac^2 \tag{2.99}$$

The relationship in Equation (2.99) is known as the *parallel axis theorem*. This theorem facilitates the calculation of the moments of inertia of a complicated cross-section, for the section can be divided into separate simpler elements of area A_e say, whose moments of inertia I_{ye} about their *own* centroidal axes are known. If then c_e is the distance of an element centroid from the y axis, we have:

$$I_y=\sum_{\text{elements}} (I_{ye}+A_e c_e^2) \tag{2.100}$$

The moments of inertia about their centroidal axes, of various sectional shapes are given in Table 2.2.

2.3.2.3 Transformation of moments of inertia

Consider a new system of centroidal axes, y' and z', formed by a rotation of α° anticlockwise about the x axis as shown in Figure

† The term 'moment of inertia' is commonly used in engineering texts because the quantity I_y defined by Equation (2.94) is directly proportional to the mechanical moment of inertia about the y axis, of a thin lamina of the same shape as the cross-section. A more precise term for I_y is the 'second moment of area'.

Table 2.2 Geometrical properties of plane sections

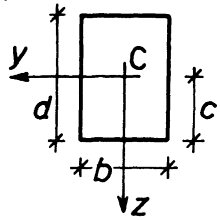
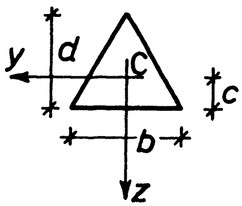
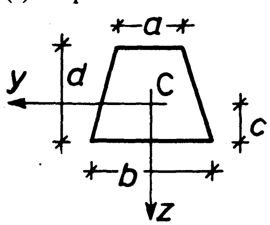
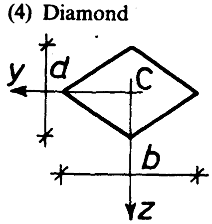
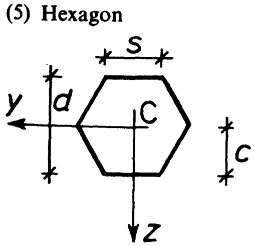
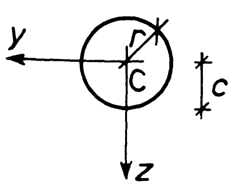
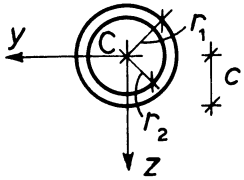
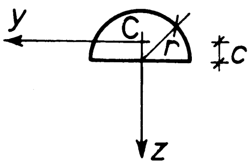
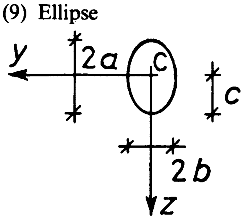
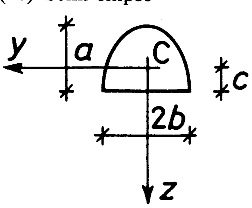
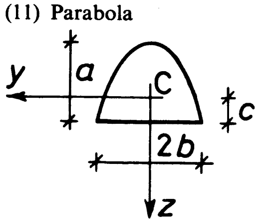
Section	*Area A*	*Position of centroid C*	*Moments of inertia*
(1) Rectangle 	$A=bd$	$c=d/2$	$I_y=bd^3/12$ $I_z=db^3/12$
(2) Triangle 	$A=bd/2$	$c=d/3$	$I_y=bd^3/36$ $I_z=db^3/48$
(3) Trapezium 	$A=d(a+b)/2$	$c=d(2a+b)/3(a+b)$	$I_y=d^3(a^2+4ab+b^2)/36(a+b)$ $I_z=d(a^3+a^2b+ab^2+b^3)/48$
(4) Diamond 	$A=bd/2$	$c=d/2$	$I_y=bd^3/48$ $I_z=db^3/48$
(5) Hexagon 	$A=0.866d^2$	$c=d/2$	$I_y=I_z=0.0601d^4$
(6) Circle 	$A=\pi r^2$ $=3.1416r^2$	$c=r$	$I_y=I_z=\pi r^4/4$ $=0.7854r^4$

Table 2.2 Geometrical properties of plane sections

Section	*Area A*	*Position of centroid C*	*Moments of inertia*
(7) Hollow circle 	$A=\pi(r_1^2-r_2^2)$ $=3.1416(r_1^2-r_2^2)$	$c=r_1$	$I_y=I_z=(\pi/4)(r_1^4-r_2^4)$ $=0.7854(r_1^4-r_2^4)$
(8) Semicircle 	$A=\pi r^2/2$ $=1.5708r^2$	$c=0.424r$	$I_y=[(\pi/8)-(8/9\pi)]r^4$ $=0.1098r^4$ $I_z=\pi r^4/8$ $=0.3927r^4$
(9) Ellipse 	$A=\pi ab$	$c=a$	$I_y=(\pi/4)ba^3=0.7854ba^3$ $I_z=(\pi/4)ab^3=0.7854ab^3$
(10) Semi-ellipse 	$A=\pi ab/2$	$c=0.424a$	$I_y=0.1098ba^3$ $I_z=0.3927ab^3$
(11) Parabola 	$A=4ab/3$	$c=2a/5$	$I_y=0.0914ba^3$ $I_z=0.2666ab^3$

2.17. Then the inertias $I_{y'}$, $I_{z'}$ and $I_{y'z'}$, being defined in the same way as I_y, I_z and I_{yz} in Equations (2.94 to 2.96), are related to I_y, I_z and I_{yz} by the equations:

$$I_{y'}=\tfrac{1}{2}(I_y+I_z)+\tfrac{1}{2}(I_y-I_z)\cos(2\alpha)-I_{yz}\sin(2\alpha) \qquad (2.101)$$

$$I_{z'}=\tfrac{1}{2}(I_y+I_z)-\tfrac{1}{2}(I_y-I_z)\cos(2\alpha)+I_{yz}\sin(2\alpha) \qquad (2.102)$$

$$I_{y'z'}=\tfrac{1}{2}(I_y-I_z)\sin(2\alpha)+I_{yz}\cos(2\alpha) \qquad (2.103)$$

Note that these transformation equations are similar in form to the transformation equations of plane stress in Equations (2.17 to 2.19), the difference being in the sign of α.

For a certain orientation of y' and z', the product of inertia $I_{y'z'}$ vanishes. Denoting these coordinates by Y and Z, then I_Y and I_Z are called the *principal moments of inertia* of the cross-section, and Y and Z are called the *principal axes*. Concerning their orientation, it can be shown in particular that one of the principal axes always coincides with an axis of symmetry in the section. Values of I_Y and I_Z for standard rolled sections are given in BS 4.[14]

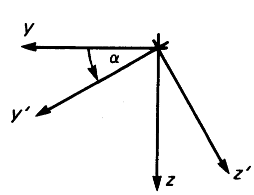


Figure 2.17

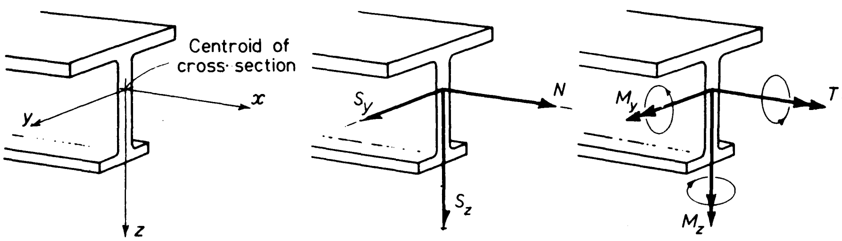


Figure 2.18

2.3.3 Stress resultants

The stresses acting across a particular cross-section of a bar under loads, are conveniently represented by their resultant forces and couples relative to the three coordinate axes x, y and z. Thus the resultants acting on the material of the bar on the negative† side of the cross-section are considered positive when acting in the directions shown in Figure 2.18 and are defined as follows:

Resultant	*Defining equation*	
Axial force N	$N=\int_A \sigma_x \, dA$	(2.104)
Bending moment about the y axis M_y	$M_y=\int_A \sigma_x z \, dA$	(2.105)
Bending moment about the z axis M_z	$M_z=-\int_A \sigma_x y \, dA$	(2.106)
Shear force in the y direction S_y	$S_y=\int_A \tau_{xy} \, dA$	(2.107)
Shear force in the z direction S_z	$S_z=\int_A \tau_{xz} \, dA$	(2.108)
Torque T	$T=\int_A (-\tau_{xy} z+\tau_{xz} y) \, dA$	(2.109)

These resultants are in equilibrium with the loads acting on that part of the bar which is on the negative side of the cross-section. Thus, if the bar is statically determinate, the resultants can be obtained directly by resolving and taking moments.

A *stress resultant diagram* represents the variation of the stress resultant with x for a specified bar loading. The diagram is drawn positive in the direction of the y and z coordinates. Thus given the beam subject to the vertical forces shown in Figure 2.19(a), the shear force (S_z) diagram and the bending moment (M_y) diagram take the form shown in Figures 2.19(b) and (c) respectively. Note that a positive bending moment M_y, causes tension on the bottom of the beam and therefore that the bending moment diagram is located on the tension side of the member. This orientation of the bending-moment diagram is very useful in reinforced concrete design leading to an immediate visual impression of where in the beam the tension reinforcement needs to be placed.

It is sometimes of interest in the case of beams to consider the value of a stress resultant (or any other parameter), at a particular point P in the beam, for various positions of a load moving across the beam. If, for example, we consider the bending moment about the y axis at P ($[M_y]_P$), caused by a unit vertical force at point x on the beam, then $[M_y]_P$ is a function of the coordinate x. The plot of $[M_y]_P$ versus x is called the *influence line* of M_y at P. Thus for the beam AB in Figure 2.20 the influence lines for $[S_z]_P$ and $[M_y]_P$ are as shown.

The stress resultants are not all independent of each other. Thus considering the rotational equilibrium about the y axis of a small element of a bar subject to a vertical distributed load q per unit length, as in Figure 2.21:

† 'Negative' means the side in the negative direction of the x axis.

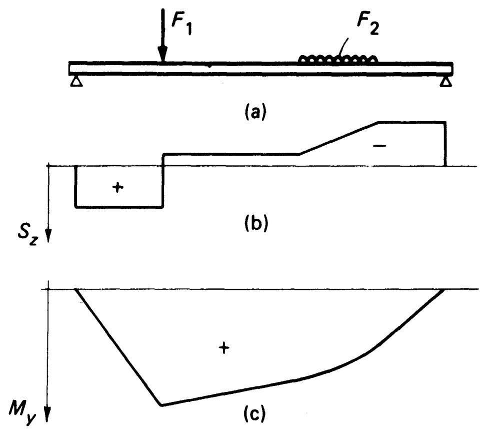


Figure 2.19

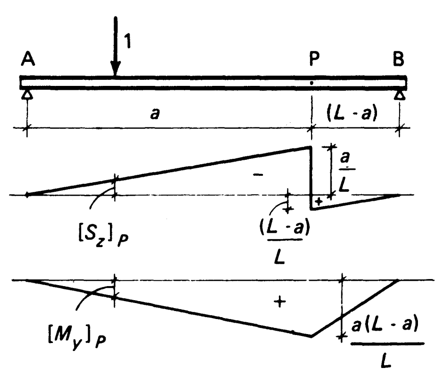


Figure 2.20

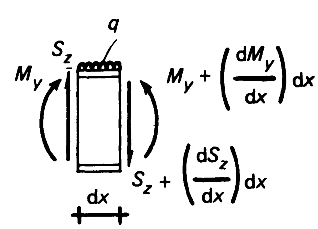


Figure 2.21

$$\mathrm{d}M_y/\mathrm{d}x = S_z \qquad (2.110)$$

Further, considering vertical equilibrium:

$$\mathrm{d}S_z/\mathrm{d}x = -q \qquad (2.111)$$

Whence, combining Equations (2.110) and (2.111) gives:

$$\mathrm{d}^2M_y/\mathrm{d}x^2 = -q \qquad (2.112)$$

A similar set of equations relates M_z, S_y and the horizontal loading on the bar.

2.3.4 Bars subject to tensile forces (ties)

Consider a bar subject to axial forces N, produced by the loading shown in Figure 2.22.

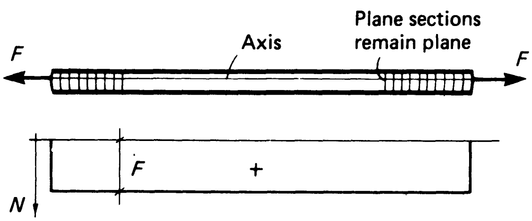


Figure 2.22

From the symmetry of the system at some distance from the loading points it can be *deduced* that plane sections originally normal to the longitudinal axis remain plane and normal to the axis after the deformation, while from the geometry of the bar, it can be assumed that the only nonzero component of stress is σ_x.[8]

The stress–strain relations corresponding to the uniaxial state of stress take the form:

$$\varepsilon_x = (\sigma_x/E) + \alpha\Delta T \qquad (2.113)$$

$$\varepsilon_y = \varepsilon_z = -(\nu\sigma_x/E) + \alpha\Delta T \qquad (2.114)$$

and it follows that at some distance from the loading points:

$$\sigma_x = N/A \qquad (2.115)$$

$$\varepsilon_x = (N/EA) + \alpha\Delta T \qquad (2.116)$$

2.3.5 Beams subject to pure bending

2.3.5.1 Beams symmetric about the vertical plane and subject to vertical loading

Consider a beam subject to a uniform bending moment M_y over part of its length, produced, for example, by the loading shown in Figure 2.23. (Note that Equation (2.110) implies that a uniform bending moment can only occur in the absence of shear forces.) From the symmetry of the system it can be *deduced* that: (1) the beam deforms in the vertical plane, and straight-line generators parallel to the longitudinal axis deform into segments of circles with a common centre; and (2) planes originally normal to the axis remain plane and normal to the axis after deformation.

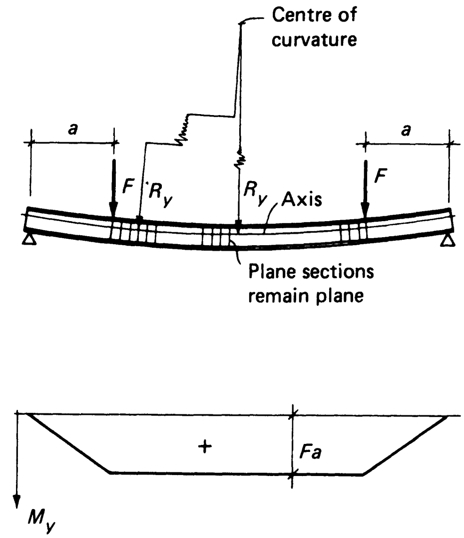


Figure 2.23

It can again be assumed that: (3) the only nonzero component of stress is σ_x.

The above three conditions are the fundamental assumptions made in the *engineering theory of the bending of beams*.

The surface containing those points in the beam at which $\varepsilon_x = 0$ is called the *neutral surface*. The intersection of the neutral surface with a cross-section produces a line called the *neutral axis*.

From the geometry of the deformation, the uniaxial stress–strain relations in Equations (2.113, 2.114), and the requirement of axial equilibrium ($N = 0$), it follows that:

(1) The neutral axis is given by the equation:

$$z = 0 \qquad (2.117)$$

i.e. it is a horizontal straight line, coincident with the y coordinate line, and passing through the centroid of the section.

(2) $$\sigma_x = \frac{M_y z}{I_y} \qquad (2.118)$$

and

$$\frac{1}{R_y} = \frac{M_y}{EI_y} \qquad (2.119)$$

where R_y is the vertical radius of curvature of the beam axis.

2.3.5.2 Composite beams

Suppose the beam in the previous subsection is made of two materials of Young's modulus E_1 and E_2 respectively comprising areas A_1 and A_2 of the total cross-section, as in Figure 2.24. The three conditions of the engineering theory of the bending of beams discussed in the previous subsection still apply. It therefore follows that:

(1) The neutral axis is a horizontal straight line passing through a point C′ called the *equivalent centroid of the cross-section*.

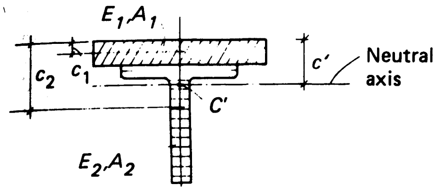


Figure 2.24

This is defined as being such that the first moment of *Young's modulus times area* about any axis passing through it is zero. Thus if c' is the distance of C′ from the upper boundary of the beam and c_1 and c_2 are the distances of the respective centroids of the areas A_1 and A_2 from the upper boundary, then:

$$c' = \frac{E_1 A_1 c_1 + E_2 A_2 c_2}{E_1 A_1 + E_2 A_2} \tag{2.120}$$

(2) $$[\sigma_x]_{A_1} = \frac{M_y z}{I'_y} \qquad [\sigma_x]_{A_2} = \frac{E_2}{E_1}\frac{M_y z}{I'_y} \qquad (2.121,\ 2.122)$$

and

$$\frac{1}{R_y} = \frac{M_y}{E_1 I'_y} \tag{2.123}$$

where $[\sigma_x]_{A_1}$ represents the axial stress in the area A_1, etc. I'_y is the *equivalent moment of inertia* of the cross-section defined as:

$$I'_y = \int_{A_1} (z^2)\,\mathrm{d}A_1 + \frac{E_2}{E_1}\int_{A_2} (z^2)\,\mathrm{d}A_2 \tag{2.124}$$

where $\int_{A_1} (\ )\,\mathrm{d}A_1$ represents an integral taken over the area A_1, etc. In the above equations, the coordinates are relative to axes y and z passing through the equivalent centroid of the section.

2.3.5.3 Reinforced concrete beams

A reinforced concrete beam behaves as a composite beam, except that where the concrete is in tension (i.e. below the neutral axis for positive bending about the y axis) its stress-bearing capacity is taken to be zero (Figure 2.25). Otherwise the conditions of the engineering theory of the bending of beams still apply.

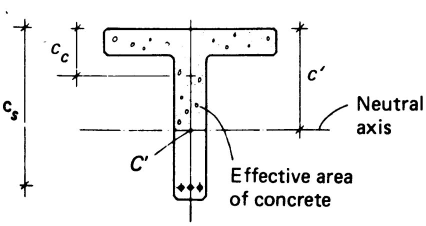


Figure 2.25

Let the subscripts c and s denote parameters associated respectively with the concrete and the steel. It then follows that:

(1) The neutral axis is a horizontal straight line passing through the equivalent centroid whose distance c' from the upper boundary of the beam is given by:

$$c' = \frac{E_c A_c c_c + E_s A_s c_s}{E_c A_c + E_s A_s} \tag{2.125}$$

(Note that since A_c and c_c are themselves functions of c', Equation (2.125) is an implicit equation.)

(2) $$[\sigma_x]_c = \frac{M_y z}{I'_y} \qquad [\sigma_x]_s = \frac{E_s}{E_c}\frac{M_y z}{I'_y} \qquad (2.126,\ 2.127)$$

and

$$\frac{1}{R_y} = \frac{M_y}{E_c I'_y} \tag{2.128}$$

where

$$I'_y = \int_{A_c} (z^2)\,\mathrm{d}A_c + \frac{E_s}{E_c}\int_{A_s} (z^2)\,\mathrm{d}A_s \tag{2.129}$$

Example 2.2. A rectangular reinforced concrete beam with a single layer of reinforcement is shown in Figure 2.26. For this section:

$$c' = \frac{E_s A_s}{E_b\, b}\left[\left(1 + \frac{2E_c}{E_s}\frac{b(d-e)}{A_s}\right)^{1/2} - 1\right] \tag{2.130}$$

$$I'_y = \frac{bc'^3}{3} + \frac{E_s}{E_c} A_s[d-(c'+e)]^2 \tag{2.131}$$

Note that the ratio $E_s : E_c$ is generally taken to be 15.

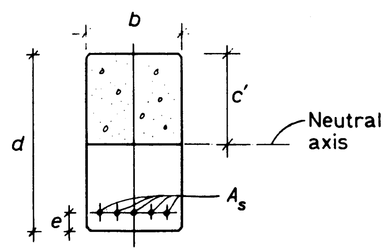


Figure 2.26

2.3.5.4 Beams of asymmetric section subject to both vertical and horizontal loading

Consider again a beam of homogeneous material. The general case of pure bending occurs when the beam is of asymmetric section and is subject to uniform bending moments M_y and M_z (Figure 2.27) over part of its length.

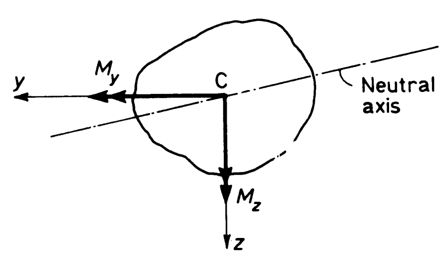


Figure 2.27

From the symmetry of the system it can be *deduced* that straight-line generators parallel to the axis of the beam deform into curves of constant horizontal and vertical curvature. The other conditions discussed in section 2.3.5.1 still apply.

From the geometry of the deformation, the uniaxial stress–strain relations and the requirement that $N=0$, it follows that:

(1) The neutral axis is given by the equation:

$$(M_yI_z+M_zI_{yz})\,z-(M_zI_y+M_yI_{yz})\,y=0 \tag{2.132}$$

i.e. it is a straight line passing through the centroid of the section, as shown in Figure 2.27.

(2) $$\sigma_x=\frac{(M_yI_z+M_zI_{yz})\,z-(M_zI_y+M_yI_{yz})\,y}{(I_yI_z-I_{yz}^2)} \tag{2.133}$$

$$\frac{1}{R_y}=\frac{(M_yI_z+M_zI_{yz})}{E(I_yI_z-I_{yz}^2)} \quad \frac{1}{R_z}=\frac{-(M_zI_y+M_yI_{yz})}{E(I_yI_z-I_{yz}^2)} \tag{2.134, 2.135}$$

where R_z is the horizontal radius of curvature of the beam axis.

Note:

(1) If the loading is vertical so that $M_z=0$, Equation (2.135) indicates that the deformed beam is curved horizontally, i.e. $R_z\neq 0$.
(2) If y and z are principal axes, so that $I_{yz}=0$, Equations (2.134, 2.135) indicate that the curvature about each axis is proportional only to the bending moment about that axis.

In some cases, where a standard commercial section is mounted obliquely, as in Figure 2.28(a) for example, $I_{y'}$, $I_{z'}$ and $I_{y'z'}$ will be known relative to the axes y', z', while the bending moments will be known about the axes y and z. In order to use the results in Equations (2.132 to 2.135) it is preferable to work in terms of the y' and z' axes and resolve the bending moments into equivalent moments about these axes, as in Figure 2.28(b).

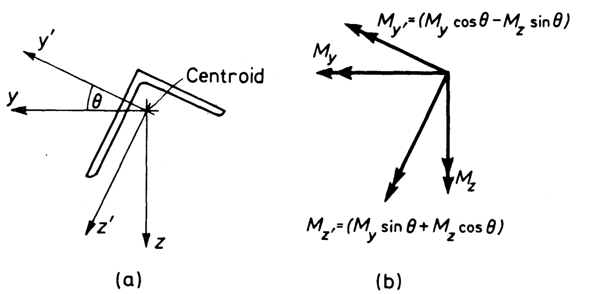


Figure 2.28

2.3.6 Beams subject to combined bending and shear

Practical loading arrangements on beams generally produce a combination of bending and shear stress resultants as, for example, in Figure 2.19.

2.3.6.1 Beams symmetric about the vertical plane and subject to vertical loading

Consider a point in a beam at which both M_y and S_z are nonzero. The presence S_z then implies the existence of the shear stresses τ_{xz} on the cross-section and corresponding shear strains γ_{xz}, and much of the symmetry of the deformation of a beam under a uniform bending moment is lost. In particular, plane sections no longer remain plane.

The following approximate analysis of the problem is due to St Venant.[15] It is assumed that the direct stresses σ_x and curvature $(1/R_y)$ are the same as they would be if M_y were acting alone. They are therefore given by Equations (2.118, 2.119). The shear stresses in the beam are then obtained by considering the longitudinal equilibrium of the element of length dx shown shaded in the cross-sectional view of Figure 2.29. Thus employing Equations (2.110) and (2.118), namely $dM_y/dx=S_z$ and $\sigma_x=M_yz/I_y$, it can be shown that the *mean* longitudinal shear stress τ on the surface ABCD is given by:

$$\tau=\frac{S_zA_ec_e}{b_eI_y} \tag{2.136}$$

where A_e is the cross-sectional area of the element, c_e is the distance of its centroid from the neutral axis, and b_e is the length of the curve joining AB (Figure 2.29).

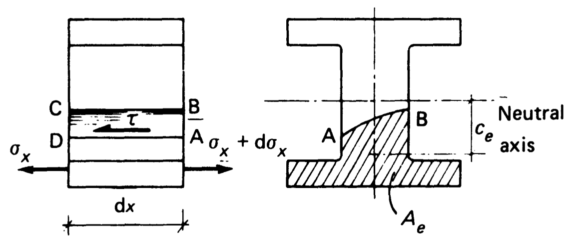


Figure 2.29

τ can then be related to the shear stresses τ_{xy} and τ_{xz} on the cross-section as follows. If the cut surface ABCD is a horizontal plane (i.e. it is a z-coordinate surface) then τ is the mean value of the shear stress component τ_{zx} on that surface. Whence, since $\tau_{zx}=\tau_{xz}$, it follows that τ is also the mean value of τ_{xz} on the line AB. For thin sections, we assume that τ_{xz} is uniformly distributed across the width so that:

$$\tau_{xz}=\tau \tag{2.137}$$

Thus for the rectangular section shown in Figure 2.30, Equation (2.136) gives the following parabolic distribution of shear stress on the cross-section:

$$\tau_{xz}=\frac{3S_z}{2bd^3}(d^2-4z^2) \tag{2.138}$$

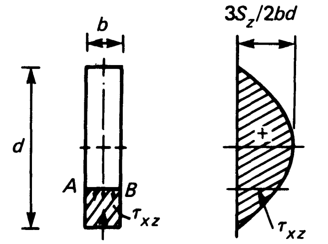


Figure 2.30

If the cut surface ABCD is a vertical plane (a y-coordinate surface) then τ is the mean value of the shear stress component τ_{yx} on that surface, or the mean value of τ_{xy} on the line AB. Thus for an I-section, the shear stresses in the flanges are as shown in Figure 2.31.

2.3.6.2 Composite beams

The existence of the longitudinal shear stress τ (Figure 2.29) is of

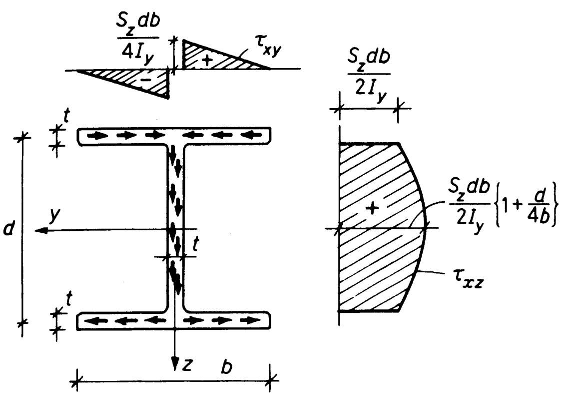


Figure 2.31

special significance in built-up composite beams, because this stress has to be transmitted between the separate components of the beams by means of suitable bonds such as welds, rivets or shear connectors.

Thus consider a beam composed of two materials of Young's modulus E_1 and E_2 respectively comprising areas A_1 and A_2 of the total cross-section (Figure 2.32). The position of the neutral axis and the equivalent moment of inertia of the cross-section are again given by Equations (2.120) and (2.124), whence, employing the assumptions of St Venant's theory, it can be deduced that the mean longitudinal shear stress at the interface AB is given by:

$$\tau = \frac{S_z A_1 (c' - c_1)}{b I'_y} \tag{2.139}$$

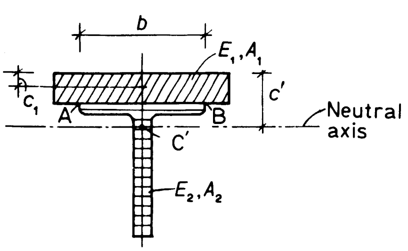


Figure 2.32

while the corresponding longitudinal shear force/unit length of beam F is given by:

$$F = b\tau \tag{2.140}$$

If the beam were composed, say, of a concrete slab connected to a steel T-section joist, then F would be transmitted by stud shear connectors of the type shown in Figure 2.33 which would be welded on to the top face of the T-section. Supposing that the factored shear strength of each connector were known experimentally to be F_s, then the connectors would need to be distributed at a concentration of F/F_s per unit length of beam.

2.3.6.3 The shear centre (beams asymmetric about the vertical plane)

In a beam of asymmetric cross-section the shear stresses given by St Venant's theory contribute to a torque T. Consider, for

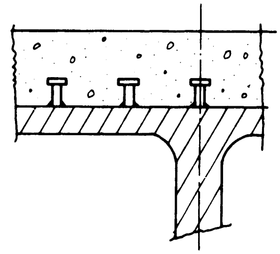


Figure 2.33

example, the shear stresses produced in the channel section shown in Figure 2.34. They are statically equivalent to the stress resultants S_f acting in the two flanges, and S_w in the web, where:

$$S_w = S_z \tag{2.141}$$

$$S_f = \frac{S_z b^2 dt}{4I_y} \tag{2.142}$$

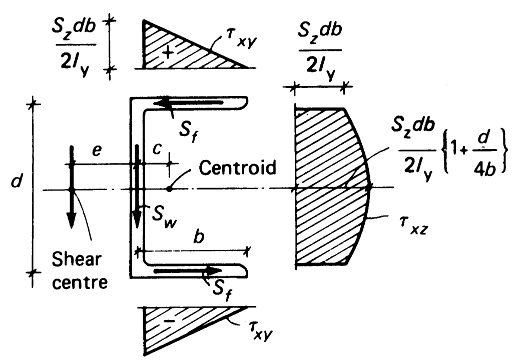


Figure 2.34

and because of the asymmetry of the section, they produce a torque T acting about the longitudinal axis of the channel given by

$$T = S_z c + \frac{S_z b^2 d^2 t}{4I_y} \tag{2.143}$$

An important assumption of St Venant's theory is that the beam deflects vertically without twist. Thus, it can be deduced that if the loading on the beam is such that it produces the torque T, then twisting does not, in fact, occur. (If the loading did not produce T then some twisting of the beam would be necessary in order to modify the torque obtained in Equation (2.143).) T can be applied by positioning the vertical loading so that its resultant at any cross-section lies at a suitable distance from the centroid. Thus the torque in the channel can be produced by the loading shown in Figure 2.35. The point at

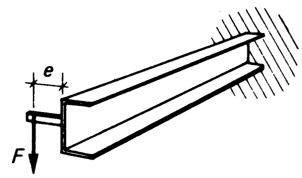
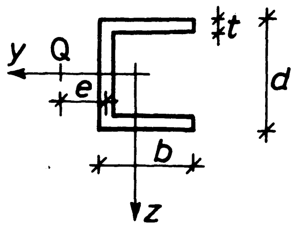


Figure 2.35

Table 2.3 Shear centres of the walled sections

Section	*Position of shear centre Q*

(1) Channel



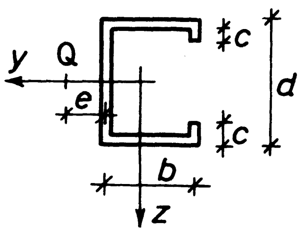
$$e=d\left(\frac{H_{yz}}{I_y}\right)$$

where H_{yz} is the product of inertia of the half section (above the y axis).

If t is uniform:

$$e=\frac{b^2d^2t}{4I_y}$$

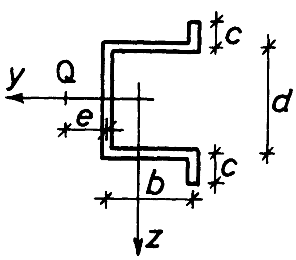
(2) Lipped channel



Values of (e/d)

c/d \ b/d	1.0	0.8	0.6	0.4	0.2
0.0	0.430	0.330	0.236	0.141	0.055
0.1	0.477	0.380	0.280	0.183	0.087
0.2	0.530	0.425	0.325	0.222	0.115
0.3	0.575	0.470	0.365	0.258	0.138
0.4	0.610	0.503	0.394	0.280	0.155
0.5	0.621	0.517	0.405	0.290	0.161

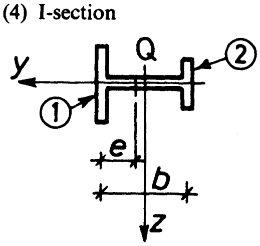
(3) Hat-section



Values of (e/d)

c/d \ b/d	1.0	0.8	0.6	0.4	0.2
0.0	0.430	0.330	0.236	0.141	0.055
0.1	0.464	0.367	0.270	0.173	0.080
0.2	0.474	0.377	0.280	0.182	0.090
0.3	0.453	0.358	0.265	0.172	0.085
0.4	0.410	0.320	0.235	0.150	0.072
0.5	0.355	0.275	0.196	0.123	0.056
0.6	0.300	0.225	0.155	0.095	0.040

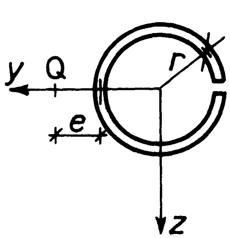
(4) I-section



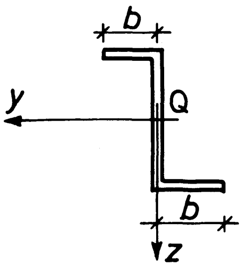
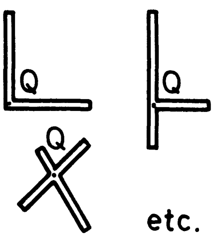
$$e=\frac{bI_2}{I_1+I_2}$$

where I_1 and I_2 respectively denote the moments of inertia about the y axis of flange 1 and flange 2

(5) Split circle



$$e=r$$

Section	*Position of shear centre Q*
(6) Z-section  	Shear centre coincides with centroid
(7) Sections with elements intersecting at a single point  	Shear centre lies at point of intersection

which the vertical resultant crosses the neutral axis is then called the *shear centre*, and for the channel section it is located at a distance e from the web (Figure 2.34) where

$$e=\frac{b^2d^2t}{4I_y} \tag{2.144}$$

The positions of the shear centres of various cross-sectional shapes are shown in Table 2.3.

The *shear axis* of the beam is defined as the line passing through the shear centres of its cross-sections, and by definition, the resultants of all lateral forces acting on the beam must pass through this axis if the beam is to deflect without twist.

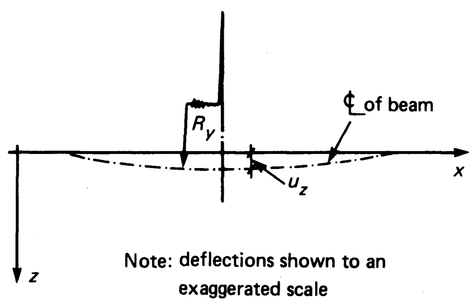


Figure 2.36

2.3.7 Deflection of beams

According to St Venant's theory, the curvature of a beam subject to combined bending and shear is given by Equation (2.119) thus: $1/R_y=M_y/EI_y$. Suppose u_z is the corresponding vertical deflection of the longitudinal axis of the beam, then from the geometry of the deformation (Figure 2.36), it can be shown that:

$$\frac{1}{R_y}=-\frac{d^2u_z}{dx^2}\Bigg/\left(1+\left(\frac{du_z}{dx}\right)^2\right)^{3/2} \tag{2.145}$$

In practice, the slopes of beams are extremely small and the denominator of the right-hand side of Equation (2.145) can be taken to be equal to unity, whence, combining Equations (2.119) and (2.145) gives the following differential equation:

$$\frac{d^2u_z}{dx^2}+\frac{M_y}{EI_y}=0 \tag{2.146}$$

called the *differential equation of beams*. For statically determinate beams, where M_y can be found as a function of x, this second-order equation can be solved subject to boundary conditions by double integration. The solution $u_z(x)$ is then the deflected shape of a beam produced by the applied loading. Examples of the boundary conditions for particular cases are shown in Figure 2.37. Special techniques, such as the step function method[16a] and the moment-area method[15] have been devised to simplify the analysis.

The differential equation of beams can be expressed in two further forms using the results of Equations (2.110) and (2.112). Thus from Equation (2.110) we have:

$$\frac{d^3u_z}{dx^3}+\frac{S_z}{EI_y}=0 \tag{2.147}$$

while from Equation (2.112) we have:

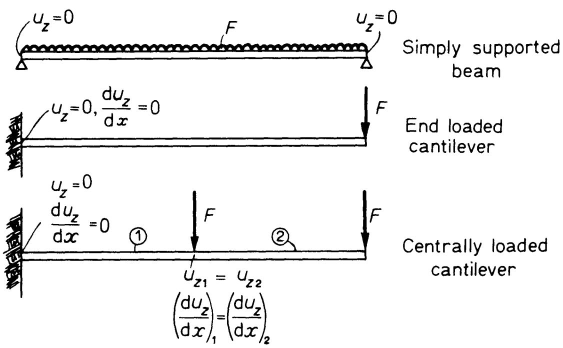


Figure 2.37

$$\frac{d^4u_z}{dx^4}-\frac{q}{EI_y}=0 \tag{2.148}$$

Equation (2.148), expressing the deflections of beams in terms of the lateral loading, is directly equivalent to the three-dimensional Navier Equations (2.61 to 2.63), and can be solved if the boundary conditions are expressed in terms of the displacements. The solution of this equation as opposed to Equation (2.146), is necessary when a beam is statically indeterminate, i.e. when M_y cannot be found in advance. Examples of the required displacement boundary conditions for particular cases are shown in Figure 2.38.

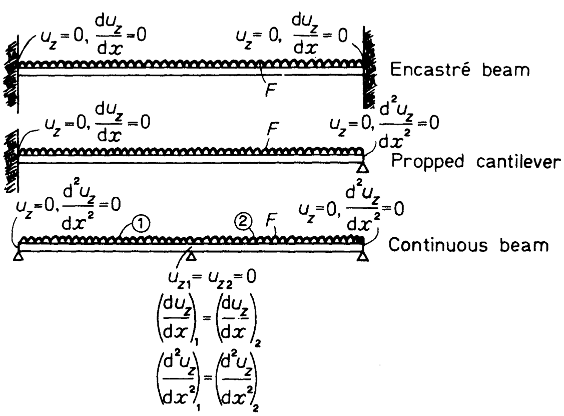


Figure 2.38

An interesting modification of Equation (2.148) occurs when a beam rests on an elastic foundation. Suppose the stiffness of the foundation is k per unit length of beam. Then in addition to the vertical applied loading q, the foundation resists the deflection of the beam with distributed forces equal to ku_z per unit length. Equation (2.148) then takes the form:

$$\frac{d^4u_z}{dx^4}+ku_z-\frac{q}{EI_y}=0 \tag{2.149}$$

Examples of the solution of this equation are given by Hetényi.[16b]

2.3.8 Bars subject to a uniform torque

2.3.8.1 Bars of circular cross-section

Consider a bar subject to a uniform torque T produced, for example, by the loading shown in Figure 2.39.

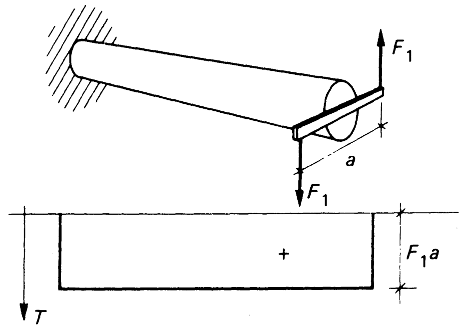


Figure 2.39

From the symmetry of the system it can be deduced that: (1) the bar twists about its longitudinal axis; (2) planes originally normal to the axis remain plane and normal to the axis and rotate like rigid laminae, and (3) the rotation θ of any plane is proportional to its distance along the beam.

From the geometry of the deformation and the shear stress–strain relations in Equations (2.49 to 2.51), it follows that:

$$\tau_{xt}=\frac{Tr}{J} \tag{2.150}$$

$$\frac{d\theta}{dx}=\frac{T}{GJ} \tag{2.151}$$

where τ_{xt} is the shear stress on the cross-section at a distance r from the axis, and tangential to the circle of radius r (Figure 2.40). J is a sectional constant, equal in this case to the polar moment of inertia I_p about the longitudinal axis.

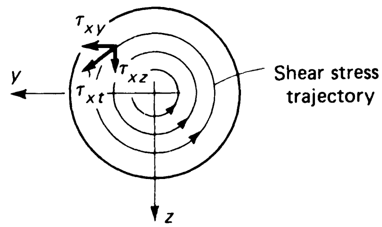


Figure 2.40

The quantity $d\theta/dx$ being the rate of change of rotation with x is called the *twist* of the bar, and is clearly uniform when the bar is subject to uniform torque.

2.3.8.2 Bars of arbitrary cross-section

The three assumptions of section 2.3.8.1 can be shown to lead to impossible values of τ_{xt} at the boundaries of an arbitrary section, since in order to satisfy longitudinal equilibrium conditions, τ_{xt} must be tangential to those boundaries (Figure 2.41).

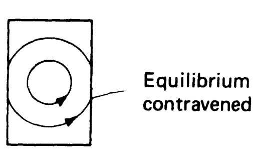


Figure 2.41

The theory for the analysis of bars of arbitrary section is again due to St Venant.[8] Thus the assumption in the previous subsection that plane sections remain plane is relaxed, and a point P is assumed to have an axial displacement u_x given by:

$$u_x = \frac{d\theta}{dx}\psi(y, z) \tag{2.152}$$

u_x is called the *warping* of the section, and is directly proportional to the twist, but is independent of x. The shear stresses τ_{xy} and τ_{xz} are then expressed in terms of a stress function $\phi(y, z)$ by the equations:

$$\tau_{xy} = \partial\phi/\partial z \quad \tau_{xz} = -\partial\phi/\partial y \tag{2.153, 2.154}$$

so that the internal equilibrium Equations (2.11) to (2.13) are identically satisfied. Satisfaction of the compatibility Equations (2.40) and (2.42) then leads to the following equation:

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -2G\left(\frac{d\theta}{dx}\right) \tag{2.155}$$

Equilibrium conditions require that ϕ is constant along the boundaries of the section, and if the section is solid ϕ can be conveniently taken as zero along the boundaries, whence it can be shown that:

$$T = 2\int_A \phi \, dA \tag{2.156}$$

Equations (2.155) and (2.156) are solved simultaneously, either numerically, or experimentally,[8] and the shear stresses corresponding to T are obtained from Equations (2.153) and (2.154). The results can be expressed in the following form:

$$[\tau_{xb}]_{max} = T/k \tag{2.157}$$

$$d\theta/dx = T/GJ \tag{2.158}$$

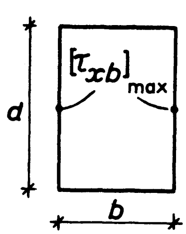
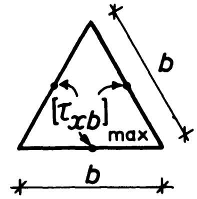
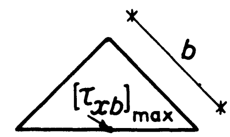
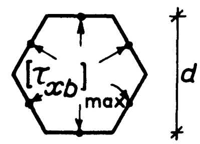
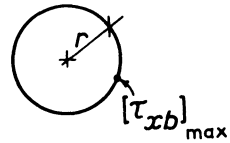
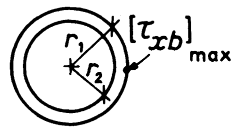
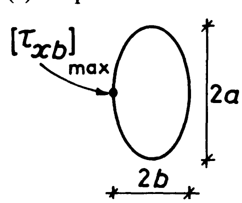
where $[\tau_{xb}]_{max}$ is the maximum shear stress on the boundary of the section and is tangential to the boundary. k and J are constants, and their values for various cross-sectional shapes are shown in Table 2.4.

For the narrow rectangular section shown in Figure 2.42:

$$k = t^2d/3 \quad J = t^3d/3 \tag{2.159, 2.160}$$

and the maximum shear stress occurs along the boundaries of greatest length. These results can be used to determine the

Table 2.4 Torsional constants

Section	d/b	k	J
(1) Rectangle 	1.0	$0.208\,(b^2d)$	$0.1406\,(b^3d)$
	1.2	$0.219\,(b^2d)$	$0.166\,(b^3d)$
	1.5	$0.231\,(b^2d)$	$0.196\,(b^3d)$
	2.0	$0.246\,(b^2d)$	$0.229\,(b^3d)$
	2.5	$0.258\,(b^2d)$	$0.249\,(b^3d)$
	3.0	$0.267\,(b^2d)$	$0.263\,(b^3d)$
	4.0	$0.282\,(b^2d)$	$0.281\,(b^3d)$
	5.0	$0.291\,(b^2d)$	$0.291\,(b^3d)$
	10.0	$0.312\,(b^2d)$	$0.312\,(b^3d)$
	∞	$1/3\ (b^2d)$	$1/3\ (b^3d)$
(2) Equilateral triangle 		$b^3/20$	$\sqrt{3}b^4/80$
(3) Right isosceles triangle 		$0.0554\,b^3$	$0.0261\,b^4$
(4) Hexagon 		$0.217\,Ad$	$0.133\,Ad^2$
(5) Circle 		$\pi r^3/2$	$\pi r^4/2$
		$1.5708\,r^3$	$1.5708\,r^4$
(6) Hollow circle 		$\frac{\pi}{2}\left(\frac{r_1^4 - r_2^4}{r_1}\right)$	$\frac{\pi}{2}(r_1^4 - r_2^4)$
(7) Ellipse 		$\pi ab^2/2$	$\pi a^3b^3/(a^2 + b^2)$

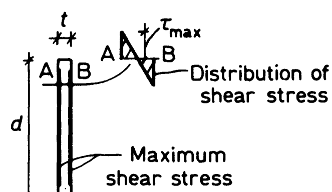


Figure 2.42

torsional properties of a thin-walled open-section bar, supposing that the cross-section can be divided into narrow rectangular elements of thickness t_e and d_e, for it can be shown that to a first approximation:

$$J=\sum_{\text{elements}}\frac{t_e^3 d_e}{3} \qquad (2.161)$$

Thus for the I-section shown in Figure 2.43:

$$J=\frac{2d_f t_f^3+d_w t_w^3}{3} \qquad (2.162)$$

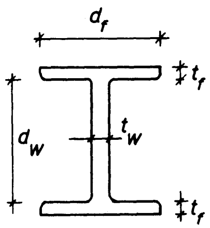


Figure 2.43

The maximum shear stress $[\tau_{xb}]_e$ along the boundaries of a particular element are given by:

$$[\tau_{xb}]_e=T/k_e \qquad (2.163)$$

where

$$k_e=J/t_e \qquad (2.164)$$

$[\tau_{xb}]_e$ however, is not the maximum shear stress on the cross-section, for this now occurs at the re-entrant corners. Thus, in a constant-thickness channel section (Figure 2.44(a)) $[\tau_{xb}]_{max}$ occurs at point P, and is related to $[\tau_{xb}]_e$ and the radius of the corner as shown in Figure 2.44(b).[8]

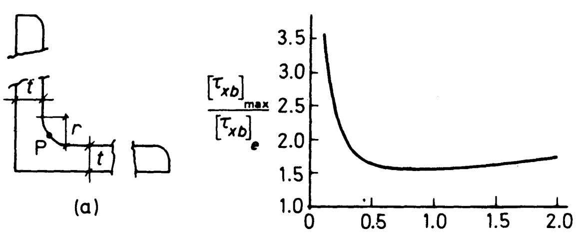


Figure 2.44

In a thin-walled closed-section bar, such as the tube of varying wall thickness t shown in Figure 2.45, the shear stress τ_{xb} is uniform across the thickness at any point and is tangential to the surface of the tube. It is given by:

$$\tau_{xb}=T/2At \qquad (2.165)$$

where A is the gross cross-sectional area.
J in Equation (2.158) is given by:

$$J=4A^2\bigg/\left(\oint\frac{ds}{t}\right) \qquad (2.166)$$

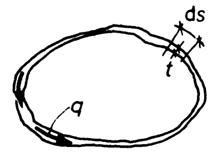


Figure 2.45

where ds is an element of length round the tube (Figure 2.45). A further quantity q called the *shear flow* is defined at a point in the tube wall by the equation

$$q=\tau_{xb}t \qquad (2.167)$$

It is then apparent from Equation (2.165) that the shear flow is independent of t.

In multicell thin-walled bars, as shown in Figure 2.46, the concept of circulatory shear flows q_1, q_2, q_3 is introduced, a concept which automatically satisfies the conditions of longitudinal equilibrium at junctions such as A. The shear flow at point B, for example, is then given by (q_1-q_2). The shear flows and the twist of the bar corresponding to a certain applied torque are calculated from the four simultaneous equations:

$$T=2q_1A_1+2q_2A_2+2q_3A_3 \qquad (2.168)$$

$$\frac{d\theta}{dx}=\frac{1}{2A_1G}\oint_1\frac{q}{t}ds_1=\frac{1}{2A_2G}\oint_2\frac{q}{t}ds_2=\frac{1}{2A_3G}\oint_3\frac{q}{t}ds_3 \qquad (2.169, 2.170, 2.171)$$

where $\oint_1$ represents the contour integral taken round cell 1, etc.

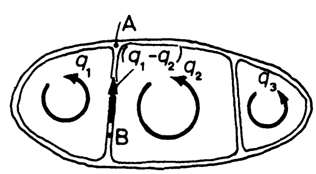


Figure 2.46

2.3.9 Nonuniform torsion

Nonuniform torsion in a bar is defined to occur when the twist $d\theta/dx$ varies along its length. This situation arises when the warping assumed in St Venant's theory is restrained at a rigid support, or when the torque exerted by the applied loading is nonuniform.

The nature of the modification necessary to St Venant's theory can be appreciated by considering the nonuniform torsion of the I-section cantilever shown in Figure 2.47. Since $d\theta/dx$ is not constant, the flanges are curved in the z plane. Considering the flanges as subsidiary beams, they contain shear forces $[S_z]_f$ which are related to this curvature. The torque T therefore includes an extra component $[S_z]_f d$. $[S_z]_f$ is given by Equation (2.147) as:

$$[S_z]_f=-E[I_y]_f\frac{d^3u_z}{dx^3} \qquad (2.172)$$

where $[I_y]_f$ is the moment of inertia of each flange about the y axis, and u_z is its displacement. Whence noting that:

$$u_z=\pm\frac{d}{2}\theta \qquad (2.173)$$

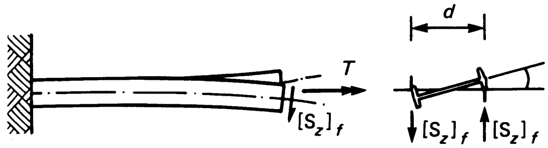
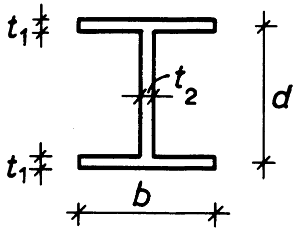
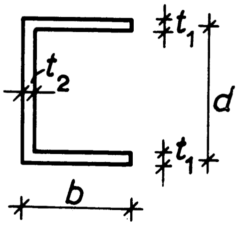
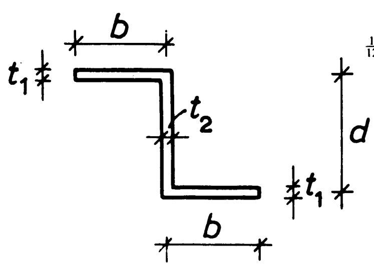
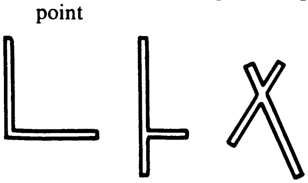


Figure 2.47

Table 2.5 Warping factors

Section	Γ
(1) I-section 	$\frac{1}{24}b^3d^2t_1$
(2) Channel section 	$\frac{1}{12}b^3d^2t_1\left(\frac{3bt_1+2dt_2}{6bt_1+dt_2}\right)$
(3) Z-section 	$\frac{1}{12}b^3d^2t_1\left(\frac{bt_1+2dt_2}{2bt_1+dt_2}\right)$
(4) Thin-walled sections with elements intersecting at a single point 	0

the additional torque component becomes:

$$-EI_y\frac{d^2}{4}\frac{d^3\theta}{dx^3}$$

where I_y is the *total* moment of inertia of the cross-section about the y axis. Combining this with the torque required for the uniform torsion of the bar, we obtain:

$$T = GJ\frac{d\theta}{dx} - EI_y\frac{d^2}{4}\frac{d^3\theta}{dx^3} \qquad (2.174)$$

Equation (2.174) can be expressed in the more general form:

$$T = GJ\frac{d\theta}{dx} - E\Gamma\frac{d^3\theta}{dx^3} \qquad (2.175)$$

where Γ is a constant called the *warping factor*. Its values for various cross-sectional shapes are given in Table 2.5. The differential Equation (2.175) can be solved for various values of T applied to the bar, subject to boundary conditions in θ. Examples of these boundary conditions are shown in Figure 2.48.

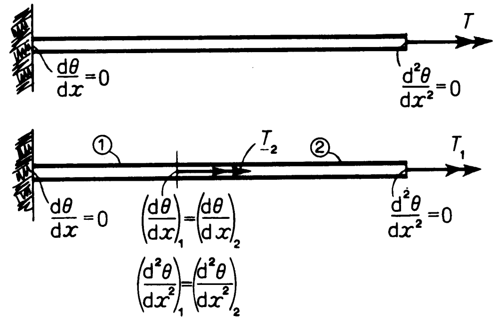


Figure 2.48

2.3.10 Bars subject to compressive forces (columns)

2.3.10.1 Short columns

If the geometry of a bar is such that its length is less than about 5 times its lateral dimensions, then it is usually stable under compressive forces. If therefore it is subjected to an axial compressive force F, then $N = -F$ and the corresponding stress σ_x is given by Equation (2.115) as: $\sigma_x = N/A$. If further, the bar is subjected to bending moments M_y and M_z acting about the principal axes y and z, then by superposition:

$$\sigma_x = \frac{N}{A} + \frac{M_y z}{I_y} - \frac{M_z y}{I_z} \qquad (2.176)$$

and the neutral axis is given by the equation:

$$\frac{M_y z}{I_y} - \frac{M_z y}{I_z} + \frac{N}{A} = 0 \qquad (2.177)$$

Combined compressive forces and bending moments occur in the bar if the compressive force F is eccentrically positioned as shown in Figure 2.49. Thus if the resultant due to F passes at a distance n and m from the y and z axes respectively, then:

$$N = -F \quad M_y = -Fn \quad M_z = +Fm \qquad (2.178)$$

and:

$$\sigma_x = -F\left(\frac{1}{A} + \frac{nz}{I_y} + \frac{my}{I_z}\right) \qquad (2.179)$$

The neutral axis is then given by the equation:

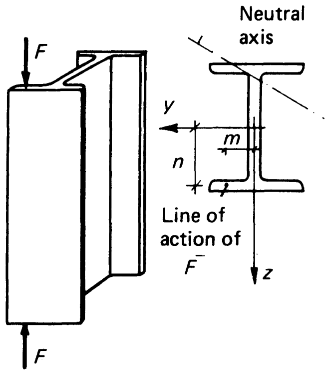


Figure 2.49

$$\frac{nz}{r_y^2}+\frac{my}{r_z^2}+1=0 \tag{2.180}$$

where r_y and r_z are the *radii of gyration* of the cross-section defined respectively by the equations:

$$r_y^2=\frac{I_y}{A}\quad r_z^2=\frac{I_z}{A} \tag{2.181}$$

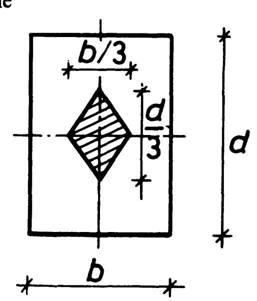
Note that if the location of the neutral axis is known, then the maximum and minimum stresses on the section are located at those points which are at the greatest perpendicular distance from this axis. Their positions can easily be found graphically.

It is apparent from Equation (2.180) that the location of the neutral axis depends only on the coordinates n and m defining the eccentricity of F. If this eccentricity is such that the neutral axis falls outside the section, then the stress σ_x is negative (or compressive) at all points in the section. This situation arises if the stress resultant lies within an area called the *core* of the section. The dimensions of the cores of regular sections can be found analytically and some examples are given in Table 2.6.

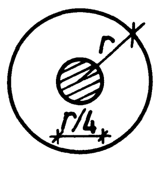
Table 2.6 Cores of sections

Section

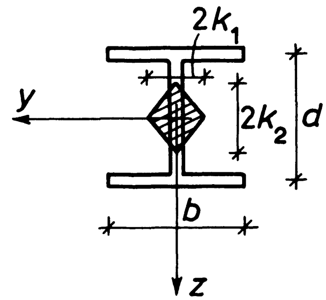
(1) Rectangle



(2) Circle



(3) I-section



$k_1=2r_z^2/b$

$k_2=2r_y^2/d$

2.3.10.2 Long columns

If the length of a bar is greater than about 5 times its lateral dimensions, it can become unstable under compressive forces. Consider, for example, the pin-ended bar subject to an axial compressive force F shown in Figure 2.50. If u_z is the lateral displacement in the z direction of a particular cross-section, then the moment M_y exerted by F at the section is Fu_z. Thus from Equation (2.146) we have the differential equation

$$\frac{d^2u_z}{dx^2}+\frac{Fu_z}{EI_y}=0 \tag{2.182}$$

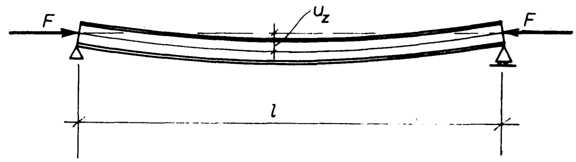


Figure 2.50

One solution of Equation (2.182) is $u_z=0$, i.e. the bar remains straight. However, further nonzero solutions for u_z occur for particular values of F called the *eigenvalues*. The lowest eigenvalue is the critical load F_{cr} of the bar, and can be regarded as the maximum load that can be carried before failure by lateral instability. It can be shown that F_{cr} is given by:

$$F_{cr}=\frac{\pi^2EI_y}{l^2} \tag{2.183}$$

while the corresponding deflected shape of the bar is sinusoidal and of arbitrary amplitude, taking the following form:

$$u_z=A\sin\left(\frac{\pi x}{l}\right) \tag{2.184}$$

The value for the critical load was first obtained by Euler, and a pin-ended bar subject to axial compression is often called an *Euler strut*.

Dividing Equation (2.183) by the area of the bar leads to the following expression for the *critical buckling stress* σ_{cr}:

$$\sigma_{cr}=\pi^2E/\lambda^2 \tag{2.185}$$

where λ $(=l/r_y)$ is called the *slenderness ratio.*

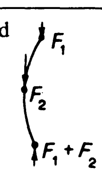
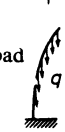
When, as in most cases, $I_y \neq I_z$, the strut buckles first about the minor principal axis, about that axis for which the moment of inertia of the section is a minimum.

The critical buckling loads of struts with other than pin-ended boundary conditions are given in Table 2.7. The corresponding *effective lengths* l_e are then defined so that the critical stresses can be given by an equation analogous to Equation (2.185) namely

$$\sigma_{cr} = \pi^2 E/\lambda_e^2 \tag{2.186}$$

where $\lambda_e = l_e/r_y$. Values for l_e are included in the table.

Table 2.7 Critical buckling loads of struts

All struts are of length l; $I_z > I_y$				
Lower end boundary condition	*Upper end boundary condition*	*Mode*	P_{cr}	l_e
(1) Hinge along y axis	Hinge along y axis		$\pi^2 EI_y/l^2$	l
(2) Clamped	Clamped		$4\pi^2 EI_y/l^2$	$0.5\,l$
(3) Clamped	Hinge along y axis		$20.19\, EI_y/l^2$	$0.7\,l$
(4) Clamped	Free		$\pi^2 EI_y/4l^2$	$2.0\,l$
(5) Hinge along z axis	Hinge along z axis		Smaller of $4\pi^2 EI_y/l^2$ or $\pi^2 EI_z/l^2$	$0.5\,l$
(6) Hinge along z axis	Hinge along y axis		$20.19\, EI_y/l^2$	$0.7\,l$
Special loading cases				
(7) Pin-ended strut under end load P_1 and central load P_2			$(F_1+F_2)_{cr} = \pi^2 EI/(kl)^2$ where $k \simeq 1/(2-c^2)$ $c = F_1/(F_1+F_2)$	
(8) Cantilever strut under uniformly distributed load q/unit length			$(q_{cr})l = \pi^2 EI/(1.122\,l)^2$	

The above type of buckling is called *flexural buckling*, and occurs when the cross-section of the strut has two axes of symmetry. For unsymmetrical sections, buckling may be accompanied by torsion as well as flexure, producing a correspondingly reduced critical load. Results for such cases are given by Bleich.[3]

2.3.10.3 Formulae for the strength of columns

The plot of σ_{cr} versus λ for various column lengths is the hyperbola shown in Figure 2.51. Clearly, when λ is very small, the critical stress becomes much greater than the yield stress σ_Y of the material, and the failure of the column is brought about by the yielding of the material rather than by flexural buckling. If the columns were perfectly straight and the axial load had no eccentricity then the ultimate stresses σ_u would be given by the upper curve in Figure 2.51, i.e. the elastic buckling hyperbola intersected by the horizontal 'squash' line. However, tests show that the strengths of real columns are considerably reduced by initial imperfections when $\sigma_{cr} \simeq \sigma_Y$ as indicated by the lower curve in the figure. The following semi-empirical formulae have been devised to account for this, giving the ultimate stresses of columns in terms of their geometrical and material properties.

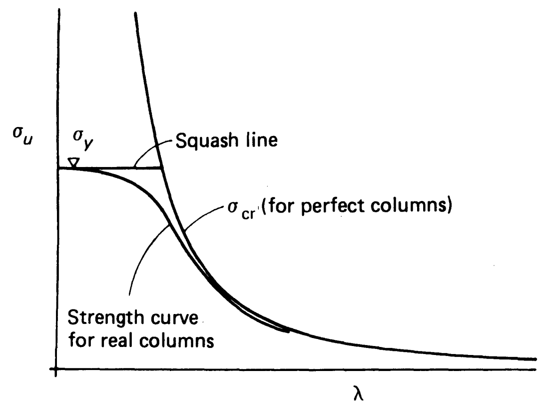


Figure 2.51

The Rankine formula.[17] A simple interaction formula relating σ_u, σ_Y and σ_{cr} is as follows:

$$\frac{1}{\sigma_u} = \frac{1}{\sigma_{cr}} + \frac{1}{\sigma_Y} \tag{2.187}$$

gives:

$$\sigma_u = \frac{\sigma_Y}{1 + \dfrac{\sigma_Y \lambda^2}{\pi^2 E}} \tag{2.188}$$

The interaction curve is tangential to the squash line at $\lambda = 0$, and to the buckling hyperbola at $\lambda = \infty$.

The Johnson parabola.[17] The formula:

$$\sigma_u = \sigma_Y \left(1 - \frac{\sigma_Y \lambda^2}{4\pi^2 E}\right) \tag{2.189}$$

gives a parabolic interaction curve in the nonelastic range which is tangential to the squash line at $\lambda=0$, and to the buckling hyperbola at the point $\sigma_{cr}=\frac{1}{2}\sigma_Y$.

The secant formula. The secant formula is derived assuming that the axial forces on the column have an initial *eccentricity e* (Figure 2.52(a)). In this case it can be shown that:

$$\sigma_u=\frac{\sigma_Y}{1+\eta\sec[(\pi/2)\sqrt{(\sigma_u/\sigma_{cr})}]} \tag{2.190}$$

where η is given by:

$$\eta=ec/r_y^2 \tag{2.191}$$

c is the distance from the neutral axis to the extreme fibre of the section.

The Perry–Robertson formula. Assuming that the column has an initial *curvature* and that its maximum misalignment is e (Figure 2.52(b)), Ayrton and Perry derived the following formula:

$$\sigma_u=\tfrac{1}{2}[\sigma_Y+(1+\eta)\sigma_{cr}]-\left[\left(\frac{\sigma_Y+(1+\eta)\sigma_{cr}}{2}\right)^2-\sigma_Y\sigma_{cr}\right]^{1/2} \tag{2.192}$$

where η is again given by Equation (2.191).

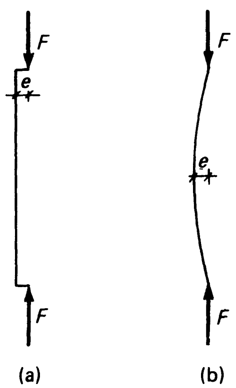


Figure 2.52

Robertson showed by experiment that a good but conservative prediction of the real strengths of columns can be obtained by making η proportional to λ, as follows:

$$\eta=0.003\,\lambda \tag{2.193}$$

Later experiments by Dutheil[17] led to the modified expression

$$\eta=0.3\,(\lambda/100v)^2 \tag{2.194}$$

2.3.10.4 *Codes of practice for the design of columns*

Section 2.3.10.3 summarizes the bases of simple empirical formulae for the strengths of columns. Current and projected codes of practice are somewhat more complicated, attempting to allow for the effects of variations in cross-sectional geometry and of residual stresses due to rolling and welding.

The *British* codes of practice are based on the Perry–Robertson formula. In the current standard for the design of steel bridges,[18] compression members are designed for η in Equation (2.191) which is linearly related to λ and a parameter α as follows:

$$\eta=0 \qquad (\lambda<\lambda_0) \tag{2.195}$$

$$\eta=0.001\alpha(\lambda-\lambda_0) \quad (\lambda>\lambda_0) \tag{2.196}$$

where λ_0 is the slenderness ratio below which the members are assumed to reach their full squash load. This is given as $0.2\,\lambda_1$ where λ_1 $(=\pi\sqrt{E/\sigma_Y})$ is the slenderness ratio for which the critical stress is equal to the yield stress. Four curves for σ_u are presented, curves A, B, C and D corresponding to $\alpha=2.5$, 4.5, 6.2 and 8.3 respectively. These are shown in a nondimensional plot in Figure 2.53. The curves appropriate for various cross-sections and fabrication methods are then selected according to

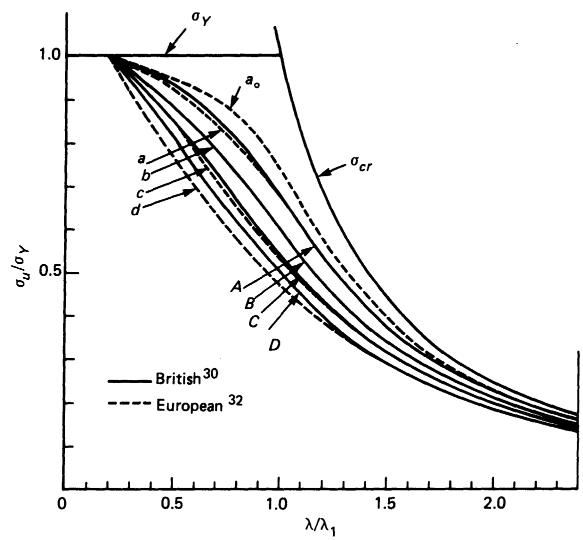


Figure 2.53 British and European column strength curves

Table 2.8.† The revised standard for steelwork in buildings,[19] adopts a similar approach, with slight differences in α for the different cases.

The *European Recommendations for Steel Construction*,[20] published by the European Convention for Structural Steelwork (ECSS) employ three basic column strength curves *a*, *b* and *c* again describing the strengths of groups of rolled and welded columns with various cross-sections. These curves are included as broken lines in Figure 2.53. The additional curves a_0 and d respectively deal with heat-treated sections in high-strength steel, and with sections with particularly thick plates (>40 mm). For welded sections the effective value of the yield stress is reduced by 6%. An extended account of the reasoning behind the Recommendations is given in Chapters 2 and 3 of the Second International Colloquium report.[21]

The current *American* codes of practice are based on the Johnson parabola. Thus the American Institute of Steel Construction[22] recommend that the *allowable* stresses are obtained by dividing the interaction curve given by Equation (2.189) by a safety factor ϕ which depends on the slenderness ratio. Thus defining λ_2 to be the slenderness ratio for which $\sigma_{cr}=\frac{1}{2}\sigma_Y$, then

† Extracts from BS 5400: Part 3: 1982 are reproduced by permission of the British Standards Institution, 2 Park Street, London, W1A 2BS from whom complete copies of the standard can be obtained.

Table 2.8 Selection of British column strength curves. British Standards Institution (1982) *Steel, concrete and composite bridges,* BS 5400: Part 3. BSI, Milton Keynes)

	Members fabricated by welding (excluding local welding of battens, lacing, etc.)	*All other members (including stress relieved welded members)*
$r_y/c \geqslant 0.7$	curve *B*	curve *A*
$r_y/c = 0.60$	curve *C*	curve *B*
$r_y/c = 0.50$	curve *C*	curve *B*
$r_y/c \leqslant 0.45$	curve *C*	curve *C*
All-rolled sections with flange thickness > 40 mm	curve *D*	
Hot-finished hollow sections	curve *A*	

Notes: (a) For intermediate values of r_y/c, linear interpolation may be used between the curves given.
(b) c is defined as for Equation (2.191).

$$\phi = \tfrac{5}{3} + \tfrac{3}{8}\left(\frac{\lambda}{\lambda_2}\right)^{1/2} - \tfrac{1}{8}\left(\frac{\lambda}{\lambda_2}\right)^{3/2} \quad (\lambda < \lambda_2) \tag{2.197}$$

$$\phi = \tfrac{23}{12} \qquad (\lambda > \lambda_2) \tag{2.198}$$

For slender bracing and secondary members for which $\lambda > 120$, the allowable stresses may be divided by $(1.6 - \lambda/200)$, giving stresses similar to those of the Rankine formula. The Structural Stability Research Council (SSRC)[23] describe three column-strength curves (1), (2) and (3) each one representing the computed strength of a group of rolled or welded sections with realistic residual stresses and an initial bow of $l/1000$. These are shown in Figure 2.54.

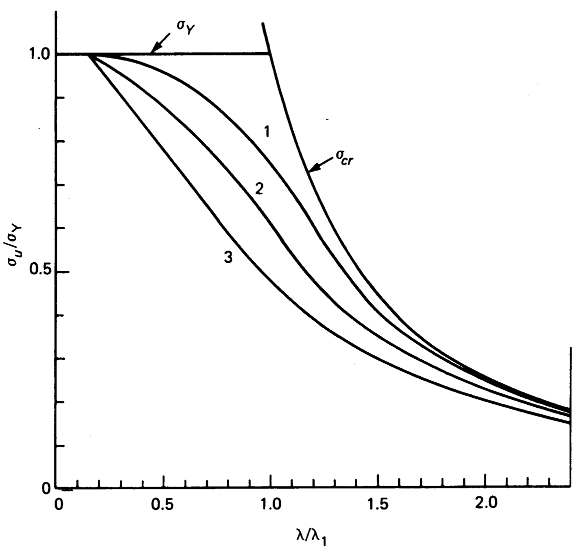


Figure 2.54 American column strength curves

2.3.11 Virtual work and strain energy of frameworks

The state of stress and strain at all points in a framework can be expressed in terms of the stress resultants at those points, using the appropriate equations of the previous sections and the stress–strain relations. The internal virtual work done in a framework corresponding to the general expression in Equation (2.76) is then given by:

$$W_i^* = \sum_{\text{bars}} \int_0^l \left(\frac{NN^*}{EA} + \frac{M_y M_y^*}{EI_y} + \frac{M_z M_z^*}{EI_z} + \frac{k_y S_y S_y^*}{GA} + \frac{k_z S_z S_z^*}{GA} + \frac{TT^*}{G} \right) \mathrm{d}x \tag{2.199}$$

where k_y and k_z are dimensionless form factors depending on the shape of the bar cross-section at each point in the framework. Values of the form factors for some common cross-sections are given in Table 2.9.

Table 2.9 Form factors

Section	k_y	k_z
1 Rectangle	1.20	
2 Circle	1.11	
3 Hollow circle	2.00	
4 I-section or hollow rectangle (approx.)	A/A_{flanges}	A/A_{webs}

Similarly the internal strain energy of a framework corresponding to the expression in Equation (2.78) is given by:

$$U = \sum_{\text{bars}} \int_0^l \left(\frac{N^2}{2EA} + \frac{M_y^2}{2EI_y} + \frac{M_z^2}{2EI_z} + \frac{k_y S_y^2}{2GA} + \frac{k_z S_z^2}{2GA} + \frac{T^2}{2GJ} \right) \mathrm{d}x \tag{2.200}$$

2.3.12 Note on the limitations of the engineering theory of the bending of beams (ETBB)

As noted in section 2.3.6, the basic assumptions of the ETBB, while quite correct when the beam is subject to pure bending, become invalid when the beam is also subject to shear. In particular, we can no longer assume that plane sections remain plane.

Some indication of the error involved in using the ETBB is obtained by analysing a thin-walled deep cantilever beam. Treating this as a plane stress problem, a complete solution is possible subject only to certain assumptions regarding the fixity at the encastre end.[8] Thus it can be shown that if the cantilever is loaded by a single vertical load F at its end so that the shear stress resultant is uniform along the length, the direct and shear stresses given by the ETBB are *exact*. However, the deflections u_x and u_z are given by:

$$u_x = \frac{F}{2EI_y}(-2lx + x^2)z + \frac{\nu F z^3}{6EI_y} - \frac{Fz^3}{6GI_y} \tag{2.201}$$

$$u_z = \frac{F}{6EI_y}(3lx^2 - x^3) + \frac{\nu F}{2EI_y}(l - x)z^2 + \frac{Fd^2x}{8GI_y} \tag{2.202}$$

and the corresponding deflected shape of the beam is composed

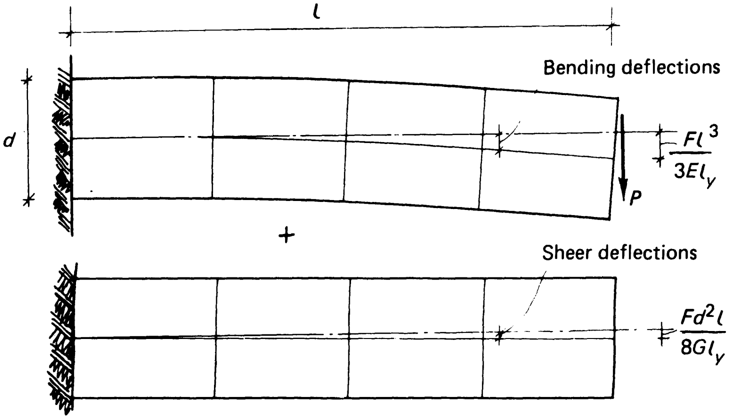


Figure 2.55

of two components as shown in Figure 2.55. One is the curved shape predicted by the ETBB, while the other is a linear vertical displacement due to the shear with the original plane cross-section taking up an S-shape in side view.

If the cantilever is loaded by a uniformly distributed load F/unit length so that the shear-stress resultant varies with x then the *stresses* given by the ETBB are also slightly inaccurate. However, it can be shown that the error is small, provided the span of the beam is large compared with its depth. Further the curvature of the beam is modified from Equation (2.119) to:

$$\frac{1}{R_y}=\left[\frac{M_y}{EI_y}+\frac{F}{EI_y}\frac{d^2}{4}\left(\frac{4}{5}+\frac{v}{2}\right)\right] \tag{2.203}$$

where the second term on the right-hand side represents the effect of the shear forces.

The preceding discussion concerns the behaviour of the webs of beams. However, in the flanges as well, it can be shown that plane sections no longer remain plane when beams are subject to shear. This phenomenon is called *shear lag*. It can be conveniently illustrated by the T-section cantilever shown in Figure 2.56(a). According to St Venant's theory (section 2.3.6), the forces in the flange are transmitted by longitudinal shear across the section A–B, so that the flange can be considered to behave like the cantilever plate shown in Figure 2.56(b) subjected to the uniformly distributed axial load along its centreline. It is then clear that the corresponding displacements u_x and the axial stress σ_x are nonuniform across the width of the flange. The analysis of shear lag for practical cases is complex, and the topic is dealt with at some length by Williams.[24]

Further departures from the ETBB occur when beams become geometrically unstable. This instability can take the form of local compressive buckling of the flanges,[3] local shear buckling of the webs[3] and overall torsional buckling.[3]

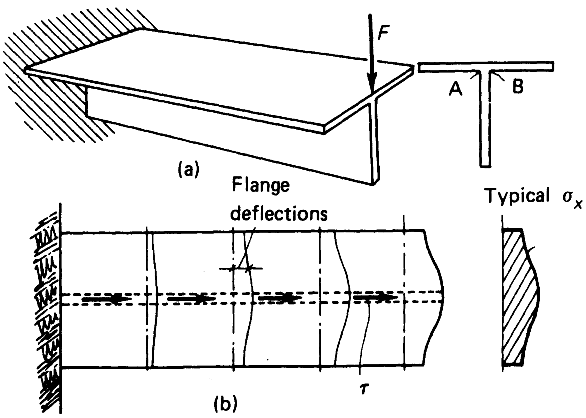


Figure 2.56

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3

Theory of Structures

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3.1 Introduction

3.1.1 Basic concepts

The 'Theory of Structures' is concerned with establishing an understanding of the behaviour of structures such as beams, columns, frames, plates and shells, when subjected to applied loads or other actions which have the effect of changing the state of stress and deformation of the structure. The process of 'structural analysis' applies the principles established by the Theory of Structures, to analyse a given structure under specified loading and possibly other disturbances such as temperature variation or movement of supports. The drawing of a bending moment diagram for a beam is an act of structural analysis which requires a knowledge of structural theory in order to relate the applied loads, reactive forces and dimensions to actual values of bending moment in the beam. Hence 'theory' and 'analysis' are closely related and in general the term 'theory' is intended to include 'analysis'.

Two aspects of structural behaviour are of paramount importance. If the internal stress distribution in a structural member is examined it is possible, by integration, to describe the situation in terms of 'stress resultants'. In the general three-dimensional situation, these are six in number: two bending moments, two shear forces, a twisting moment and a thrust. Conversely, it is, of course, possible to work the other way and convert stress-resultant actions (forces) into stress distributions. The second aspect is that of deformation. It is not usually necessary to describe structural deformation in continuous terms throughout the structure and it is usually sufficient to consider values of displacement at selected discrete points, usually the joints, of the structure.

At certain points in a structure, the continuity of a member, or between members, may be interrupted by a 'release'. This is a device which imposes a zero value on one of the stress resultants. A hinge is a familiar example of a release. Releases may exist as mechanical devices in the real structure or may be introduced, in imagination, in a structure under analysis.

In carrying out a structural analysis it is generally convenient to describe the state of stress or deformation in terms of forces and displacements at selected points, termed 'nodes'. These are usually the ends of members, or the joints and this approach introduces the idea of a structural element such as a beam or column. A knowledge of the forces or displacements at the nodes of a structural element is sufficient to define the complete state of stress or deformation within the element providing the relationships between forces and displacements are established. The establishment of such relationships lies within the province of the theory of structures.

Corresponding to the basic concepts of force and displacement, there are two important physical principles which must be satisfied in a structural analysis. The structure as a whole, and every part of it, must be in equilibrium under the actions of the force system. If, for example, we imagine an element, perhaps a beam, to be removed from a structure by cutting through the ends, the internal stress resultants may now be thought of as external forces and the element must be in equilibrium under the combined action of these forces and any applied loads. In general, six independent conditions of equilibrium exist; zero sums of forces in three perpendicular directions, and zero sums of moments about three perpendicular axes. The second principle is termed 'compatibility'. This states that the component parts of a structure must deform in a compatible way, i.e. the parts must fit together without discontinuity at all stages of the loading. Since a release will allow a discontinuity to develop, its introduction will reduce the total number of compatibility conditions by one.

3.1.2 Force–displacement relationships

A simple beam element AB is shown in Figure 3.1 The application of end moments M_A and M_B produces a shear force Q throughout the beam, and end rotations θ_A and θ_B. By the stiffness method (see page 3/11), it may be shown that the end moments and rotations are related as follows:

$$\left.\begin{aligned} M_A &= \frac{4EI\theta_A}{l} + \frac{2EI\theta_B}{l} \\ M_B &= \frac{4EI\theta_B}{l} + \frac{2EI\theta_A}{l} \end{aligned}\right\} \tag{3.1}$$

Or, in matrix notation,

$$\begin{bmatrix} M_A \\ M_B \end{bmatrix} = \frac{2EI}{l}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix}$$

which may be abbreviated to,

$$\mathbf{S} = \mathbf{k}\boldsymbol{\theta} \tag{3.2}$$

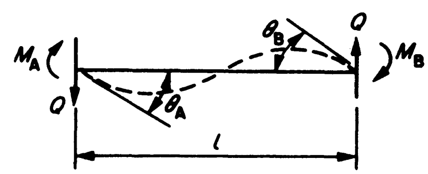


Figure 3.1

Equation (3.2) expresses the force–displacement relationships for the beam element of Figure 3.1. The matrices $\mathbf{S}$ and $\boldsymbol{\theta}$ contain the end 'forces' and displacements respectively. The matrix $\mathbf{k}$ is the *stiffness matrix* of the element since it contains end forces corresponding to *unit* values of the end rotations.

The relationships of Equation (3.2) may be expressed in the inverse form:

$$\begin{bmatrix} \theta_A \\ \theta_B \end{bmatrix} = \frac{l}{6EI}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} M_A \\ M_B \end{bmatrix}$$

or

$$\boldsymbol{\theta} = \mathbf{fS} \tag{3.3}$$

Here the matrix $\mathbf{f}$ is the *flexibility matrix* of the element since it expresses the end displacements corresponding to *unit* values of the end forces.

It should be noted that an inverse relationship exists between $\mathbf{k}$ and $\mathbf{f}$

i.e.

$$\mathbf{kf} = I$$

or,

$$\mathbf{k} = \mathbf{f}^{-1} \tag{3.4}$$

or,

$$\mathbf{f} = \mathbf{k}^{-1}$$

The establishment of force–displacement relationships for structural elements in the form of Equations (3.2) or (3.3) is an important part of the process of structural analysis since the element properties may then be incorporated in the formulation of a mathematical model of the structure.

3.1.3 Static and kinematic determinacy

If the compatibility conditions for a structure are progressively reduced in number by the introduction of releases, there is reached a state at which the introduction of *one further* release would convert the structure into a mechanism. In this state the structure is *statically determinate* and the nodal forces may be calculated directly from the equilibrium conditions. If the releases are now removed, restoring the structure to its correct condition, nodal forces will be introduced which cannot be determined solely from equilibrium considerations. The structure is *statically indeterminate* and compatibility conditions are necessary to effect a solution.

The structure shown in Figure 3.2(a) is hinged to rigid foundations at A, C and D. The continuity through the foundations is indicated by the (imaginary) members, AD and CD. If the releases at A, C and D are removed, the structure is as shown in Figure 3.2(b) which is seen to consist of two closed rings. Cutting through the rings as shown in Figure 3.2(c) produces a series of simple cantilevers which are statically determinate. The number of stress resultants released by each cut would be three in the case of a planar structure, six in the case of a space structure. Thus, the degree of statical indeterminacy is 3 or 6 times the number of rings. It follows that the structure shown in Figure 3.2(b) is 6 times statically indeterminate whereas the structure of Figure 3.2(a), since releases are introduced at A, C and D, is 3 times statically indeterminate. A general relationship between the number of members m, number of nodes n, and degree of static indeterminacy n_s, may be obtained as follows:

$$n_s = \frac{6}{3}(m - n + 1) - r \tag{3.5}$$

where r is the number of releases in the actual structure

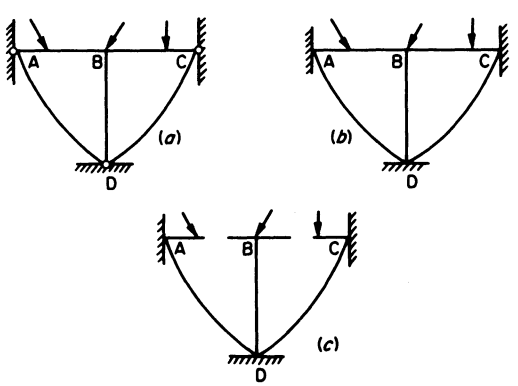


Figure 3.2

Turning now to the question of *kinematical determinacy*; a structure is defined as kinematically determinate if it is possible to obtain the nodal displacements from compatibility conditions without reference to equilibrium conditions. Thus a fixed-end beam is kinematically determinate since the end rotations are known from the compatibility conditions of the supports.

Again, consider the structure shown in Figure 3.2(b). The structure is kinematically determinate except for the displacements of joint B. If the members are considered to have infinitely large extensional rigidities, then the rotation at B is the only unknown nodal displacement. The degree of kinematical indeterminacy is therefore 1. The displacements at B are *constrained* by the assumption of zero vertical and horizontal displacements. A *constraint* is defined as a device which constrains a displacement at a certain node to be the same as the corresponding displacement, usually zero, at another node. Reverting to the structure of Figure 3.2(a), it is seen that three constraints, have been removed by the introduction of hinges (releases) at A, C and D. Thus rotational displacements can develop at these nodes and the degree of kinematical indeterminacy is increased from 1 to 4.

A general relationship between the numbers of nodes n, constraints c, releases r, and the degree of kinematical indeterminacy n_k is as follows,

$$n_k = \frac{6}{3}(n - 1) - c + r \tag{3.6}$$

The coefficient 6 is taken in three-dimensional cases and the coefficient 3 in two-dimensional cases. It should now be apparent that the modern approach to structural theory has developed in a highly organised way. This has been dictated by the development of computer-orientated methods which have required a re-assessment of basic principles and their application in the process of analysis. These ideas will be further developed in some of the following sections.

3.2 Statically determinate truss analysis

3.2.1 Introduction

A structural frame is a system of bars connected by joints. The joints may be, ideally, *pinned* or *rigid*, although in practice the performance of a real joint may lie somewhere between these two extremes. A *truss* is generally considered to be a frame with pinned joints, and if such a frame is loaded only at the joints, then the members carry axial tensions or compressions. *Plane* trusses will resist deformation due to loads acting in the plane of the truss only, whereas *space* trusses can resist loads acting in any direction.

Under load, the members of a truss will change length slightly and the geometry of the frame is thus altered. The effect of such alteration in geometry is generally negligible in the analysis.

The question of statical determinacy has been mentioned in the previous section where a relationship, Equation (3.5) was stated from which the degree of statical indeterminacy could be determined. Although this relationship is of general application, in the case of plane and space trusses, a simpler relationship may be established.

The simplest plane frame is a triangle of three members and three joints. The addition of a fourth joint, in the plane of the triangle, will require two additional members. Thus in a frame having j joints, the number of members is:

$$n = 2(j - 3) + 3 = 2j - 3 \tag{3.7}$$

A truss with this number of members is statically determinate, providing the truss is supported in a statically determinate way. Statically determinate trusses have two important properties. They cannot be altered in shape without altering the length of one or more members, and, secondly, any member may be altered in length without inducing stresses in the truss, i.e. the truss cannot be *self stressed* due to imperfect lengths of members or differential temperature change.

The simplest space truss is in the shape of a tetrahedron with four joints and six members. Each additional joint will require three more members for connection with the tetrahedron, and thus:

$$n=3(j-4)+6=3j-6 \tag{3.8}$$

A space truss with this number of members is statically determinate, again providing the support system is itself statically determinate. It should be noted that in the assessment of the statical determinacy of a truss, member forces and reactive forces should all be considered when counting the number of unknowns. Since equilibrium conditions will provide two relationships at each joint in a plane truss (there is a space truss), the simplest approach is to find the total number of unknowns, member forces and reactive components, and compare this with 2 or 3 times the number of joints.

3.2.2 Methods of analysis

Only brief mention will be made here of the methods of statically determinate analysis of trusses. For a more detailed treatment the reader is referred to Jenkins[1] and Coates, Coutie and Kong.[2]

The *force diagram* method is a graphical solution in which a vector polygon of forces is drawn to scale proceeding from joint to joint. It is necessary to have not more than two unknown forces at any joint, but this requirement can be met with a judicious choice of order. The two conditions of overall equilibrium of the plane structure imply that the force vector polygon will form a closed figure. The method is particularly suitable for trusses with a difficult geometry where it is convenient to work to a scale drawing of the outline of the truss.

The method of *resolution at joints* is suitable for a complete analysis of a truss. The reactions are determined and then, proceeding from joint to joint, the vertical and horizontal equilibrium conditions are set down in terms of the member forces. Since two equations will result at each joint in a plane truss, it is possible to determine not more than two forces for each pair of equations. As an illustration of the method, consider the plane truss shown in Figure 3.3. The truss is symmetrically loaded and the reactions are clearly 15 kN each.

Consider the equilibrium of joint A,

vertically, $P_{AE}\cos 45° = R_A$; hence $P_{AE} = 15\sqrt{2}$ kN (compression)

horizontally, $P_{AC} = P_{AE}\cos 45°$; hence $P_{AC} = 15$ kN (tension)

It should be noted that the arrows drawn on the members in Figure 3.3 indicate the directions of forces acting *on the joints*. It is also seen that the directions of the arrows at joint A, for example, are consistent with equilibrium of the joint. Proceeding to joint C it is clear that $P_{CE} = 10$ kN (tension), and that $P_{CD} = P_{AC} = 15$ kN (tension). The remainder of the solution may be obtained by resolving forces at joint E, from which $P_{ED} = 5\sqrt{2}$ kN (tension) and $P_{EF} = 20$ kN (compression).

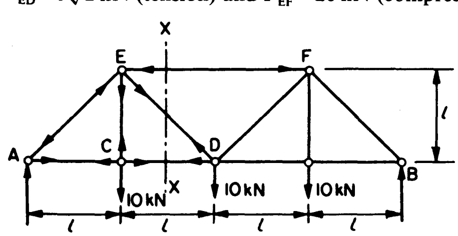


Figure 3.3

The *method of sections* is useful when it is required to determine forces in a limited number of the members of a truss. Consider, for example, the member ED of the truss in Figure 3.3. Imagine a cut to be made along the line XX and consider the vertical equilibrium of the part to the left of XX. The vertical forces acting are R_A, the 10 kN load at C and the vertical component of the force in ED. The equation of vertical equilibrium is:

$$15-10=P_{ED}\cos 45° \quad \text{hence } P_{ED}=5\sqrt{2}\text{ kN}$$

Since a downwards arrow on the left-hand part of ED is required for equilibrium, it follows that the member is in tension. The method of tension coefficients is particularly suitable for the analysis of space frames and will be outlined in the following section.

3.2.3 Method of tension coefficients

The method is based on the idea of systematic resolution of forces at joints. In Figure 3.4, let AB be any member in a plane truss, T_{AB} = force in member (tension positive), and L_{AB} = length of member.

We define:

$$T_{AB}=L_{AB}t_{AB} \tag{3.9}$$

where t_{AB} = tension coefficient.

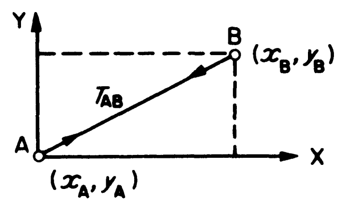


Figure 3.4

That is, the tension coefficient is the actual force in the member divided by the length of the member. Now, at A, the component of T_{AB} in the X-direction:

$$=T_{AB}\cos \text{BAX}$$

$$=T_{AB}\frac{(x_B-x_A)}{L_{AB}}=t_{AB}(x_B-x_A)$$

Similarly the component of T_{AB} in the Y-direction:

$$=t_{AB}(y_B-y_A)$$

At the other end of the member the components are:

$$t_{AB}(x_A-x_B),\ t_{AB}(y_A-y_B)$$

If at A the external forces have components X_A and Y_A, and if there are members AB, AC, AD etc. then the equilibrium conditions for directions X and Y are:

$$\left.\begin{aligned}t_{AB}(x_B-x_A)+t_{AC}(x_C-x_A)+t_{AD}(x_D-x_A)+\ \ldots\ +X_A=0\\ t_{AB}(y_B-y_A)+t_{AC}(y_C-y_A)+t_{AD}(y_D-y_A)+\ \ldots\ +Y_A=0\end{aligned}\right\}\tag{3.10}$$

Similar equations can be formed at each joint in the truss. Having solved the equations, for the tension coefficients, usually a very simple process, the forces in the members are determined from Equation (3.9).

The extension of the theory to space trusses is straightforward. At each joint we now have three equations of equilibrium, similar to Equation (3.10) with the addition of an equation representing equilibrium in the Z direction:

$$t_{AB}(z_B - z_A) + t_{AC}(z_C - z_A) + \ldots + Z_A = 0 \tag{3.11}$$

The method will now be illustrated with an example. The notation is simplified by writing AB in place of t_{AB} etc. A fabular presentation of the work is recommended.

Example 3.1. A pin-jointed space truss is shown in Figure 3.5. It is required to determine the forces in the members using the method of tension coefficients. We first check that the frame is statically determinate as follows:

Number of members = 6
Number of reactions = 9

Total number of unknowns = 15

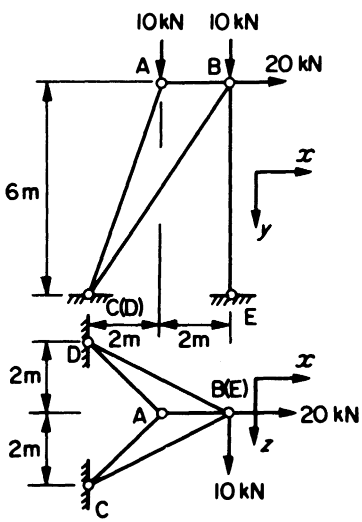


Figure 3.5

The number of equations available is 3 times the number of joints, i.e. $3 \times 5 = 15$. Hence, the truss is statically determinate. In counting the number of reactive components, it should be observed that all components should be included even if the particular geometry of the truss dictates (as in this case at E) that one or more components should be zero.

The solution is set out in Tables 3.1 and 3.2 where it should be noted that, in deriving the equations, the origin of coordinates is taken at the joint being considered. Thus, each tension coefficient is multiplied by the projection of the member on the particular axis.

The methods of truss analysis just outlined are suitable for 'hand' analysis, as distinct from computer analysis, and are useful in acquiring familiarity and understanding of structural behaviour. Much analysis of this kind is now carried out on computers (mainframe, mini- and microcomputers) where the stiffness method provides a highly organized and suitable basis. This topic will be further considered under the heading of the stiffness method.

Table 3.1

Joint	*Direction*	*Equations*	*Solutions*
A	x	$-2AC - 2AD + 2AB = 0$	$AC = AD = -\frac{10}{12}$
	y	$6AC + 6AD + 10 = 0$	$AB = -\frac{10}{6}$
	z	$2AC - 2AD = 0$	$-4BC - 4BD + \frac{10}{3} + 20 = 0$ $2BC - 2BD + 10 = 0$
C	x	$-4BC - 4BD - 2AB + 20 = 0$	$BC = \frac{10}{24}$
	y	$6BC + 6BD + 6BE + 10 = 0$	$BD = \frac{130}{24}$
	z	$-2BD + 2BC + 10 = 0$	Hence $BE = -\frac{15}{2}$

Table 3.2

Member	*Length* (m)	*Tension coefficient*	*Force* (kN) *(tension +)*
AB	2	$-\frac{10}{6}$	−3.33
AC	6.62	$-\frac{10}{12}$	−5.52
AD	6.62	$-\frac{10}{12}$	−5.52
BC	7.48	$\frac{10}{24}$	+3.12
BD	7.48	$\frac{130}{24}$	+40.5
BE	6	$-\frac{15}{2}$	−45.0

3.3 The flexibility method

3.3.1 Introduction

The idea of statical determinacy was introduced previously (see page 3/4) and a relationship between the degree of statical indeterminacy and the numbers of members, nodes and releases was stated in Equation (3.5). A statically determinate structure is one for which it is possible to determine the values of forces at all points by the use of equilibrium conditions alone. A statically indeterminate structure, by virtue of the number of members or method of connecting the members together, or the method of support of the structure, has a larger number of forces than can be determined by the application of equilibrium principles alone. In such structures the force analysis requires the use of compatibility conditions. The flexibility method provides a means of analysing statically indeterminate structures.

Consider the propped cantilever shown in Figure 3.6(a). Applying Equation (3.5) the degree of statical indeterminacy is seen to be:

$$n_s = 3(2 - 2 + 1) - 2 = 1$$

(Note that two releases are required at B, one to permit angular rotation and one to permit horizontal sliding, and also that an additional foundation member is inserted connecting A and B.) The structure can be made statically determinate by removing the propping force R_B or alternatively by removing the fixing moment at A. We shall proceed by removing the reaction R_B. The structure thus becomes the simple cantilever shown in Figure 3.6(b). The application of the load w produces the deflected shape, shown dotted, and in particular a deflection u at the free end B. Note also that it is now possible to determine the bending moment at A = $wl^2/2$, by simple statical principles. The

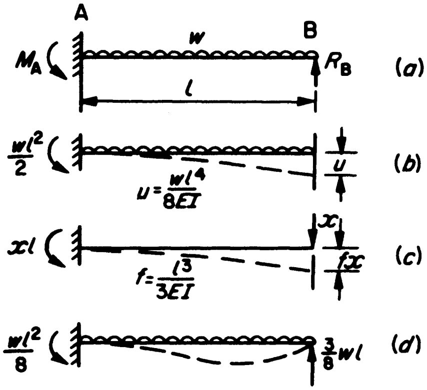


Figure 3.6 Basis of the flexibility method

deflection u may be obtained from elementary beam theory as $wl^4/8EI$. We now remove the applied load w and apply the, unknown, redundant force x at B. It is unnecessary to know the sense of the force x; in this case we have assumed a downwards direction for positive x. The application of the force x produces a displacement at B which we shall call fx; i.e. a unit value of x would produce a displacement f. The compatibility condition associated with the redundant force x is that the final displacement at B should be zero, i.e.:

$$u+fx=0 \tag{3.12}$$

and substituting values of u and f

$$x=-\tfrac{3}{8}wl$$

The process may be regarded as the superposition of the diagrams Figures 3.6(b) and (c) such that the final displacement at B is zero. The addition of the two systems of forces will also give values of bending moment throughout the beam, e.g. at A:

$$M_A=\frac{wl^2}{2}+xl$$

$$=\frac{wl^2}{2}-\tfrac{3}{8}wl^2 \qquad =\frac{wl^2}{8}$$

The actual values of reactions are as shown in Figure 3.6(d).

The displacement f is called a 'flexibility influence coefficient'. In general f_{rs} is the displacement in direction r in a structure due to unit force in direction s. The subscripts were omitted in the above analysis since the force and displacement considered were at the same position and in the same direction.

3.3.2 Evaluation of flexibility influence coefficients

As seen in the above example, flexibility coefficients are displacements calculated at specified positions, and directions, in a structure due to a prescribed loading condition. The loading condition is that of a single unit load replacing a redundant force in the structure. It should be remembered that at this stage the structure is, or has been made, statically determinate.

For simplicity we restrict our attention to structures in which flexural deformations predominate. The extension to other types of deformation is straightforward.[3] In the case of pure flexural deformation we may evaluate displacements by an application of Castigliano's theorem or use the principle of virtual work.[3] In either case a convenient form is:

$$\Delta_i=\int M\partial M/\partial F_i \; \frac{ds}{EI} \tag{3.13}$$

in which Δ_i is the displacement required, M is a function representing the bending moment distribution and F_i is a force, real or virtual, applied at the position and in the direction designated by i. It follows that $\partial M/\partial F_i$ can be regarded as the bending moment distribution due to unit value of F_i.

Consider the cantilever beam shown in Figure 3.7(a). Forces x_1 and x_2 act on the beam and it is required to determine influence coefficients corresponding to the positions and directions defined by x_1 and x_2. From now on we work with *unit* values of x_1 and x_2 and draw bending moment diagrams, as in Figure 3.7(b) and (c), due to unit values of x_1 and x_2 separately.

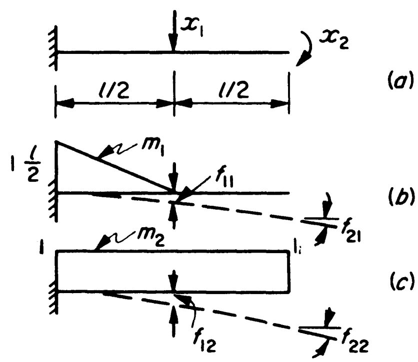


Figure 3.7 Evaluation of flexibility coefficients

These are labelled m_1 and m_2. Consider the application of unit force at x_1 ($x_2=0$). Displacements will occur in the directions of x_1 and x_2. Applying Equation (3.13) the displacement in the direction of x_1 will be:

$$f_{11}=\int m_1 m_1 \frac{ds}{EI}$$

and in the direction of x_2: (3.14)

$$f_{21}=\int m_2 m_1 \frac{ds}{EI}$$

Similarly, when we apply $x_2=1$, $x_1=0$, we obtain:

$$f_{22}=\int m_2 m_2 \frac{ds}{EI}$$

and: (3.15)

$$f_{12}=\int m_1 m_2 \frac{ds}{EI}$$

The general form is:

$$f_{rs}=\int m_r m_s \frac{ds}{EI} \tag{3.16}$$

The evaluation of Equation (3.16) requires the integration of the product of two bending moment distributions over the complete structure. Such distributions can generally be represented by simple geometrical figures such as rectangles, triangles and parabolas and standard results can be established in advance. Table 3.3 gives values of product integrals for a range of combinations of diagrams. It should be noted that in applying Equation (3.16) in this way, the flexural rigidity EI is assumed constant over the length of the diagram.

We may now use Table 3.3 to obtain values of the flexibility coefficients for the cantilever beam under consideration. Using Equations (3.14) and (3.15) with Figures 3.7(b) and (c) we obtain:

$$f_{11}=\frac{1}{3}\cdot\frac{l}{2}\cdot\frac{l}{2}\cdot\frac{l}{2}\cdot\frac{1}{EI}=\frac{l^3}{24EI}$$

$$f_{21}=\frac{1}{2}\cdot\frac{l}{2}\cdot 1\cdot\frac{l}{2}\cdot\frac{1}{EI}=\frac{l^2}{8EI}$$

$$f_{22}=l\cdot 1\cdot 1\cdot\frac{1}{EI}=\frac{l}{EI}$$

$$f_{12}=\frac{1}{2}\cdot\frac{l}{2}\cdot\frac{l}{2}\cdot 1\cdot\frac{1}{EI}=\frac{l^2}{8EI}$$

It is seen that f_{21} and f_{12} are numerically equal, a result which could be established using the Reciprocal Theorem. This is a useful property since in general $f_{rs}=f_{sr}$ and the effect is to reduce the number of separate calculations required. It should be further noted that whilst $f_{21}=f_{12}$, f_{21} is an angular displacement and f_{12} a linear displacement.

The evaluation of the flexibility coefficients f_{rs} provides the displacements at selected points in the structure due to unit values of the associated, redundant, forces. Before the compatibility conditions can be written down, it remains to calculate displacements (u) at corresponding positions due to the actual applied load. The basic equation (Equation 3.13) is applied once more. Now the bending moment distribution M is that due to the applied loads and we will re-designate this m_0. As before, $\partial M/\partial F_i=m_i$, and thus:

$$u_i=\int m_0 m_i\frac{ds}{EI} \tag{3.17}$$

The table of product integrals, Table 3.3, can be used for evaluating the u_i in the same way as the f_{rs}.

Table 3.3

Product integrals $\int_0^l m_r\, m_s\, ds$ (EI uniform)

m_s \ m_r	a, l	a, l	a, l, b
c, l	lac	$\frac{l}{2}ac$	$\frac{l}{2}(a+b)c$
c, l	$\frac{l}{2}ac$	$\frac{l}{3}ac$	$\frac{l}{6}(2a+b)c$
c, l	$\frac{l}{2}ac$	$\frac{l}{6}ac$	$\frac{l}{6}(a+2b)c$
c, l, d	$\frac{l}{2}a(c+d)$	$\frac{l}{6}a(2c+d)$	$\frac{l}{6}\{a(2c+d)+b(2d+c)\}$
c, l	$\frac{2}{3}lac$	$\frac{l}{3}ac$	$\frac{l}{3}(a+b)c$

In cases where the bending moment diagrams do not fit the standard values given in Table 3.3 or where a member has a stepped variation in EI, the member may be divided into segments such that the standard results can be applied and the total displacement obtained by addition. In cases where the standard results cannot be applied, e.g. a continuous variation in EI, the integration can be carried out conveniently by the use of Simpson's rule:

$$\int m_r m_s\frac{ds}{EI}\simeq\frac{a}{3}(h_1+4h_2+2h_3+4h_4+\ \ldots\ +h_n)$$

where a = width of strip

$$h_i=\frac{m_r m_s}{EI}\text{ at section i.}$$

In using Simpson's rule it should be remembered that the number of strips must be even, i.e. n must be odd.

3.3.2.1 Sign convention

A flexibility coefficient will be positive if the displacement it represents is in the same sense as the applied, unit, force. The bending moment expressions must carry signs based on the type of curvature developing in the structure. Since the integrand in Equation (3.16) is always the product of two bending moment expressions, it is only the relative sign which is of importance. A useful convention is to draw the diagrams on the tension (convex) sides of the members and then the relative signs of m_r and m_s can readily be seen. In Figure 3.7(b) and (c), both the m_1 and m_2 diagrams are drawn on the top side of the member. Their product is therefore positive. Naturally, the product of one diagram and itself will always be positive. This follows from simple physical reasoning since the displacement at a point due to an applied force at the same point will always be in the same sense as the applied force.

3.3.3 Application to beam and rigid frame analysis

The application of the theory will now be illustrated with two examples.

Example 3.2. Consider the three-span continuous beam shown in Figure 3.8(a). The beam is statically indeterminate to the second degree and we shall choose as redundants the internal bending moments at the interior supports B and C. The beam is made statically determinate by the introduction of moment releases at B and C as in Figure 3.8(b). We note that the application of the load W now produces displacements in span BC only, and in particular rotations u_1 and u_2 at B and C. The bending moment diagram (m_0) is shown in Figure 3.8(c).

We now apply unit value of x_1 and x_2 in turn. The deflected shapes and the flexibility coefficients, in the form of angular rotations, are shown at (d) and (e). The bending moment diagrams m_1 and m_2 are shown at (f) and (g).

Using the table of product integrals (Table 3.3), we find:

$$EIf_{11}=\frac{2}{3}l$$

$$EIf_{22}=\frac{2}{3}l$$

$$EIf_{12}=EIf_{21}=\frac{l}{6}$$

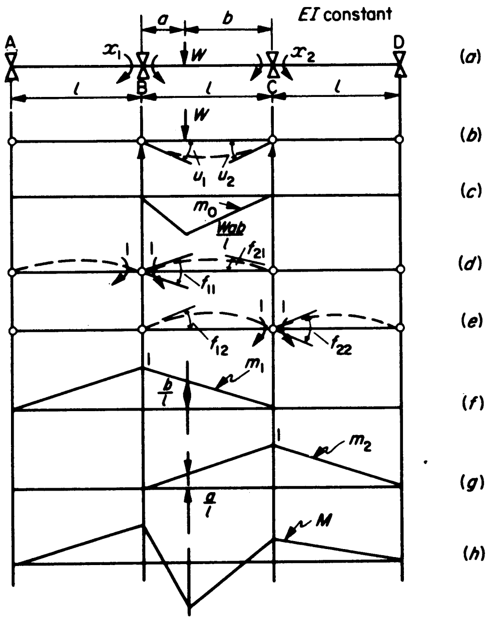


Figure 3.8 Flexibility analysis of continuous beam

$$EIu_1 = -\frac{a}{6}\left(1+\frac{2b}{l}\right)\frac{Wab}{l} - \frac{b}{3}\cdot\frac{b}{l}\cdot\frac{Wab}{l}$$

$$= -\frac{Wab}{6l}(a+2b)$$

and

$$EIu_2 = -\frac{Wab}{6l}(b+2a)$$

The required compatibility conditions are, for continuity of the beam:

at B, $f_{11}x_1 + f_{12}x_2 + u_1 = 0$
at C, $f_{21}x_1 + f_{22}x_2 + u_2 = 0$

or, in matrix form:

$$\mathbf{FX} + \mathbf{U} = \mathbf{0} \tag{3.18}$$

i.e.:

$$\frac{l}{6EI}\begin{bmatrix}4 & 1\\ 1 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{Wab}{16EIl}\begin{bmatrix}(a+2b)\\ (b+2a)\end{bmatrix}$$

and the solutions are:

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix} = \frac{Wab}{15l^2}\begin{bmatrix}(2a+7b)\\ (2b+7a)\end{bmatrix}$$

The actual bending moment distribution may now be determined by the addition of the three systems, i.e. the applied load and the two redundants. The general expression is:

$$M = m_0 + m_1x_1 + m_2x_2 \tag{3.19}$$

In particular:

$$M_B = x_1 = \frac{Wab}{15l^2}(2a+7b)$$

$$M_C = x_2 = \frac{Wab}{15l^2}(2b+7a)$$

and the bending moment under the load W is:

$$M_W = -\frac{Wab}{l} + \frac{b}{l}x_1 + \frac{a}{l}x_2$$

$$= -\frac{2Wab}{15l^3}(4l^2 + 5ab)$$

The final bending moment diagram is shown in Figure 3.8(h).

Example 3.3. A portal frame ABCD is shown in Figure 3.9(a). The frame has rigid joints at B and C, a fixed support at A and a hinged support at D. The flexural rigidity of the beam is twice that of the columns.

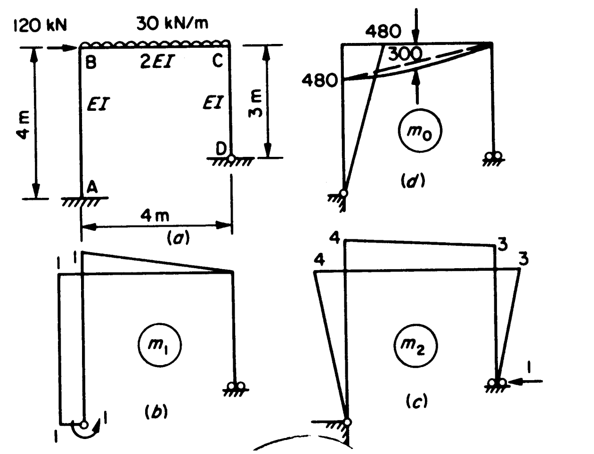


Figure 3.9

The frame has two redundancies and these are taken to be the fixing moment at A and the horizontal reaction at D. The bending moment diagrams corresponding to the unit redundancies, m_1 and m_2 and the applied load, m_0, are shown at (b), (c) and (d) in Figure 3.9.

Using the table of product integrals, Table 3.3, we obtain:

$$f_{11} = \int m_1^2\frac{ds}{EI} = \frac{14}{3EI}$$

$$f_{22} = \int m_2^2\frac{ds}{EI} = \frac{55}{EI}$$

$$f_{12} = f_{21} = \int m_1m_2\frac{ds}{EI} = \frac{35}{3EI}$$

$$u_1 = \int m_0m_1\frac{ds}{EI} = -\frac{1320}{EI}$$

$$u_2 = \int m_0m_2\frac{ds}{EI} = -\frac{4600}{EI}$$

Thus the compatibility equations are:

$$\frac{1}{3}\begin{bmatrix}14 & 35\\ 35 & 165\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = +\begin{bmatrix}1320\\ 4600\end{bmatrix}$$

from which

$$x_1 = +157\,\text{kNm}$$

and

$$x_2 = +\ \ 50\,\text{kN}$$

The bending moment at any point in the frame may now be determined from the expression:

$$M = m_0 + m_1x_1 + m_2x_2$$

e.g.:

$$M_{BA} = 480 - 1(+157) - 4(+50) = 123\,\text{kNm}$$

and

$$M_{CD} = 3x_2 = 150\,\text{kNm}$$

3.3.4 Application to truss analysis

The analysis of statically indeterminate trusses follows closely on that established for rigid frames; however, the problem is simplified due to the fact that for each system of loading investigated, the axial forces are constant within the lengths of the members and thus the integration is considerably simplified. We are now concerned with deformations in the members due to axial forces only and the flexibility coefficients are:

$$f_{rs} = \sum p_r\ p_s \frac{l}{AE} \qquad (3.20)$$

and

$$u_i = \sum p_0\ p_i \frac{l}{AE} \qquad (3.21)$$

in which the p_r system of forces is due to unit tension in the rth redundant member and similarly for p_s and p_i. The p_0 system of forces is that due to the applied load system acting on the statically determinate structure (i.e. with the redundant members omitted). Equations (3.20) and (3.21) should be compared with Equations (3.16) and (3.17) in the flexural case.

Example 3.4. The plane truss shown in Figure 3.10 has two redundancies which we will choose as the forces in members AE and EC. AE is constant for all the members and equal to 1×10^6 kN. The member EC is $l/10\,000$ short in manufacture and has to be forced into position. The member force systems p_0, p_1 and p_2 are found from a simple statical analysis and are listed in Table 3.4.

The flexibility coefficients may now be obtained as follows:

$$f_{11} = \sum p_1 p_1 \frac{l}{AE} = \frac{2l}{AE}(1+\surd 2)$$

$$f_{22} = f_{11}$$

$$f_{12} = f_{21} = \sum p_1 p_2 \frac{l}{AE} = \frac{l}{2AE}$$

$$u_1 = \sum p_1 p_0 \frac{l}{AE} = \frac{Wl}{AE}(1 + 1/\surd 2)$$

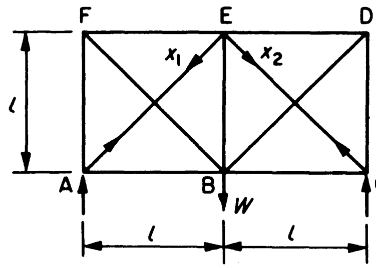


Figure 3.10

Table 3.4

Member	*Length*	p_0/w	p_1	p_2
AB	l	0	$-1/\surd 2$	0
BC	l	0	0	$-1/\surd 2$
CD	l	$-1/2$	0	$-1/\surd 2$
DE	l	$-1/2$	0	$-1/\surd 2$
EF	l	$-1/2$	$-1/\surd 2$	0
AF	l	$-1/2$	$-1/\surd 2$	0
FB	$\surd(2)l$	$1/\surd 2$	1	0
BE	l	0	$-1/\surd 2$	$-1/\surd 2$
BD	$\surd(2)l$	$1/\surd 2$	0	1
AE	$\surd(2)l$	0	1	0
EC	$\surd(2)l$	0	0	1

Ignoring, for the moment, the effect of the shortness in length of member EC, the compatibility equations are:

$$f_{11}x_1 + f_{12}x_2 + u_1 = 0$$

$$f_{21}x_1 + f_{22}x_2 + u_2 = 0$$

Clearly the symmetry will produce $x_1 = x_2$ and thus:

$$x_1 = x_2 = -W\frac{(2+\surd 2)}{(5+4\surd 2)}$$

The effect of the prestrain caused by the forced fit of member EC may be obtained by putting:

$$U = -\begin{bmatrix} 0 \\ 10^{-4}l \end{bmatrix} \qquad (3.22)$$

and then solving $\mathbf{FX} + \mathbf{U} = \mathbf{0}$ obtaining:

$$x_1 = \frac{-200}{(47+32\surd 2)}\,\text{kN}$$

$$x_2 = \frac{800(1+\surd 2)}{(47+32\surd 2)}\,\text{kN}$$

The forces in the other members may now be obtained from $p = p_0 + p_1x_1 + p_2x_2$.

The sign of the lack of fit in Equation (3.22) should be studied carefully and it should be noted that the convention for the signs of forces is tension-positive throughout.

3.3.5 Comments on the flexibility method

For a more detailed treatment of the flexibility method the reader may consult any of the standard texts, e.g. Jenkins[1] and

Coates, Coutie and Kong.[2] The method has declined in popularity in recent years due to the widespread adoption of computerized methods based on *stiffness* concepts. In the context of automatic computation, the stiffness method, which will be considered in the next section, offers considerable advantages over the flexibility method. Methods based on flexibility offer some advantage for hand computation in structures with low (1 or 2) degrees of statical indeterminacy or with lack of fit, temperature change or flexible supports. The concept of flexibility influence coefficients is also useful in determining stiffness coefficients, e.g. in nonprismatic members.

3.4 The stiffness method

3.4.1 Introduction

This method has been very extensively developed in recent years and now forms the basis of most structural analysis carried out on digital computers. The method of 'slope-deflection' is an example of the application of the general stiffness method.

Consider the structure shown in Figure 3.11(a) which is fixed at A and C and has a rigid joint at B. The degree of kinematical indeterminacy, from Equation (3.6), is:

$$n_k = 3(n-1) - c + r$$

$$= 3(3-1) - 5 + 0$$

$$= 1$$

The five constraints are the zero displacements, three at C and two at B, related to the fixed point A. The single unknown

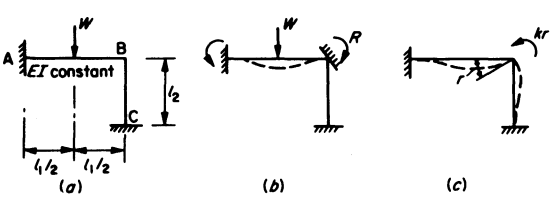


Figure 3.11 Basis of the stiffness method

displacement, nodal degree of freedom is, of course, the rotation of the joint B.

The procedure is to clamp the joint B so constraining the nodal degree of freedom r. On applying the load W, a constraining force, R, will be required at B to prevent the rotation of the joint. The constraining force R is now applied to the, otherwise unloaded, structure with its sign reversed and the nodal degree of freedom released. The result is a rotation of joint B through angle r. The external moment required to effect this rotation is kr where k is the stiffness of the structure for this particular displacement. Thus, for equilibrium:

$$kr = R \tag{3.23}$$

From the table of fixed-end moments, Table 3.5:

$$R = \frac{Wl_1}{8}$$

and from the force–displacement relationships of Equation (3.1)

$$k = \frac{4EI}{l_1} + \frac{4EI}{l_2}$$

Thus:

$$4EI\left(\frac{1}{l_1} + \frac{1}{l_2}\right) r = \frac{Wl_1}{8}$$

Hence:

$$r = \frac{Wl_1^2 l_2}{32EI(l_1 + l_2)}$$

The member forces are now obtained by adding the two systems (b) and (c) in Figure 3.11, e.g.:

$$M_{BA} = \frac{Wl_1}{8} - \frac{4EI(r)}{l_1} = \frac{Wl_1}{8}\left(1 - \frac{l_2}{l_1 + l_2}\right)$$

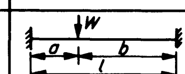
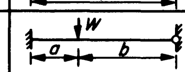
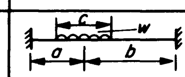
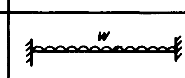
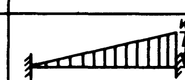
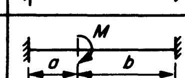
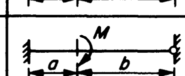
$$= \frac{Wl_1^2}{8(l_1 + l_2)}$$

and

$$M_{BC} = -\frac{4EI(r)}{l_2} = -\frac{Wl_1^2}{8(l_1 + l_2)}$$

Note that in the above, clockwise moments are considered positive.

Table 3.5 Fix-end moments for uniform beams (clockwise moments positive)

M_{FL}	Loading	M_{FR}
$-\frac{Wab}{l}\left(\frac{b}{l}\right)$		$\frac{Wab}{l}\left(\frac{a}{l}\right)$
$-\frac{Wab}{2l^2}(a+2b)$		0
$-\frac{wc}{12l^2}\left[12ab^2+c^2(a-2b)\right]$		$\frac{wc}{12l^2}\left[12a^2b+c^2(b-2a)\right]$
$-\frac{wl^2}{12}$		$\frac{wl^2}{12}$
$-\frac{wl^2}{30}$		$\frac{wl^2}{20}$
$-\frac{5}{96}wl^2$		$\frac{5}{96}wl^2$
$\frac{Mb}{l^2}(2a-b)$		$\frac{Ma}{l^2}(2b-a)$
$\frac{M}{2l^2}(l^2-3b^2)$		0

3.4.2 Member stiffness matrix

In the stiffness method, a structure is considered to be an assemblage of discrete elements, beams, columns, plates, etc. and the method requires a knowledge of the stiffness characteristics of the elements. In the 'finite element' method (see page 3/14) an artificial discretization of the structure is adopted. As an

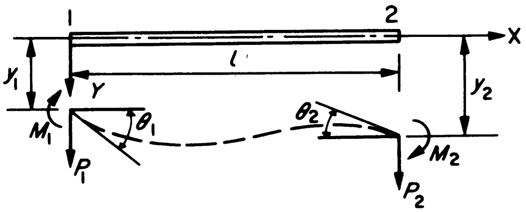


Figure 3.12 Structural beam element

example of the determination of stiffness influencing coefficients we shall consider the simple beam element shown in Figure 3.12. We neglect any axial deformation.

The expression for the bending moment in the beam with origin at end 1 and deflections y positive downwards is:

$$EI\mathrm{d}^2y/\mathrm{d}x^2 = P_1x - M_1$$

Integrating

$$EI\mathrm{d}y/\mathrm{d}x = \frac{P_1x^2}{2} - M_1x + C_1$$

$$= EI\theta_1 \text{ for } x=0$$

Hence:

$$C_1 = EI\theta_1$$

$$= EI\theta_2 \text{ for } x=l$$

Hence:

$$EI(\theta_2-\theta_1) = \frac{P_1l^2}{2} - M_1l \tag{3.24}$$

Integrating again:

$$EIy = \frac{P_1x^3}{6} - M_1\frac{x^2}{2} + EI\theta_1x + C_2$$

$$= EIy_1 \text{ for } x=0$$

Therefore:

$$C_2 = EIy_1$$

$$= EIy_2 \text{ for } x=l$$

Hence:

$$EI(y_2-y_1) - EI\theta_1 l = P_1\frac{l^3}{6} - M_1\frac{l^2}{2} \tag{3.25}$$

Solving equations (3.24) and (3.25) for M_1 and P_1:

$$M_1 = \frac{4EI\theta_1}{l} + \frac{6EIy_1}{l^2} + \frac{2EI\theta_2}{l} - \frac{6EIy_2}{l^2} \tag{3.26}$$

and

$$P_1 = \frac{6EI\theta_1}{l^2} + \frac{12EIy_1}{l^3} + \frac{6EI\theta_2}{l^2} - \frac{12EIy_2}{l^3} \tag{3.27}$$

Two further relationships between the forces and displacements are obtained from statical equilibrium as follows:

For vertical equilibrium, $P_1 + P_2 = 0$

Hence:

$$P_2 = -P_1 \tag{3.28}$$

Taking moments about end 1:

$$M_2 = -M_1 - P_2l$$

$$= \frac{2EI\theta_1}{l} + \frac{6EIy_1}{l^2} + \frac{4EI\theta_2}{l} - \frac{6EIy_2}{l^2} \tag{3.29}$$

Equations (3.26)–(3.29) may be combined in the matrix form:

$$\begin{bmatrix} M_1 \\ P_1 \\ M_2 \\ P_2 \end{bmatrix} = \frac{EI}{l^3}\begin{bmatrix} 4l^2 & 6l & 2l^2 & -6l \\ 6l & 12 & 6l & -12 \\ 2l^2 & 6l & 4l^2 & -6l \\ -6l & -12 & -6l & 12 \end{bmatrix}\begin{bmatrix} \theta_1 \\ y_1 \\ \theta_2 \\ y_2 \end{bmatrix}$$

$$\text{or } \mathbf{S} = \mathbf{k}\boldsymbol{\Delta} \tag{3.30}$$

The matrix **k** is the stiffness matrix of the beam, and **S** and **Δ** are the matrices of member forces and nodal displacements respectively. Equation (3.30) expresses the force–displacement relationships for the beam in the *stiffness* form as distinct from the *flexibility* form. The symmetry of the matrix should be noted as consistent with the symmetry exhibited by flexibility coefficients (see page 3/9).

3.4.3 Assembly of structure stiffness matrix

The stiffness method involves the solution of a set of linear simultaneous equations, representing equilibrium conditions, which may be expressed in the form:

$$\mathbf{Kr} = \mathbf{R} \tag{3.31}$$

Equation (3.31) is similar in form to Equation (3.23) with the important difference that now we are concerned with a multiple degree of freedom system as distinct from a single unknown displacement. **K** is the structure stiffness matrix, **r** is a matrix of nodal displacements and **R** a matrix of applied nodal forces.

The process of assembling the matrix **K** is one of transferring individual element stiffnesses into appropriate positions in the matrix **K**. Naturally, this has been the subject of considerable organization for digital computer analysis and the subject is well documented.[3] Some aspects of a computerized approach will be considered later but the basic process will be illustrated here using a simple example. Consider the structure shown in Figure 3.13(a). The two beams are rigidly connected together at B where there is a spring support with stiffness k_s. End A is hinged and end C fixed. The structure has three degrees of freedom, rotations r_1 and r_3 at A and B and a vertical displacement r_2 at B. The stiffness matrix for each beam has the form of Equation (3.30) from which **k** may be written in the general form:

$$\mathbf{k} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \tag{3.32}$$

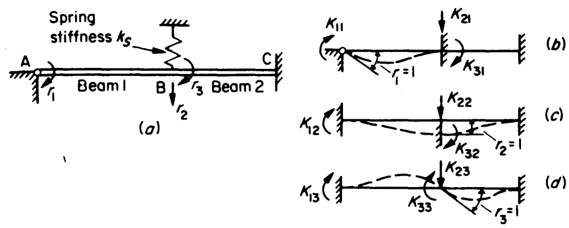


Figure 3.13

where $k_{11}=4EI/l$; $k_{12}=6EI/l^2$, etc.

We apply unit value of each degree of freedom in turn as shown in Figure 3.13(b), (c) and (d). It should be noted that when $r_1=1$ is applied, r_2 and r_3 are constrained at zero value and similarly with $r_2=1$ and $r_3=1$. The force systems necessary to achieve the unit values of the degrees of freedom are also shown at (b), (c) and (d). The equilibrium conditions are clearly:

$$K_{11}r_1+K_{12}r_2+K_{13}r_3=R_1$$

$$K_{21}r_1+K_{22}r_2+K_{23}r_3=R_2$$

$$K_{31}r_1+K_{32}r_2+K_{33}r_3=R_3$$

i.e. $\mathbf{Kr}=\mathbf{R}$

where $\mathbf{R}$ is the matrix of applied loads. Clearly, the forces shown in Figure 3.13(b), (c) and (d) constitute the elements of the stiffness matrix $\mathbf{K}$ and this may now be assembled by inspection. Using the individual beam elements from Equation (3.30) with the notation of Equation (3.32):

$$\mathbf{K}=\begin{array}{|c|c|c|}\hline (k_{11})_1 & -(k_{12})_1 & (k_{13})_1 \\ \hline -(k_{12})_1 & (k_{44})_1+(k_{22})_2+k_s & (k_{23})_2-(k_{14})_1 \\ \hline (k_{13})_1 & (k_{23})_2-(k_{14})_1 & (k_{33})_1+(k_{11})_2 \\ \hline \end{array} \tag{3.33}$$

and more specifically:

$$\mathbf{K}=\begin{array}{|c|c|c|}\hline 4\left(\frac{EI}{l}\right)_1 & -6\left(\frac{EI}{l^2}\right)_1 & 2\left(\frac{EI}{l}\right)_1 \\ \hline -6\left(\frac{EI}{l^2}\right)_1 & 12\left(\frac{EI}{l^3}\right)_1+12\left(\frac{EI}{l^3}\right)_2+k_s & 6\left(\frac{EI}{l^2}\right)_2-6\left(\frac{EI}{l^2}\right)_1 \\ \hline 2\left(\frac{EI}{l}\right)_1 & 6\left(\frac{EI}{l^2}\right)_2-6\left(\frac{EI}{l^2}\right)_1 & 4\left(\frac{EI}{l}\right)_1+4\left(\frac{EI}{l}\right)_2 \\ \hline \end{array} \tag{3.34}$$

3.4.4 Stiffness transformations

The member stiffness matrix $\mathbf{k}$ in Equation (3.30) is based on a coordinate system which is convenient for the member, i.e. origin at one end and X-axis directed along the axis of the beam. Such a coordinate system is termed 'local' as distinct from the 'global' coordinate system which is used for the complete structure. This subject is considered in detail in a number of texts[2,3] and we shall give only a brief indication of the type of computation required.

Consider a three-dimensional coordinate system $\bar{X}\bar{Y}\bar{Z}$ (global) which is obtained by rotation of the (local) coordinate system XYZ. In the local system the force–displacement relationships for a beam element may be expressed in the partitioned matrix form:

$$\begin{bmatrix}\mathbf{S}_1\\ \mathbf{S}_2\end{bmatrix}=\begin{bmatrix}\mathbf{k}_{11} & \mathbf{k}_{12}\\ \mathbf{k}_{21} & \mathbf{k}_{22}\end{bmatrix}\begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix} \tag{3.35}$$

in which the subscripts refer to ends 1 and 2.

The stiffness expressed in the coordinate system $\bar{X}\bar{Y}\bar{Z}$ may be obtained as follows:

$$\begin{bmatrix}\bar{\mathbf{S}}_1\\ \bar{\mathbf{S}}_2\end{bmatrix}=\begin{bmatrix}\lambda\mathbf{k}_{11}\lambda^T & \lambda\mathbf{k}_{12}\lambda^T\\ \lambda\mathbf{k}_{21}\lambda^T & \lambda\mathbf{k}_{22}\lambda^T\end{bmatrix}\begin{bmatrix}\mathbf{r}_1\\ \mathbf{r}_2\end{bmatrix} \tag{3.36}$$

in which λ is a matrix of direction cosines as follows:

$$\boldsymbol{\lambda}=\begin{bmatrix}\lambda_{\bar{x}x} & \lambda_{\bar{x}y} & \lambda_{\bar{x}z} & 0 & 0 & 0\\ \lambda_{\bar{y}x} & \lambda_{\bar{y}y} & \lambda_{\bar{y}z} & 0 & 0 & 0\\ \lambda_{\bar{z}x} & \lambda_{\bar{z}y} & \lambda_{\bar{z}z} & 0 & 0 & 0\\ 0 & 0 & 0 & \lambda_{\bar{x}x} & \lambda_{\bar{x}y} & \lambda_{\bar{x}z}\\ 0 & 0 & 0 & \lambda_{\bar{y}x} & \lambda_{\bar{y}y} & \lambda_{\bar{y}z}\\ 0 & 0 & 0 & \lambda_{\bar{z}x} & \lambda_{\bar{z}y} & \lambda_{\bar{z}z}\end{bmatrix} \tag{3.37}$$

where $\lambda_{\bar{x}x}=\cos \bar{X}OX$, etc.

3.4.5 Some aspects of computerization of the stiffness method

The remarkable increase in popularity of the stiffness method is due to the widespread availability of relatively cheap computing power. The method is of limited practical use *except* on computers. The stiffness method is eminently suitable for computers because the setting up of the data describing the structure and loading system to be analysed is a comparatively simple operation. Although there is then generally considerable numerical computation to do, this is done by the computer. Thus the human effort required is minimized and the likelihood of errors being made also reduced. With the phenomenal development of cheap and powerful microcomputers, which are quite suitable for analysing most 'run-of-the-mill' structures, it is quite likely that in the very near future almost all structural analysis will be carried out on computers.

It will be useful to look briefly at the more important aspects of adapting the stiffness method for use on computers. The method may be viewed as a succession of six stages:

(1) Define the nodal degrees of freedom of the structure (n) (Equation (3.6)), the nodal 'coordinates'. The total number determines the size of the structure stiffness matrix $\mathbf{K}$. The ordering is a matter of convenience but in some programs a judicial ordering of coordinates is necessary to reduce the 'band width' of $\mathbf{K}$. An array $\mathbf{K}$ ($n\times n$) is now generated in the computer and all elements are zeroed. This is necessary since component stiffnesses are going to be added-in to this array thus 'accumulating' the stiffnesses element by element.
(2) The individual structural elements are now defined and their force–displacement relationships expressed in stiffness matrices, $\mathbf{k}$ (Equation (3.30)); $\mathbf{S}=\mathbf{k\Delta}$. The dimensions of these matrices will depend on the type of element used but for most of the common elements (beam, column, pin-jointed truss member, etc.) the standard matrices are pub-

lished in the textbooks. The element stiffnesses are now transformed from local to global coordinates using matrix transformations as in Equation (3.36).

(3) The transformed stiffnesses are now transferred into appropriate locations of the structure stiffness matrix **K**. Suppose we are to transfer the stiffnesses of a particular element and suppose this element has two coordinates numbered 1 and 2. If the coordinates in the actual structure which correspond to 1 and 2 of the element are, say, *i* and *j* then the transfer of stiffnesses is carried out as follows:

$$\mathbf{k}_{11}\rightarrow\mathbf{k}_{ii}$$
$$\mathbf{k}_{12}\rightarrow\mathbf{k}_{ij}$$
$$\mathbf{k}_{21}\rightarrow\mathbf{k}_{ji}$$
$$\mathbf{k}_{22}\rightarrow\mathbf{k}_{jj}$$

There is considerable economy in organization and programming if the above procedure is applied to 'groups' of coordinates, e.g. *all* the displacements at one node. This can be achieved by *partitioning* the element stiffness matrices.

(4) Once **K** has been set up, the applied load matrix **R** is generated. This is simply a column matrix containing the applied (nodal) loads arranged in the same order as the nodal coordinates. If the structure is carrying loads other than at the defined nodes, then such loads must be converted to statically equivalent nodal loads. In rigid frames, for example, this is easily done using the standard values of 'fixed-end' effects. If a concentrated load does not coincide with the defined nodal coordinates then it is a simple matter, as an alternative, to introduce a node at the load point. This procedure, although it increases the size of the system to be solved, does have the advantage of yielding the displacements developing at the load point.

(5) The computer now solves the linear simultaneous equations (Equation (3.31)) $\mathbf{Kr}=\mathbf{R}$ to produce the nodal displacements **r**.

(6) Lastly, the element forces are obtained from Equation (3.30) $\mathbf{S}=\mathbf{k}\Delta$. In this last operation, some logical organization is clearly needed to extract the element nodal displacements Δ from the structure displacement **Sr**.

The foregoing is a description of the fundamental basis of the stiffness method applied on computers. Of course, it is possible to incorporate many refinements and devices to simplify the input and output, to check the results and to make changes in data without having to re-input all data.

In its most general form the stiffness method is used to analyse complex structures in which not only simple elements such as beams and columns are used but 'continua' such as plates and shells. This is the 'finite element' method which will now be examined briefly.

3.4.6 Finite element analysis

This extremely powerful method of analysis has been developed in recent years and is now an established method with wide applications in structural analysis and in other fields. Space permits only the most brief introduction here but the method is extensively documented elsewhere.[4-6] We have discussed the application of the stiffness method to framed structures in which the structural elements, beams and columns, have been connected at the nodes and the method observes the correct conditions of displacement compatibility and equilibrium at the nodes. The finite element method was developed, originally, in order to extend the stiffness method to the analysis of elastic continua such as plates and shells and indeed to three-dimensional continua. The first step in the process is to divide the structure into a finite number of discrete parts called 'elements'. The elements may be of any convenient shape, e.g. a thin plate may be represented by triangular or rectangular elements, and the discretization may be *coarse*, with a small number of elements, or *fine*, with a large number of elements. The connection between elements now occurs not only at the nodal points but along boundary lines and over boundary faces.

The procedure ensures, as for framed structures, that equilibrium and compatibility conditions are satisfied at the nodes but the regions of connection between nodes are constrained to adopt a chosen form of displacement function. Thus, compatibility conditions along the interfaces between elements may not be completely satisfied and a degree of approximation is generally introduced. Once the geometry of the elements has been determined and the displacement function defined, the stiffness matrix of each element, relating nodal forces to nodal displacements, can be obtained. The remainder of the structural analysis follows the established procedures similar to those for framed structures. Naturally the best choice of element and discretization pattern, the precise conditions occurring at the interfaces and the accuracy of the solution, are matters which have received a great deal of attention in the literature.

A central stage in the process is the adoption of a suitable displacement function for the element chosen, and the subsequent evaluation of the element stiffnesses. This will be illustrated with one of the simplest possible elements, a triangular plate element for use in a plane stress situation.

3.4.6.1 Triangular element for plant stress

A triangular element ijk is shown in Figure 3.14. Under load, the displacement of any point within the element is defined by the displacement components *u*, *v*. In particular the nodal displacements are:

$$\Delta=\{u_i u_j u_k v_i v_j v_k\} \tag{3.38}$$

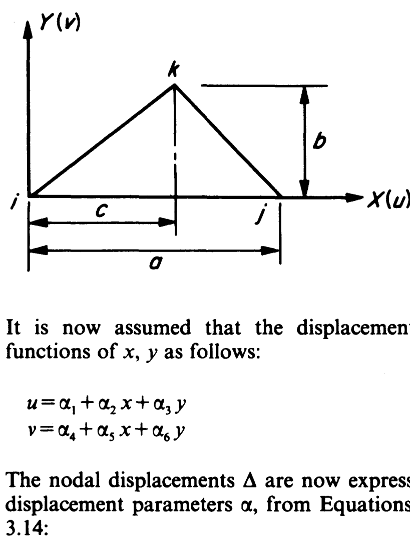


Figure 3.14

It is now assumed that the displacements *u*, *v* are linear functions of *x*, *y* as follows:

$$\begin{aligned} u&=\alpha_1+\alpha_2 x+\alpha_3 y\\ v&=\alpha_4+\alpha_5 x+\alpha_6 y \end{aligned} \tag{3.39}$$

The nodal displacements Δ are now expressed in terms of the displacement parameters α, from Equations (3.39) and Figure 3.14:

$$\begin{bmatrix} u_i\\ u_j\\ u_k\\ v_i\\ v_j\\ v_k \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 1 & a & 0 & 0 & 0 & 0\\ 1 & c & b & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & a & 0\\ 0 & 0 & 0 & 1 & c & b \end{bmatrix} \begin{bmatrix} \alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6 \end{bmatrix} \tag{3.40}$$

or, $\Delta=\mathbf{A}\boldsymbol{\alpha}$

The strains in the element are functions of the derivatives of u and v as follows:

$$\boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \partial u/\partial x \\ \partial v/\partial y \\ \partial u/\partial y + \partial v/\partial x \end{bmatrix} \tag{3.41}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{bmatrix} \tag{3.42}$$

i.e.:

$$\boldsymbol{\varepsilon} = \mathbf{B}\boldsymbol{\alpha} = \mathbf{B}\mathbf{A}^{-1}\boldsymbol{\Delta} \tag{3.43}$$

from Equation (3.40).

It should be noted that the matrix **B** in Equation (3.42) contains only constant terms and it follows that the strains are constant within the element.

The stress–strain relationships for plane stress in an isotropic material with Poisson's ratio v and Young's modulus E are:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \frac{E}{(1-v^2)} \begin{bmatrix} 1 & v & 0 \\ v & 1 & 0 \\ 0 & 0 & \frac{1}{2}(1-v) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

i.e.:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\mathbf{B}\mathbf{A}^{-1}\boldsymbol{\Delta} \tag{3.44}$$

Matrix **D** is the 'elasticity' matrix relating stress and strain. To obtain the element stiffness we employ the principle of virtual work and apply arbitrary nodal displacements $\bar{\boldsymbol{\Delta}}$ producing virtual strains in the element:

$$\bar{\boldsymbol{\varepsilon}} = \mathbf{B}\mathbf{A}^{-1}\bar{\boldsymbol{\Delta}} \tag{3.45}$$

The virtual strain energy in the element, from Equation (2.78) of Chapter 2, is:

$$\int_{vol} \bar{\boldsymbol{\varepsilon}}^T \boldsymbol{\sigma} \, dV$$

where V = volume of triangular element = $tab/2$, t = thickness

Substituting for $\bar{\boldsymbol{\varepsilon}}^T$ and $\boldsymbol{\sigma}$ from Equations (3.45) and (3.44) respectively, the virtual strain energy is:

$$\int_{vol} [\mathbf{B}\mathbf{A}^{-1}\bar{\boldsymbol{\Delta}}]^T \mathbf{D}\mathbf{B}\mathbf{A}^{-1}\boldsymbol{\Delta} \, dV$$

Now since all the matrices contain constant terms only and are thus independent of x and y, the expression for the virtual strain energy may be written:

$$\bar{\boldsymbol{\Delta}}^T\{[\mathbf{A}^{-1}]^T \mathbf{B}^T \mathbf{D}\mathbf{B}\mathbf{A}^{-1} V\}\boldsymbol{\Delta}$$

The external work is the product of the virtual displacements $\bar{\boldsymbol{\Delta}}$ and the nodal forces **S**, hence equating external virtual work and internal virtual strain energy:

$$\bar{\boldsymbol{\Delta}}^T \mathbf{S} = \bar{\boldsymbol{\Delta}}^T\{[\mathbf{A}^{-1}]^T \mathbf{B}^T \mathbf{D}\mathbf{B}\mathbf{A}^{-1} V\}\boldsymbol{\Delta}$$

The virtual displacements are quite arbitrary and in particular may be taken to be represented by a unit matrix, thus:

$$\begin{aligned} \mathbf{S} &= \{[\mathbf{A}^{-1}]^T \mathbf{B}^T \mathbf{D}\mathbf{B}\mathbf{A}^{-1} V\}\boldsymbol{\Delta} \\ &= \mathbf{k}\boldsymbol{\Delta}, \text{ from Equation (3.30)} \end{aligned}$$

Thus:

$$\mathbf{k} = [\mathbf{A}^{-1}]^T \mathbf{B}^T \mathbf{D}\mathbf{B}\mathbf{A}^{-1} V \tag{3.46}$$

Before the matrix multiplications required in Equation (3.46) can be performed we need to find $\mathbf{A}^{-1}$. This is easily determined as:

$$\mathbf{A}^{-1} = \frac{1}{ab} \begin{bmatrix} ab & 0 & 0 & 0 & 0 & 0 \\ -b & b & 0 & 0 & 0 & 0 \\ (c-a) & -c & a & 0 & 0 & 0 \\ 0 & 0 & 0 & ab & 0 & 0 \\ 0 & 0 & 0 & -b & b & 0 \\ 0 & 0 & 0 & (c-a) & -c & a \end{bmatrix}$$

Hence finally, with $\lambda_1 = \frac{1}{2}(1-v)$ and $\lambda_2 = \frac{1}{2}(1+v)$ we obtain the stiffness matrix for the plane stress triangular element as shown in equation (3.47) below.

It is neither necessary nor economical to carry out these operations by hand; the computation of the element stiffness and, indeed, the entire computational process is easily programmed for the digital computer.

Computer 'packages' for finite element analysis of structures are highly developed, very powerful and readily available. Because of the comparatively heavy demands on computer storage, the use of the packages is generally confined to mainframe computers. A good example of a finite element system which is used very extensively is PAFEC.[6] The more important topics which should be studied in pursuing finite element analysis include: (1) shape (displacement) functions; (2) conforming and nonconforming elements; (3) isoparametric elements; and (4) automatic mesh generation.

$$\mathbf{k} = \frac{Et}{2(1-v^2)ab} \begin{bmatrix} b^2+\lambda_1(c-a)^2 & & & & & \\ -b^2-\lambda_1 c(c-a) & b^2+\lambda_1 c^2 & & & \text{Symmetric} & \\ \lambda_1 a(c-a) & -\lambda_1 ac & \lambda_1 a^2 & & & \\ -\lambda_2 b(c-a) & \lambda_1 cb + vb(c-a) & -\lambda_1 ab & \lambda_1 b^2+(c-a)^2 & & \\ \lambda_1 b(c-a)+vcb & -\lambda_2 bc & \lambda_1 ab & -\lambda_1 b^2 - c(c-a) & \lambda_1 b^2+c^2 & \\ -vab & vab & 0 & a(c-a) & -ac & a^2 \end{bmatrix} \tag{3.47}$$

3.5 Moment distribution

3.5.1 Introduction

Although the stiffness method, described in the previous section has the merit of simplicity, the solution of the equilibrium equations (3.31) is generally a matter for the digital computer since only for the simplest structures can a hand solution be contemplated. An alternative procedure which is eminently suitable for hand computation is the method of moment distribution which is essentially an iterative solution of the equations of equilibrium.

As in the general stiffness method, we first imagine all the degrees of freedom, joint rotations and joint translations, to be constrained. We ignore axial effects in members and consider flexure only. The constraints are imagined to be clamps applied to the joints to prevent rotation and translation. The forces required to effect the constraints are applied artificially and in the moment distribution processes these clamping forces are systematically released so as to allow the structure to achieve an equilibrium state. It is important to note that in the method as generally applied, the rotational joint restraints are relaxed by one process and the translational restraints by another. Finally the principle of superposition is used to combine the separate results.

It is necessary to assemble certain standard results before we can consider the actual process.

3.5.2 Distribution factors, carry-over factors and fixed-end moments

For the time being we confine our attention to prismatic members. The treatment of nonuniform section members will be touched on later.

Standard member stiffnesses are required and these are illustrated in Figure 3.15. The member end forces are those required to produce the deflected forms shown. Diagrams (a) and (b) relate to rotation without translation (sway), and diagrams (c) and (d) relate to sway without rotation. The results in diagrams (a) and (c) may be deduced from the stiffness matrix in Equation (3.30). The other results may be obtained easily from elementary beam theory, e.g. in Figure 3.15(b), taking the origin of x at the left-hand end and y positive downwards:

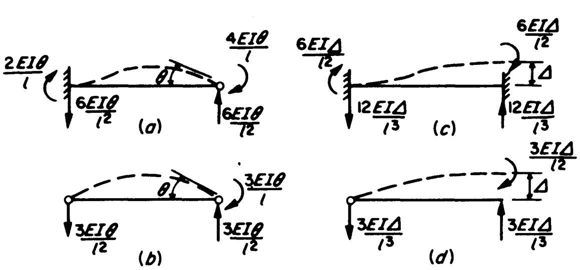


Figure 3.15

$EId^2y/dx^2=\dfrac{Mx}{l}$, where M is the moment, to be determined, at the right-hand end,

$$EIdy/dx=\frac{M}{l}\frac{x^2}{2}+C_1$$

$$=EI\theta \text{ for } x=l;\text{ hence } C_1=EI\theta-M\frac{l}{2}$$

$$EIy=\frac{M}{l}\frac{x^3}{6}+\left(EI\theta-M\frac{l}{2}\right)x+C_2$$

$$=0 \text{ for } x=0;\text{ hence } C_2=0$$

$$=0 \text{ for } x=l;\text{ hence, } M=\frac{3EI\theta}{l}$$

When loads are applied to members which are constrained at the joints, fixed-end moments are required to prevent the end rotations. This is another standard type of result which is required in the moment distribution method. Table 3.5 lists fixed-end moments for a selection of loading cases on uniform section beams. Again, these results may be obtained from elementary beam theory. It should be noted that the sign convention is that a moment is *positive* if tending to produce clockwise rotation of the end of the member at which it acts. This convention is different to, and should not be confused with, the sign convention for constructing bending moment diagrams which must be based on the curvature produced in the member.

As an illustration of the basic process, consider the structure ABC shown in Figure 3.11. This structure was analysed by the stiffness method previously. Joint B is considered to be clamped and thus a system of fixed-end moments is set up in member AB. The end moments in the members are shown in line 1 of Table 3.6. The constraining moment at joint B is seen to be $Wl_1/8$ clockwise and we imagine this moment to be removed by the application of a moment $-Wl_1/8$. The subsequent rotation of joint B, anticlockwise through angle θ, will develop moments in both members. Referring to Figure 3.15 the moments induced will be:

$$M_{BA}=-\frac{4EI\theta}{l_1};\ M_{AB}=-\frac{2EI\theta}{l_1}$$

$$M_{BC}=-\frac{4EI\theta}{l_2};\ M_{CB}=-\frac{2EI\theta}{l_2}$$

For equilibrium of joint B, the applied moment $-Wl_1/8$ must equal the sum of the moments absorbed by the two members meeting at the joint:

$$-\frac{Wl_1}{8}=-\frac{4EI\theta}{l_1}-\frac{4EI\theta}{l_2}=-4EI\theta\left(\frac{I}{l_1}+\frac{I}{l_2}\right)$$

and it is seen that the moment is 'distributed' to the members in proportion to their I/l values.

Thus:

$$M_{BA}=\frac{-Wl_1}{8}\frac{I/l_1}{(I/l_1+I/l_2)}=\frac{-Wl_1}{8}\left(\frac{l_2}{l_1+l_2}\right)$$

and:

$$M_{BC}=\frac{-Wl_1}{8}\frac{I/l_2}{(I/l_1+I/l_2)}=\frac{-Wl_1}{8}\left(\frac{l_1}{l_1+l_2}\right)$$

The moments induced at A and C are from Figure 3.15, one-half of those induced at B and the factor of one-half is termed the *carry over* factor. This set of moments is shown in line 2 of Table 3.6.

Joint B is now 'in balance' and since it was the only joint which was clamped we have reached an equilibrium state and no further distribution of moments is required. The final set of

Table 3.6

Stage	*Operation*	M_{AB}	M_{BA}	M_{BC}	M_{CB}
1	Fixed-end moments	$-Wl_1/8$	$+Wl_1/8$	0	0
2	Distribution at B	$-\frac{Wl_1}{16}\left(\frac{l_2}{l_1+l_2}\right)$	$-\frac{Wl_1}{8}\left(\frac{l_2}{l_1+l_2}\right)$	$-\frac{Wl_1}{8}\left(\frac{l_1}{l_1+l_2}\right)$	$-\frac{Wl_1}{16}\left(\frac{l_1}{l_1+l_2}\right)$
3	Total moments	$-\frac{Wl_1}{16}\left(\frac{2l_1+3l_2}{l_1+l_2}\right)$	$\frac{Wl_1^2}{8(l_1+l_2)}$	$-\frac{Wl_1^2}{8(l_1+l_2)}$	$-\frac{Wl_1^2}{16(l_1+l_2)}$

moments is obtained in line 3 of Table 3.6, by the addition of lines 1 and 2. This result is the same as that obtained from pure stiffness considerations. It should be noted that the zero sum of moments M_{BA} and M_{BC} indicates that joint B is in rotational equilibrium.

Two further points should be noted before we consider the moment distribution process in more detail. Referring to Figure 3.16, of the three members connected at joint A, member AD is hinged at the end remote from A whereas the other two members are fixed. Since D is hinged no moment can exist there and hence there is no carry-over to D. Furthermore, the moment–rotation relationship is different for a member pinned at the remote end, as may be seen by comparing Figures 3.15(a) and (b). In calculating distribution factors this is taken account of by taking $\frac{3}{4}(I/l)$ for such members as compared with I/l for members fixed at the remote end.

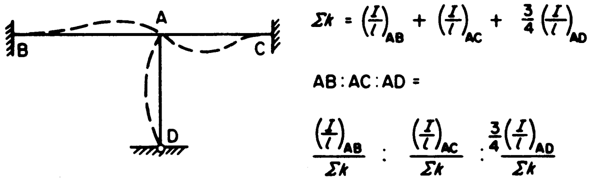


Figure 3.16 Distribution factors at typical joint

3.5.3 Moment distribution without sway

As an example of a structure with two degrees of freedom of joint rotation and no sway, consider the frame shown in Figure 3.17, EI (beams) = 3 × EI (columns).

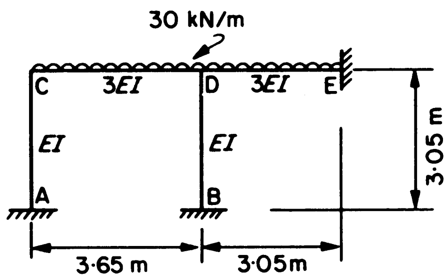


Figure 3.17

Table 3.7 Moment distribution for frame shown in Figure 3.17

	Joint	*A*	*C*		*D*			*B*	*E*
	Distribution factors		0.285	0.715	0.386	0.154	0.460		
	end moments	AC	CA	CD	DC	DB	DE	BD	ED
(1)	Fixed-end moments			−33.3	+33.3		−23.3		+23.3
(2)	Distribution at C		+9.5	+23.8					
(3)	Carry-over to A and D	+4.75			+11.9				
(4)	Distribution at D				−8.45	−3.38	−10.07		
(5)	Carry-over to C, B and E			−4.23				−1.69	−5.04
(6)	Distribution at C		+1.20	+3.03					
(7)	Carry-over to A and D	+0.60			+1.52				
(8)	Distribution at D				−0.59	−0.23	−0.70		
(9)	Carry-over to C, B and E			−0.30				−0.12	−0.35
(10)	Distribution at C		+0.09	+0.21					
(11)	Carry-over to A and D	+0.05			+0.11				
(12)	Distribution at D				−0.04	−0.02	−0.05		
(13)	Carry-over to C, B and E				May be neglected				
(14)	Total moments (kNm)	+5.40	+10.79	−10.79	+37.75	−3.63	−34.12	−1.81	+17.91

The fixed-end moments are, $(wl^2/12)$,

$$M_{FCD} = -30 \times \frac{3.65^2}{12};\ M_{FDC} = +30 \times \frac{3.65^2}{12} = 33.3\ \text{kNm}$$

$$F_{FDE} = -30 \times \frac{3.05^2}{12};\ M_{FED} = +30 \times \frac{3.05^2}{12} = 23.3\ \text{kNm}$$

and the distribution factors are:

$$\text{at C, CD:CA} = \frac{3/3.65}{(1/3.05)+(3/3.65)} : \frac{1/3.05}{(1/3.05)+(3/3.65)} = 0.715:0.285$$

at D, DC:DB:DE =

$$\frac{3/3.65}{(3/3.65)+(1/3.05)+(3/3.05)} : \frac{1/3.05}{(3/3.65)+(1/3.05)+(3/3.05)} : \frac{3/3.05}{(3/3.65)+(1/3.05)+(3/3.05)}$$

$$= 0.386:0.154:0.460$$

The moment distribution is carried out in Table 3.7. It should be noted that after each distribution at a joint the distributed moments are underlined to indicate that the joint is balanced at that stage. At step 4, the out-of-balance moment to be distributed at D is $+33.3+11.9-23.3=+21.9$; hence the distributed moments should total -21.9.

3.5.4 Moment distribution with sway

This process will be illustrated with reference to Example 3.3 (page 3/9), for which the structure is shown in Figure 3.9. We first ignore any horizontal movement (sway) of the joints B and C and carry out a moment distribution.

The fixed-end moments are $wl^2/12 = \pm 40$ kNm; and the distribution factors are:

$$\text{BA:BC} = \tfrac{1}{3}:\tfrac{2}{3}$$

$$\text{CB:CD} = \tfrac{2}{3}:\tfrac{1}{3} \text{ (noting } \tfrac{3}{4}I/l \text{ for CD)}$$

The result of this (no sway) moment distribution is given in line 3 of Table 3.8. We now consider the horizontal equilibrium of the beam BC, Figure 3.18(a), and find that a force F_1 is required to maintain equilibrium. F_1 may be calculated by evaluating the horizontal shear forces at the tops of the columns as follows:

$$F_1 = 120 + \frac{(20+10)}{4} - \frac{20}{3} = 120.8\ \text{kN}$$

This force cannot exist in practice and what happens is that the beam BC deflects to the right and a new set of bending moments is set up with the effect that the out-of-balance horizontal force F_1 is removed. We consider the effect of this sway separately. Referring to Figure 3.18(b), a movement to the right of Δ, without joint rotation, requires column moments as shown. From Figure 3.15(c) and (d), these column moments are,

$$M_{FBA} = M_{FAB} = -6\left(\frac{EI}{l^2}\right)\Delta_{AB}$$

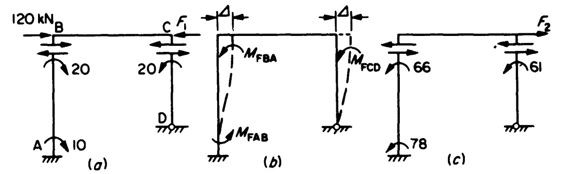


Figure 3.18

$$M_{FCD} = -3\left(\frac{EI}{l^2}\right)\Delta_{CD} \quad (\text{note } M_{FDC}=0)$$

We cannot evaluate these moments unless Δ is known but we could proceed with an arbitrary value of Δ, and carry out a distribution to produce rotational equilibrium of the joints B and C. In fact, it is seen that any arbitrary values of moments can be used providing these are in the correct proportions between the two columns. The ratio in this example is:

$$\text{AB:CD} = \left(\frac{I}{l^2}\right)_{AB} : \frac{1}{2}\left(\frac{I}{l^2}\right)_{CD}$$

If we adopt

$$M_{FBA} = M_{FAB} = -90$$

and

$$M_{FCD} = -80$$

the moments are in the correct proportion. A second moment distribution is now carried out, using these values of fixed-end moments, and the result is shown in line 1 of Table 3.8. This set of moments is consistent with an applied horizontal force F_2, Figure 3.18(c), and:

$$F_2 = \frac{66+78}{4} + \frac{61}{3} = 56.3\ \text{kN}$$

Table 3.8

	Joint	A	B		C	
	End moments	AB	BA	BC	CB	CD
(1)	Arbitrary sway	−78	−66	+66	+61	−61
(2)	Corrected [(1) × λ]	−167	−141	+141	+131	−131
(3)	No sway moments	+10	+20	−20	+20	−20
(4)	Final moments [(2)+(3)]	−157	−121	+121	+151	−151

Now F_2 has to be scaled to equal F_1 and the scaling factor is $F_1/F_2 = \lambda = 120.8/56.3 = 2.14$.

The corrected moments are given in line 2 of Table 3.8 and the final moments are in line 4 obtained by adding lines 2 and 3.

3.5.5 Additional topics in moment distribution

Space has permitted only a brief introduction to the method of moment distribution. Additional topics which should be studied by reference to the standard texts,[3,4] are as follows:

(1) *Frames with multiple degrees of freedom for sway.* These are handled by carrying out an arbitrary sway distribution

for each sway in turn. Equilibrium conditions are then used to relate the out-of-balance forces and obtain the correction factors for each sway mode.

(2) *Treatment of symmetry.* In cases of symmetry the moment distribution process can be considerably shortened. Two cases arise and should be studied, systems in which it is known that the final set of moments is symmetrical and systems in which the final moments form an anti-symmetrical system.

(3) *Nonprismatic members.* If the flexural rigidity (EI) of a member varies within its length, then the effect is to change the values of end stiffnesses, carry-over factor and fixed end moments. A suitable general method for handling this situation is to evaluate end flexibilities by the use of Simpson's rule and then convert the flexibilities into stiffnesses.

3.6 Influence lines

3.6.1 Introduction and definitions

It is frequently necessary to consider loads which may occupy variable positions on a structure. For example, in bridge design it is important to determine the maximum effects due to the passage of a specified train or system of loads. In other cases the total load on a structure may be comprised of different loads which may be applied in various combinations and this again is a problem of variability of load or load position. The effect of varying a load position may be studied with the help of *influence lines*.

An influence line shows the variation of some resultant action or effect such as bending moment, shear force, deflection, etc. at a particular point as a unit load traverses the structure. It is important to observe that the effect considered is at a fixed position, e.g. *bending moment at* C, and that the independent variable in the influence line diagram is *the load position*. The following is a summary of influence line theory. For a more detailed treatment the reader should consult Jenkins.[1]

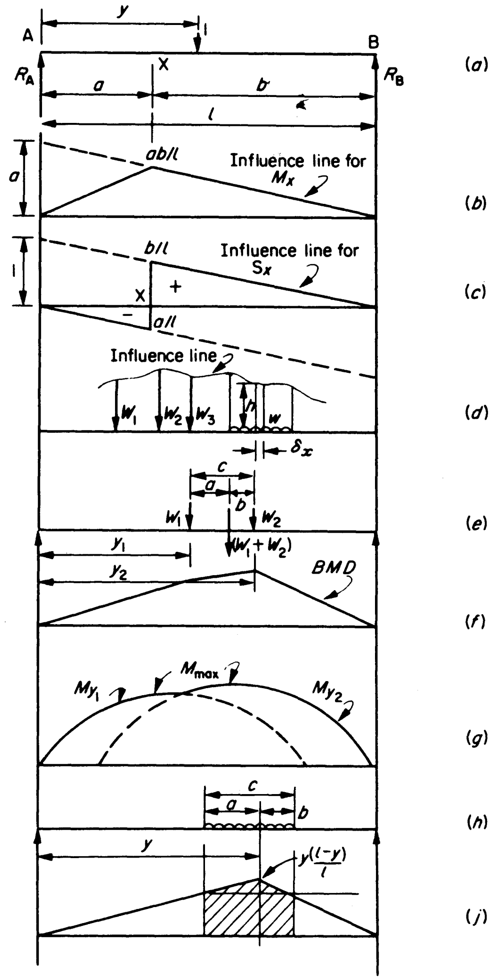


Figure 3.19 Influence lines and related diagrams for simply supported beams

3.6.2 Influence lines for beams

Consider the simply-supported beam AB, Figure 3.19, carrying a single unit load occupying a variable position distant y from A. We require to obtain influence lines for bending moment and shear force at a fixed point X distant a from A and b from B.

If the unit load lies between X and B:

$$M_x = R_A \cdot a = 1\frac{(l-y)}{l}a \tag{3.48}$$

If the unit load acts between A and X:

$$M_x = R_B \cdot b = 1 \cdot y/l \cdot b \tag{3.49}$$

Equations (3.48) and (3.49) are linear in y and when plotted in the regions to which they relate, form a triangle as shown in Figure 3.19(b). We note that, in both cases, substitution of $y = a$ gives $M_x = 1 \cdot ab/l$. Thus the influence line for M_x is a triangle with a peak value ab/l at the section X.

Turning now to the influence line for shearing force at X. For unit load between X and B:

$$S_x = R_A = \frac{l-y}{l} \tag{3.50}$$

(and now we have implied a sign convention for shear force namely that S_x is positive if the resultant force to the left of the section is upwards).

Where $y = a$, $S_x = b/l$

For unit load between A and X:

$$S_x = -R_B = -y/l \tag{3.51}$$

when $y = a$, $S_x = -a/l$

We note that Equations (3.50) and (3.51) give different values of S_x for $y = a$ and moreover the signs are opposite. This means that the shear force influence line contains a discontinuity at X as shown in Figure 3.19(c).

In using influence lines with a given system of loads and having determined the locations of the loads on the span, the total effect is evaluated as:

$$\sum(W \times \text{ordinate}), \text{ for concentrated loads} \tag{3.52}$$

and:

$$\int whdx = w \text{ (area under influence line)} \tag{3.53}$$

for distributed loads (Figure 3.19(d)).

The maximum effect produced at a given position is of interest in the design process. In the case of concentrated loads, from Equation (3.52), this is obtained when:

$$\sum(W \times \text{ordinate}) \text{ is a maximum}$$

The process of locating the loads to produce the maximum value is best done by trial and error. It follows from the straight-line nature of a bending moment diagram due to concentrated loads, that the maximum bending moment at a section will be obtained when one of the loads acts at the section. This may be illustrated by reference to the two-load system shown at (e) in Figure 3.19. The shape of the bending moment diagram is as shown at (f) and at (g) is drawn a diagram which shows the maximum value of bending moment at any section in the beam. This is the *maximum bending moment envelope* M_{max} which is seen to consist of two intersecting parabolic curves M_{y1} and M_{y2}.

The curve M_{y1} represents the maximum bending moment at all sections in the beam when this is obtained with load W_1 placed at the section. The curve M_{y2} represents the maximum bending moment at all sections in the beam when this is obtained with load W_2 at the section. It is seen that W_1 should be placed at the section towards the left-hand end of the beam, and W_2 at the section towards the right-hand end of the beam.

The expressions for M_{y1} and M_{y2} are as follows:

$$\left.\begin{aligned} M_{y1} &= (W_1+W_2)\frac{y_1}{l}(l-y_1-a) \\ M_{y2} &= (W_1+W_2)\frac{(l-y_2)}{l}(y_2-b) \end{aligned}\right\} \tag{3.54}$$

In the case of a distributed load which has a length greater than the span, then for an influence line of type (b) in Figure 3.19, the whole span would be loaded, whereas for an influence line of type (c) one would place the left-hand end of the load at X thus avoiding the introduction of a negative effect on the maximum positive value. For a short distributed load, as at (h), for maximum effect at y, the load must be placed so that the shaded area in (j) is a maximum.

The rule for this is:

$$y/l = a/c \tag{3.55}$$

3.6.3 Influence lines for plane trusses

In the analysis of plane trusses, the influence line is useful in representing the variations in forces in members of the truss.

Figure 3.20(a) shows a Warren girder AB of span 20 m. For the unit load acting at any of the lower chord joints, the force in member 1 is:

$$P_1 = \frac{AR_A}{2\sqrt{3}}$$

The peak value occurs when the unit load is at C, and thus:

$$P_{1\,max} = \frac{2}{\sqrt{3}} \times \frac{4}{5} \times 1 = \frac{8}{5\sqrt{3}}$$

The influence line for P_1 is shown at (b).

For member 2, if the unit load lies between A and E, we take:

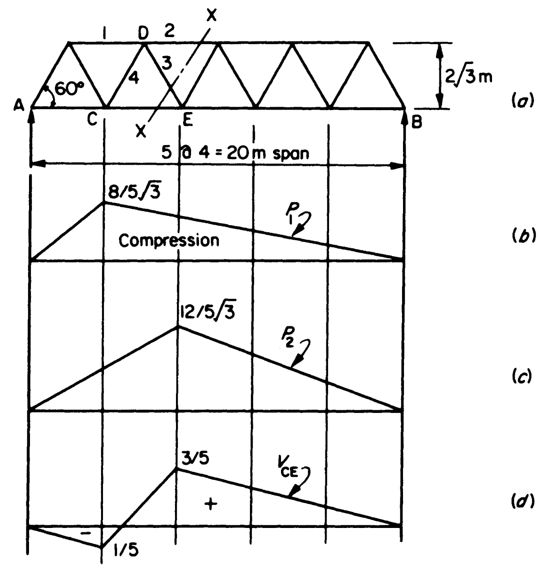


Figure 3.20 Influence lines for plane truss

$$P_2 = \frac{12R_B}{2\sqrt{3}}$$

or, if the unit load lies between E and B we take:

$$P_2 = \frac{8R_A}{2\sqrt{3}}$$

The result is a triangle with peak value $12/5\sqrt{3}$ at E, as shown in diagram (c).

It should be noted that both the P_1 and P_2 influence lines indicate compression for all positions of the unit load.

For members 3 and 4 it is useful to note that these members carry the vertical shear force in the panel CE, and we proceed by drawing the influence line for V_{CE} as at (d).

Considering now the force in member 3 and the section XX in diagram (a), it is clear that the relationship is:

$$P_3 = \frac{V_{CE}}{\sin 60°}$$

and that P_3 is tensile when V_{CE} is positive and compressive when V_{CE} is negative.

3.6.4 Influence lines for statically indeterminate structures

The use of influence lines in representing the effects of variable-position loads in statically determinate beams and trusses has been outlined. The concept is, of course, of general application. When dealing with statically indeterminate structures it is convenient to introduce some additional theorems to assist the analysis. It is possible to relate influence line shapes to deflected shapes of structures under particular forms of applied force. This involves an application of Mueller-Breslau's principle, which we shall look at in this section. The application of this principle can take the form of a model analysis, to which a simple form or model of the structure is made and particular distortions of the model produce scaled versions of influence lines.

With the enormous increase in computing power now available there is little need to use models in this way and it is generally more economical to produce influence lines by computer. It should be noted that it is always possible to construct influence lines by repeated analysis of the structure under a unit applied load, changing the load position for each analysis and thus producing a succession of ordinates to the influence line sought. This latter approach will be illustrated in section 3.6.8.

We now look at two important theorems concerned with influence lines.

3.6.5 Maxwell's reciprocal theorem

Consider the propped cantilever shown in Figure 3.21 to be subjected to a load W at A, producing displacements f_{11} and f_{21} as shown at (a), and then separately to be subjected to a moment M at B producing displacements f_{12} and f_{22} as at (b). Assuming a linear load–displacement relationship we may use the principle of superposition and obtain the combined effects of W and M by adding (a) and (b). Clearly it will be immaterial in which order the forces are applied. Applying W first and then M, the work done by the loads will be:

$$(\tfrac{1}{2}Wf_{11})+(\tfrac{1}{2}Mf_{22}+Wf_{12}) \qquad (3.56)$$

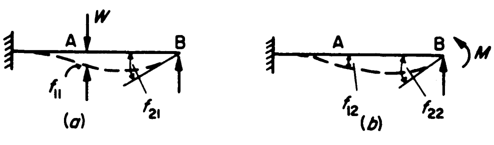


Figure 3.21

The first bracket in Equation (3.56) contains the work done during the application of W and the second bracket the work done (by both M and W) during the application of M.

In a similar way, if the order is reversed, the work done is:

$$(\tfrac{1}{2}Mf_{22})+(\tfrac{1}{2}Wf_{11}+Mf_{21}) \qquad (3.57)$$

From Equations (3.56) and (3.57) it is evident that:

$$Wf_{12}=Mf_{21} \qquad (3.58)$$

If the applied actions are taken to have unit values, then Equation (3.58) simplifies to:

$$f_{12}=f_{21} \qquad (3.59)$$

Equation (3.59) is a statement of Maxwell's reciprocal theorem. A more general theorem, of which Maxwell's is a special case, is due to Betti. This latter theorem states that if a system of forces P_i produces displacements p_i at corresponding positions and another set of forces Q_i, at similar positions to P_i, produces displacements q_i, then:

$$P_1q_1+P_2q_2+\ \dots\ +P_nq_n=Q_1p_1+Q_2p_2+\ \dots\ +Q_np_n \qquad (3.60)$$

3.6.6 Mueller-Breslau's principle

This principle is the basis of the indirect method of model analysis. It is developed from Maxwell's theorem as follows. Consider the two-span continuous beam shown in Figure 3.22(a). On removal of the support at C and the application of a unit load at C, a deflected shape, shown dotted in Figure

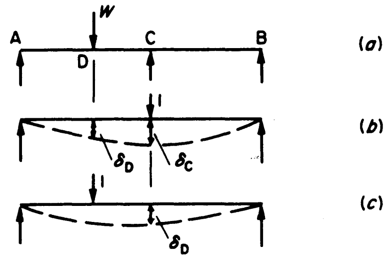


Figure 3.22

3.22(b), is obtained. If a unit load now occupies any arbitrary position D, as at (c), then from Maxwell's theorem the deflection at C will be δ_D. In other words, the deflected form (b) is the influence line for deflection of C.

Now the force at C to move C through $\delta_C=1$

Hence, the force at C to move C through $\delta_D=1\times\delta_D/\delta_C$.

If a unit load acts at D, producing a deflection δ_D at C, then the upwards force needed to restore C to the level of AB is $1\times\delta_D/\delta_C$. Hence, the reaction at C for unit load at D is $1\times\delta_D/\delta_C$. Since D is an arbitrary point in the beam then it is seen that the deflected shape due to unit load at C, Figure 3.22(b), is to some scale, the influence line for R_C. The scale of the influence line is determined from the knowledge that the actual ordinate at C should equal unity. Hence, the ordinates should all be divided by δ_C.

This result leads to Mueller-Breslau's principle which may be stated as follows:

> 'The ordinates of the influence line for a redundant force are equal to those of the deflection curve when a unit load replaces the redundancy, the scale being chosen so that the deflection at the point of application of the redundancy represents unity.'

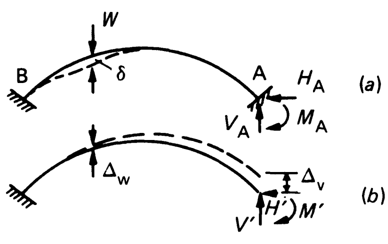


Figure 3.23

3.6.7 Application to model analysis

Consider the fixed arch shown in Figure 3.23(a). The arch has three redundancies which may be taken conveniently as H_A, V_A and M_A. We make a simple model of the arch to a chosen linear scale and pin this to a drawing board. End B is fixed in position and direction and the undistorted centreline is transferred to the drawing paper. We then impose a *purely* vertical displacement Δ_v at A and transfer the distorted centreline to the drawing paper. The distortion produced will require force actions at A, V', H' and M'. Let the displacement of a typical load point be Δ_w. Applying Equation (3.60) to the two systems of forces:

$$V_A(\Delta_v)+H_{A-}(0)+M_A(0)+W(\Delta_w)=V'(0)+H'(0)+M'(0)+0(\delta)$$

Hence:

$$V_A\Delta_v+W\Delta_w=0$$

and if $W=1$:

$$V_A = \frac{-\Delta_w}{\Delta_v} \tag{3.61}$$

Similarly, we impose a *purely* horizontal displacement Δ_H and obtain:

$$H_A = \frac{-\Delta'_w}{\Delta_H} \tag{3.62}$$

then a pure rotation θ and obtain:

$$M_A = -\frac{\Delta''_w}{\theta} \tag{3.63}$$

In Equations (3.62) and (3.63) the displacements Δ'_w and Δ''_w represent the arch displacements due to the imposed horizontal and rotational displacements respectively. In each case the deflected shape, suitably scaled, gives the influence line for the corresponding redundancy.

3.6.7.1 Sign convention

The negative sign in Equations (3.61) to (3.63) leads to the following convention for signs. On the assumption that a reaction is positive if in the direction of the imposed displacement, then a load W will give a positive value of the reaction if the influence line ordinate at the point of application of the load is opposite to the direction of the load. This is evident in Figure 3.23(b) where the upward deflection Δ_w, being opposed to the direction of the load W, is consistent with a positive (upwards) direction for V_A.

3.6.7.2 Scale of the model

It should be noted that when using relationships (3.61) and (3.62) the ratios Δ_w/Δ_v and Δ'_w/Δ_H are dimensionless and thus the linear scale of the model does not affect the influence line ordinates. On the other hand, when using Equation (3.63) in obtaining an influence line for bending moment, Δ_w/θ has the dimensions of length and thus the model displacements must be multiplied by the linear scale factor.

In performing the model analysis, quite large displacements can be used providing the linear relation between load and displacement is maintained. Hence, the indirect method is sometimes called the 'large displacement' method.

3.6.8 Use of the computer in obtaining influence lines

With adequate computing facilities it is generally more economical to proceed directly to the computation of influence line ordinates by the analysis of the structure under a unit load, the unit load occupying a succession of positions. The actual method of analysis is immaterial but for bridge-type structures often the flexibility method offers some advantage especially if the structural members are 'nonprismatic'. An example of this type of computation is shown in Figure 3.24 where influence lines for bending moments at the interior supports of a five-span continuous beam are given. The beam is taken to be uniform in section over its length and, due to the symmetry of the spans, unit load positions need only be taken over one-half of the structure as shown.

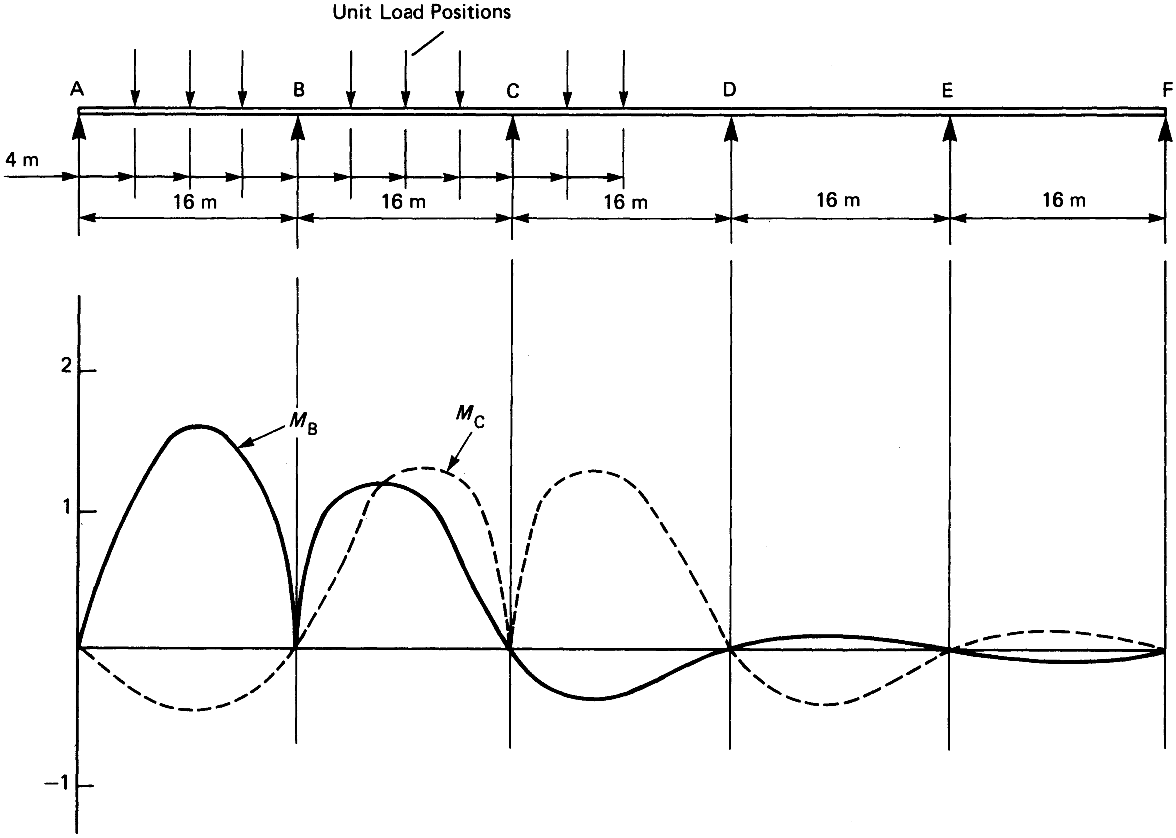


Figure 3.24 Influence lines for bending moments in a continuous beam obtained by computer analysis

3.7 Structural dynamics

3.7.1 Introduction and definitions

Structural vibrations result from the application of *dynamic* loads, i.e. loads which vary with time. Loads applied to structures are often time-dependent although in most cases the rate of change of load is slow enough to be neglected and the loads may be regarded as *static*. Certain types of structure may be susceptible to dynamic effects; these include structures designed to carry moving loads, e.g. bridges and crane girders, and structures required to support machinery. One of the most severe and destructive sources of dynamic disturbance of structures is, of course, the earthquake.

The dynamic behaviour of structures is generally described in terms of the displacement–time characteristics of the structure, such characteristics being the subject of vibration analysis. Before considering methods of analysis it is helpful to define certain terms used in dynamics.

(1) *Amplitude* is the maximum displacement from the mean position.
(2) *Period* is the time for one complete cycle of vibration.
(3) *Frequency* is the number of vibrations in unit time.
(4) *Forced vibration* is the vibration caused by a time-dependent disturbing force.
(5) *Free vibrations* are vibrations after the force causing the motion has been removed.
(6) *Damping*. In structural vibrations, damping is due to: (a) internal molecular friction; (b) loss of energy associated with friction due to slip in joints; and (c) resistance to motion provided by air or other fluid (drag). The type of damping usually assumed to predominate in structural vibrations is termed *viscous* damping in which the force resisting motion is proportional to the velocity. Viscous damping adequately represents the resistance to motion of the air surrounding a body moving at low speed and also the internal molecular friction.
(7) *Degrees of freedom*. This is the number of independent displacements or *coordinates* necessary to completely define the deformed state of the structure at any instant in time. When a single coordinate is sufficient to define the position of any section of the structure, the structure has a *single degree of freedom*. A continuous structure with a distributed mass, such as a beam, has an infinite number of degrees of freedom. In structural dynamics it is generally satisfactory to transform a structure with an infinite number of degrees of freedom into one with a finite number of freedoms. This is done by adopting a *lumped mass* representation of the structure, as in Figure 3.25. The total mass of the structure is considered to be *lumped* at specified points in the structure and the motion is described in terms of the displacements of the lumped masses. The accuracy of the analysis can be improved by increasing the number of lumped masses. In most cases sufficiently accurate results can be obtained with a comparatively small number of masses.

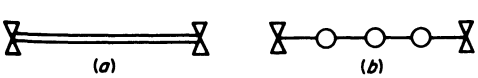


Figure 3.25

3.7.2 Single degree of freedom vibrations

The portal frame shown in Figure 3.26 is an example of a structure with a single degree of freedom providing certain assumptions are made. If it is assumed that the entire mass of the structure (M) is located in the girder and that the girder has an infinitely large flexural rigidity and further, that the columns have infinitely large extensional rigidities, then the displacement of the mass M resulting from the application of an exciting force $P(t)$, is defined by the transverse displacement y. The girder moves in a purely horizontal direction restrained only by the flexure of the columns.

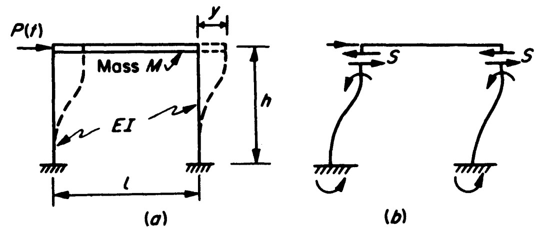


Figure 3.26

From Newton's second law of motion:

Force = mass × acceleration

i.e.:

$$\sum P = M\ddot{y} \tag{3.64}$$

Now from Figure 3.26(b), the force resisting motion is:

$$2S = 2\left(\frac{12EIy}{h^3}\right)$$

$$= 24\frac{EIy}{h^3} \tag{3.65}$$

Thus Equation (3.64) becomes:

$$P(t) - 24\frac{EIy}{h^3} = M\ddot{y}$$

or:

$$M\ddot{y} + 24\frac{EIy}{h^3} = P(t) \tag{3.66}$$

If the effect of damping is included then the equation of motion, Equation (3.66) is modified by the inclusion of a term $c\dot{y}$ where c is a constant. It should be noted that since the effect of damping is to resist the motion, then the term $c\dot{y}$ is added to the left-hand side of Equation (3.66). Thus:

$$M\ddot{y} + c\dot{y} + 24\frac{EIy}{h^3} = P(t) \tag{3.67}$$

Equation (3.67) may be generalized for any single degree of freedom structure by observing that the stiffness of the structure, i.e. force required for unit displacement horizontally, is given by:

$$k = 24\frac{EI}{h^3} \tag{3.68}$$

Combining Equations (3.67) and (3.68) we obtain the general single degree of freedom equation of motion:

$$M\ddot{y} + c\dot{y} + ky = P(t) \tag{3.69}$$

If in Equation (3.69) $P(t)=0$, we have a state of *free vibration* of the structure. The governing equation becomes:

$$M\ddot{y}+c\dot{y}+ky=0 \tag{3.70}$$

The situation envisaged by Equation (3.70) would arise if the beam were given a horizontal displacement and then released. The resulting vibrations would depend on the amount of damping present, measured by the coefficient c.

The solution of Equation (3.70) is:

$$y=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t} \tag{3.71}$$

where A_1 and A_2 are the constants of integration, to be evaluated from initial conditions, and λ_1 and λ_2 are the roots of the auxiliary equation:

$$M\lambda^2+c\lambda+k=0 \tag{3.72}$$

or, substituting:

$$\left.\begin{array}{l} p^2=k/M \\ \text{and} \\ 2n=c/M \end{array}\right\} \tag{3.73}$$

Equation (3.72) becomes:

$$\lambda^2+2n\lambda+p^2=0 \tag{3.74}$$

Hence:

$$\lambda=-n\pm\sqrt{(n^2-p^2)} \tag{3.75}$$

Four cases arise:

Case 3.1 $p^2<n^2$

Here (n^2-p^2) is always positive and $<n^2$ and thus λ_1 and λ_2 are real and negative.

Equation (3.71) takes the form:

$$y=e^{-nt}(A_1e^{\sqrt{(n^2-p^2)}t}+A_2e^{-\sqrt{(n^2-p^2)}t}) \tag{3.76}$$

The relationship between y and t of Equation (3.76) is shown in Figure 3.27(a) and it is seen that the displacement y gradually returns to zero, no vibrations taking place.

Now, since $n^2>p^2$, then:

$$\frac{c^2}{4M^2}>\frac{k}{M}$$

or

$$c>2\sqrt{(Mk)} \tag{3.77}$$

A structure exhibiting these characteristics is said to be *over-damped.*

Case *3.2* $p^2=n^2$

From Equation (3.75), $\lambda-n$ (twice)
and hence,

$$y=e^{-nt}(A_1+A_2t) \tag{3.78}$$

Again, no vibrations result and Equation (3.78) has the form shown in Figure 3.27(a).

From Equation (3.73) the value of c for this condition is given by:

$$c_c=2\sqrt{(Mk)} \tag{3.79}$$

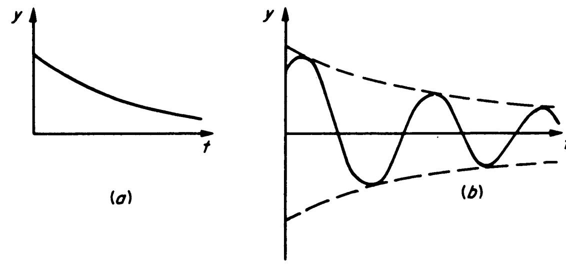


Figure 3.27

This is termed *critical damping* and the critical damping coefficient c_c is the value of the damping coefficient at the boundary between vibratory and nonvibratory motion. The critical damping coefficient is a useful measure of the damping capacity of a structure. The damping coefficient of a structure is usually expressed as a percentage of the critical damping coefficient.

Case 3.3 $p^2>n^2$

Here $c<c_c$ and the structure is *underdamped.*

From Equation (3.75), $\lambda=-n\pm i\sqrt{(p^2-n^2)}$

Hence:

$$y=e^{-nt}(A_1e^{i\sqrt{(p^2-n^2)}t}+A_2e^{-i\sqrt{(p^2-n^2)}t})$$

or, putting:

$$(p^2-n^2)=q^2$$

$$y=e^{-nt}(A_1e^{iqt}+A_2e^{-iqt})$$

or

$$y=e^{-nt}(A\cos qt+B\sin qt) \tag{3.80}$$

A typical displacement–time relationship for this condition is shown in Figure 3.27(b).

An alternative form for Equation (3.80) is:

$$y=Ce^{-nt}\sin(qt+\beta) \tag{3.81}$$

where C and β are new arbitrary constants

The period $T=\dfrac{2\pi}{q}=\dfrac{2\pi}{p\sqrt{\{1-(n/p)^2\}}}$

The period is constant but the amplitude decreases with time. The decay of amplitude is such that the ratio of amplitudes at intervals equal to the period is constant, i.e.:

$$\frac{y_{(t)}}{y_{(t+T)}}=e^{nT}$$

and $\log e^{nT}=nT=\delta$

δ is called the *logarithmic decrement*, and is a useful measure of damping capacity.

The percentage critical damping

$$=100\frac{c}{c_c}$$

$$=100\frac{\delta}{pT}$$

This is of the order of 4% for steel frames and 7% for concrete frames.

Case 3.4 $c=0$

In the absence of damping, Equation (3.70) becomes:

$$M\ddot{y}+ky=0 \tag{3.82}$$

The solution of which is:

$$y=A_1e^{\lambda_1 t}+A_2e^{\lambda_2 t}$$

where, from Equation (3.72):

$$\lambda_1=ip$$

$$\lambda_2=-ip$$

Thus:

$$y=A\sin pt+B\cos pt \tag{3.83}$$

The period is, $T=\frac{2\pi}{p}$

where p is the *natural circular frequency*

The *natural frequency* is $f=\frac{1}{T}=\frac{p}{2\pi}$

3.7.3 Multi-degree of freedom vibrations

Vibration analysis of systems with many degrees of freedom is a complex subject and only a brief indication of one useful method will be given here. For a more comprehensive and detailed treatment, the reader should consult one of the standard texts.[7]

For a system represented by lumped masses, the governing equations emerge as a set of simultaneous ordinary differential equations equal in number to the number of degrees of freedom. Mathematically the problem is of the *eigenvalue* or *characteristic value* type and the solutions are the *eigenvalues* (frequencies) and the *eigenvectors* (modal shapes). We shall consider the evaluation of mode shapes and fundamental, undamped, frequencies by the process of matrix iteration using the flexibility approach (see page 3/6). The method to be described, leads automatically to the lowest frequency, the fundamental, this being the one of most interest from a practical point of view. The alternative method using a stiffness matrix approach leads to the highest frequency.

Consider the simply-supported, uniform cross-section beam shown in Figure 3.28(a). The mass/unit length is w and we will regard the total mass of the beam to be lumped at the quarter-span points as shown in Figure 3.28(b). We may ignore the end masses $wl/8$ since they are not involved in the motion, and consider the three masses

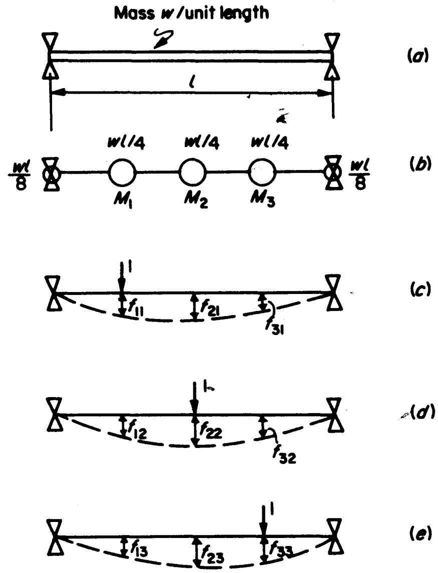


Figure 3.28

$$M_1=M_2=M_3=wl/4.$$

The appropriate flexibilities, f_{ij}, are shown at (c), (d) and (e). Using the flexibility method previously described, we may obtain a flexibility matrix as follows:

$$\mathbf{F}=\begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}=\frac{l^3}{256EI}\begin{bmatrix} 3.00 & 3.67 & 2.33 \\ 3.67 & 5.33 & 3.67 \\ 2.33 & 3.67 & 3.00 \end{bmatrix} \tag{3.84}$$

It should be noted that f_{ij} is the displacement of mass M_i due to unit force acting at mass M_j. Thus, if the forces acting at the positions of the lumped masses are $F_{1,2,3}$ and the corresponding displacements are $y_{1,2,3}$, then:

$$\left.\begin{aligned} y_1&=f_{11}F_1+f_{12}F_2+f_{13}F_3 \\ y_2&=f_{21}F_1+f_{22}F_2+f_{23}F_3 \\ y_3&=f_{31}F_1+f_{32}F_2+f_{33}F_3 \end{aligned}\right\} \tag{3.85}$$

For free, undamped vibrations, F_i is an inertia force $=-M_i\ddot{y}_i$.

Thus:

$$\left.\begin{aligned} y_1+f_{11}M_1\ddot{y}_1+f_{12}M_2\ddot{y}_2+f_{13}M_3\ddot{y}_3=0 \\ y_2+f_{21}M_1\ddot{y}_1+f_{22}M_2\ddot{y}_2+f_{23}M_3\ddot{y}_3=0 \\ y_3+f_{31}M_1\ddot{y}_1+f_{32}M_2\ddot{y}_2+f_{33}M_3\ddot{y}_3=0 \end{aligned}\right\} \tag{3.86}$$

The solutions take the form:

$$y_i=\delta_i\cos(pt+a) \tag{3.87}$$

Hence:

$$\ddot{y}_i=-p^2y_i \tag{3.88}$$

Thus, Equations (3.86) become:

$$\left.\begin{aligned}\delta_1-f_{11}M_1p^2\delta_1-f_{12}M_2p^2\delta_2-f_{13}M_3p^2\delta_3=0\\ \delta_2-f_{21}M_1p^2\delta_1-f_{22}M_2p^2\delta_2-f_{23}M_3p^2\delta_3=0\\ \delta_3-f_{31}M_1p^2\delta_1-f_{32}M_2p^2\delta_2-f_{33}M_3p^2\delta_3=0)\end{aligned}\right\} \tag{3.89}$$

or:

$$\mathbf{\Delta}=p^2\mathbf{FM\Delta} \tag{3.90}$$

where:

$$\mathbf{\Delta}=\begin{bmatrix}\delta_1\\ \delta_2\\ \delta_3\end{bmatrix};\qquad \mathbf{M}=\begin{bmatrix}M_1 & 0 & 0\\ 0 & M_2 & 0\\ 0 & 0 & M_3\end{bmatrix}$$

The unknowns in Equation (3.90) are the displacement amplitudes δ_i and the frequency p; p has as many values as there are equations in the system, and for every value of p (eigenvalue) there corresponds a set of y (eigenvector).

We adopt an iterative procedure for the solution of Equation (3.90) and first of all rewrite the equations in the form:

$$\mathbf{FM\Delta}=\frac{1}{p^2}\mathbf{\Delta} \tag{3.91}$$

We start with an assumed vector $\mathbf{\Delta}_0$, thus:

$$\mathbf{FM\Delta}_0=\frac{1}{p^2}\mathbf{\Delta}_0$$

Putting $\mathbf{FM\Delta}_0=\mathbf{\Delta}_1$

$$\mathbf{\Delta}_1\simeq\frac{1}{p^2}\mathbf{\Delta}_0 \text{ giving } p^2\simeq\frac{\mathbf{\Delta}_0}{\mathbf{\Delta}_1}$$

We cannot form $\mathbf{\Delta}_0/\mathbf{\Delta}_1$ since each $\mathbf{\Delta}$ is a column matrix, so we take the ratio of corresponding elements in $\mathbf{\Delta}_0$ and $\mathbf{\Delta}_1$ and form the ratio δ_0/δ_1. It is best to use the numerically greatest δ for this purpose.

Continuing the process:

$$\mathbf{FM\Delta}_1\simeq\frac{1}{p^2}\mathbf{\Delta}_1 \text{ giving } p^2=\delta_1/\delta_2$$

$$=\mathbf{\Delta}_2$$

and again:

$$\mathbf{FM\Delta}_2\simeq\frac{1}{p^2}\mathbf{\Delta}_2$$

$$=\mathbf{\Delta}_3 \text{ giving } p^2=\delta_2/\delta_3$$

It can be shown that this iterative process converges to the largest value of $1/p^2$ and hence yields the lowest (fundamental mode) frequency.

Applying the iterative scheme to the beam of Figure 3.28, and assuming:

$$\mathbf{\Delta}_0=\begin{bmatrix}1\\ 2\\ 1\end{bmatrix}$$

then, $\mathbf{\Delta}_1=\mathbf{FM\Delta}_0$

$$\text{where } \mathbf{FM}=\frac{l^3}{256EI}\begin{bmatrix}3.00 & 3.67 & 2.33\\ 3.67 & 5.33 & 3.67\\ 2.33 & 3.67 & 3.00\end{bmatrix}\begin{bmatrix}wl/4 & 0 & 0\\ 0 & wl/4 & 0\\ 0 & 0 & wl/4\end{bmatrix}$$

$$=\frac{wl^4}{1024EI}\begin{bmatrix}3.00 & 3.67 & 2.33\\ 3.67 & 5.33 & 3.67\\ 2.33 & 3.67 & 3.00\end{bmatrix}$$

$$\text{Thus: } \mathbf{\Delta}_1=\frac{wl^4}{1024EI}\begin{bmatrix}12.67\\ 18.00\\ 12.67\end{bmatrix}=\frac{12.67wl^4}{1024EI}\begin{bmatrix}1.00\\ 1.42\\ 1.00\end{bmatrix}$$

$$\text{Hence: } p_1^2=\frac{\delta_0}{\delta_1}=\frac{2\times 1024EI}{12.67\times 1.42wl^4}$$

$$=114\frac{EI}{wl^4}$$

A second iteration gives:

$$\mathbf{\Delta}_2=\mathbf{FM\Delta}_1=\frac{wl^4}{1024EI}\begin{bmatrix}3.00 & 3.67 & 2.33\\ 3.67 & 5.33 & 3.67\\ 2.33 & 3.67 & 3.00\end{bmatrix}\frac{12.67wl^4}{1024EI}\begin{bmatrix}1.00\\ 1.42\\ 1.00\end{bmatrix}$$

$$=12.67\left(\frac{wl^4}{1024EI}\right)^2\begin{bmatrix}10.54\\ 14.91\\ 10.54\end{bmatrix}$$

Hence:

$$p_2^2=\frac{\delta_1}{\delta_2}=\frac{12.67\times 1.42wl^4}{1024EI}\times\frac{1}{12.67(wl^4/1024EI)^2\times 14.91}$$

$$=\ 97.5\frac{EI}{wl^4}$$

This result is very close to that produced by an exact method, i.e. $97.41EI/wl^4$.

3.8 Plastic analysis

3.8.1 Introduction

The plastic design of structures is based on the concept of a *load factor* (N), where

$$N=\frac{\text{Collapse load}}{\text{Working load}}=\frac{W_c}{W_w}, \tag{3.92}$$

A structure is considered to be on the point of collapse when finite deformation of at least part of the structure can occur without change in the loads. The simple plastic theory is based on an idealized stress–strain relationship for structural steel as shown in Figure 3.29. A linear, elastic, relationship holds up to a stress σ_y, the *yield stress*, and at this value of stress the material is considered to be in a state of perfect plasticity, capable of infinite strain, represented by the horizontal line AB continued indefinitely to the right. For comparison the dotted line shows the true relationship.

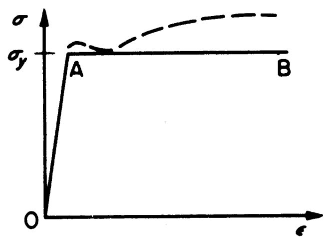


Figure 3.29

The term '*plastic analysis*' is generally related to steel structures for which the relationship indicated in Figure 3.29 is a good approximation. The equivalent approach when dealing with concrete structures is generally termed 'ultimate load analysis' and requires considerable modification to the method described here.

The stress–strain relationship of Figure 3.29 will now be applied to a simple, rectangular section, beam subjected to an applied bending moment M (Figure 3.30).

Under purely elastic conditions, line OA of Figure 3.29, the stress distribution over the cross-section of the beam will be as shown in Figure 3.30(b) and the limiting condition for elastic behaviour will be reached when the maximum stress reaches the value σ_y. As the applied bending moment is further increased, material within the depth of the section will be subjected to the yield stress σ_y and a condition represented by Figure 3.30(c) will exist in which part of the cross-section is plastic and part plastic. On further increase of the applied bending moment ultimately condition (d) will be reached in which the entire cross-section is plastic. It will not be possible to increase the applied bending moment further and any attempt to do so will result in increased curvature, the beam behaving as if hinged at the plastic section. Hence, the use of the term *plastic hinge* for a beam section which has become fully plastic.

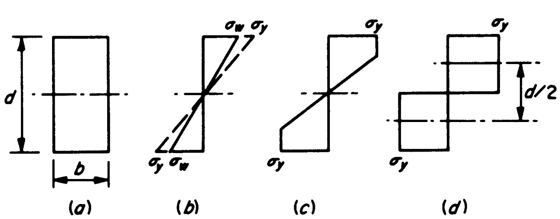


Figure 3.30

The moment of resistance of the fully plastic section is, from Figure 3.30(d):

$$M_p = b\frac{d}{2}\sigma_y\frac{d}{2} = \frac{bd^2\sigma_y}{4}$$

$$= Z_e\sigma_w \qquad (3.93)$$

where Z_p = plastic section modulus

In contrast, the moment of resistance at working stress σ_w is, from Figure 3.30(b):

$$M_w = b\frac{d}{2}\frac{\sigma_w}{2}\frac{2}{3}d = \frac{bd^2}{6}\sigma_w \qquad (3.94)$$

$$= Z_e\sigma_w$$

where Z_e = elastic section modulus

The ratio Z_p/Z_e is the shape factor of the cross-section. Thus the shape factor for a rectangular cross-section is 1.5.

The shape factor for an I-section, depth d and flange width b, is given approximately by:

$$\left(\frac{1+x/2}{1+x/3}\right)$$

where $x = \dfrac{t_w d}{2t_f b}$ and t_w and t_f are the web and flange thicknesses respectively

Values of plastic section moduli for rolled universal sections are given in steel section tables.

3.8.2 Theorems and principles

The definition of collapse, which follows from the assumed basic stress–strain relationship of Figure 3.29, has already been given. If the structural analysis is considered to be the problem of obtaining a correct bending moment distribution at collapse, then such a bending moment distribution must satisfy the following three conditions:

(1) *Equilibrium condition:* the reactions and applied loads must be in equilibrium.
(2) *Mechanism condition:* the structure, or part of it, must develop sufficient plastic hinges to transform it into a mechanism.
(3) *Yield condition:* at no point in the structure can the bending moment exceed the full plastic moment of resistance.

In elastic analysis of structures where several loads are acting, e.g. dead load, superimposed load and wind load, it is permissible to use the principle of superposition and obtain a solution based on the addition of separate analyses for the different loads. In plastic theory the principle of superposition is not applicable and it must be assumed that all the loads bear a constant ratio to one another. This type of loading is called 'proportional loading'. In cases where this assumption cannot be made, a separate plastic analysis must be carried out for each load system considered.

For cases of proportional loading, the uniqueness theorem states that the collapse load factor N_c is uniquely determined if a bending moment distribution can be found which satisfies the three collapse conditions stated.

The collapse load factor N_c may be approached indirectly by adopting a procedure which satisfies two of the conditions but not necessarily the third. There are two approaches of this type:

(a) We may obtain a bending moment distribution which satisfies the equilibrium and mechanism conditions, (1) and (2); in these circumstances it can be shown that the load factor obtained is either greater than or equal to the collapse load factor N_c. This is the 'minimum principle' and a load factor obtained by this approach constitutes an 'upper bound' on the true value.
(b) We may obtain a bending moment distribution which satisfies the equilibrium and yield conditions, (1) and (3), and in these circumstances it can be shown that the load factor obtained is either less than or equal to the collapse load factor N_c. This is the 'maximum principle' and its application produces a 'lower bound' on the true value.

It should be observed that whilst method (a) is simpler to use in practice, it produces an apparent load factor which is either correct or too high and thus an incorrect solution is on the unsafe side. A most useful approach is to employ both principles in turn and obtain upper and lower bounds which are sufficiently close to form an acceptable practical solution.

3.8.3 Examples of plastic analysis

This section contains some examples of plastic analysis based on the minimum principle. The method employed is termed the 'reactant bending moment diagram method'.

Example 3.5. The structure is a propped cantilever beam of uniform cross-section, carrying a central load W, as shown in Figure 3.31(a). The bending moment distribution under elastic conditions is shown in Figure 3.31(b) and it should be noted that the maximum bending moment occurs at the fixed end A.

As the load W is increased, plasticity will develop first at end A. As the load is further increased, end A will eventually become fully plastic with a stress distribution of the type shown in Figure 3.30(d) and the bending moment at A, M_A, will equal M_p the fully plastic moment of the beam. Further increase of load will have no effect on the value of M_A but will increase M_B until it also reaches the value M_p. The resulting bending moment distribution will now be as shown in Figure 3.31(c).

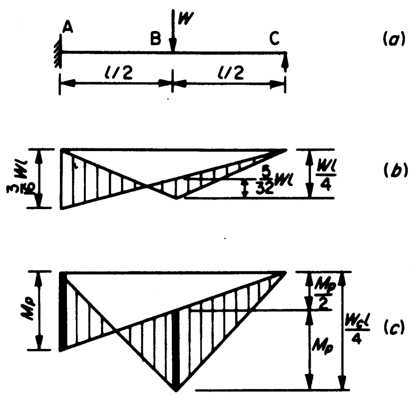


Figure 3.31

The geometry of the diagram produces a relationship between the load at collapse, W_c, and the plastic moment of resistance of the beam M_p, as follows:

$$\frac{W_c l}{4} = M_p + M_p/2$$

or:

$$W_c = 6\frac{M_p}{l} \tag{3.95}$$

If the working load is W_w then the load factor is given by:

$$N = \frac{W_c}{W_w} \tag{3.96}$$

Example 3.6. This is again a propped cantilever but here the load is uniformly distributed (Figure 3.32(a)). At collapse the bending moment diagram will be as shown in Figure 3.32(b) with plastic hinges at A and C. It should be noted that C is not at the centre of the beam. The location of the plastic hinge at C

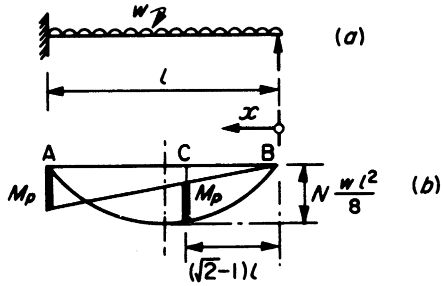


Figure 3.32

and the relationship between the load and the value of M_p may be obtained by differentiation as follows.

At C:

$$M_p = \left(N\frac{wlx}{2} - N\frac{wx^2}{2}\right) - M_p\frac{x}{l}$$

i.e.:

$$M_p = N\frac{wlx(l-x)}{2\,(l+x)} \tag{3.97}$$

$$\frac{dM_p}{dx} = N\frac{wl}{2}\frac{\{(l+x)(l-2x)-x(l-x)\}}{(l+x)^2}$$

$$= 0 \text{ for } M_{p\,max}$$

Hence: $x^2 + 2xl - l^2 = 0$

i.e.:

$$x = l(\sqrt{2}-1) = 0.414l$$

which locates the point C.

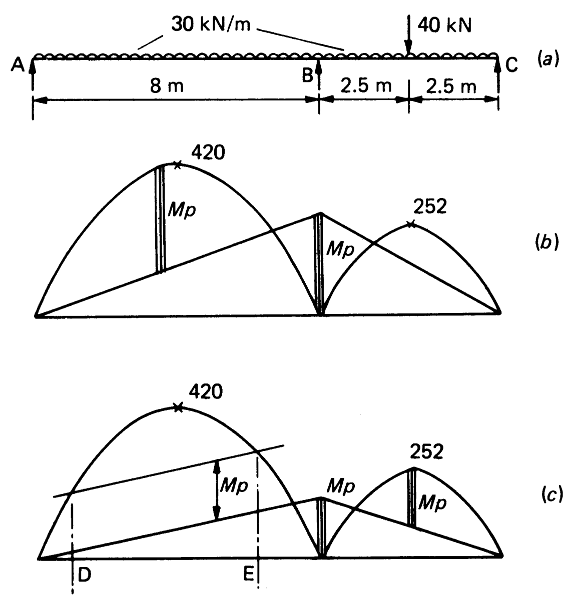


Figure 3.33

Also, substituting in Equation (3.97) for x:

$$M_p = \frac{Nwl^2(\sqrt{2}-1)}{2\quad\sqrt{2}}(2-\sqrt{2})$$

$$= \left(\frac{Nwl^2}{8}\right) 4(3-2\sqrt{2})$$

$$= 0.686\left(\frac{Nwl^2}{8}\right)$$

Example 3.7. A two-span continuous beam is shown in Figure 3.33. The loads shown are maximum working loads and it is required to determine a suitable universal beam (UB) section such that $N = 1.75$ with a yield stress $\sigma_y = 250\ \mathrm{N/mm^2}$. Effects of lateral instability are ignored for the purposes of this example.

With factored loads, the free bending moments are:

$$1.75 \times 30 \times \frac{8^2}{8} = 420\ \mathrm{kNm}$$

$$1.75 \times 30 \times \frac{5^2}{8} + 1.75 \times 40 \times \frac{5}{4} = 252\ \mathrm{kNm}$$

For collapse to occur in span AB, Figure 3.33(b)

$$420 \times 0.686 = M_p = 288\ \mathrm{kNm}$$

For collapse in BC, assuming the span hinge in BC to occur at the centre (Figure 3.33(c)):

$$252 = \frac{3}{2}M_p;\quad M_p = 168 < 288$$

Hence the beam must be designed for $M_p = 288$ kNm

$$= Z_p\sigma_y$$

Hence:

$$Z_p = \frac{288 \times 10^6}{250 \times 10^3}\ \mathrm{cm^3} = 1152\ \mathrm{cm^3}$$

From section tables, select 406 × 178 UB 60 ($Z_p = 1194\ \mathrm{cm^3}$).

This design may be compared with elastic theory from which we obtain $M_{max} = 198$ kNm, $Z_e = 1200\ \mathrm{cm^3}$ (using $\sigma_w = 165\ \mathrm{N/mm^2}$). A suitable section would be 457 × 152 UB 67 ($Z_e = 1250\ \mathrm{cm^3}$) or, 406 × 178 UB 74 ($Z_e = 1324\ \mathrm{cm^3}$).

The plastic design may be improved by choosing different sections for spans AB and BC:

For BC, $M_{PBC} = 168$ giving $Z_p = \frac{168}{250} \times \frac{10^6}{10^3} = 672\ \mathrm{cm^3}$

Select 356 × 171 UB 45 ($Z_p = 773.7\ \mathrm{cm^3}$)

For AB, $M_{PAB} \simeq 420 - \frac{1}{2}M_{PBC}$

$$= 420 - \tfrac{1}{2} \times \frac{773.7 \times 10^3 \times 250}{10^6}$$

$$= 420 - 96.7 = 323\ \mathrm{kNm}$$

$$\therefore Z_p = \frac{323}{250} \times \frac{10^6}{10^3} = 1293\ \mathrm{cm^3}$$

Select 406 × 178 UB 67.

The weights of steel used in the different designs may be compared.

First plastic design	780 kg
Elastic design	871 kg
Second plastic design	761 kg

As an alternative to the second plastic design the lower value of M_p could be used, based on collapse in BC (356 × 171 UB 45, $Z_p = 773.7$, $M_p = 193$ kNm), and flange plates welded on to the beam in the region DE, Figure 3.33(c).

The additional M_p required at the plated section

$$= 420 - \tfrac{3}{2} \times 193$$
$$= 130\ \mathrm{kNm}$$

Using plates 150 mm wide top and bottom, the plastic moment of resistance of the plates is approximately:

$$2\left(150 \times t \times 250 \times \frac{356}{2}\right) \times 10^{-6}$$

$$= 13.4\,t$$

where t = plate thickness in millimetres

Hence:

$$t = \frac{130}{13.4} \simeq 10\ \mathrm{mm})$$

Example 3.8. Here we consider the plastic analysis of a portal frame type structure as in Figure 3.34(a) and (b). At (a) the frame has pinned supports and at (b) fixed supports. A simple form of loading is used for illustration of the principles.

The frame is made statically determinate by the removal of H_A in both cases, and by the removal of M_A and M_E in case (b). The 'free' bending moment diagram is then as in diagram (c) and the reactant bending moment diagrams are as at (d) for H_A and at (e) for M_A and M_E combined. We now seek combinations of the diagrams which will satisfy the conditions of equilibrium, mechanism and yield (see page 3/27). We consider first the case of the two-hinged frame.

Diagram (f)
This is consistent with a pure sideway mode of collapse. From the geometry of the diagram:

$$M_p = \frac{Hh}{2} \tag{3.98}$$

The yield condition will be satisfied providing:

$$\frac{wl}{4} \leqslant \frac{Hh}{2} \tag{3.99}$$

Diagram (g)
This is a combined mechanism involving collapse of the beam and sidesway. From the geometry of the diagram:

At D:

$$M_p = Hh \mp H_A h$$

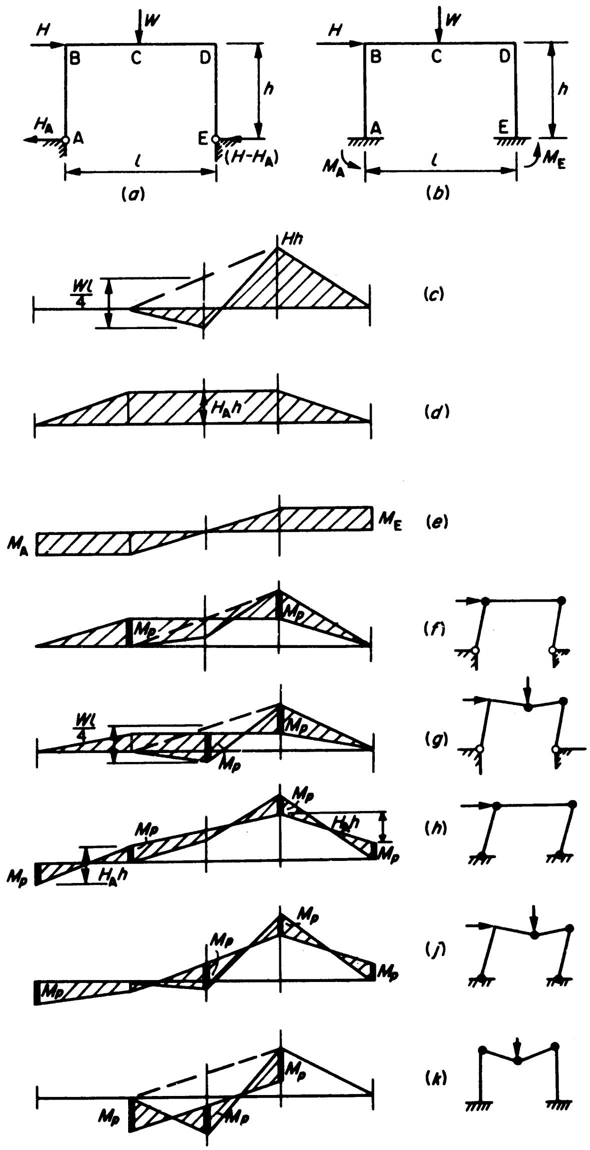


Figure 3.34

At C:

$$M_p = \frac{Wl}{4} - \frac{Hh}{2} \pm H_A h$$

Adding:

$$2M_p = \frac{Wl}{4} + \frac{Hh}{2}$$

or:

$$M_p = \frac{Wl}{8} + \frac{Hh}{4} \tag{3.100}$$

In the case of the frame with fixed feet, there are three possible modes of collapse. The corresponding bending moment diagrams are constructed at (h), (j) and (k) and the results are as follows:

Diagram (h):

$$M_p = \frac{H_A h}{2}$$

$$M_p = Hh - H_A h - M_p$$

Hence:

$$M_p = \frac{Hh}{4} \tag{3.101}$$

Diagram (j):

$$M_p = \frac{Wl}{4} - \frac{Hh}{2} \pm H_A h$$

$$M_p = Hh \mp H_A h - M_p$$

Adding:

$$3M_p = \frac{Wl}{4} + \frac{Hh}{2}$$

or:

$$M_p = \frac{Wl}{12} + \frac{Hh}{6} \tag{3.102}$$

Diagram (k)
This mode is the same as the collapse of a fixed end beam; the columns are not involved in the collapse apart from providing the resisting moment M_p at B and D. From the geometry of the diagram:

$$M_p = \frac{Wl}{8} \tag{3.103}$$

Example 3.9. Here we consider a pitched roof frame, a structure which is eminently suitable for design by plastic methods. The frame is shown in Figure 3.35(a). The given loads are already factored and we are to find the required section modulus on the basis of a yield-stress $\sigma_y = 280$ N/mm², neglecting instability tendencies and the reduction in plastic moment of resistance due to axial forces.

The bending moment diagram at collapse is shown in Figure 3.35(b). The free bending moment diagram, EFGB, is drawn to scale after evaluating values of moment at intervals along the rafter members. The reactant line (H_A diagram) is then drawn by trial and error so that the maximum moment in the region BC is equal to the moment at D. This moment is the required M_p for the frame and is found to be:

$$M_p = 52\ \text{kNm} = \sigma_y Z_p$$

from which:

$$Z_p = \frac{52 \times 10^3 \times 10^3}{280 \times 10^3} = 186\ \text{cm}^3$$

Horne[8] and Baker and Heyman[9] should be consulted for a more

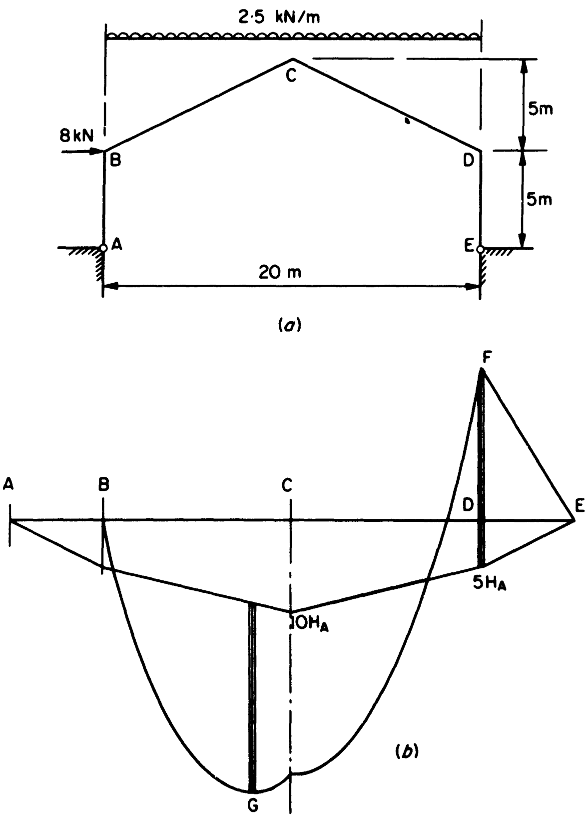


Figure 3.35

detailed study of plastic analysis. Among the topics deserving of further study are:

(1) Use of the principle of virtual work in obtaining relationships between applied loads and plastic moments of resistance.
(2) Effects of strain hardening.
(3) Evaluation of shape factors for various cross-sections.
(4) Application of the maximum principle in obtaining lower bounds.
(5) Numbers of independent mechanisms.
(6) Shakedown.
(7) Effects of axial forces.
(8) Moment carrying capacity of columns.
(9) Behaviour of welded connections.

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4

Materials

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Standards and codes of practice referred to in Chapter 4/27

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4.1 Introduction

A good working knowledge of the materials used in civil engineering is very important to the engineer and in this book the characteristics and properties of many materials are described appropriately in other chapters as indicated below.

Material	*Chapter*
Soils	9
Rocks	8 and 10
Reinforcement	12
Steel	13
Aluminium	14
Bricks and masonry	15
Timber	16
Bituminous materials	23 (also 17 and 24)

This chapter is concerned with materials which are not covered elsewhere in the book and considers in detail only: concrete (pages **4**/3 to **4**/18), plastics and rubbers (pages **4**/18 to **4**/24) and paint (pages **4**/24 to **4**/26).

The authors gratefully acknowledge permission by Peter Pullar Strecker to include or update parts of his text from the 3rd Edition of the reference book (1974).

In the field of materials especially, the solution of problems often requires a full understanding of technologies outside the engineer's normal experience. Fortunately specialist help is usually readily available in the UK, although the enquirer does not always know where to look for it. Many sources are listed by the Construction Industry Research and Information Association (CIRIA)[1] *Guide to sources of construction information.* A selection of useful organizations and their addresses is as follows.

Aluminium Federation Ltd, Broadway House, Calthorpe Road, Five Ways, Birmingham B15 1TN.
Asbestos Information Centre, 40 Piccadilly, London WlV 9PA.
Association of Bronze and Brass Founders, 136 Hagley Road, Birmingham B16 9PN.
Brick Development Association, Woodside House, Winkfield, Windsor, Berks SL4 2DX.
British Aggregate Construction Materials Industries, 156 Buckingham Palace Road, London SW1W 9TR.
British Cast Iron Research Association, Alvechurch, Birmingham B48 7QB.
British Cement Association, Wexham Springs, Slough, Berks SL3 6PL.
British Ceramic Research Ltd, Queens Road, Penkhull, Stoke-on-Trent, Staffs ST4 7LQ.
British Constructional Steelwork Association Ltd, 35 Old Queen Street, London SW1H 9HZ.
British Glass Industry Research Association, Northumberland Road, Sheffield S10 2UA.
British Non-ferrous Metals Federation, 10 Greenfield Crescent, Edgbaston, Birmingham B15 3AU.
British Rubber Manufacturers' Association Ltd, 90–91 Tottenham Court Road, London, W1P 0BR.
British Standards Institution, 2 Park Street, London W1A 2BS.
British Steel Corporation, Corporate Research Laboratories, Swinden House, Moorgage, Rotherham S60 3AR.
British Wood Preserving Association, 150 Southampton Row, London WC1B 5AL.
Building Centres: London, Manchester, Bristol, Peterborough, Durham, Glasgow.
Building Research Establishment, Garston, Watford, Herts WD2 7JR.
Cement and Concrete Association, *see* British Cement Association.
Clay Pipe Development Association, Drayton House, 30 Gordon Street, London WC1H 0AN.
Concrete Pipe Association, 60 Charles Street, Leicester LE1 1FB.
Construction Industry Research and Information Association (CIRIA), 6 Storey's Gate, London SW1P 3AU.
Copper Development Association, Orchard House, Mutton Lane, Potters Bar, Herts EN6 3AP.
Flat Glass Council, 44–48 Borough High Street, London SE1 1XB.
Institution of Mining and Metallurgy, 44 Portland Place, London W1N 4BR.
Lead Development Association, 34 Berkeley Square, London W1X 6AJ.
National Physical Laboratory, Teddington, Middlesex TW11 0LW.
Paint Research Association, Waldegrave Road, Teddington, Middlesex TW11 8LD.
RAPRA Technology Ltd, Shawbury, Shrewsbury, Shropshire SY4 4NR.
Steel Construction Institute, Silwood Park, Ascot, Berks SL5 7QN.
Stone Federation, 82 New Cavendish Street, London W1M 8AD.
Timber Research and Development Association, Stocking Lane, Hughenden Valley, High Wycombe, Buckinghamshire HP14 4ND.
Zinc Development Association, 34 Berkeley Square, London W1X 6AJ.

4.1.1 Standards and codes of practice

British and some other standards and codes referred to in this chapter are listed separately in the bibliography.

4.2 Concrete

4.2.1 Cement

Hydraulic cement, i.e. a cement which hardens because of chemical reactions between the cement and water is the main, and often the only, binder used in concrete for civil engineering purposes. Portland cement or one of its variants is usually used, but high-alumina cement has advantages for some applications. The following list of cements is likely to be encountered in civil engineering. The relevant British Standards governing properties are given in the headings.

4.2.1.1 Ordinary Portland cement (OPC): BS 12

This is the most commonly used form of cement. It is made by heating together raw materials containing alumina and calcium. Clay and chalk or limestone are common sources. During the heating process the materials fuse to form clinker which is subsequently ground to a fine powder, gypsum usually being added at this stage to control the setting characteristics of the cement. Portland cements normally comprise four main phases or chemical compounds: tricalcium silicate, dicalcium silicate, tricalcium aluminate and calcium ferroaluminate. For convenience, these phases are usually given a shorthand notation of C_3S, C_2S, C_3A and C_4AF. This powder resulting from the grinding of clinker is the cement in its final form. The fineness of grinding, the raw materials and the conditions of the fusing process influence the nature and the reactivity of the cement, fine cement hardening more quickly than coarse cement of the same composition. The quality of British cement, although varying according to its source, usually exceeds the BS requirements by a considerable margin.

4.2.1.2 Rapid-hardening Portland cement (RHPC): BS 12

This is similar to OPC in composition but it is more finely ground. It gains strength more quickly than OPC, though the final strength is only slightly increased. Heat is generated more quickly during the hydration of the cement. This may have advantages in cold weather, or in precasting operations. The difference in strength development between OPC and RHPC has now become less marked.

4.2.1.3 Low-heat Portland cement: BS 1370

This cement is less reactive than OPC because it differs in composition, but it is nevertheless more finely ground than OPC. Heat is generated more slowly on hydration and lower concrete temperatures are reached. Early and eventual strengths are less than with OPC and the initial setting time is greater. This cement is made only to order in the UK.

4.2.1.4 Sulphate-resisting Portland cement: BS 4027

This cement is similar to OPC but the proportions of the cement phases are different and it is less prone to attack by sulphates principally by having a controlled low C_3A content. Heat may be generated more slowly than with OPC, but a little more quickly than with low-heat Portland cement.

4.2.1.5 Portland blast-furnace cement: BS 146

This cement is made by grinding together OPC clinker with granulated blast-furnace slag (see later). The granulated blast-furnace slag content must be less than 65% of the total weight. This cement is less reactive than OPC and gains strength a little more slowly. It has advantages in generating heat less quickly than OPC and in being more resistant than OPC to attack from sulphates. Portland blast-furnace cement is not widely available in the UK. (Low-heat Portland blast-furnace cement contains more slag but is manufactured only to order in the UK; BS 4246 governs its composition and properties.) Combination at the concrete mixer of Portland cement with ground granulated blast-furnace slag is more commonly used to achieve similar performance. By this method a wider range of OPC:slag ratios is readily achievable. These combinations are likely to be available in most parts of the UK.

4.2.1.6 Portland PFA cement: BS 6588

This cement is manufactured by intergrinding or combining at the cement plant pulverized fuel ash (PFA), complying with BS 3892, Part 1 (see later) with ordinary Portland cement. The PFA content should be between 15 and 35% by weight. The rate of strength development is slower than that of the respective Portland cement source. The cement may generate heat less quickly and be more chemically resistant in some circumstances.

Combination of PFA with ordinary Portland cement at the concrete mixer can produce concrete with a similar performance to that using this cement.

4.2.1.7 Pozzolanic cement with PFA as pozzolana: BS 6610

As for BS 6588 but the PFA content is between 35 and 50%. This cement is not referred to in BS 8110 or BS 5328 and is therefore unlikely to be used in reinforced concrete or other slender structural elements. The lower heat of hydration is useful property in massive structures.

4.2.1.8 White Portland cement

This cement is similar to OPC but with selected raw materials and processing to remove the normal OPC grey coloration; it would also comply with BS 12 for setting time and early and eventual strength.

4.2.1.9 Supersulphated cement: BS 4248

This cement is made from granulated blast-furnace slag, gypsum and not more than 5% of OPC clinker. It is more resistant to sulphate attack than sulphate-resisting cement, and it is not attacked by weak acids. This cement is much finer though less reactive than OPC, but eventual strengths are at least as high. It is not currently available in the UK. Good control of concrete mix is essential and its use has largely been superseded by other cement–slag combinations.

4.2.1.10 Water-repellent cement

This is made from OPC and stearates. It is used to reduce water permeability especially in screeds and rendering.

4.2.1.11 Masonry cement: BS 5224

This cement is made by mixing OPC with plasticizers and a fine powder (often whiting). It is used to give plasticity to bricklaying and rendering mortars, especially where the local sand is harsh.

4.2.1.12 High-alumina cement: BS 915

This cement is chemically different from OPC and its varieties. Concrete made with it has different properties from OPC concrete. High-alumina cement is very reactive and produces very high early strengths (the eventual strength may be reached in less than 1 day) but the initial setting is slower than with all varieties of Portland cement.

High-alumina cement is very resistant to attack from sulphates and is more resistant to acid attack than any variety of Portland cement but is attacked by alkalis. At temperatures above 700°C, high-alumina cement forms a ceramic bond with suitable aggregates and it can therefore be used for refactory concrete. Under moist conditions at temperatures of 40° to 100°C conversion takes place and high-alumina cement loses strength. Cement in this condition is less resistant to chemical attack.

It is widely believed that high-alumina cement should not be used in contact with hardened Portland cement. The scientific basis for this is, however, less well founded. Mixtures of unhardened Portland and high-alumina cements lead to very rapid 'flash' setting. This phenomenon has some practical applications where almost instantaneous setting is wanted, but the quality of the resulting concrete will be in most respects inferior to either Portland cement concrete or high-alumina cement concrete.

High-alumina cement concrete is not permitted for use in structural concrete in BS 8110. Applications such as floor toppings, hardstandings are still permissible.

4.2.1.13 Other cementing materials

Ground granulated blast-furnace slag. This is a by-product of the manufacture of iron from iron ore. The molten slag is removed from the furnace and quenched rapidly (granulation). Subsequent grinding can be either after combination with Portland cement clinker or more commonly of the granulated slag alone. The slag is composed mainly of calcium and magnesium silicates and alumino-silicates. Although some small strength gain or hardening would take place in water, the strengths developed are not likely to be sufficient for construction. Blending with a Portland cement produces a much faster and useful strength gain. Combinations of ground granulated blast-furnace slag and Portland cements have been used for many years both in the UK and overseas. An increase in the use and interest in these materials has taken place over recent years in the UK and BS 6699 gives composition and performance requirements. It is widely available in the UK.

Pozzolanas. Natural or artificial materials containing amorphous silica in a reactive form. The silica can react with lime to produce cementing compounds giving useful strength properties. This lime can be either hydrated lime or the calcium hydroxide produced during the hydration of Portland cements. The original pozzolana was volcanic ash from Pozzuoli, Italy. Using pozzolanas as a cementing component in Portland cement concretes can be useful to reduce heat of hydration or to improve resistance to some chemicals. Early age strength development may be affected unless the concrete is proportioned to allow for it.

Pulverized fuel ash (PFA). This is the most common pozzolana used in Portland cement concrete. It is electrostatically precipitated from the exhaust fumes of coal-fired power stations burning pulverized coal. It is widely available in the UK, and performance and compositional requirements are given in BS 3892, Part 1 (for use in structural concrete) and BS 3892, Part 2 (for miscellaneous uses in concrete).

Condensed silica fume. A high-purity silica pozzolana which has a very fine particle size much smaller than that of cement or PFA (mean particle size approximately 1 μm). Condensed silica fume is so fine it can be used to fill the interstices between cement particles and it reacts rapidly with the cement hydration products. Condensed silica flume is a by-product of the production of silicon and ferro-silicon being collected by cooling and filtering of furnace gases. Condensed silica flume can be used to produce very high strengths and good chemical resistance.

4.2.1.14 Non-UK standards

Many other national standards exist for Portland cements and combinations of Portland cements with blast-furnace slag or PFA. These standards cover similar ranges of materials to those in the British Standards given in the preceding pages although the overlap will not be complete for each country. Methods or terminology of classification vary for each country but common principles exist, e.g. sulphate-resisting cements are always low in C_3A content but the actual limiting value will be different.

Standards issued by the American Society for Testing Materials (ASTM) are widely used outside the US. Their standard C-150 has five main categories of Portland cement and a summary of these types is given in Table 4.1.

Other national standards for Portland cements which are likely to be encountered more widely are issued by Deutsches Institut für Normung (DIN) and in Japan as Japanese Industrial Standards (JIS). A wide range of cement specifications are incorporated within these standards and, hence, are not reproduced here.

4.2.2 Aggregates

Aggregates form more than three-quarters of the volume of concrete and the selection and proportioning of coarse and fine aggregates greatly influence the properties of both fresh and hardened concrete. The choice of grading, maximum aggregate size and aggregate:cement ratio are subjects for concrete mix design and are dealt with below. In this section the selection of aggregate type will be covered. Broadly, aggregates can be classified according to density as normal (particle density 2000 to 3000 kg/m^3), lightweight (less than 2000 kg/m^3) and heavy aggregates (greater than 3000 kg/m^3). Typical properties of concretes made with a range of aggregates are given in Table 4.2.

Table 4.1 Cement type classification in ASTM C-150

Type	*Use*	*Special requirements*
I	Where other special types not needed	
II	General use, moderate sulphate resistance or moderate heat of hydration	Max. C_3A (8%)
III	For high early strength	
IV	For low heat of hydration	Max. C_3S (35%) Min. C_2S (40%) Max. C_3A (7%)
V	For high sulphate resistance	Max C_3A (5%) Max. C_4AF + 2 C_3A (20%)

4.2.2.1 Normal aggregates

These usually consist of natural materials, hard crushed rock or crushed or natural gravel and their corresponding sands, but artificial materials like crushed brick and blast-furnace slag can also be used. The specific gravity of these materials usually lies between 2.6 and 2.7. Because satisfactory concrete for most purposes can be made with a very wide range of aggregates, local sources of supply usually determine which aggregate will be used. Where very high strength, resistance to skidding, good appearance or other special properties are required, appropriate aggregates will have to be selected, preferably on the basis of previous experience.

For example, the low-speed skidding resistance of concrete roads is affected by the hardness of the sand but only slightly by the polished-stone value of the coarse aggregate.Thus, a hard sand should be chosen for concrete which is to form the surface of a concrete pavement.

Some aggregates have undesirable influences on important concrete properties or are themselves unsound. They should be used with caution, if at all. Examples are aggregates with high drying shrinkages, which may lead to poor durability in exposed concrete, aggregates which react with alkalis in the cement paste, aggregates which are readily oxidized, aggregates which can cause surface staining, and aggregates made from weathered, partially decomposed, rocks.

Other aggregates, although making reasonably satisfactory hardened concrete, for most purposes, may give the fresh concrete poor handling characteristics. Aggregates with flat, flakey, very angular or hollow particles tend to have this effect. In general, aggregates with well-rounded particles in the case of gravels, or near-cubical particles in the case of crushed rock, produce concrete with better workability and fewer voids than aggregates with angular particles.

Natural sands have advantages over crushed rock sands because their particles tend to be more rounded and they contain less very fine material (of 150 μm or less), but crushed rock sands may be preferable, e.g. where the grading of locally occurring natural sands is poor, where the colour of natural sands would be unsatisfactory in weathered concrete (many sands weather to a yellowish colour) or where resistance to slipping is important. General requirements for aggregates to be used in concrete are given in BS 882.

Table 4.2 Properties of concrete using different aggregates

Aggregate	*Typical range of dry density* Aggregate (kg/m³)	Concrete (kg/m³)	*Compressive strength* (N/mm²)	*Drying shrinkage* (%)	*Thermal conductivity at 5% moisture content* (W/m°C)
Flint gravel or crushed rock	1350–1600	2200–2500	20–80	0.03–0.08	1.6–2.2
Crushed limestone	1350–1600	2200–2400	20–80	0.03–0.04	1.6–2.0
Crushed brick	1100–1350	1700–2150	15–30	—	0.85–1.50
Expanded clay, shale or slate and sintered pulverized fuel ash	300–1050	1350–1800	15–60	0.02–0.12	0.55–0.95
Foamed slag	500–950	1700–2100	15–60	0.04–0.10	0.85–1.40
Expanded clay, shale or slate and sintered pulverized fuel ash	300–1050	700–1300	2–7	0.03–0.07	0.24–0.50
Foamed slag	500–950	950–1500	2–7	0.03–0.07	0.30–0.65
Pumice	500–900	650–1450	2–15	0.04–0.08	0.21–0.63
Exfoliated vermiculite and expanded perlite	60–250	400–1100	0.5–7	0.20–0.35	0.15–0.39
Clinker	700–1050	1050–1500	2–7	0.04–0.08	0.35–0.65

4.2.2.2 Lightweight aggregates

These consist of various artificial and natural materials with specific gravities of between 0.1 and 1.2. They are used to make lightweight concrete for structural and insulating applications. In general, concrete made with lightweight aggregates has better fire resistance than dense concrete, but greater shrinkage and moisture movement.

Examples of lightweight aggregates are given below.

(1) *Sintered PFA* is made by heating pellets of PFA until they fuse to form hard spherical lumps.
(2) *Expanded clay, shale, slate and perlite* are made by heating suitable grades of these materials to their fusion temperature (about 1000°C) when they simultaneously fuse and are blown by gases generated within the material.
(3) *Pumice* is a natural lightweight aggregate consisting of a frothy volcanic glass.
(4) *Clinker* consists of fused lumps of fuel residues. To be suitable for use as a concreting aggregate it must be low in sulphates and residual fuel. Limits are given in BS 1156.
(5) *Foamed blast-furnace slag* is made by treating molten blast-furnace slag with water so that the steam which is generated blows the slag. Standards for this material are given in BS 877.
(6) *Exfoliated vermiculite* is made by heating vermiculite (a micalike mineral found in Africa and America) to a temperature of about 700°C when it expands to form a very light material.

Of these aggregates the sintered PFA, and the expanded clay, shale and slate and perlite are the most likely to be encountered.

4.2.2.3 Heavy aggregates

These consist either of natural or artificial materials and are used to make high-density concrete for radiation shielding or ballasting.

Examples of heavy aggregates are barytes, which is a naturally occurring rock consisting of 95% barium sulphate (specific gravity about 4.1; density of concrete up to 3700 kg/m³); iron ores such as magnetite, goethite, limonite and ilmenite (specific gravity about 3.4 to 5.3, density of concrete up to 4200 kg/m³) iron or steel shot (specific gravity 7.7; concrete density up to 5500 kg/m³); lead shot (specific gravity 11.4; concrete density up to 7000 kg/m³) and scrap-iron stampings and punchings. Provided the materials are sound and free from oil, satisfactory concrete of good structural strength can be made, especially if prepared by a method such as prepacking to avoid segregation. Consideration of the higher-density effect on mixing and batching facilities is important.

4.2.2.4 Contaminants, unsound aggregates and reactive aggregates

Aggregates may contain impurities which upset the hydration of the cement or coatings which interfere with bond, or the aggregates themselves may be unstable. To some extent, impurities and surface coating can be removed by suitable treatments, but aggregates which are unsound or reactive must be avoided.[2] Unsound or reactive particles may occur naturally with the aggregate source and may be detected by careful examination of the supply. It is also possible for a small percentage of contamination to occur during transportation or storage of aggregate.

Organic impurities. These may or may not delay or prevent the hydration of the cement and it is best to compare the strength of the concrete made with the contaminated aggregate with the strength of concrete made from similar but clean aggregate. Sugar, sugar-like substances and humic acid are among common contaminants which are known to retard or prevent cement hydration. Products of wood degradation such as 'cellibiose' have a similar effect.

Clay and fine material. These can contaminate aggregates either as a coating on the coarse aggregate or as a constituent of the fine aggregate. As coatings, these materials interfere with bond and therefore reduce concrete strength. As constituents of the mix they are less troublesome unless the quantity is great enough to require the addition of extra water to make the concrete workable. Clay, silt and fine material should not form more than 1% by weight of coarse aggregate, 3% by weight of gravel sand or 15% by weight of crushed rock sand (BS 882).

Salt is usually present in marine deposited or extracted aggregates and in small quantities it is harmless. Efficient washing of the aggregates before use in concrete is capable of reducing the salt to an acceptable level. The salt content should, however, be limited to the levels in Table 4.3 taken from BS 882:1983.

In addition to the limits given in Appendix C of BS 882, there is an overall limit given for the chloride ion from all sources calculated as a percentage by weight of cement given in Table 6.4 of BS 8110.

Table 4.3 Maximum chloride content of aggregates

Type or use of concrete	*Maximum total chloride content expressed as percentage of chloride ion by mass of combined aggregate*
Pre-stressed concrete Steam-cured structural concrete	0.02
Concrete made with cement complying with BS 4027 or BS 4248	0.04
Concrete containing embedded metal and made with cement complying with BS 12	0.06 for 95% of test results, with no result greater than 0.08

Note:
Marine aggregate and some inland aggregate contain chlorides. Both should be selected carefully and may need efficient washing to achieve the limit required for use in pre-stressed concrete.

Nondurable particles. These are sometimes found in aggregates which are otherwise satisfactory. Examples of such particles are lumps of clay, shale, wood or coal. Being soft, they are easily eroded and will lead to pitting or spalling of the concrete surface. If more than about 5% of such particles are present in the aggregate they will also cause strength to be reduced. Although no limits are given for these in BS 882, generally such particles should not form more than 1% of the aggregate by weight. The actual significance of the particles in the structure will be affected by the nature of the structure, e.g. a concrete paving will be more affected by soft particles floating to the surface than will a wall.

Reactive particles. Reactive particles found in some aggregates may be soluble in, or react with, water or the hydrating cement paste. Mica and sulphates, e.g. gypsum, react with cement paste, and iron sulphides, e.g. pyrites and marcasite, react with air and water to form products which then react with the cement paste and cause staining or pop-outs.

Unsound material. This may form the whole of the aggregate or unsound particles may merely contaminate it. Unsoundness is the property of some aggregates to expand or contract excessively as a result of freezing and thawing, wetting and drying, or temperature changes. Such movements can be large enough to cause the aggregate itself to break down or they may disrupt concrete made with it. Examples of unsound aggregates are rocks with very high water absorption, porous cherts, limestones and other sedimentary rocks if they contain laminae of clay, and some shales. Foreknowledge of how such aggregates behave in concrete is the only reliable guide, but freezing and thawing tests may give some indication of an aggregate's unsoundness.

Reactive aggregates. Reactive aggregates are those which react chemically with the cement paste, the most common reaction being between reactive silica and alkalis (in the form of sodium and potassium ions). Reactive silicas occur in opaline and chalcedonic cherts, siliceous limestone, rhyolites, andesite and phyllites. The actual susceptibility of particular aggregate sources needs to be assessed by tests or previous experience. The silica forms a gel with the alkali and this gel expands continuously as it absorbs water, exerting enough force to disrupt the surrounding cement paste in some cases.[1] This phenomenon of alkali silica reactions is well known and recorded. It was first identified some 46 years ago by Stanton in the US. Since then, workers in other countries around the world notably Denmark, Iceland, Germany and South Africa have identified similar reactions. It was believed until recently that the combination of high alkali cements together with reactive aggregates did not occur in the UK. However, a number of cases of alkali silica reaction have now been reported in UK structures built over many years. It is not clear at this time what the extent of these occurrences are or what significance they will have in structural performance. Guidance is available on minimizing the risks of the reaction.[3]

4.2.3 Admixtures

Relatively small quantities of other materials called admixtures can be added to concrete to modify its properties in either fresh or hardened state. There are several classes of admixtures which are listed below.

The British Standard for admixtures BS 5075 is in separate parts for each class of admixture.

4.2.3.1 Water-reducing admixtures and workability aids (BS 5075, Part 1)

These materials are also commonly called plasticizers and have the effect of making concrete more workable for a given water content. They can also reduce the water:cement ratio for a constant workability and can therefore be used to improve strength development.

These materials can also entrain a little air in the concrete or, if used in too high a dosage, can cause retardation of the cement setting. If used as a result of trial mixes or in accordance with the manufacturer's recommendations these side-effects should not be significant under normal site conditions.

Plasticizers for mortars are used to give plasticity or cohesion. They function by entraining large amounts of air which, as a side-effect, reduces strength. This modification to mortars should be carried out using only admixtures specifically formulated for the particular use.

4.2.3.2 Superplasticizers and high-range water-reducing admixtures (BS 5075, Part 3)

These more specialized admixtures perform similar functions to normal plasticizers but with increased effectiveness. Very high workability or flowing concrete is a common application. Because of their very effective action on the fluid properties of the concrete, much closer control of the initial mix design and subsequent batching is needed to prevent excessive bleeding or segregation of the mix. Many of the general superplasticizing admixtures have a relatively limited activity and the concrete workability may fall back to normal levels after approximately 30 to 45 min.

4.2.3.3 Air-entraining agents (BS 5075, Part 2)

These are widely used admixtures, especially for paving concrete. Their importance is related to the capacity of concrete containing a small amount of air in the form of well-distributed small bubbles to have greater resistance to the destructive action of freezing and thawing when the concrete is saturated than similar concrete made without air-entraining agents. The freezing and thawing action is made more severe when de-icing salts are used, or can be brought on to the surfaces by vehicles. In such circumstances the use of air entrainment is strongly recommended in codes of practice. This increased durability is gained at the expense of some strength and it is therefore important to control the amount of entrained air between close limits.

The amount of air that will be entrained with a given addition of an air-entraining agent is influenced by the grading of the sand, the workability of the concrete, the type of mixer and the duration of mixing. Trial mixes are essential to establish how much of each agent is to be added. Frequent regular measurements must be made throughout the work to ensure that the correct air content is being maintained (see page **4**/17). Some difficulty may be experienced when using fine sands, sands with an organic or carbon content or when PFA and ground granulated blast-furnace slag materials are incorporated in the mix constituents.

As well as being more resistant to damage from de-icing salts, air-entrained concrete is somewhat more cohesive than concrete made without an air-entraining agent and tends to have slightly higher workability, a factor which can be used partly to offset the strength reduction.

4.2.3.4 Accelerators and 'antifreezes' (BS 5075, Part 1)

These are used to hasten the hardening of concrete, particularly in cold weather. The term 'antifreeze' is misleading because these admixtures merely lessen the period when frost damage is likely; they do not prevent concrete from freezing. Since the prohibition of the use of chloride-based accelerators as a result of corrosion of embedded steel, other proprietary products, often based on calcium formate, have been developed. Such admixtures are much less efficient at accelerating the strength development and therefore are less attractive to use. There may also remain some uncertainty about the risks of inducing corrosion. Alternative procedures for protecting concrete or mortars from frost, such as heated materials and adequate protection for the formed work, may be preferable.

4.2.3.5 Retarders (BS 5075, Part 1)

These have the effect of delaying the onset of hardening and usually also of reducing the rate of the reaction when it starts. Ultimate strengths are unaffected by retardation for several hours but may be reduced if the addition of retarder is excessive. Accidental overdosage may cause retardation of a few days or it may prevent hardening altogether. The fear that this may happen is probably one of the reasons why retarders are seldom used in the UK. Nevertheless, retarders can be beneficial where large volumes of concrete have to be poured in one operation or where high ambient temperature conditions prevail which lead to rapid setting. Care must be taken in this situation that the rapid set is not the result of rapid moisture loss by evaporation. Trial mixes are essential to determine the dosage at which the retarder is to be used.

4.2.3.6 Mixed admixtures

Mixed admixtures containing a variety of materials are available. Examples are combinations of an air-entrainment admixture with water-reducing admixture, or water-reducing and retarding admixtures.

4.2.3.7 Other admixtures

These include waterproofers, viscosity modifiers, resin bonding agents, fungicides, etc. They may be useful for specific applications, but the claims made for them should be supported by impartial test results. This applies particularly to the permanence of the effects claimed.

Pigments may be incorporated in concrete mixes. If bright or pastel shades are wanted, white cement and light-coloured sand must be used for the basic concrete, but low-key colours and dark shades can be obtained with ordinary concrete. The pigments must be stable in cement, fast to light and resistant to being washed out by weathering. Requirements are given in BS 1014.

Although a number of organic pigments can be used in concrete, the most commonly used are iron oxides for red, brown, yellow and black, and chromium oxide for green. Synthetic iron oxides have better staining power than natural ones and are available in a greater colour range. Although more expensive than natural oxides, they may be cheaper in use. Carbon black gives a more intense black than iron oxide, but because it is often greasy it is difficult to disperse and has the reputation of being easily washed out. Pigment additions vary typically from about 2 to 10% or more by cement weight. Some strength reduction should be expected with the larger rates of addition.

4.2.4 Concrete mix design

4.2.4.1 General

The purpose of concrete mix design is to choose and proportion the ingredients used in a concrete mix to produce economical concrete which will have the desired properties both when fresh and when hardened. The variables which can be controlled are: (1) water:cement ratio; (2) maximum aggregate size; (3) aggregate grading; (4) aggregate:cement ratio; and (5) use of admixtures.

Interactions between the effects of the variables complicate mix design and successive adjustments following trial mixes are usually necessary. Experience built up by ready-mix concrete producers should enable them to produce suitable mix designs more quickly than this. Many different methods of mix design have been developed, one relatively simple method is given by Teychenné, Franklin and Erntroy.[5]

4.2.4.2 Water:cement ratio

Many of the most important properties of fully compacted hardened concrete and strength in particular are for normal concrete virtually decided by the water:cement ratio of the mix. The importance of this parameter is due to the fact that any excess of water over that needed to hydrate the cement (about 25% by weight) forms voids in the concrete, thus reducing its density. The reduced density leads to reduced compressive, tensile and bond strengths, lower durability, lower resistance to abrasion and greater permeability to water. Excess water cannot be eliminated altogether because it is needed to lubricate the mix and make it workable, but it should be kept to a minimum.

Figure 4.1 shows how strength is influenced by water:cement ratio and the first step in concrete mix design is to fix the water:cement ratio from a knowledge of the strength required. The shape of the curves will be similar for all types of Portland cements but the actual relationship between strength and water:cement ratio will be different for each cement source.

4.2.4.3 Workability

When the concrete is fresh it must be workable or fluid enough to be compacted easily under the conditions in which it will be