Programming the Dynamic Analysis of Structures

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SPON PRESS
Taylor & Francis Group

First published 2002 by Spon Press
11 New Fetter Lane, London EC4P 4EE

Simultaneously published in the USA and Canada
by Spon Press
29 West 35th Street, New York, NY 10001

Spon Press is an imprint of the Taylor & Francis Group

Publisher's Note:
This book has been prepared from camera-ready-copy supplied by the author.

Camera-ready-copy prepared by the author.
Printed and bound by T.J. International, Padstow, Cornwall.

British Library Cataloguing in Publication Data
A catalogue record for this book is available from the British Library

Library of Congress Cataloging in Publication data
A catalog record for this book has been requested

ISBN 0-419-15610-0

Dedicated with affection to the younger generation

(Arun, Ranjana, Ramendra, Sashidhar, Sudeep, Sujaatha, Sumeeta, and Vinod).

CONTENTS

PREFACE

The subject of structural dynamics is an important branch of structural engineering. Its importance has increased with the need for more and more flexible structures such as tall buildings, long span bridges, cable roofs, etc., many founded in earthquake prone areas of the world. Many books covering either the theoretical aspects of structural dynamics or the mathematical aspects of eigen-value problems or the finite element method or listing of finite element programs are available, but rarely in the same book. The calculations involved in structural dynamics are by their very nature numerically intensive and the use of a digital computer is almost obligatory, even for the solution of simple problems. The existing books do give some simple programs, almost as an afterthought, but rarely in any detail.

The object of this book is to provide, in a single comprehensive volume, both the theory and the associated computer programs. Only methods amenable to automatic computation are included and all hand calculation methods are omitted. Full details of the theory of solution methods for linear eigen-value problems, finite element and finite strip methods are given. All concepts are explained in detail and illustrated by numerous fully worked out numerical examples. Corresponding elementary programs follow many of the methods. These programs have two main aims. Firstly to teach the reader the steps needed to convert theory to computer programs and secondly to provide programs which the reader can use to check their working of numerical examples. These programs are finally translated to 'full blown' programs in FORTRAN-90 for the eigen-value solution of two-dimensional rigid-jointed frames, plane grids, elastic plates, with the dynamic stiffness matrices established using finite element and finite strip methods. Programs are also provided for the solution by direct integration of differential equations. It is hoped that the book will be welcomed both by students studying courses in structural dynamics and also by practicing engineers.

As in any such undertaking, many people have helped and I express my sincere thanks to all. To my colleague Mr Robert Watson for some of the diagrams, to Mrs Tessa Bryden for enthusiastic secretarial assistance, to the late Ernie Hinton for enthusiastic review of the initial proposal and subsequent unstinting encouragement and provision of his programs, to Sheila, Arun and Ranjana for constantly asking me 'When is the book going to finish?', to the many editors at Spon Press for mild pressure and for being very understanding. Finally when deciding on the 'balance' between various topics of the book, to Ovid for the advice 'Medio tutissimus ibis' (you will go most safely in the middle).

P. Bhatt
2nd October (Mahatma Gandhi's birthday) 2001, Glasgow

CHAPTER 1

SINGLE DEGREE OF FREEDOM SYSTEM - I

1.1 INTRODUCTION

Structures in practice are subjected to a variety of forces, both static and dynamic. Static forces, such as gravity forces, remain constant with time. Structures are also subjected to dynamic forces, which vary with respect to time. Some of these forces can act over a long period of time while others act over a relatively short period. Typical of forces which act over a long period of time are vehicular loading on a bridge and wind loads on buildings. On the other hand, forces due to an earthquake or an explosion act over a fairly short period of time. In the vast majority of design situations, the dynamic forces affect the serviceability limit state. For example, vibrations could cause discomfort to users of a structure such as a building or a bridge. In some cases vibrations could lead to the malfunctioning of delicate apparatus. In extreme cases, dynamic forces could affect the ultimate limit state by causing collapse due to violent shaking during an earthquake or cause fatigue failure of joints and components. As will be shown later, dynamic properties of a structure are governed mainly by the mass and stiffness of the structure. Many design advances of recent years, such as the use of higher strength materials, the use of welding or friction grip bolts in steel structures, the wide spread use of prestressed concrete, the emergence of flexible structures such as long span bridges and tall buildings, have all made structures more sensitive to dynamic forces. Many of the changes in construction practice have also reduced the inherent damping present in structures, making them more susceptible to vibration. It is for these reasons that the study of structural dynamics has assumed great importance. The nature of the dynamic forces that act on a structure vary widely. Some, such as the force due to a rotating machine, can be described almost completely as a function of time both in magnitude and direction. Such forces are deterministic forces. On the other hand, forces due to wind can only be described in terms of statistical properties, such as mean and standard deviation. Such forces are called stationary random forces. Forces due to an earthquake are even more complicated. Each earthquake is almost unique. Earthquake forces cannot be described even in statistical terms. Such forces are called non-stationary random forces. In this book, only analysis of structures subjected to deterministic force is considered.

1.2 INERTIAL FORCE

According to Newton's second law of motion, the resultant force acting on a particle and the corresponding acceleration are related by the equation

Resultant force = Mass x Acceleration

If a particle is in static equilibrium, then the acceleration is equal to zero and, hence, the resultant force should also be equal to zero. It should be remembered that since both force and acceleration are vector quantities, the relationship is a vector equation.

In contrast to a problem in statics, while considering equilibrium in a dynamic problem, acceleration is not equal to zero. Newton's second law can be restated as

Resultant force on the mass - Mass x Acceleration = 0

This shows that a dynamic problem can be treated as an equivalent static problem by including, in addition to external forces acting on the mass, an additional force equal to - (mass x acceleration). This additional force is called inertial force. This way of looking at equilibrium under dynamic situation is called D'Alembert's principle.

1.3 DAMPING FORCE

When a structure is vibrating, it moves relative to the surrounding medium, such as air as in the case of most structures or water as in the case of structures such as oil rigs. The surrounding medium resists motion of the structure and causes additional forces to act on the structure. These forces generated by the relative motion with respect to the surrounding medium are called damping forces. It should be appreciated that damping forces can also arise due to relative motion between parts of the structure such as at a bolted joint in a steel structure or across cracks in a reinforced concrete structure.

1.4 SINGLE DEGREE OF FREEDOM SYSTEM

Consider the simple 'structure' shown in Fig. 1.1. The mass M is attached to the support through a 'weightless' spring of stiffness K and a dashpot simulating damping normally present in all structures. It is assumed for simplicity, that the mass can move only horizontally. In other words, the structure has only one degree of freedom of movement.

Figure 1.2 shows some real structures modelled for mathematical purposes as single degree freedom systems. Fig. 1.2a shows a simply supported beam of span L and flexural rigidity EI, supporting a concentrated mass at midspan. The 'spring' stiffness K is the force required to cause unit displacement at the midspan of a simply supported beam. Therefore $K = 48\ EI/L^3$.

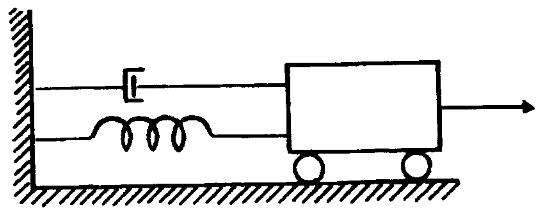

Fig. 1.1 Mass-spring-damper System

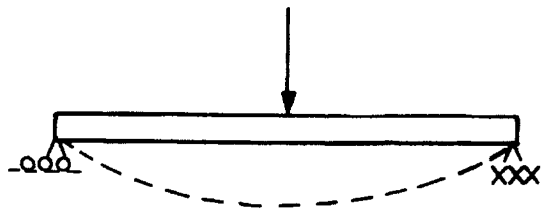

Fig. 1.2a A simply supported beam

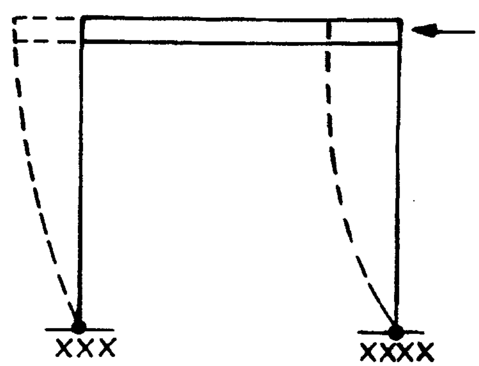

Fig. 1.2b Single bay portal frame

Figure 1.2b is a single bay portal. The height of the columns is H and their flexural rigidity is EI. It is assumed that the beam is sufficiently rigid to prevent rotation of the columns at the top so that the motion of the mass can be described by the sway displacement at the top of columns. In this case, the 'spring' stiffness K is the force required to cause unit sway displacement of the two legs of the portal frame.

Fixed feet columns, $K = 2$ legs x $\{12\ EI/H^3]$.

Pinned feet column, $K = 2$ legs x $\{3\ EI/H^3]$.

The assumption that the spring is 'weightless' is made purely to simplify the problem at this early stage of discussion. In reality, the columns and beams, which contribute to spring stiffness, are not weightless.

In Fig. 1.1, the damping present in the structure is shown by a dashpot. It should be appreciated that in real structures, in general, there are no identifiable dashpots causing damping. A dashpot is only a simple way of modelling the presence of damping in structures.

The rest of this chapter is devoted to the study of single degree of freedom systems. The study of SDOF systems is important because it brings out many of the important properties affecting the behaviour of structures subjected to dynamic loads. Very often in practice, an SDOF system is the simplest idealisation used for the preliminary study of quite complex structures. In addition, as will be shown in Chapter 3, multi-degree freedom systems can be 'reduced' to a series of SDOF systems, thus facilitating the study of complex systems.

1.5 MATHEMATICAL STUDY OF THE SDOF SYSTEM

The differential equation governing the behaviour of the system shown in Fig. 1.1 is established quite simply by using the concept of dynamic equilibrium. Consider the forces acting on the free body shown in Fig. 1.3.

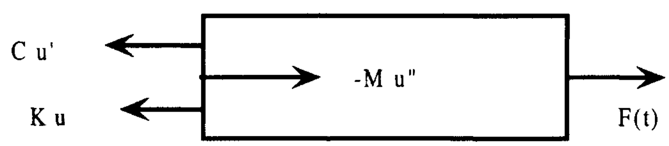

Fig. 1.3 Forces on the free body

The forces acting on the mass are the spring force, damping force, inertial force and external force. It should be remembered that the damping force always opposes motion. The dynamic equilibrium requires that the sum of the forces is equal to zero. Assuming that the positive direction of motion is to the right, let the displacement of the mass be u(t). The forces acting on the mass are

i. Spring force acting on the mass = K u(t)

ii. Inertial force = -M $\dfrac{d^2u}{dt^2}$

Note that the inertial force acts in the positive direction.

iii. Damping force

It is conventional to assume that damping force is proportional to the velocity of the mass. Therefore

Damping force = C $\dfrac{du}{dt}$

C = Coefficient of viscous damping.
iv. External force = $F(t)$, which is a function of time, t.
Summing up the forces to zero for dynamic equilibrium, we have

$$M\frac{d^2u}{dt^2} + C\frac{du}{dt} + Ku = F(t) \tag{1.1}$$

For convenience in writing mathematical expressions, let

$$K/M = \omega^2,\ C/M = 2\beta \tag{1.2}$$

The differential equation (1.1) can therefore be expressed as

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \omega^2 u = \frac{F(t)}{M} \tag{1.3}$$

1.6 INFLUENCE OF GRAVITATIONAL FORCES

Consider the system shown in Fig. 1.4. The system is identical in all respects to that shown in Fig. 1.1, except that the motion is vertical. Under static conditions, the displacement of the mass is equal to the extension Δ of the spring due to the weight W. Therefore, Δ is equal to W/ K. Under dynamic conditions, when considering the forces acting on the mass, in addition to the forces shown in Fig. 1.3, we have to include the weight, W = Mg, g = acceleration due to gravity. The equilibrium equation is therefore given by

$$M\frac{d^2(u+\Delta)}{dt^2} + C\frac{d(u+\Delta)}{dt} + K(u+\Delta) = W + F(t) \tag{1.4}$$

In the above equation, the displacement u is measured from the static position. Therefore, (u + Δ) is the total displacement. Since Δ is a constant and KΔ = W, the above equation simplifies to equation (1.1). This shows that equation 1.1 is applicable to cases where acceleration due to gravity is to be included provided that the dynamic displacement is measured from the static position of rest as the origin.

1.7 SOLUTION OF THE DIFFERENTIAL EQUATION

The differential equation (1.1) is an ordinary differential equation with constant coefficients. The solution is obtained as the sum of a complementary solution and the particular integral.

Complementary solution is the solution to the equation when F(t) is equal to zero. Therefore, complementary function is the solution of the equation

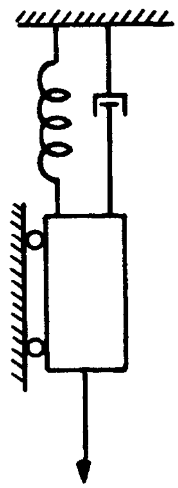

Fig. 1.4 Mass-spring-damper system

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \omega^2 u = \frac{F(t)}{M} \quad (1.5)$$

The solution to $u(t)$ in the above equation is obtained by assuming that

$$u(t) = A\, e^{\alpha t} \quad (1.6)$$

where A is an arbitrary constant and α is yet to be determined.
Differentiating $u(t)$ with respect to time t, we have

$$u'(t) = \alpha A e^{\alpha t} = \alpha\, u(t) \quad (1.7)$$

$$u''(t) = \alpha^2 A e^{\alpha t} = \alpha^2 u(t) \quad (1.8)$$

Substituting equations (1.7) and (1.8) in equation (1.5), and simplifying

$$(\alpha^2 + 2\,\alpha\beta + \omega^2)\, u(t) = 0 \quad (1.9)$$

Since u(t) is not equal to zero, the terms inside the bracket must be zero

$$\alpha^2 + 2\,\alpha\beta + \omega^2 = 0 \quad (1.10)$$

This is called the characteristic equation of the differential equation. Since the characteristic equation is a quadratic equation, the roots are given by

$$\alpha_1,\, \alpha_2 = -\,\beta \pm \sqrt{(\beta^2 - \omega^2)} \quad (1.11)$$

The complementary equation is thus given by

$$u(t) = A_1 e^{\alpha_1 t} + A_2 e^{\alpha_2 t} \quad (1.12)$$

The constants A_1 and A_2 are determined from the initial conditions prescribing the displacement and velocity at t = 0.

The particular integral is the solution of the equation, when the force F(t) is present but without any reference to the initial conditions.

Because the complementary function part of the solution depends on the initial boundary conditions and exists even when the external force F(t) is zero, the complementary function is called the natural motion solution. Similarly, because the particular integral part of the solution depends on the external force, this solution is called the forced motion solution.

Depending on the presence of damping and external force, the solution to the differential equation is obtained for two distinct cases as follows.

i. Free vibration: Free vibration refers to the case when the external force is equal to zero and the motion results when the system is disturbed from its state of rest and allowed to vibrate. If damping is equal to zero, then such a motion is called undamped free vibration. On the other hand, if damping is present, then the resulting motion is called damped free vibration.

ii. Forced vibration: When vibration takes place due to an external vibratory force acting on the system, then the resulting motion is described as forced vibration. As in the case of free vibration, depending on the presence or absence of damping, one can have damped forced vibration or undamped forced vibration respectively.

1.8 SOLUTION TO THE FREE VIBRATION PROBLEM

The solution to $u(t)$ is given by the equations (1.11) and (1.12). Depending on the sign of $(\omega^2 - \beta^2)$ in equation (1.11), three possible cases arise, as follows.

i. $\beta < \omega$. This is called an underdamped case for reasons to be explained in the next section.

Setting $\omega_d = \sqrt{(\omega^2 - \beta^2)}$, then from equation (1.11),

$\alpha_1, \alpha_2 = -\beta \pm i\,\omega_d$, $i = \sqrt{-1}$

The roots α_1, α_2 are complex. Substituting for α_1, α_2 in equation (1.12) u(t) is given by

$$u(t) = A_1\, e^{(-\beta + i\,\omega_d)t} + A_2\, e^{(-\beta - i\omega_d)t} \tag{1.13}$$

where A_1 and A_2 are conjugate complex constants to be determined from boundary conditions.

Noting that

$$\cos \omega_d t = 0.5\{e^{i\,\omega_d t} + e^{-i\omega_d t}\}$$

$$\sin \omega_d t = -0.5i\{e^{i\,\omega_d t} - e^{-i\omega_d t}\}$$

$$u(t) = e^{-\beta t}\{B_1 \cos\omega_d t + B_2 \sin\omega_d t\} \tag{1.14}$$

where B_1 and B_2 are real constants of integration to be determined so as to satisfy the initial conditions.

ii. $\beta = \omega$. This is called the critically damped case. In this case there are two real repeated roots $\alpha_1 = \alpha_2 = -\beta$. The solution is given by

$$u(t) = e^{-\beta t}\{B_1 + B_2\, t\} \tag{1.15}$$

where B_1 and B_2 are integration constants.

iii. $\beta > \omega$. This is called the over damped case. In this case there are two real roots. The solution is given by

$$u(t) = A_1\, e^{(-\beta + \omega_d)t} + A_2\, e^{(-\beta - \omega_d)t} \tag{1.16}$$

The hyperbolic sine and cosine functions are related to the exponential functions by

$$\cosh\omega_d t = 0.5\,\{e^{\omega_d t} + e^{-\omega_d t}\}$$
$$\sinh\omega_d t = 0.5\{e^{\omega_d t} - e^{-\omega_d t}\}$$

$$u(t) = e^{-\beta t}\{B_1 \cosh\omega_d t + B_2 \sinh\omega_d t\} \tag{1.17}$$

The solutions to the above three cases are discussed in more detail in the next section.

1.9 UNDAMPED FREE VIBRATION

The displacement $u(t)$ is given by equation (1.14). The constants B_1 and B_2 are determined from initial conditions. If the system is disturbed from its initial stationary position by giving an initial displacement of u_0 and an initial velocity of u_0' at t = 0, then $u_0 = B_1$ and $u_0' = B_2\,\omega$.

$$u(t) = u_0 \cos\omega t + \{\frac{\dot{u}_0}{\omega}\} \sin\omega t \tag{1.18}$$

The above expression can be expressed more elegantly as follows. Let

$$u_0 = R\cos\theta\ ,\ \frac{\dot{u}_0}{\omega} = R\sin\theta,$$
$$R = \sqrt{\{u_0^2 + (\frac{\dot{u}_0}{\omega})^2}\ ,\ \theta = \tan^{-1}(\frac{\dot{u}_0}{\omega u_0}) \tag{1.19}$$

$$u(t) = R(\cos\omega t\ \cos\theta + \sin\omega t\ \sin\theta) = R\cos(\omega t - \theta) \tag{1.20}$$

This indicates that the motion is described by a cosine curve with an amplitude equal to R and a time lag of $t_o = \theta/\omega$.

It is perhaps worth noting that the frequency of vibration is independent of the amplitude R. This is of course true only if the amplitude is not large enough to invalidate the basic assumptions involved in the derivation of the equations of motion.

Since the trigonometric cosine and sine functions are periodic functions with a period of 2π, the motion is periodic with a period T. The system vibrates with a period of T or a frequency f. The three quantities ω, f and T are related as follows

$$\text{circular frequency } \omega = \sqrt{\frac{K}{M}}\ \omega \text{ radians/seconds} \tag{1.21a}$$

$$\text{frequency } f = \frac{1}{T} = \frac{\omega}{2\pi} \text{ Hertz} \tag{1.21b}$$

$$\text{period } T = \frac{1}{f} \text{ seconds} \tag{1.21c}$$

1.9.1 Units Used in Dynamic Analysis

In vibration problems, both mass and force units occur. It is therefore important that a consistent system of units is used. It is suggested that the following consistent system be adopted. By definition, a force of 1 N acting on a mass of 1 kg produces an acceleration of 1 m/sec^2. Therefore, consistency of units is satisfied when length is expressed in metres, force in Newtons, mass in kilograms and time in seconds.

1.9.2 Example

A simply supported beam of 3 m span supports a load at midspan of 100 kN. Calculate the natural frequency of vibration. It is given that the second moment of area I of the beam is equal to 11710 cm^4, Young's modulus E = 210 kN/mm^2.
Solution: Express all parameters in terms of units specified in Section 1.9.1.
I = 11710 cm^4 = 11710 x 10^{-8} m^4, E = 210 kN/mm^2 = 210 x 10^3 x 10^6 N/m^2

$$EI = 24.591 \times 10^6 \text{ Nm}^2,\ L = \text{span} = 3 \text{ m}$$

As the beam is a simply supported and carries a mass at the midspan,

$$K = 48\ EI/L^3 = 48 \times [24.591 \times 10^6]/3^3 = 43.717 \times 10^6 \text{ N/m}$$

Load W acting at midspan = 100 kN = 100 x 10^3 N
Assuming that the gravitational constant g $\approx$ 10 m/sec/sec, the corresponding mass M = W/g. Therefore

$$M = 100 \times 10^3 / g = 10 \times 10^3 \text{ kg}$$

$$\omega = \sqrt{\{K/M\}} = 66.11 \text{ radians/second}$$

$$f = \frac{\omega}{2\pi} = 10.52 \text{ Hertz},\ T = \frac{1}{f} = 0.095 \text{ seconds}$$

Figure 1.5 shows a plot of variation of $u(t)$ with t. Calculations were made on the assumption that $u_0' = 0$. As can be seen, the motion is a simple harmonic and repeats itself indefinitely.

1.10 DAMPED FREE VIBRATION

Damped free vibrations are similar to undamped free vibrations considered in Section 1.9, except that the damping constant C is not equal to zero. We have to consider three separate cases as follows.

1.10.1 Underdamped System ($\beta < \omega$)

In this case, the solution is given by equation (1.14). Assuming that the system is given at time t = 0, $u(t) = u_o$ and $u'(t) = u_o'$ then

$$u_o = B_1 \text{ and } u_o' = -\beta B_1 + B_2\, \omega_d$$

$$B_1 = u_o,\ B_2 = \{u_o' + \beta\, u_o\}/\, \omega_d$$

$$u(t) = e^{-\beta t}\,[u_o \cos\omega_d t + \{(u_o' + \beta\, u_o)/\omega_d\}\sin\omega_d t] \qquad (1.22)$$

Using steps similar to those used in deriving equation (1.19)

$$u(t) = e^{-\beta t} R \cos(\omega_d t - \theta) = e^{-\beta t} R \cos(\omega_d t - \theta) \qquad (1.23)$$

$$R^2 = u_o^2 + \{(u_o' + \beta\, u_o)/\omega_d\}^2,\ \tan\theta = \{u_o' + \beta\, u_o\}/(u_o\, \omega_d)$$

As can be seen, $\cos(\omega_d t - \theta)$ is periodic with a period $T_d = \omega_d/(2\pi)$ but the presence of $e^{-\beta t}$ term damps out the vibration. As the value of C is increased, the vibrations are damped out at a greater rate per cycle.

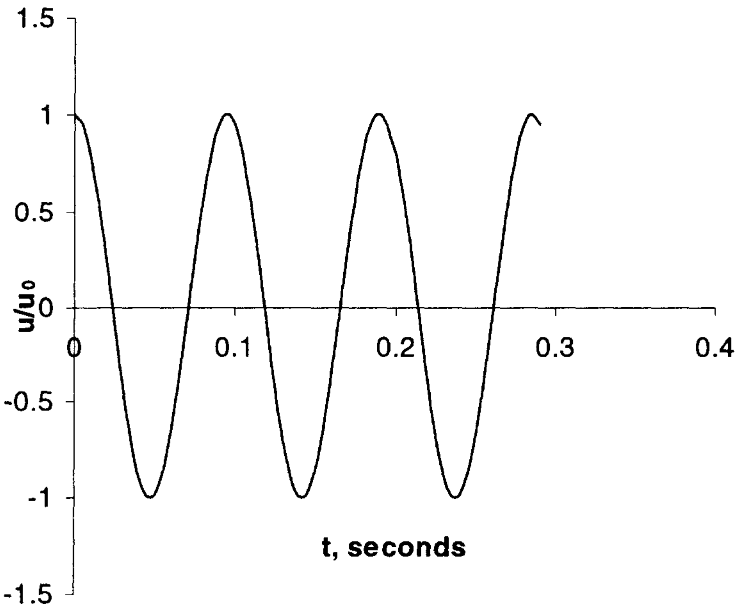

Fig. 1.5 Undamped free vibration

1.10.2 Critically damped case ($\beta = \omega$)

The solution to u(t) is given by equation (1.15). Assuming that $u(t) = u_o$ and $u'(t) = u_o'$ at t = 0,

$$B_1 = u_o,\ B_2 = \beta\, u_o + u_o'$$

$$u(t) = e^{-\beta t}\{u_o(1 + \beta\, t) + u_o'\, t\} \qquad (1.24)$$

As can be seen, vibratory motion is completely damped out. The smallest amount of damping constant C required to damp out all vibration is called critical damping C_{cr}.

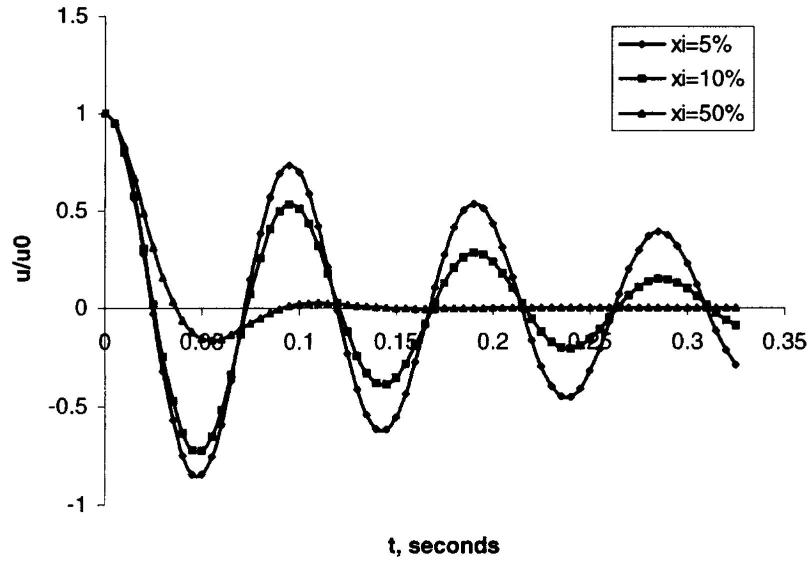

Fig. 1.6 Damped free vibrations

1.11 CRITICAL DAMPING

Critical damping C_{cr} can be expressed in terms of M and K as follows. At critical damping $\beta = \omega$, but $\beta = C_{cr}/(2M)$ and $\omega = \sqrt{(K/M)}$. Therefore

$$C_{cr} = 2\sqrt{(KM)} \tag{1.25}$$

Using the expression for C_{cr}, the expression for ω_d in the case of an under damped system is given by

$$\omega_d = \sqrt{(\omega^2 - \beta^2)} = \omega\sqrt{\{1 - (\beta/\omega)^2\}}$$

Substituting for $\beta = C/(2M)$, $\omega = \sqrt{(K/M)}$, $C_{cr} = 2\sqrt{(KM)}$

$$(\beta/\omega)^2 = C^2/(4KM) = (C/C_{cr})^2$$

Letting $C/C_{cr} = \xi$, damping ratio and $\beta = \xi\,\omega$

$$\omega_d = \omega\sqrt{\{1 - \xi^2\}} \tag{1.26}$$

Fig. 1.6 shows a plot of the variation of $u(t)$ with t. Calculations were made for the same simply supported beam considered in Section 1.9.2 assuming that $u_0^{'} = 0$ and using damping ratio ξ of 5%, 10% and 50%.

1.12 DAMPING IN STRUCTURES

Damping in structures has various origins. The simplest is friction due to relative movement between parts of structures, for example at bolted joints in steel structures and at cracked surfaces in the case of concrete structures. This type of frictional damping is called Coulomb damping. It is independent of velocity. On the other hand when a body moves in a fluid, such as air or water, then the resistance is normally proportional to the square of velocity. In the interest of simplicity, damping force is often assumed to be proportional to velocity. This type of damping is called linear 'Viscous damping'. In addition to the above external causes, there is also energy dissipated due to internal friction caused by slipping and sliding of particles at internal planes during deformation. The phenomenon of damping in structures is thus very complex involving many causes. It is almost impossible to determine what proportion of damping can be assigned to a particular aspect. Therefore in practice the value of damping present is assumed as a percentage of critical damping. Some experimental data obtained from measurements on actual structures, such as buildings and bridges, exist. These values provide reasonable guidance for practical calculation. Table 1.1 shows typical values of C/C_{cr} met in practice. As can be seen, in practice the damping present is such that $C/C_{cr} < 15\%$. If $C/C_{cr} = 0.15$, then $\omega_d = 0.99\ \omega$. Therefore it can be concluded that the effect of a small amount of damping is mainly to damp out the vibrations but leaving the frequency of vibration practically unaltered from the undamped value.

Table 1.1 Damping ratios in practical structures

	$C/C_{cr}\%$	
Type and condition of structure	Working stress	Near yield
Welded steel, prestressed concrete, well reinforced concrete	2-3	5-7
Reinforced concrete with considerable cracking	3-5	7-10
Bolted steel, wood structures with nailed or bolted joints	5-7	10-15

In practical calculations it is safe to ignore damping, because it has the effect of reducing the stresses under dynamic loading. As will be shown in Chapter 2, in the case of sudden dynamic loading, such as wind gusts or earthquake disturbance, there is generally insufficient time for damping to have any significant effect. However, it is desirable to include it in the case of continuous dynamic loading.

1.13 OVERDAMPED SYSTEM

As indicated in Section 1.12, the amount of damping present in practical situations is very small. The case of an overdamped system where ($\beta > \omega$) does not commonly occur in problems of structural engineering interest. The general expression for displacement is given by equation 1.17.

Assuming that the system is given at time t = 0, $u(t) = u_o$ and $u'(t) = u_o'$ then

$$u_o = B_1 \text{ and } u_o' = -\beta B_1 + B_2\, \omega_d$$
$$B_1 = u_o,\ B_2 = \{u_o' + \beta u_o\}/\omega_d$$
$$u(t) = e^{-\beta t}\,[u_o \cosh\omega_d t + \{(u_o' + \beta u_o)/\, \omega_d\}\sinh\omega_d t] \tag{1.27}$$

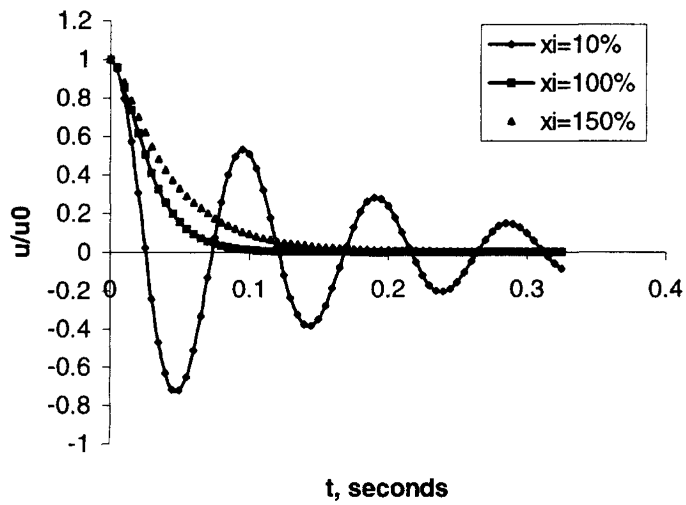

Fig. 1.7 Underdamped, critically damped and overdamped systems

It is interesting to note that from equation (1.27), that although both cosh ω_d t and sinh ω_d t, tend to infinity as t tends to infinity, but because of the presence of the $e^{-\beta t}$ term, vibrations are damped out. Fig.1.7 shows a plot of displacement of the simply supported beam considered before for three cases of damping of viz. underdamped with $\xi = 0.10$, critically damped with $\xi = 1.0$ and an overdamped system with $\xi = 1.5$. It is worth noting that the displacements of an overdamped system are larger than that of the critically damped system, although in both cases all vibratory motion is completely suppressed and displacements fade out exponentially.

1.14 MEASUREMENT OF DAMPING

In the case of underdamped systems, which are of practical interest, a simple measure of damping is the ratio of displacements at times one cycle apart. If T_d is the period of vibration, using equation (1.14), the ratio between displacements T_d apart can be calculated as follows

$$u(t) = e^{-\beta t}\{B_1 \cos\omega_d t + B_2 \sin\omega_d t\}$$
$$u(t + T_d) = e^{-\beta(t + T_d)}\,\{B_1 \cos\omega_d(t + T_d) + B_2 \sin\omega_d(t + T_d)\}$$

$$T_d = 2\pi / \omega_d,\ \omega_d = \omega\sqrt{\{1 - (C/C_{cr})^2\}}$$
$$\cos\omega_d(t + T_d) = \cos(\omega_d t + 2\pi) = \cos\omega_d t$$
$$\sin\omega_d(t + T_d) = \sin(\omega_d t + 2\pi) = \sin\omega_d t$$
$$u(t + T_d) = e^{-\beta(t + T_d)}\{B_1 \cos\omega_d t + B_2 \sin\omega_d t\}$$

Therefore the ratio of displacements T_d apart is given by

$$u(t)/u(t + T_d) = e^{\beta T d}$$
$$\beta T_d = \log_e [u(t)/u(t + T_d)]$$

The term $\log_e [u(t)/u(t+T_d)]$ is called log decrement δ.
As

$$\beta = \xi\,\omega,\ \omega_d = \omega\sqrt{\{1 - \xi^2\}},\ T_d = 2\pi / \omega_d$$
$$\delta = \beta\, T_d = [2\pi\, \xi]/\sqrt{\{1 - \xi^2\}}$$

Since the damping ratio ξ is generally less than 0.10,

$$\delta \approx 2\,\pi,\ \xi \approx \delta/(2\pi) \tag{1.28}$$

Note that if $\xi = 0.10$, then $\delta = 0.63$ and the ratio of displacements at times T_d apart is equal to $e^{\beta Td} = 1.88$. Since $1/1.88 \approx 0.5$, as a rough rule of thumb, 10% critical damping reduces the amplitude by 50% per cycle. From the point of view of calculating the damping ratio from an actual displacement versus time plot, if the ratio of displacements n cycles apart is used, then

$$u(t)/u(t + nT_d) = e^{\beta n\, Td}$$
$$n\,\beta T_d = \log_e [u(t)/u(t + nT_d)$$

$$n\,\delta = \log_e [u(t)/u(t + nT_d)] \tag{1.29}$$

This allows a damping ratio to be calculated to a better accuracy than if the ratio of displacements just T_d apart are used.

In general, using accelerometers, it is easier to measure acceleration at a point in a structure rather than the corresponding displacement, because measurement of displacement requires a datum. Assuming that the system is given at time t = 0, u(t) = u_o and u'(t) = 0 then, using equation (1.23),

$$u(t) = e^{-\beta t} R \cos(\omega_d t - \theta)$$
$$R^2 = u_o^2 + \{\beta\, u_o\}\ \omega_d\}^2$$
$$\tan\theta = \{u_o' + \beta\, u_o\}/(u_o\, \omega_d)$$

Differentiating displacement twice with respect to t, acceleration is given by

$$U''(t) = e^{-\beta t} R\{(\beta^2 - \omega_d^2) \cos\omega_d(t - t_o) + 2\,\beta\, \omega_d \sin\omega_d(t - t_o)\}$$

Using the same steps as used in deriving equation (1.22) in Section 1.9, the acceleration is given by

$$u''(t) = \omega^2 e^{-\beta t} R \cos(\omega_d t - \theta + \varphi)$$
$$R^2 = u_o^2 + \{\beta\, u_o\}/\omega_d\}^2$$
$$\tan\varphi = (-2\beta\, \omega_d)/\omega^2$$

This shows that plots of $u(t)$ and $u(t)''/\omega^2$, will be identical except for a phase shift of φ. This means that acceleration–time plot rather than displacement–time plot can be used to calculate damping present in real structures.

1.15 SUMMARY OF FREE VIBRATION ANALYSIS

The results of free vibration response can be summarised as follows:
i. With zero damping, motion is purely simple harmonic.
ii. When damping is below critical damping, the motion is still periodic but there is amplitude decay.
iii. As damping is increased up to critical damping all vibratory motion is completely damped out.
iv. The factors, which affect the free vibration response are mass M, stiffness K and damping C of the system.
v. Damping present in practical situations is in less than 10% of critical damping. The effect of light damping is to cause amplitude decay leaving the frequency of vibration practically unaltered.
vi. The frequency of vibration in the case of lightly damped system is given by

$$\omega \approx \sqrt{(K/M)},\ f \approx \omega/(2\pi),\ T \approx 2\pi/\omega.$$

vii. Logarithmic decrement δ is given by

$$\delta = \log_e [(u(t)/u(t + T_d)] \approx 2\pi\ (C/C_{cr}).$$

1.16 SYSTEMS SUBJECTED TO HARMONIC EXCITATION

In the previous sections, motion in the absence of external forces was considered. As an introduction to the study of SDOF systems subjected to external force and also to bring out the important concept of resonance, the underdamped system shown in Fig. 1.1 subjected to an external sinusoidal force $F = F_o \sin\Omega t$ will be studied. It is interesting to mention in passing that because a general periodic force can be expressed as a Fourier series in terms of sine and cosine functions, the results obtained in this section are of more general interest than might appear at first glance.

The differential equation to be solved is

$$M\frac{d^2u}{dt^2} + C\frac{du}{dt} + Ku = F(t) = F_0 \sin\Omega t \tag{1.30}$$

The solution to the differential equation (1.30) is the sum of complementary solution (also called natural motion solution or starting transient) and particular integral (also called forced motion solution or steady state solution). In the case of an underdamped system, the complementary solution is given by

Complementary Function = $e^{-\beta t}\{B_1 \cos\omega_d t + B_2 \sin\omega_d t\}$

The particular integral is assumed to be given by

Particular Integral = $\{D_1 \cos\Omega t + D_2 \sin\Omega t\}$

where D_1 and D_2 are constants.

Substituting the P.I. in equation (1.30)

$$-\Omega^2 (D_1 \cos\Omega t + D_2 \sin\Omega t) + 2\beta\Omega(-D_1 \sin\Omega t + D_2 \cos\Omega t)$$
$$+ (D_1 \cos\Omega t + D_2 \sin\Omega t) = (F/M) \sin\Omega t$$

Equating the coefficients of terms in $\sin\Omega t$ and $\cos\Omega t$ we have

$$-2\beta\,\Omega\, D_1 + (\omega^2 - \Omega^2)\, D_2 = F_o/M$$
$$2\,\beta\Omega\, D_2 + (\omega^2 - \Omega^2)\, D_1 = 0$$

Adopting the notation $r = \Omega/\omega$ and since $\beta/\omega = C/C_{cr} = \xi$ and $K/M = \omega^2$, D_1 and D_2 can be expressed as follows

$$D_1 = -\frac{F_o}{K}\frac{2r\xi}{R^2}, \quad D_2 = \frac{F_o}{K}\frac{(1 - r^2)}{R^2}, \; R^2 = (1-r^2)^2 + (2r\xi)^2$$

$$PI = \frac{F_o}{KR^2}\left[(1-r^2)\sin\Omega t - 2r\xi\cos\Omega t\right]$$

This can be simplified further by setting

$$(1 - r^2) = R\cos\theta,\; 2r\xi = R\sin\theta,\; \tan\theta = (2r\xi)/(1 - r^2)$$
$$PI = (F_o/KR)\, si\,(\Omega t - \theta)$$

$$u(t) = e^{-\beta t}\{B_1\cos\omega_d t + B_2\sin\omega_d t\} + (F_o/KR)\sin(\Omega t - \theta) \qquad (1.31)$$

If starting from rest, then

$$B_1 = (Fo/KR)\sin\theta$$
$$B_2 = (Fo/KR)\,[(\beta/\omega_d)\sin\theta - (\Omega/\omega_d)\cos\theta]$$

The complementary solution represents damped free vibration. In the presence of damping, the effect of this is quickly damped out. Remembering that damping equal to only 10% critical damping reduces the amplitude by 50% per cycle, the presence of the $e^{-\beta t}$ term in the complementary solution ensures rapid decay of this part of the solution. This is why the complementary solution is also called starting transient. Once the effect of the starting transient has disappeared, we are left with the particular integral part of the solution. This is why the particular integral is also called steady state solution. Therefore the steady state motion is given by

$$u(t) = (F_o/KR)\sin(\Omega t - \theta) \qquad (1.32)$$

Since the applied force $F = Fo\sin\Omega t$, it is clear that the frequency of vibration is the same as the frequency of the applied force except that the motion lags behind the applied force by $t_o = \theta/\Omega$. It should be noted that since

$$\tan\theta = (2r\xi)\,/\,(1-r^2),\; t_0 = 0 \text{ if } \xi = 0$$

Therefore damping present in the system causes the lag between the displacement and the applied force.

1.17 DYNAMIC MAGNIFICATION FACTOR

The steady state response is given by

$$u(t) = Fo/(KR)]\,\sin\Omega(t - t_o)$$
$$u(t)_{max} = (Fo/KR)$$

Since Fo/K = maximum static deflection Δ_{st}, the maximum dynamic deflection can be expressed as

$$u(t)_{max} = \Delta_{st}/R$$

Therefore the maximum dynamic deflection is 1/R maximum static deflection. 1/R is called dynamic magnification factor (DMF).

$$DMF = \frac{1}{\sqrt{[(1-r^2)^2 + (2r\xi)^2]}} \tag{1.33}$$

where $r = \Omega/\omega$, the ratio of applied to undamped natural frequency of the system.

Evidently DMF is a function of frequency ratio r and damping ratio ξ. Fig. 1.8, shows a plot of DMF versus r for various values of ξ.

For a given value of damping ratio, DMF is a maximum when

$$d(DMF)/dr = 0$$

Carrying out the differentiation and simplifying

$$1 - r^2 - 2\xi^2 = 0,\ r = \sqrt{(1 - 2\xi^2)}$$

For lightly damped system (i.e. $\xi = < 0.10$), the maximum DMF occurs when r is almost equal to 1 and the maximum DMF is equal to $0.5/\xi$. If $\xi = C/C_{cr} = 0.10$, the maximum dynamic displacement is five times the corresponding maximum static displacement.

Figure 1.8 shows that for values of r less than about 0.5, DMF remains fairly near unity. This corresponds to a quasi-static situation and dynamic effects can be safely ignored and only static analysis carried out.

Similarly if r is greater than about 1.5, the dynamic magnification is less than unity. This is the case where the structure is being isolated from the effects of vibration. The structure will show little response to forces pulsating at frequencies above the resonant frequency. The region where it is important to consider dynamic is $0.5 < r < 1.5$.

1.17.1 Response Near Resonance

In the previous section, it was shown that if the frequency of applied force is the same as the frequency of the system, then large displacements can result. In the case of undamped system, the solution to the differential equation is given by

$$u'' + \omega^2 u = (1/M)Fo \sin\Omega t$$

The solution is given by

$$u(t) = \{B_1 \cos\omega t + B_2 \sin\omega t\} + \{D_1 \cos\Omega t + D_2 \sin\Omega t\}$$

If $\omega = \Omega$, then because of the repeated nature of the Complementary Function and Particular Integral part of the solution, the solution is given by

$$u(t) = \{B_1 \cos\omega t + B_2 \sin\omega t\} + t\{D_1 \cos\omega t + D_2 \sin\omega t\}$$

In order to satisfy the differential equation, the values of constants D_1 and D_2 become

$$D_2 = 0,\ D_1 = -F_0/(2M\omega)$$

Therefore

$$u(t) = \{B_1 \cos\omega t + B_2 \sin\omega t\} - F_0/(2M\omega)\ t \cos\Omega t$$

Clearly, because of the presence of the term t, the displacements can become very large. However it also takes time to build up a large amplitude. Hence it is a safe procedure to accelerate a machine through a resonant frequency so long as the normal working frequency is well above the resonant frequency.

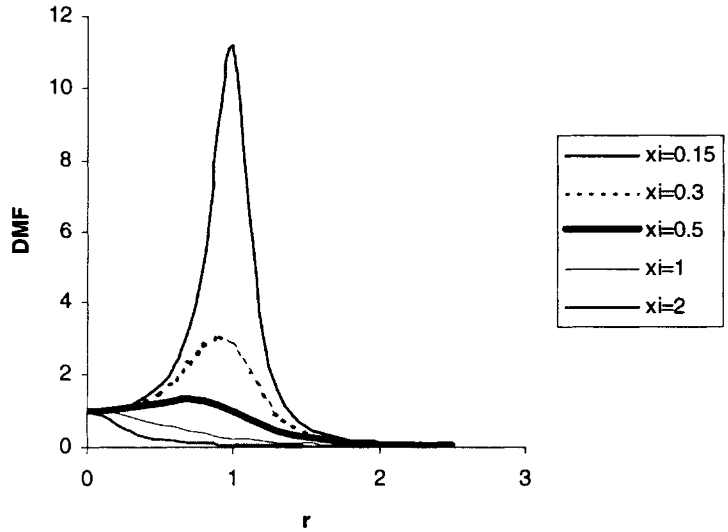

Fig. 1.8 Dynamic magnification factor versus frequency ratio

1.18 RESPONSE TO BASE EXCITATION

In Section 1.17, the SDOF system subjected to an external force was studied. As an introduction to the study of systems subjected to forces arising from the acceleration of the foundation such as that due to seismic disturbance, the SDOF system shown in Fig. 1.1 subjected to foundation movement will be studied. Let the displacement of the base be u_b and the displacement of the mass be u_m. The forces acting on the mass are

a. Inertial force which depends purely on the acceleration of the mass = $-M\,u''_m$

b. Damping force which depends on the relative velocity of the mass with respect to the base = $-C(u'_m - u'_b)$

c. Spring force which depends on the extension of the spring = $-K(u_m - u_b)$

Using D'Alembert's principle, the equation of equilibrium is given by

$$M\,u''_m + C\,(u'_m - u'_b) + K\,(u_m - u_b) = 0 \tag{1.34}$$

Adding - Mu"$_b$ to both sides of the equation

$$M\,(u''_m - u''_b) + C\,(u'_m - u'_b) + K\,(u_m - u_b) = -M\,u''_b$$

Setting $u = u_m - u_b$, the differential equation can be written as

$$Mu'' + Cu' + Ku = -Mu''_b \tag{1.35}$$

where u is the relative displacement of the mass with respect to the foundation. As can be seen, equation (1.35) is identical to equation (1.1) except that $F(t)$ has been replaced by $-Mu_b''$. In other words, the analysis of systems subjected to acceleration of the base is similar to the analysis of systems subjected to external force.

CHAPTER 2

SINGLE DEGREE OF FREEDOM SYSTEM - II

2.1 INTRODUCTION

In Chapter 1, the response of a single degree of freedom (SDOF) system to free and forced vibration under a harmonic force was investigated. It was shown that for lightly damped systems, the frequency is dependent mainly on the mass and stiffness of the system. It was also shown that if the frequency of the applied force is nearly equal to the natural frequency of the system then the system resonates resulting in very large displacements. In this chapter the response of the SDOF system to general loads will be investigated and methods both analytical and numerical will be described for solving the differential equation viz.

$$\frac{d^2u}{dt^2} + 2\beta\frac{du}{dt} + \omega^2 u = \frac{F(t)}{M} \tag{2.1}$$

$\omega^2 = K/M$, $\xi = C/C_{cr}$, $\omega_d = \omega\sqrt{(1 - \xi^2)}$, $\beta = \xi\,\omega$

2.2 LAPLACE TRANSFORM METHOD

The differential equation (2.1) is best solved using the Laplace transform method especially because the forcing function F(t) is, in general, discontinuous and the constants of integration are determined on the basis of initial conditions. As some readers might not be familiar with the method, the procedure is set out in some detail along with some simple examples.

It is perhaps worth reminding the reader that the Laplace transform method is similar to Macaulay's Method (also called Singularity Functions Method) used in the solution of beam deflection problems when the lateral load on the beam is discontinuous.

The Laplace transform $\bar{\Phi}(s)$ of a function $\Phi(t)$ is defined by

$$\bar{\Phi}(s) = \int_0^\infty \Phi(t)\, e^{-st}\, dt \tag{2.2}$$

where s is complex.

2.2.1 Laplace Transform: Examples

The basic step of calculating the Laplace transform is illustrated by a few simple examples.

Example 1: Calculate the Laplace transform of $\Phi(t) = e^{at}$.
Using equation (2.2),

$$\bar{\Phi}(s) = \int_0^\infty \Phi(t)\, e^{-st}\, dt = \int_0^\infty e^{at}\, e^{-st}\, dt = \int_0^\infty e^{-(s-a)t}\, dt$$

$$= -\frac{1}{(s-a)} e^{-(s-a)t} \Big|_0^\infty = \frac{1}{(s-a)}$$

The last step is obtained by noting that e^{-t} tends to zero as t tends to infinity.
As a corollary, if $\Phi = (e^{at} - e^{bt})$, then from the result in Example 1,

$$\bar{\Phi}(s) = \frac{1}{(s-a)} - \frac{1}{(s-b)} = \frac{(a-b)}{(s-a)(s-b)}$$

Example 2: Calculate the Laplace transform of $\Phi(t) = t$.
Using equation (2.2),

$$\bar{\Phi}(s) = \int_0^\infty \Phi(t)\, e^{-st}\, dt = \int_0^\infty t\, e^{-st}\, dt$$

Integrating by parts,

$$\bar{\Phi}(s) = e^{-st}\, [-\frac{t}{s} - \frac{1}{s^2}] \Big|_0^\infty \frac{1}{s^2}$$

2.3 INVERSE LAPLACE TRANSFORM

Examples in Section 2.2.1, showed how to calculate the Laplace transform for a given function. In the solution of differential equations using this method, it is necessary to calculate the original function, if only the transform of the function is given. Unfortunately this ‘Inverse’ process is not straightforward. Fortunately, with the help of ready-made tables, which tabulate for a large number of well-known functions corresponding to transforms, the reverse process can be accomplished.

2.3.1 Inverse Laplace Transform: Examples

A simple example to illustrate the inverse process.

Example 1: Calculate the function $\Phi(t)$, given that the Laplace transform is

$$\bar{\Phi}(s) = \frac{s}{(s-a)(s-b)}$$

Using the concept of partial fractions, the transform can be written as

$$\bar{\Phi}(s) = \frac{s}{(s-a)(s-b)} = \frac{1}{(a-b)}\left[\frac{a}{(s-a)} - \frac{b}{(s-b)}\right]$$

Using the result from Example 1 in Section 2.2.1, the function corresponding to each of the partial fractions can be determined. Therefore

$$\Phi(t) = \frac{1}{(a-b)}\left[a\,e^{at} - be^{bt}\right]$$

2.4 LAPLACE TRANSFORM OF DERIVATIVES

The Laplace transforms of derivatives of $\Phi(t)$ are obtained as follows.

i. First derivative of Φ (t):

The Laplace transform of the first derivative of $\Phi(t)$ is, by definition, given by

$$\bar{\Phi}'(s) = \int_0^\infty \frac{d\Phi(t)}{dt}\, e^{-st}\, dt$$

Integrating by parts

$$\bar{\Phi}'(s) = e^{-st}\,\Phi(t)\,\Big|_0^\infty + s\int_0^\infty \Phi(t)\; e^{-st}\; dt$$

The integral on the right-hand side is equal to $\bar{\Phi}$ and since e^{-st} tends to zero as t approaches infinity

$$\bar{\Phi}'(s) = -\Phi(0) + s\,\bar{\Phi}(s) \tag{2.3}$$

ii Second derivative of Φ (t):

The Laplace transform of the second derivative of $\Phi(t)$ is, by definition, given by

$$\bar{\Phi}''(s) = \int_0^\infty \frac{d^2\Phi(t)}{dt^2}\; e^{-st}\; dt$$

Integrating by parts once

$$\bar{\Phi}''(s) = \Phi'(t)\,e^{-st}\,\Big|_0^\infty + s\int_0^\infty \frac{d\Phi(t)}{dt}\; e^{-st}\; dt$$

The integral on the right-hand side is evidently $\bar{\Phi}'(s)$. Substituting for $\bar{\Phi}'(s)$ and in addition noting that since e^{-st} tends to zero as t approaches infinity

$$\bar{\Phi}''(s) = -\Phi'(0) - s\,\Phi(0) + s^2\,\bar{\Phi}(s) \tag{2.4}$$

2.5 SOLUTION OF DIFFERENTIAL EQUATION

In order to solve the differential equation (2.1), by the Laplace transform method, we take the Laplace transform of both sides of the above equation. Substituting for the Laplace transform of $u'(t)$ and $u''(t)$ in terms of the Laplace transform of $u(t)$ and its initial conditions $u(0)$ and $u'(0)$, we have

$$\{-u'(0) - s\,u(0) + s^2\,\bar{u}(s)\} + 2\beta\,\{-u(0) + s\,\bar{u}(s)\} + \omega^2\,\bar{u}(s) = \frac{\bar{F}(s)}{M}$$

Simplifying

$$\{s^2 + 2\beta\,s + \omega^2\}\bar{u}(s) = u'(o) + (2\beta + s)\,u(0) + \frac{\bar{F}(s)}{M} \tag{2.5}$$

Factorising, $\{s^2 + 2\beta\,s + \omega^2\}$ as (s-a) (s-b), where,

$$a = -\beta + j\,\omega_d\,, \quad b = -\beta - j\,\omega_d\,, j = \sqrt{-1}\,, \omega_d = \sqrt{(\omega^2 - \beta^2)}$$

The Laplace transform of u becomes

$$\bar{u}(s) = u'(o)\,\bar{g}(s) + (2\beta + s)\,u(0)\,\bar{g}(s) + \frac{1}{M}\,\bar{F}(s)\,\bar{g}(s) \tag{2.6}$$

$$\bar{g}(s) = \frac{1}{(s-a)(s-b)}$$

If the system is undamped, then $\beta = 0$ and

$$\bar{g}(s) = \frac{1}{(s^2 - \omega^2)}$$

The solution to u(t) is obtained by taking the inverse transform of both sides of equation (2.6).

2.6 SOME USEFUL RESULTS

Application of Laplace transform for the solution of practical problems is facilitated by the introduction of some special functions and some important 'theorems'. These are discussed in this section.

2.6.1 Unit Step Function

In studying the response of SDOF systems to general loading, the unit step function (also called Heaviside step function) is useful for defining discontinuous loading. As shown in Fig. 2.1, it is defined as follows

$H(t - \tau) = 0$, if $t < \tau$ and $H(t - \tau) = 1$, if $t \geq \tau$ (2.7)

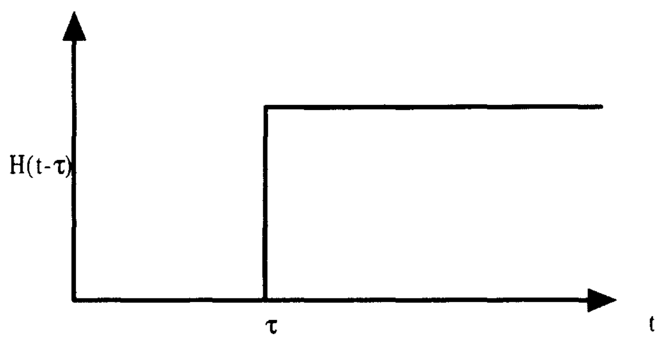

Fig. 2.1 Unit step function

Unit step function is similar to uniformly distributed load in the case of beam problems, where the uniformly distributed load does not start from the origin. Substituting for $\Phi(t) = H(t - \tau)$, the Laplace transform of unit step function is given by

$$\bar{\Phi} = \int_0^\infty H(t-\tau)e^{-st}\,dt = \int_\tau^\infty e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_\tau^\infty = \frac{1}{s}e^{-s\tau} \qquad (2.8)$$

Note that because $H(t - \tau) = 0$, if $t < \tau$, the lower limit of integration changes from 0 to τ.

Note that if an impulse lasts only over the period $t = t_1$ to $t = t_2$, then the corresponding $\Phi(t)$ is given by

$\Phi(t) = H(t - t_1) - H(t - t_2)$

The corresponding Laplace transform is given by

$$\bar{\Phi} = \int_0^\infty \{H(t-t_1) - H(t-t_2)\}e^{-st}\,dt = \frac{1}{s}\{e^{-st_1} - e^{-st_2}\} \quad (2.9)$$

2.6.2 Shift Theorems

There are two useful theorems that facilitate the evaluation of Laplace transforms and their respective inverses.

2.6.2.1 Shift Theorem 1

If $\Phi(t) = \Psi(t)\, e^{at}$, then

$$\bar{\Phi}(s) = \int_0^\infty \Phi(t)\, e^{-st}\, dt = \int_0^\infty [\Psi(t)\, e^{at}]\, e^{-st}\, dt = \int_0^\infty \Psi(t)\, e^{-(s-a)t}\, dt$$

If $\bar{\Psi}(s) = \int_0^\infty \Psi(t)\, e^{-st}\, dt$, then comparing the two expressions, one can see that

$$\bar{\Phi}(s) = \bar{\Psi}(s-a)$$

The theorem can be stated as follows

$$\Phi(t) = \Psi(t)\, e^{at},\ \bar{\Phi}(s) = \bar{\Psi}(s-a) \tag{2.10}$$

As a simple application of Shift Theorem 1, let $\Psi = t^2$. Using the definition for Laplace transform and also successively integrating by parts

$$\bar{\Psi}(s) = \int_0^\infty t^2\, e^{-st}\, dt = -\frac{1}{s} t^2\, e^{-st} \Big|_0^\infty + \frac{2}{s}\int_0^\infty t\, e^{-st}\, dt$$

$$= \frac{2}{s}\Big[-\frac{1}{s} t\, e^{-st} \Big|_0^\infty \Big] + \frac{2}{s^2}\int_0^\infty e^{-st}\, dt$$

$$= \frac{2}{s^2}\Big[\int_0^\infty e^{-st}\, dt \Big] = \frac{2}{s^2}\Big[-\frac{1}{s} e^{-st} \Big|_0^\infty \Big] = \frac{2}{s^3}$$

Therefore if $\Psi(t) = t^2$, then $\bar{\Psi}(s) = \dfrac{2}{s^3}$. From the Shift Theorem 1, the Laplace transform of $t^2 e^{-at}$ is equal to $\dfrac{2}{(s-a)^3}$.

2.6.2.2 Shift Theorem 2

If $\Phi(t) = \Psi(t-a)\, H(t-a)$, then

$$\bar{\Phi}(s) = \int_0^\infty \Phi(t)\, e^{-st}\, dt = \int_0^\infty \Psi(t-a)\, H(t-a)\, e^{-st}\, dt$$

$$= \int_a^\infty \Psi(t-a)\, e^{-st}\, dt$$

Note that in the last integral the lower limit changes from 0 to a. This is because $H(t-a) = 0$ for $t < a$.

$$\bar{\Phi}(s) = \int_0^\infty \Psi(t-a)\, e^{-st}\, dt$$

Substituting $(t - a) = u$, then

$$\bar{\Phi}(s) = e^{-as} \int_0^\infty \Psi(u) \quad e^{-su} \; du \; = \; e^{-as} \; \bar{\Psi}(u)$$

Therefore, the second shift can be stated as follows. If *Φ(t)* is a given function, then the Laplace transform of the product of *Φ(t - a) H(t - a)* is equal to $\bar{\Phi} \; e^{-as}$.

For example, the Laplace transform of t^2 is equal to $\frac{2}{s^3}$. Therefore the Laplace transform of $(t-a)^2 \; H(t-a)$ is given by $\frac{2}{s^3} \; e^{-as}$

2.6.3 Convolution Theorem

Convolution theorem is useful for evaluating the **inverse transform** of the product of the transforms of individual functions. This is particularly important in obtaining the particular integral of ordinary differential equations. The theorem says that if *Φ(t)* and *Ψ(t)* are two functions whose Laplace transforms are, respectively, $\bar{\Phi}$ and $\bar{\Psi}$, then the **inverse transform** of the product $\bar{\Phi} \; \bar{\Psi}$ is given by

$$\int_0^t \Phi(\tau) \; \Psi(t - \tau) \, d\tau \; = \; \int_0^t \Phi(t - \tau) \; \Psi(\tau) \, d\tau$$

The proof of this theorem is quite simple. By definition

$$\bar{\Psi} \; = \; \int_0^\infty \Psi \; e^{-st} dt \qquad \therefore \; \bar{\Phi}\bar{\Psi} \; = \; \bar{\Phi} \int_0^\infty \Psi \; e^{-st} \; dt$$

Note that *Ψ* is a function of t but $\bar{\Phi}$ is a function of s. Taking $\bar{\Phi}$ which is a function of s inside the integral sign

$$\bar{\Phi} \; \bar{\Psi} \; = \; \int_0^\infty \Psi \; [\bar{\Phi} \; e^{-st} \;] dt \tag{2.11}$$

In order to avoid confusion with t later on, we can, in the above integral, change the variable t to u without making any difference. Therefore

$$\bar{\Phi} \; \bar{\Psi} \; = \; \int_0^\infty \Psi \; [\bar{\Phi} \; e^{-su} \;] \; du$$

However from the second shift theorem, $\bar{\Phi} \; e^{-su}$ is equal to the Laplace transform of *Φ(t - u) H(t - u)*. Therefore

$$\bar{\Phi} \; e^{-su} = \; \int_0^\infty \Phi(t - u) \; H(t - u) \; e^{-st} \; dt$$

Substituting the above integral into the original expression for $\bar{\Phi} \; \bar{\Psi}$, we have

$$\bar{\Phi} \; \bar{\Psi} \; = \; \int_0^\infty \Psi \; [\; \int_0^\infty \; \Phi(t - u) \; H(t - u) \; e^{-st} \; dt \;] \; du$$

Reorganizing the above expression as follows (which is permissible in this case), we get

$$\bar{\Phi}\,\bar{\Psi} = \int_0^{\infty}\left[\int_0^{\infty} \Phi(t-u)\, H(t-u)\, \Psi\, du\right] e^{-st}\, dt$$

We notice now that the variable in the inside integral is u If t is the variable, then

$H(t-u) = 0$, if $t < u$ and $H(t-u) = 1$, *if* $t \geq u$ (2.12a)

However if u is treated as the variable, then

$H(t-u) = 0$, if $u > t$ and $H(t-u) = 1$, if $u \leq t$ (2.12b)

Since in the inside integral, u is the variable, using equation (2.12b), the upper limit changes from infinity to t because $H(t - u)$ is zero for all values of $u > t$.

Introducing this change, the expression for the product $\bar{\Phi}\,\bar{\Psi}$ can be written as

$$\bar{\Phi}\,\bar{\Psi} = \int_0^{\infty}\left[\int_0^{t} \Phi(t-u)\, \Psi\, du\right] e^{-st}\, dt$$

This shows that the right-hand side is nothing but the Laplace transform of inner integral inside square brackets. Therefore we can say that the **inverse** Laplace transform of $\bar{\Phi}\,\bar{\Psi}$ is equal to

$$\int_0^{t} \Phi(t-u)\, \Psi\, du$$

In a similar way we can show that the inverse Laplace transform of $\bar{\Phi}\,\bar{\Psi}$ is also equal to

$$\int_0^{t} \Phi\, \Psi(t-u)\, du$$

This result is known as convolution integral theorem and will be used in later sections to derive the particular integral part of the solution to the differential equation.

2.7 SUMMARY OF SOME RESULTS

As already remarked, using a table of functions and their transforms, knowing $\bar{\Phi}(s)$, one can determine the corresponding $\Phi(t)$. Table 2.1 gives some standard results.

The following results (see Example 1 in Section 2.2.1) will be useful in applying Laplace transforms to the solution of the differential equation.

If $a = -\beta + j\,\omega_d$, $b = -\beta - j\,\omega_d$, $(a - b) = 2j\,\omega_d$, then if

(i) $\bar{\Phi}(s) = \dfrac{1}{(s-a)(s-b)}$,

$$\Phi(t) = e^{-\beta t}\frac{[e^{j\omega_d t} - e^{-j\omega_d t}]}{2j\omega_d} = \frac{e^{-\beta t}}{\omega_d}\sin \omega_d t$$

If $\beta = 0$ for an undamped system, then

$$\Phi(t) = \frac{1}{\omega}\sin\omega t$$

(ii) $\bar{\Phi}(s) = \dfrac{s}{(s-a)(s-b)}$

$$\Phi(t) = e^{-\beta t}\frac{[(-\beta + j\,\omega_d)e^{j\omega_d t} - (-\beta - j\,\omega_d)e^{-j\omega_d t}]}{2j\omega_d}$$

$$\Phi(t) = e^{-\beta t}[\frac{e^{j\omega_d t} + e^{-j\omega_d t}}{2} - \frac{\beta}{\omega_d}\frac{e^{j\omega_d t} - e^{-j\omega_d t}}{2j}] = e^{-\beta t}[\cos\,\omega_d t - \frac{\beta}{\omega_d}\sin\,\omega_d t]$$

If $\beta = 0$ for an undamped system, then

$$\Phi(t) = \cos\omega\,t$$

(iii) Shift Theorem 1: $\Phi(t) = \Psi(t)\,e^{at}$, $\bar{\Phi}(s) = \bar{\Psi}(s-a)$

(iv) Shift Theorem 2: The Laplace transform of $\Phi(t - a)\,H(t - a) = \bar{\Phi}\ e^{-as}$

(v) Convolution Theorem: The **inverse transform** of the product $\bar{\Phi}\ \bar{\Psi}$ is given by $\int_0^t \Phi(\tau)\,\Psi(t-\tau)d\tau$ or $\int_0^t \Phi(t-\tau)\,\Psi(\tau)d\tau$

Table 2.1 Laplace transforms

Function	Transform
$\frac{1}{(a-b)}[e^{at} - e^{bt}]$	$\frac{1}{(s-a)(s-b)}$
$\frac{1}{(a-b)}[ae^{at} - be^{bt}]$	$\frac{s}{(s-a)(s-b)}$
H(t - τ)	$\frac{e^{-\tau s}}{s}$
F(t - τ).H(t - τ)	$e^{-\tau s}\,\bar{F}(s)$
$\int_0^t \Phi(\tau)\,g(t-\tau)\,d\tau$	$\bar{\Phi}(s)\ \bar{g}(s)$
$\frac{t^n}{n!}$	$\frac{1}{s^{(n+1)}}$

2.8 GENERAL SOLUTION OF DUHAMEL INTEGRAL

Using equation (2.6) and the results of Sections 2.7, the solution to the differential equation (2.1) can be written as

$$u(t)=e^{-\beta t}\{\cos\omega_d t+\frac{\beta}{\omega_d}\sin\omega_d t\}\,u(0)+e^{-\beta t}\{\frac{1}{\omega_d}\sin\omega_d t\}\,u'(0)$$

$$+\text{ Inverse transform of }[\frac{1}{M}\bar{F}(s)\ \bar{g}(s)]$$

where

$$\bar{g}(s)=\frac{1}{(s-a)(s-b)},\quad a=-\beta+j\omega_d\ ,\ b=-\beta-j\omega_d$$

$$g(t)=e^{-\beta t}\{\frac{1}{\omega_d}\sin\omega_d t\},\quad g(t-\tau)=e^{-\beta(t-t)}\frac{1}{\omega_d}\sin\omega_d(t-\tau)$$

From the Convolution theorem (Section 2.6.3), the **inverse** Laplace transform of $\bar{\Phi}(s)\ \bar{g}(s)$ is $\int_0^t\Phi(\tau)\,g(t-\tau)\,d\tau$. Therefore the inverse Laplace transform of $\frac{1}{M}\bar{F}(s)\,\bar{g}(s)$ is

$$\frac{1}{M}\int_0^t F(\tau)\,g(t-\tau)\,d\tau=\frac{1}{M\,\omega_d}\int_0^t F(\tau)\ e^{-\beta(t-\tau)}\sin\omega_d(t-\tau)\ d\tau$$

The solution to the differential equation (2.1) is

$$u(t)=e^{-\beta t}\{\cos\omega_d t+\frac{\beta}{\omega_d}\sin\omega_d t\}\ u(0)+e^{-\beta t}\{\frac{1}{\omega_d}\sin\omega_d t\}\,u'(0)$$

$$+\ \frac{1}{M\,\omega_d}\int_0^t F(\tau)\ e^{-\beta(t-\tau)}\sin\omega_d(t-\tau)\ d\tau \tag{2.13}$$

In many cases it is simpler to operate on the above integral as it is. However, using the relationship for compound angles,

$$\sin\omega_d(t-\tau)=\sin\omega_d t\,\cos\omega_d\tau-\cos\omega_d t\,\sin\omega_d\tau$$

$$\frac{1}{M}\int_0^t F(\tau)\ g(\tau-t)\ d\tau=\frac{1}{M\ \omega_d}\int_0^t F(\tau)\ e^{-\beta(t-\tau)}\ \sin\ \omega_d(t-\tau)\ d\tau$$

$$=\ \frac{e^{-\beta t}}{M\ \omega_d}\int_0^t F(\tau)\ e^{\beta\tau}\ [\sin\ \omega_d t\ \cos\omega_d\tau\ -\ \cos\ \omega_d t\ \sin\omega_d\tau\ ]d\tau$$

$$=\frac{e^{-\beta t}}{M\ \omega_d}[\sin\omega_d\,t\int_0^t F(\tau)\,e^{\beta\tau}\ \cos\omega_d\,\tau\,d\tau\ -\ \cos\omega_d\,t\int_0^t F(\tau)\ e^{\beta\tau}\ \sin\omega_d\tau\ d\tau] \tag{2.14}$$

The above integrals are commonly known as Duhamel integrals. This representation is generally convenient for numerical evaluation of the integral.